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RAHMAN SH.

GEOMETRY OF HYPERSURFACES OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD

The purpose of the paper is to study the notion of CR-submanifold and the existence of some structures on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold. We study the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of Goldberg S.I., Yano K. [6]. We discuss the integrability of distributions on M and geometry of their leaves. We have tries to relate this result with those before obtained by Goldberg V., Rosca R. devoted to Sasakian manifold and conformal connections.

Key words and phrases: CR-submanifold, quasi-Sasakian manifold, quarter symmetric non metric connection, integrability conditions of the distributions.

Department of Mathematics, Maulana Azad National Urdu University Polytechnic Darbhanga (Centre), 846001, Bihar, India
E-mail: shamsur@rediffmail.com

INTRODUCTION

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are respectively given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [5] S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\varphi X - u(X)\varphi Y,$$

where u is a 1-form and φ is a tensor field of type $(1,1)$. Some properties of quarter symmetric connections are studied in [7]. In [8, 9] S. Rahman studied Transversal hypersurfaces of almost hyperbolic contact manifolds with a quarter symmetric non metric connections respectively.

The concept of CR-submanifold of a Kahlerian manifold has been defined by A. Bejancu [3]. Later A. Bejancu, N. Papaghiue [4] introduced and studied the notion of semi-invariant submanifold of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kahlerian manifold. However the existence of the structure vector field implies some important changes.

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The paper is organized as follows. In the first section we recall some results and formulae for the later use. In the second section we prove the existence of a Kahler structure on M and the existence of a globally metric frame f-structure in sence of S.I. Goldberg, S.I. Yano. The third section is concerned with integrability of distributions on M and geometry of their leaves. In section 4 the study of conformal connections with respect to the quarter symmetric non metric connection in a quasi-Sasakian manifold is considered.

1 PRELIMINARIES

Let $\bar{M}$ be a real $2n+1$ dimensional differentiable manifold, endowed with an almost contact metric structure (f, ξ, η, g). Then we have

$$\begin{aligned} &(a)\ f^2 = -I + \eta \otimes \xi, \quad (b)\ \eta(\xi) = 1, \quad (c)\ \eta \circ f = 0, \quad (d)\ f(\xi) = 0, \\ &(e)\ \eta(X) = g(X, \xi), \quad (f)\ g(fX, fY) = g(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{1}$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $F(\bar{M})$ the algebra of differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of sections of a vector bundle E over $\bar{M}$.

The Niyembuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] + f[X, fY]$$

for all $X, Y \in \Gamma(T\bar{M})$ and the fundamental 2-form Φ is given by $\Phi(X, Y) = g(X, fY)$ for all $X, Y \in \Gamma(T\bar{M})$. The curvature tensor field of $\bar{M}$, denoted by $\bar{R}$ with respect to the Levi-Civita connection $\bar{\nabla}$, is defined by $\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z$ for all $X, Y, Z \in \Gamma(T\bar{M})$.

Definition 1. *(a) An almost contact metric manifold $\bar{M}$ (f, ξ, η, g) is called normal if*

$$N_f(X, Y) + 2d\eta(X, Y)\xi = 0 \qquad \text{for all} \quad X, Y \in \Gamma(T\bar{M}),$$

or equivalently ([1]) $(\bar{\nabla}_{fX} f)Y = f(\bar{\nabla}_X f)Y - g((\bar{\nabla}_X \xi, Y)$ for all $X, Y \in \Gamma(T\bar{M})$.
(b) The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Phi = d\eta = 0$.

Let $\bar{M}$ be an almost contact metric manifold $\bar{M}$. According to [1] we say that $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field and

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi \qquad \text{for all} \quad X, Y \in \Gamma(T\bar{M}). \tag{2}$$

Next we define a tensor field F of type $(1, 1)$ by $FX = -\bar{\nabla}_X \xi$ for all $X \in \Gamma(T\bar{M})$.

Lemma 1. *Let $\bar{M}$ be a quasi-Sasakian manifold. Then for all $X, Y \in \Gamma(T\bar{M})$ we have*

$$\begin{aligned} &(a)\ (\bar{\nabla}_\xi f)X = 0, \quad (b)\ f \circ F = F \circ f, \quad (c)\ g(FX, Y) + g(X, FY) = 0, \\ &(d)\ F\xi = 0, \quad (e)\ \eta \circ F = 0, \quad (f)\ (\bar{\nabla}_X F)Y = \bar{R}(\xi, X)Y. \end{aligned} \tag{3}$$

The tensor field f defined on $\bar{M}$ is an f-structure in sense of Yano that is $f^3 + f = 0$.

Definition 2. *The quasi-Sasakian manifold $\bar{M}$ is said to be of rank $2p+1$ iff*

$$\eta \wedge (d\eta)^p \neq 0 \quad \text{and} \quad (d\eta)^{p+1} = 0.$$

On other hand, a quarter symmetric non metric connection ∇ on M is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X. \tag{4}$$

Using (4) in (2), we have

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi + \eta(Y)X - \eta(X)\eta(Y)\xi, \tag{5}$$

$$\bar{\nabla}_X \xi = -FX + fX. \tag{6}$$

Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^{\perp}$ the normal vector bundle to M. The Gauss and Weingarten formulas of hypersurfaces of a quarter symmetric non metric connections are

$$(a)\ \bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad (b)\ \bar{\nabla}_X N = -AX, \tag{7}$$

where A is the shape operator with respect to the section N. It is known that for all $X, Y \in \Gamma(TM)$

$$B(X,Y) = g(AX,Y). \tag{8}$$

Because the position of the structure vector field with respect to M is very important we prove the following result.

Theorem 1. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the structure vector field ξ is normal to M then $\bar{M}$ is cosympletic manifold and M is totally geodesic immersed in $\bar{M}$.*

Proof. Because $\bar{M}$ is quasi-Sasakian manifold, then it is normal and $d\Phi = 0$ ([2]). By direct calculation using (7) (b), we infer for all $X, Y \in \Gamma(T\bar{M})$

$$\begin{aligned} d\eta(X,Y) &= \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} = \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\}, \\ 2d\eta(X,Y) &= g(AY,X) - g(AX,Y) = 0. \end{aligned} \tag{9}$$

From (7) (b) and (9) we deduce for all $X, Y \in \Gamma(T\bar{M})$

$$\begin{aligned} 0 = d\eta(X,Y) &= \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} \\ &= \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\} = g(Y, \bar{\nabla}_X\xi) = -g(AX,Y) = 0, \end{aligned} \tag{10}$$

which proves that M is totally geodesic. From (10) we obtain $\bar{\nabla}_X\xi = 0$ for all $X \in \Gamma(T\bar{M})$. By using (6), (3) (b) and (1) (d) from the above relation we state for all $X \in \Gamma(T\bar{M})$

$$-f(\bar{\nabla}_{fX}\xi) + fX = \bar{\nabla}_X\xi, \tag{11}$$

because $fX \in \Gamma(T\bar{M})$ for all $X \in \Gamma(T\bar{M})$. Using (11) and the fact that ξ is a not Killing vector field, we deduce $d\eta \neq 0$.

Next we consider only the hypersurface which are tangent to ξ. Denote by $U = fN$ and from (1) (f), we deduce $g(U,U) = 1$. Moreover, it is easy to see that $U \in \Gamma(TM)$. Denote by $D^{\perp} = Span(U)$ the 1-dimensional distribution generated by U, and by D the orthogonal complement of $D^{\perp} \oplus (\xi)$ in TM. It is easy to see that

$$fD = D, \quad D^{\perp} \subseteq TM^{\perp}, \quad TM = D \oplus D^{\perp} \oplus (\xi), \tag{12}$$

where $\oplus$ denote the orthogonal direct sum. According with [1] from (12) we deduce that M is a CR-submanifold of $\bar{M}$. ☐

Definition 3. *A CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is called CR-product if both distributions $D \oplus (\xi)$ and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifold of M.*

Denote by P the projection morphism of TM to D and using the decomposion in (10) we deduce for all $X \in \Gamma(T\bar{M})$ that

$$X = PX + a(X)U + \eta(X)\xi, \qquad fX = fPX + a(X)fU + \eta(fX)\xi,$$

therefore $fX = fPX - a(X)fU$. Since

$$U = fN, \qquad fU = f^2N = -N + \eta(N)\xi = -N + g(N,\xi)\xi = -N,$$

where a is a 1-form on M defined by $a(X) = g(X,U)$, $X \in \Gamma(TM)$. From (12) using (1) (a) we infer for all $X \in \Gamma(TM)$

$$fX = tX - a(X)N, \tag{13}$$

where t is a tensor field defined by $tX = fPX$, $X \in \Gamma(TM)$. It is easy to see that

$$(a)\ t\xi = 0, \quad (b)\ tU = 0. \tag{14}$$

2 INDUCED STRUCTURES ON A HYPERSURFACE OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD

The purpose of this section is to study the existence of some induced structure on a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold. Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. From (1) (a), (13) and (14) we obtain $t^3 + t = 0$, that is the tensor field t defines an f-structure on M in sense of Yano [10]. Moreover, from (1) (a), (13), (14) we infer for all $X \in \Gamma(TM)$

$$t^2X = -X + a(X)U + \eta(X)\xi. \tag{15}$$

Lemma 2. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$ the tensor field t satisfies for all $X, Y \in \Gamma(TM)$*

$$(a)\ g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y), \quad (b)\ g(tX,Y) + g(X,tY) = 0. \tag{16}$$

Proof. From (1) (f), and (13) we deduce for all $X, Y \in \Gamma(TM)$

$$\begin{aligned} g(X,Y) - \eta(X)\eta(Y) &= g(fX,fY) = g(tX - a(X)N, tY - a(Y)N) \\ &= g(tX,tY) - a(Y)g(tX,N) - a(X)g(N,tY) \\ &\quad + a(X)a(Y)g(N,N) = g(tX,tY) + a(X)a(Y), \\ g(tX,tY) &= g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y), \\ g(tX,Y) + g(X,tY) &= g(fX + a(X)N, Y) + g(X, fY + a(Y)N) \\ &= g(fX,Y) + a(X)g(N,Y) + g(X,fY) + a(Y)g(X,N) \\ &= g(fX,Y) + g(X,fY) = 0. \end{aligned}$$

□

Lemma 3. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*

$$(a)\ FU = fA\xi, \qquad (b)\ FN = A\xi, \qquad (c)\ [U,\xi] = 0. \tag{17}$$

Proof. We take $X = U$ and $Y = \xi$ in (2) $f(\bar{\nabla}_U \xi) = -\bar{\nabla}_N \xi - U$. Then using (1) (a), (6), (7) (b), we deduce the assertion (a). The assertion (b) follows from (1) (a), (3) (b) and (7) (b) we derive

$$\bar{\nabla}_\xi U = (\bar{\nabla}_\xi f)N + f\bar{\nabla}_\xi N = -fA\xi = -FU = \bar{\nabla}_U \xi,$$
$$[U,\xi] = \bar{\nabla}_U \xi - \bar{\nabla}_\xi U = \bar{\nabla}_U \xi - \bar{\nabla}_U \xi = 0,$$

which prove assertion (c). □

By using the decomposition $T\bar{M} = TM \oplus TM^\perp$, we deduce

$$FX = \alpha X - \eta(AX)N \qquad \text{for all} \quad X \in \Gamma(T\bar{M}),$$

where α is a tensor field of type (1, 1) on M, since $g(FX,N) = -g(X,FN) = -g(X,A\xi) = -\eta(AX)$ for all $X \in \Gamma(T\bar{M})$. By using (5), (6), (7), (13) and (15) we obtain following theorem.

Theorem 2. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then the covariant derivative of a tensors t, a, η and α are given by*

$$\begin{aligned}
&(a)\ (\nabla_X t)Y = g(FX,fY)\xi - g(X,Y)\xi - a(Y)AX + B(X,Y)U + \eta(Y)[\alpha tX + X - \eta(AX)U],\\
&(b)\ (\nabla_X a)Y = B(X,tY) + \eta(Y)\eta(AtX),\\
&(c)\ (\nabla_X \eta)Y = g(Y,\nabla_X \xi),\\
&(d)\ (\nabla_X \alpha)Y = R(\xi,X)Y + B(X,Y)A\xi - \eta(AY)AX \text{ for all } X,Y \in \Gamma(TM)
\end{aligned} \tag{18}$$

respectively, where R is the curvature tensor field of M.

From (5), (6), (14) and (18) (a) we get the following.

Proposition 1. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$, we have for all $X \in \Gamma(TM)$*

$$(a)\ \nabla_X U = -tAX + \eta(AtX)\xi, \quad (b)\ B(X,U) = a(AX). \tag{19}$$

Theorem 3. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. The tensor field t is a parallel with respect to the Levi Civita connection ∇ on M iff for all $X \in \Gamma(TM)$*

$$(a)\ AX = \eta(AX)\xi - a(X)\xi + a(AX)U, \quad (b)\ FX = fX - \eta(AX)N + a(X)N. \tag{20}$$

Proof. Suppose that the tensor field t is parallel with respect to ∇, that is $\nabla t = 0$. By using (2) (a), we deduce for all $X,Y \in \Gamma(TM)$

$$\eta(Y)[\alpha tX + X - \eta(AX)U] - a(Y)AX + g(FX,fY)\xi + B(X,Y)U - g(X,Y)\xi = 0. \tag{21}$$

Take $Y = U$ in (21) and using (7) (b), (8), (19) (b) we infer

$$\begin{aligned}
&\eta(U)[\alpha tX + X - \eta(AX)U] - a(U)AX + g(FX,fU)\xi - g(X,U)\xi + B(X,U)U = 0,\\
&\eta(U) = 0, \quad a(U) = 1, \quad g(X,N) = 0,\\
&-AX + g(FX,fU)\xi - g(X,U)\xi + a(AX)U = 0,\\
&AX = g(FX,-N)\xi - a(X)\xi + a(AX)U\\
&\quad\ = g(X,FN)\xi - a(X)\xi + a(AX)U = g(X,A\xi)\xi - a(X)\xi + a(AX)U,\\
&AX = \eta(AX)\xi - a(X)\xi + a(AX)U
\end{aligned}$$

and the assertion (20) (a) is proved. Next let $Y = fZ$, $Z \in \Gamma(D)$ in (21) and using (1) (f), (3) (b), (17), (20) (a), we deduce for all $X \in \Gamma(TM)$

$$g(X, FZ) = 0 \Rightarrow FX = fX - \eta(AX)N + a(X)N.$$

The proof is complete. □

Proposition 2. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the assertions for all $X, Y \in \Gamma(TM)$*

$$(a)\ (\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = 0, \qquad (b)\ (\nabla_X \eta)Y = 0 \Leftrightarrow \nabla_X \xi = 0.$$

Proof. Let $X, Y \in \Gamma(TM)$. Using (8), (16) (b), (18) (b) and (19) (a) we obtain

$$\begin{aligned} g(\nabla_X U, Y) &= g(-tAX + \eta(AtX)\xi, Y) = g(-tAX, Y) + \eta(AtX)g(\xi, Y) \\ &= g(AX, tY) + \eta(AtX)\eta(Y) = (\nabla_X a)Y, \end{aligned}$$

which proves assertion (a). The assertion (b) is consequence of the fact that ξ is not a killing vector field. □

According to Theorem 2 in [6], the tensor field $\bar{f} = t + \eta \otimes U - a \otimes \xi$ defines an almost complex structure on M. Moreover, from Proposition 2 we deduce the following assertion.

Theorem 4. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the tensor fields t, a, η are parallel with respect to the connection ∇, then $\bar{f}$ defines a Kahler structure on M.*

3 INTEGRABILITY OF DISTRIBUTIONS ON A HYPERSURFACE OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD $\bar{M}$

In this section we establish conditions for the integrability of all distributions on a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$. From Lemma 3 we obtain.

Corollary 1. *On a hypersurface of a quarter symmetric non metric connection M of a quasi-Sasakian manifold $\bar{M}$ there exists a 2-dimensional foliation determined by the integral distribution $D^\perp \oplus (\xi)$.*

Theorem 5. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the following.*

(a) A leaf of $D^\perp \oplus (\xi)$ is totally geodesic submanifold of M if and only if

$$(1)\ AU = a(AU)U + \eta(AU)\xi - \xi \quad \text{and} \quad (2)\ FN = a(FN)U.$$

(b) A leaf of $D^\perp \oplus (\xi)$ is totally geodesic submanifold of $\bar{M}$ if and only if for all $X \in \Gamma(D)$

$$(1)\ AU = 0 \quad \text{and} \quad (2)\ a(FX) = a(FN) - 1 = 0.$$

Proof. (a) Let M^* be a leaf of integrable distribution $D^\perp \oplus (\xi)$ and h^* be the second fundamental form of the immersion $M^* \to M$. By using (1) (f) and (7) (b) we get for all $X \in \Gamma(TM)$

$$\begin{aligned} g(h^*(U,U),X) &= g(\bar{\nabla}_U U, X) = -g(N, (\bar{\nabla}_U f)X - g(\bar{\nabla}_U N, fX) \\ &= 0 - g(-AU, fX) = g(AU, fX) = g(AU, fX) \end{aligned} \tag{22}$$

and for all $X \in \Gamma(TM)$

$$g(h^*(U,\xi),X) = g(\bar{\nabla}_U \xi, X) = g(-FU + U, X) = g(FN, fX) + a(X), \tag{23}$$

because $g(FU,N) = 0$ and $f\xi = 0$ the assertion (a) follows from (22) and (23).

(b) Let h_1 be the second fundamental form of the immersion $M^* \to M$. It is easy to see that

$$h_1(X,Y) = h^*(X,Y) + B(X,Y)N \qquad \text{for all} \quad X, Y \in \Gamma(D^\perp \oplus (\xi)). \tag{24}$$

From (6) and (8) we deduce

$$(h_1(U,U),N) = g(\bar{\nabla}_U U, N) = a(AU), \tag{25}$$

$$g(h_1(U,\xi),N) = g(\bar{\nabla}_U \xi, N) = a(FN) - 1. \tag{26}$$

The assertion (b) follows from (23)–(26). □

Theorem 6. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then*

(a) the distribution $D \oplus (\xi)$ is integrable iff for all $X, Y \in \Gamma(D)$

$$g(AfX + fAX, Y) = 0, \tag{27}$$

(b) the distribution D is integrable iff (27) holds and for all $X \in \Gamma(D)$

$$FX = \eta(AtX)U - \eta(AX)N, \quad (\textit{equivalent with } FD \perp D),$$

(c) the distribution $D \oplus D^\perp$ is integrable iff $FX = 0$ for all $X \in \Gamma(D)$.

Proof. Let $X, Y \in \Gamma(D)$. Since ∇ is a torsion free and ξ is a Killing vector field, we infer

$$g([X,\xi],U) = g(\bar{\nabla}_X \xi, U) - g(\bar{\nabla}_\xi X, U) = g(\nabla_X \xi, U) + g(\nabla_U \xi, X) = 0. \tag{28}$$

Using (1) (a), (7) (a) we deduce for all $X, Y \in \Gamma(D)$

$$\begin{aligned} g([X,Y],U) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, U) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, fN) \\ &= g(\bar{\nabla}_Y fX - \bar{\nabla}_X fY, N) = -g(AfX + fAX, Y). \end{aligned} \tag{29}$$

Next by using (4), (5) (d) and the fact that ∇ is a metric connection we get for all $X, Y \in \Gamma(D)$

$$g([X,Y],\xi) = g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) = 2g(FX - fX, Y) = 2g(FX, Y) - 2g(fX, Y). \tag{30}$$

The assertion (a) follows from (28), (29) and assertion (b) follows from (28)–(30). Using (6) and (3) we obtain for all $X \in \Gamma(D)$

$$g([X,U],\xi) = g(\bar{\nabla}_X U, \xi) - g(\bar{\nabla}_U X, \xi) = 2g(FX, U) - 2g(fX, U). \tag{31}$$

Taking into account that for all $X \in \Gamma(D)$

$$g(FX, N) = g(FfX, fN) = g(FfX, U), \tag{32}$$

the assertion (c) follows from (30) and (31). □

Theorem 7. *Let M be a hypersurface of a quarter symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*

(a) the distribution D is integrable and its leaves are totally geodesic immersed in M if and only if for all $X \in \Gamma(D)$

$$FD \perp D \quad \text{and} \quad AX = a(AX)U - \eta(AX)\xi, \tag{33}$$

(b) the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic immersed in if and only if for $X \in \Gamma(D)$ takes place $AX = a(AX)U$ and $FU = 0$,

(c) the distribution $D \oplus D^{\perp}$ is integrable and its leaves are totally geodesic immersed in M if and only if for $X \in \Gamma(D)$ takes place $FX = 0$.

Proof. Let M_1^* be a leaf of integrable distribution D and h_1^* the second fundamental form of immersion $M_1^* \to M$. Then by direct calculation we infer

$$g(h_1^*(X,Y),U) = g(\bar{\nabla}_X Y, U) = -g(Y, \nabla_X U) = -g(AX, tY) \tag{34}$$

and for all $X, Y \in \Gamma(D)$

$$g(h_1^*(X,Y),\xi) = g(\bar{\nabla}_X Y, \xi) = g(FX,Y) - g(fX,Y). \tag{35}$$

Now suppose M_1^* is a totally submanifold of M. Then (33) follows from (34) and (35). Conversely suppose that (33) is true. Then using the assertion (b) in Theorem 6 it is easy to see that the distribution D is integrable. Next the proof follows by using (34) and (35). Next, suppose that the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic submanifolds of M. Let M_1 be a leaf of $D \oplus (\xi)$ and h_1 the second fundamental form of immersion $M_1 \to M$. By direct calculations, using (6), (7) (b), (16) (b) and (19) (c), we deduce that for all $X, Y \in \Gamma(D)$

$$g(h_1(X,Y),U) = g(\bar{\nabla}_X Y, U) = -g(AX, tY), \tag{36}$$

and for all $X \in \Gamma(D)$

$$g(h_1(X,\xi),U) = g(\bar{\nabla}_X \xi, U) = g(-FU + fU, X) = g(FU, X). \tag{37}$$

Then the assertion (b) follows from (32), (36), (37) and the assertion (a) of Theorem 6. Next let $\bar{M}_1$ be a leaf of the integrable distribution $D \oplus D^{\perp}$ and $\bar{h}_1$ is the second fundamental form of the immersion $M_1 \to M$. By direct calculation for all $X \in \Gamma(D), Y \in \Gamma(D \oplus D^{\perp})$ we get

$$g(\bar{h}_1(X,Y),\xi) = g(FX,Y) - g(fX,Y). \tag{38}$$

The assertion (c) follows from (3) (c), (32) and (38). □

4 CONTACT CONFORMAL CONNECTION ON A HYPERSURFACE OF A QUARTER SYMMETRIC NON METRIC CONNECTION IN A QUASI-SASAKIAN MANIFOLD $\bar{M}$

Let the conformal change of the metric tensor $\bar{g}$ which induces a new metric tensor, given by $\bar{\bar{g}}(X,Y) = e^{2p}\bar{g}(X,Y)$ with regard to this metric, take an affine connection, which satisfies

$$\bar{\bar{\nabla}}_X \bar{\bar{g}}(Y,Z) = \bar{\nabla}_X\{e^{2p}\bar{g}(Y,Z)\} = e^{2p}p(X)\eta(Y)\eta(Z), \tag{39}$$

where p is a scalar point function. The torsion tensor of the connection $\bar{\bar{\nabla}}$ satisfies

$$T(X,Y) = -2\bar{g}(fX,Y)U = S(X,Y) - S(Y,X), \tag{40}$$

where U is a vector field. Let

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + S(X,Y), \tag{41}$$

where S is a tensor of type $(1,2)$. Using (39), (40), (41), we have

$$\begin{aligned} \bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + p(X)\{Y - \eta(Y)\xi\} + p(Y)\{X - \eta(X)\xi\} \\ - \bar{g}(fX, fY)P + u(X)fY + u(Y)fX - \bar{g}(fX,Y)U, \end{aligned} \tag{42}$$

where $\bar{g}(P,X) = p(X)$, $\bar{g}(QX,P) = p(fX) = -q(X)$, $\bar{g}(Q,X) = q(X)$, $\bar{g}(U,X) = u(X)$.

$$\begin{aligned} (\bar{\bar{\nabla}}_X f)(Y) = (\bar{\nabla}_X f)(Y) + \{X - \eta(X)\xi\}p(fY) - p(Y)fX + \bar{g}(fX,Y)p + \bar{g}(fX,fY)fP \\ + u(fY)fX + u(Y)\{X - \eta(X)\xi\} - \bar{g}(fX,fY)U + \bar{g}(fX,Y)fU = 0. \end{aligned}$$

Using (5), the relation becomes

$$\begin{aligned} \bar{g}(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi + \eta(Y)X - \eta(X)\eta(Y)\xi - p(Y)fX \\ + \{X - \eta(X)\xi\}p(fY) + \bar{g}(fX,Y)p + \bar{g}(fX,fY)fP + u(fY)fX \\ + u(Y)\{X - \eta(X)\xi\} - \bar{g}(fX,fY)U + \bar{g}(fX,Y)fU = 0. \end{aligned}$$

Contracting with respect to X,

$$\begin{aligned} 2m\eta(Y) + 2mp(fY) - 2p(fY) + 2mu(Y) - 2u(Y) + 2\eta(U)\eta(Y) = 0, \\ 2(m-1)p(fY) + 2(m-1)u(Y) + 2\eta(Y)\{m + \eta(U)\} = 0. \end{aligned}$$

If we put $\eta(U) = -1 = u(\xi)$, then $u(Y) = q(Y) - \eta(Y)$. Thus (42) takes the form

$$\begin{aligned} \bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \{Y - \eta(Y)\xi\}p(X) + \{X - \eta(X)\xi\}p(Y) - \bar{g}(fX,fY)P \\ + \{q(X) - \eta(X)\}fY + \{q(Y) - \eta(Y)\}fX - \bar{g}(fX,Y)(Q - \xi). \end{aligned} \tag{43}$$

Then $\bar{\bar{\nabla}}_X \xi = 0 = \bar{\nabla}_X \xi + \{X - \eta(X)\xi\}p(\xi) - fX$. Using (6) in this equation, we have

$$-FX + fX + \bar{\nabla}_X \xi + \{X - \eta(X)\xi\}p(\xi) - fX = 0,$$

which implies that $FX = \{X - \eta(X)\xi\}p(\xi)$.

Proposition 3. *On a hypersurface of a quarter symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$ the affine connection $\bar{\bar{\nabla}}$ which satisfies* (40)*, is given by* (43) *with the conditions* $u(\xi) = -1 = \eta(U)$, $FX = \{X - \eta(X)\xi\}p(\xi)$.

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Рахман Ш. *Геометрія гіперповерхонь четвертинно симетричного неметричного зв'язку в квазі Сасакяновому многовиді* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 226–235.

Метою цієї статті є вивчення поняття CR-підмноговидів та існування деяких структур на гіперповерхні четвертинно симетричного неметричного зв'язку в квазі Сасакяновому многовиді. Ми досліджуємо існування структури Кахлера на M та існування глобально метричної конструкції f-структури у сенсі Гольдберга С.І., Яно К. [6]. Обговорюється інтегрованість розподілів на M і геометрія їхніх листків. Описано спроби пов'язати цей результат з отриманими раніше результатами Гольдберга В., Роска Р., які присвячені многовиду Сасакяна та конформним зв'язкам.

Ключові слова і фрази: CR-підмноговид, квазі Сасакяновий многовид, четвертинно симетричний неметричний зв'язок, умови інтегрованості розподілів.