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WIMAN'S TYPE INEQUALITY FOR ANALYTIC AND ENTIRE FUNCTIONS AND h-MEASURE OF AN EXCEPTIONAL SETS

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Let $\mathcal{E}_R$ be the class of analytic functions f represented by power series of the form $f(z)=\sum\limits_{n=0}^{+\infty}a_nz^n$ with the radius of convergence $R:=R(f)\in(0;+\infty]$. For $r\in[0,R)$ we denote the maximum modulus by $M_f(r)=\max\{|f(z)|\colon\ |z|=r\}$ and the maximal term of the series by $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$. We also denote by $\mathcal{H}_R$, $R\leq+\infty$, the class of continuous positive functions, which increase on $[0;R)$ to $+\infty$, such that $h(r)\geq 2$ for all $r\in(0,R)$ and $\int_{r_0}^R h(r)d\ln r=+\infty$ for some $r_0\in(0,R)$. In particular, the following statements are proved.

1^0. If $h\in\mathcal{H}_R$ and $f\in\mathcal{E}_R$, then for any $\delta>0$ there exist $E(\delta,f,h):=E\subset(0,R)$, $r_0\in(0,R)$ such that

$$\forall\, r\in(r_0,R)\backslash E\colon\ M_f(r)\leq h(r)\mu_f(r)\big\{\ln h(r)\ln(h(r)\mu_f(r))\big\}^{1/2+\delta}\quad\text{and}\quad\int_E h(r)d\ln r<+\infty.$$

2^0. If we additionally assume that the function $f\in\mathcal{E}_R$ is unbounded, then

$$\ln M_f(r)\leq(1+o(1))\ln(h(r)\mu_f(r))$$

holds as $r\to R$, $r\notin E$.

Remark, that assertion 1^0 at $h(r)\equiv$ const implies the classical Wiman-Valiron theorem for entire functions and at $h(r)\equiv 1/(1-r)$ theorem about the Kövari-type inequality for analytic functions in the unit disc. From statement 2^0 in the case that $\ln h(r)=o(\ln\mu_f(r))$, $r\to R$, it follows that $\ln M_f(r)=(1+o(1))\ln\mu_f(r)$ holds as $r\to R$, $r\notin E$.

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1 INTRODUCTION

Let $\mathcal{E}_R$, $0<R\leq+\infty$, be the class of analytic functions in $\mathbb{D}_R=\{z\in\mathbb{C}\colon |z|<R\}$ of the form

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n. \tag{1}$$

In particular, $\mathcal{E}:=\mathcal{E}_{+\infty}$ is the class of entire functions. We denote by $M_f(r)=\max\{|f(z)|\colon |z|=r\}$ and $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$, $r\in[0,R)$, the maximum of the modulus and maximal term of series (1), respectively. By the classical theorem of A. Wiman and G. Valiron we have (see [1, 16, 18, 19]) that for each entire function $f\in\mathcal{E}$ of the form (1), for every $\varepsilon>0$

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there exists a set $E \subset [1;+\infty)$ of finite logarithmic measure (i.e. $\int_E d\ln r < +\infty$) such that $\forall\, r \in (1,+\infty)\backslash E$:

$$M_f(r) \leq \mu_f(r)\ln^{1/2+\delta}\mu_f(r). \tag{2}$$

Regarding the statement about the Wiman inequality, Prof. I.V. Ostrovskii in 1995 formulated the following problem: *what is the best possible description of the value of an exceptional set E*? Later, the same issue was considered of a number of articles (for example, see [2,5–7,11,13–16]) in relation to many other relations obtained in the Wiman-Valiron theory.

We denote by $\mathcal{H}_R$ the class of continuous positive increasing to $+\infty$ on $[0;R), R \leq +\infty$, functions such that $h(r) \geq 2, \forall r \in (0,R)$, and

$$\int_{r_0}^{R} \frac{h(r)}{r} dr = +\infty$$

for some $r_0 \in (0,R)$. We also denote $\mathcal{H} = \mathcal{H}_{+\infty}$. The following theorem from [16] complements the classical statement on the Wiman inequality.

Theorem A ([16]). *For every functions $f \in \mathcal{E}$ and $h \in \mathcal{H}$ such that $\ln_2^+ h(r) = o(\ln_2 \mu_f(r))$, $r \to +\infty$, and for each $\varepsilon > 0$ there exists a set $E = E(\varepsilon, f, h) \subset [1;+\infty)$ of finite h-logarithmic measure (i.e. h-meas $E := \int_E h(r) d\ln r < +\infty$) such that $\forall\, r \in (1,+\infty)\backslash E$:*

$$M_f(r) \leq h(r)\mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}. \tag{3}$$

In the article [11], Theorem A was proved for a function h such that $h(r) \leq \ln r$, $r \geq r_0$, and with the factor $\ln r$ instead $h(r)$ in inequality (3). Note, that the condition $h(r) \leq \ln r$, $r \geq r_0$, implies $\ln_2^+ h(r) = o(\ln_2 \mu_f(r))$, $r \to +\infty$.

From Theorem A it follows also such statement.

Proposition 1. *If the functions $f \in \mathcal{E}$ and $h \in \mathcal{H}$ are such that $\ln^+ h(r) = o(\ln_2 \mu_f(r))$, $r \to +\infty$, then for every $\varepsilon > 0$ inequality* (2) *holds for all $r \in [1;+\infty) \setminus E(\varepsilon, f, h)$, where a set $E(\varepsilon, f, h) := E$ has finite h-logarithmic measure, i.e. h-meas $E < +\infty$.*

From the example of the entire function constructed in [11] (see also [16]) it follows that the description given in Proposition 1 of an exceptional set for the fixed entire function f is essential cannot be improved. Actually [11,16], for every $\varepsilon > 0$ there exist an entire function f and a set $E \subset [1,+\infty)$ such that for all $r \in E$

$$f(r) \geq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon} \quad \text{and} \quad \int_E \frac{(\ln \mu_f(r))^{1/2+\varepsilon}}{r} dr = +\infty.$$

On the other hand, in the paper [16] it was established also that the estimate

$$\int_E \frac{\ln^{1/2} \mu_f(r)}{r} dr < +\infty$$

of an exceptional set E in the Wiman's inequality (2) is fulfilled in some sense almost surely.

Moreover, it is known about exceptional sets in analogues of the Wiman inequality for power series of form (1) with radius of convergence equal to one. In this article, we will obtain some new estimates for the magnitude of the exceptional set, and in this case.

For every analytic function $f \in \mathcal{E}_1$ of form (1) there exists a set $E_f(\delta) \subset (0,1)$ of finite logarithmic measure on $(0,1)$, i.e.

$$\int_{E_f(\delta)} \frac{dr}{1-r} < +\infty,$$

such that for all $r \in (0,1) \setminus E_f(\delta)$ the inequality

$$M_f(r) \le \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}$$

holds ([17]). Similar inequality for analytic functions $f \in \mathcal{E}_1$ one can find in [3].

Also in [17] it is noted that for the function $g(z) = \sum\limits_{n=1}^{+\infty} \exp\{n^{\varepsilon}\} z^n$, $\varepsilon \in (0,1)$, we have

$$\varliminf_{r \to 1-0} \frac{M_g(r)}{\frac{\mu_g(r)}{1-r} \ln^{1/2} \frac{\mu_g(r)}{1-r}} \ge C > 0.$$

In [12], it was proved that by some additional conditions inequality (2) holds outside a some exceptional set for every analytic function of the form (1) in the unit disc.

The purpose of this note is to obtain for analytic functions an analogue of Theorem A from which, as a consequence, we obtain assertions about Wiman-type inequalities that take place outside exceptional sets of finite h-measure, without any additional conditions.

2 AUXILLARY PROPOSITIONS

Let $\mathcal{W}$ be a class an positive continuous increasing on $[0,+\infty)$ functions $\psi(x)$ such that

$$\int_{x_0}^{+\infty} \frac{dx}{\psi(x)} < +\infty$$

for some $x_0 \in (0,+\infty)$.

We need the following lemmas.

Lemma 1. *Let $T \in (t_0,+\infty]$, $t_0 \in (-\infty,+\infty)$, and $g_0(x)$ be positive differentiable non-decreasing on (t_0,T) function, $\psi \in \mathcal{W}$, and $h_0(x)$ be positive local integrable on $[t_0;T)$ function such that $\int_{t_0}^{T} h_0(x)\,dx = +\infty$. Then there exists a set $E_0 \subset [t_0;T)$ such that $\int_{E_0} h_0(x)\,dx < +\infty$ and*

$$\forall x \in [t_0;T) \setminus E_0\colon \quad g_0'(x) \le h_0(x)\psi(g_0(x)).$$

The proof of Lemma 1 is carried out by verbatim repetition of the arguments from article [8] (see also [4,7,9]). Indeed, denote

$$E_0 = \{x \in (t_0,T)\colon g_0'(x) > h_0(x)\psi(g_0(x))\}.$$

Then

$$\int_{E_0} h_0(x)dx \le \int_{E_0} \frac{g_0'(x)}{\psi(g_0(x))}dx \le \int_{E_0} \frac{dg_0(x)}{\psi(g_0(x))} \le \int_{g_0(E_0)\cap[0,x_0)} \frac{du}{\psi(u)} + \int_{x_0}^{+\infty} \frac{du}{\psi(u)} < +\infty.$$

Lemma 2. *Let $h \in \mathcal{H}_R$, and f be any analytic function represented by power series of form* (1) *with radii of convergence $R(f) = R \in (0, +\infty]$, and $r_0 \in (0, R)$ be such that* $\ln \mathfrak{M}(r_0) \geq 10$. *Then, there exists a set $E_0 := E(f, h) \subset (0, R)$ such that*

$$\forall\, r \in (r_0, R) \setminus E_0\colon\ \ln \mathfrak{M}_f(r) \leq 2 \ln \big(h(r)\mu_f(r)\big) \text{ and } h\text{-meas } E_0 := \int_E h(r) d \ln r < +\infty,$$

where $\mathfrak{M}_f(r) = \sum\limits_{n=0}^{+\infty} |a_n| r^n$.

Proof. Denote $g(x) = \mathfrak{M}_f(e^x)$, $g_1(x) = \ln g(x)$, $x = \ln r < \ln R$. Remark, that $g_1'(x) = \frac{g'(x)}{g(x)}$. Thus for $a = 2g_1'(x)$ we obtain

$$g(x) - \sum_{n \leq a} |a_n| e^{xn} = \sum_{n > a} |a_n| e^{xn} \leq \frac{1}{a} \sum_{n > a} n |a_n| e^{xn} \leq \frac{g'(x)}{a} = \frac{1}{2} g(x).$$

Hence,

$$g(x) \leq 2 \sum_{n \leq a} |a_n| e^{xn} \leq 2 g_1'(x) \mu_f(r).$$

By Lemma 1 with $g_0(x) = g_1(x)$, $h_0(x) = h(e^x)$, $\psi(x) = x(\ln x)^2$, there exist a set $E \subset (-\infty, \ln R)$ and $t_0 < \ln R$ such that $g_1'(x) \leq h_0(x) g_1(x) (\ln g_1(x))^2$ for all $x \in (t_0, \ln R) \setminus E$ and $\int_E h_0(x) dx < +\infty$. Therefore, $\mathfrak{M}_f(r) = g(\ln r) \leq 2\mu_f(r) h(r) g_1(\ln r)(\ln g_1(\ln r))^2$. Consequently,

$$\ln \mathfrak{M}_f(r) \leq \ln 2 + \ln \big(\mu_f(r) h(r)\big) + \ln g_1(\ln r) + 2 \ln \ln g_1(\ln r). \tag{4}$$

Since $x - \ln x - 2 \ln \ln x - \ln 2 \geq x/2$ for all $x \geq 10$, then

$$g_1(\ln r) - \ln 2 - \ln g_1(\ln r) - \ln \ln g_1(\ln r) \geq \frac{1}{2} \ln \mathfrak{M}_f(r).$$

Statement of Lemma 2 is proved, because

$$\int_{E_0} h(r) d \ln r = \int_E h(e^x) dx < +\infty,$$

where the set E_0 is the image of the set E by the mapping $r = e^x$. □

If in addition to the conditions of Lemma 2 we assume that the function f is unbounded, then we obtain the following proposition.

Proposition 2. *Let $h \in \mathcal{H}_R$ and $f \in \mathcal{E}_R$ be any unbounded analytic function represented by power series of form* (1) *with radii of convergence $R(f) = R \in (0, +\infty]$. Then, there exists a set $E_0 := E(f, h) \subset (0, R)$ such that*

$$\ln \mathfrak{M}_f(r) \leq (1 + o(1)) \ln \big(h(r)\mu_f(r)\big) \tag{5}$$

holds true as $r \to R$, $r \notin E_0$, where E_0 is the set from Lemma 2.

Indeed, by the condition that f is unbounded it follows $\mathfrak{M}_f(r) \to +\infty$, $r \to R - 0$, hence

$$\ln 2 + \ln g_1(\ln r) + 2 \ln \ln g_1(\ln r) = o\big(\ln \mathfrak{M}_f(r)\big)$$

as $r \to R$, because $g_1(\ln r) = \ln \mathfrak{M}_f(r)$. Therefore inequality (4) implies that relation (5) holds as $r \to R$, $r \notin E$, where the set E is the set from Lemma 2.

Remark 1. *If the functions $h \in \mathcal{H}_R$ and $f \in \mathcal{E}_R$ are such that $\ln h(r) = o(\ln \mu_f(r))$, $r \to R - 0$, then Proposition 2 follows*

$$\ln M_f(r) = (1 + o(1)) \ln \mu_f(r)$$

as $r \to R - 0$, $r \notin E$, h-meas$E < +\infty$. In particular, as a corollary in the case $R = +\infty$, we obtain one statement from article [2] (see also [10, p.58]).

3 MAIN RESULT

Theorem 1. *Let $h \in \mathcal{H}_R$, $R \in (0,+\infty]$. For every analytic function $f \in \mathcal{E}_R$, for every functions $\psi_j \in \mathcal{W}$, $j \in \{1,2\}$, and for any $\delta > 0$ there exist a set $E := E(\delta, f, h) \subset (0,R)$ and $r_0 \in (0,R)$ such that h-meas$E = \int_E h(r)d\ln r < +\infty$ and*

$$\forall\, r \in (r_0, R) \setminus E: \quad \mathfrak{M}_f(r) \le \mu_f(r)\sqrt{h(r)\psi_2\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)},$$

in particular,

$$\forall\, r \in (r_0, R) \setminus E: \quad \mathfrak{M}_f(r) \le h(r)\mu_f(r)\big(\ln h(r)\ln(h(r)\mu_f(r))\big)^{1/2+\delta}.$$

Remark 2. *It is easy to see that the assertion of Theorem A follows from Theorem 1.*

Proof of Theorem 1. We first repeat the reasoning from the proof of Theorem 2 in [16]. Let again $g(x) = \sum\limits_{n=0}^{\infty} |a_n|e^{nx}$, $x < \ln R$, and ξ be a discrete random variable with distribution

$$\mathbb{P}\{\xi = n\} = \frac{1}{g(x)}|a_n|e^{nx}, \quad n \ge 0.$$

Then we have the mean $\mathbf{M}\xi = g_1'(x)$ and the variance $\mathbf{D}\xi = g_1''(x)$, where $g_1(x) = \ln g(x)$.

By Bienayme-Chebyshev inequality we obtain (see also [4,8,9])

$$\mathbb{P}\big\{|\xi - g_1'(x)| < \sqrt{2g_1''(x)}\big\} = \mathbb{P}\big\{|\xi - \mathbf{M}\xi| < \sqrt{2\mathbf{D}\xi}\big\} \ge 1 - \frac{\mathbf{D}\xi}{\big(\sqrt{2\mathbf{D}\xi}\big)^2} = \frac{1}{2},$$

i.e.

$$g(x) \le 2g(x)\mathbb{P}\big\{|\xi - g_1'(x)| < \sqrt{2g_1''(x)}\big\} = 2\sum_{|n-g_1'(x)|<\sqrt{2g_1''(x)}} |a_n|e^{xn}.$$

Hence, at $\ln r = x < \ln R$ we have (see also [16, (7) on p.16])

$$g(x) \le 2\mu_f(r)\big(2\sqrt{2g_1''(x)} + 1\big).$$

Without of the loss of generality we can assume that $\psi_2(x) \ge 1$ for $x \ge 1$. We denote the sets

$$E_1 = \{x < \ln R : g_1''(x) > 2^{-7}h(e^x)\psi_2(g_1'(x)),\ g_1'(x) \ge 1\},$$

$$E_2 = \{x < \ln R : g_1'(x) > h(e^x)\psi_1(g_1(x)/2),\ g_1(x) \ge 1\}.$$

Applying Lemma 1 twice, with $\psi_0(t) = 2^{-7}\psi_2(t)$, $h_0(x) = h(e^x)$ and $g_0(x) = g_1'(x)$ in the first case, and with $\psi_0(t) = \psi_1(t/2)$, $h_0(x) = h(e^x)$ and $g_0(x) = g_1(x)$ in the second case, we obtain

$$\int\limits_{E_1\cup E_2} h(e^x)dx = \int\limits_{E_1\cup E_2} h_0(x)dx \le \int\limits_{E_1} h_0(x)dx + \int\limits_{E_2} h_0(x)dx < +\infty.$$

By E we denote the image of the set $E_1 \cup E_2$ under the mapping $r = e^x$. Then

$$h\text{-meas}\, E = \int\limits_E \frac{h(r)}{r}dr = \int\limits_{E_1\cup E_2} h(e^x)dx < +\infty.$$

Now from definitions of the sets E_j by using of inequality from Lemma 2 we get

$$\frac{\mathfrak{M}_f(r)}{2\mu_f(r)} \leq \left(2\sqrt{2g_1''(x)}+1\right) \leq \frac{1}{4}\sqrt{h(r)\psi_2\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)}+1$$
$$\leq \frac{1}{2}\sqrt{h(r)\psi_2\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)}$$

for all $r \in (r_0, R) \setminus (E \cup E_0)$. This complete the first part of Theorem 1. To prove the second part we put $\psi(t) = \psi_2(t) = t\ln^{1+\delta} t$. Then

$$\begin{aligned}
\mathfrak{M}_f(r) &\leq \mu_f(r)\sqrt{h(r)\psi_2\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)}\\
&\leq \mu_f(r)\sqrt{h(r)\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)\ln^{1+\delta}\Big(h(r)\psi_1\big(\ln\big(\mu_f(r)h(r)\big)\big)\Big)}\\
&\leq h(r)\mu_f(r)\ln^{1/2}\big(\mu_f(r)h(r)\big)\ln^{1/2+\delta/2}\big(\ln\big(\mu_f(r)h(r)\big)\big)\\
&\quad\times\ln^{1/2+\delta/2}\Big(h(r)\big(\ln^2\big(\mu_f(r)h(r)\big)\big)\Big)\\
&\leq h(r)\mu_f(r)\ln^{1/2}\big(\mu_f(r)h(r)\big)\ln^{1/2+\delta/2}\big(\ln\big(\mu_f(r)h(r)\big)\big)\\
&\quad\times\max\Big\{\ln^{1/2+\delta/2}h(r), 2\ln^{1/2+\delta/2}\ln\big(\mu_f(r)h(r)\big)\Big\}\\
&\leq h(r)\mu_f(r)\big(\ln h(r)\ln(h(r)\mu_f(r))\big)^{1/2+\delta}.
\end{aligned}$$

□

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Скасків О.Б., Куриляк А.О. *Нерівність типу Вімана для аналітичних та цілих функцій і h-міра виняткових множин* // Карпатські матем. публ. — 2020. — Т.12, №2. — С. 492–498.

Нехай $\mathcal{E}_R$ — клас аналітичних функцій f, представлених степеневими рядами вигляду $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ з радіусом збіжності $R := R(f) \in (0; +\infty]$. Для $r \in [0, R)$ через $M_f(r) = \max\{|f(z)|\colon\ |z| = r\}$ та $\mu_f(r) = \max\{|a_n|r^n\colon n \geq 0\}$ відповідно позначимо максимум модуля і максимальний член степеневого ряду. Через $\mathcal{H}_R$, $R \leq +\infty$, також позначимо клас неперервних додатних функцій, що зростають на інтервалі $[0; R)$ до $+\infty$ і таких, що $h(r) \geq 2$ для всіх $r \in (0, R)$ і $\int_{r_0}^{R} h(r) d\ln r = +\infty$ для деякого $r_0 \in (0, R)$. Доведено, зокрема, такі твердження.

1^0. Якщо $h \in \mathcal{H}_R$ і $f \in \mathcal{E}_R$, то для довільного $\delta > 0$ існують $E(\delta, f, h) := E \subset (0, R)$, $r_0 \in (0, R)$, такі що

$$\forall\, r \in (r_0, R) \backslash E\colon\ M_f(r) \leq h(r)\mu_f(r)\big\{\ln h(r) \ln(h(r)\mu_f(r))\big\}^{1/2+\delta} \quad \text{та} \quad \int_E h(r) d\ln r < +\infty.$$

2^0. Якщо додатково припустити, що функція $f \in \mathcal{E}_R$ необмежена, то співвідношення

$$\ln M_f(r) \leq (1 + o(1)) \ln(h(r)\mu_f(r))$$

виконується при $r \to R$, $r \notin E$.

Зауважимо, що з твердження 1^0 при $h(r) \equiv \mathrm{const}$ випливає класична теорема Вімана-Валірона для цілих функцій, а при $h(r) \equiv 1/(1-r)$ — теорема про нерівність типу Кеварі для аналітичних функцій в одиничному крузі. З твердження 2^0 у випадку, коли $\ln h(r) = o(\ln \mu_f(r))$, $r \to R$, отримуємо, що співвідношення $\ln M_f(r) = (1 + o(1)) \ln \mu_f(r)$ виконується при $r \to R$, $r \notin E$.

Ключові слова і фрази: нерівність Вімана, аналітична функція, максимум модуля, максимальний член, виняткова множина, h-міра.