Additional close links between balancing and Lucas-balancing polynomials

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Abstract

Using generating functions, we derive many identities involving balancing and Lucas-balancing polynomials. By relating these polynomials to Chebyshev polynomials of the first and second kind, and Fibonacci and Lucas numbers, we offer some presumably new combinatorial identities involving these famous sequences.

1 Introduction and preliminaries

For any integer $n \geq 0$, the balancing polynomials $\big(B_n(x)\big)_{n\geq 0}$ and Lucas-balancing polynomials $\big(C_n(x)\big)_{n\geq 0}$ are defined by the same second-order homogeneous linear recurrence

$$u_n(x) = 6xu_{n-1}(x) - u_{n-2}(x), \tag{1}$$

but with different initial terms $B_0(x) = 0$, $B_1(x) = 1$ and $C_0(x) = 1$, $C_1(x) = 3x$. These polynomials have been introduced as a natural extension of the popular balancing and Lucas-balancing numbers B_n and C_n, respectively, which were firstly studied in [1]. Obviously, $B_n = B_n(1)$ and $C_n = C_n(1)$. Sequences $\big(B_n\big)_{n\geq 0}$ and $\big(C_n\big)_{n\geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [16] (see entries A001109 and A001541, respectively).

The closed forms which are also called Binet's formulas for balancing and Lucas-balancing polynomials are given by

$$B_n(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{2\sqrt{9x^2-1}}, \qquad C_n(x) = \frac{\lambda^n(x) + \lambda^{-n}(x)}{2}, \tag{2}$$

where $\lambda(x) = 3x + \sqrt{9x^2-1}$.

[1]Statements and conclusions made in this article by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

For $n \geq 1$, the balancing and Lucas-balancing polynomials are given explicitly [13, 14] by

$$B_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} (6x)^{n-1-2k}, \quad C_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (6x)^{n-2k}.$$

The first polynomials are

$$B_0(x) = 0, \quad B_1(x) = 1, \quad B_2(x) = 6x, \quad B_3(x) = 36x^2 - 1,$$
$$B_4(x) = 216x^3 - 12x, \quad B_5(x) = 1296x^4 - 108x^2 + 1,$$

and

$$C_0(x) = 1, \quad C_1(x) = 3x, \quad C_2(x) = 18x^2 - 1, \quad C_3(x) = 108x^3 - 9x,$$
$$C_4(x) = 648x^4 - 72x^2 + 1, \quad C_5(x) = 3888x^5 - 540x^3 + 15x.$$

These polynomials have been studied extensively in different contexts and a variety of interesting results about them have been uncovered [2, 3, 6, 8, 9, 11, 14]. For example, in [2], the first author established direct connections of the polynomials $B_n(x)$ and $C_n(x)$ with Fibonacci numbers, Lucas numbers and Chebyshev and Legendre polynomials. By using combinatorial methods, Meng derived some symmetry identities of the structural properties of balancing numbers and balancing polynomials [11]. In [8, 9], the authors study sums of finite products of balancing and Lucas-balancing polynomials and represent them in terms of nine orthogonal polynomials. In [14], Ray studied the sequences obtained by differentiating the balancing polynomials and presented some relations between the balancing polynomials and their derivatives.

In the present study, we derive new identities for polynomials $B_n(x)$ and $C_n(x)$. Evaluating these identities at specific points, we can also establish some interesting combinatorial identities as special cases, especially those with Fibonacci and Lucas numbers. Our approach is in the spirit of [4, 7].

2 Balancing polynomial relations using ordinary generating functions

To establish our main results, we will find the ordinary (non-exponential) generating functions for the sequences in question. We will make use of the following result [12] to compute the ordinary generating functions for balancing, Lucas-balancing polynomials and their odd (even) indexed companions.

Lemma 1. *The second-order polynomial recurrence $u_n(x) = pu_{n-1}(x) + qu_{n-2}(x)$, $n \geq 2$, $p^2 + 4q \neq 0$, with initial terms $u_0(x)$ and $u_1(x)$ has generating function*

$$\sum_{n\geq 0} u_n(x) z^n = \frac{u_0(x) + \big(u_1(x) - pu_0(x)\big)z}{1 - pz - qz^2},$$

while for odd (even) indexed sequences

$$\sum_{n\geq 0} u_{2n+1}(x)z^n = \frac{u_1(x) + \big(pqu_0(x) - qu_1(x)\big)z}{1-(p^2+2q)z+q^2z^2},$$

$$\sum_{n\geq 0} u_{2n}(x)z^n = \frac{u_0(x) + \big(u_2(x) - (p^2+2q)u_0(x)\big)z}{1-(p^2+2q)z+q^2z^2}.$$

From the above lemma we obtain ordinary generating functions of the sequences $B_n(x)$, $B_{2n+1}(x)$, and $B_{2n}(x)$ as follows

$$b(x,z) = \sum_{n\geq 0} B_n(x)z^n = \frac{z}{1-6xz+z^2}, \tag{3}$$

$$b_1(x,z) = \sum_{n\geq 0} B_{2n+1}(x)z^n = \frac{1+z}{1-(36x^2-2)z+z^2},$$

$$b_2(x,z) = \sum_{n\geq 0} B_{2n}(x)z^n = \frac{6xz}{1-(36x^2-2)z+z^2}. \tag{4}$$

In the similar manner, we conclude that ordinary generating functions of the sequences $C_n(x)$, $C_{2n+1}(x)$ and $C_{2n}(x)$ can be derived as

$$c(x,z) = \sum_{n\geq 0} C_n(x)z^n = \frac{1-3xz}{1-6xz+z^2},$$

$$c_1(x,z) = \sum_{n\geq 0} C_{2n+1}(x)z^n = \frac{3x-3xz}{1-(36x^2-2)z+z^2}, \tag{5}$$

$$c_2(x,z) = \sum_{n\geq 0} C_{2n}(x)z^n = \frac{1+(1-18x^2)z}{1-(36x^2-2)z+z^2}. \tag{6}$$

We present our first findings in two theorems, which provide some relations between balancing and Lucas-balancing polynomials using their respective ordinary generating functions.

Theorem 2. *For $n \geq 1$, the following formulas hold*

$$\begin{gathered} B_n(x) - 3xB_{n-1}(x) = C_{n-1}(x), \\ 3x\big(B_{2n+1}(x) - B_{2n-1}(x)\big) = C_{2n+1}(x) + C_{2n-1}(x), \\ B_{2n}(x) - (18x^2-1)B_{2(n-1)}(x) = 6xC_{2(n-1)}(x), \\ B_{2n+1}(x) - (18x^2-1)B_{2n-1}(x) = C_{2n}(x) + C_{2(n-1)}(x), \\ 3x\big(B_{2n}(x) - B_{2(n-1)}(x)\big) = 6xC_{2n-1}(x). \end{gathered} \tag{7}$$

Proof. All stated identities can be proved directly using Binet's formulas (2). We present a proof based on generating functions. We will prove formula (7); the others may also be shown in a similar manner.

By (4) and (6), we obtain

$$\big(1-(18x^2-1)z\big)b_2(x,z)=6xzc_2(x,z).$$

Expanding both sides of the last equation as a power series in z, we then have

$$\sum_{n\geq 0}B_{2n}(x)z^n-(18x^2-1)\sum_{n\geq 0}B_{2n}(x)z^{n+1}=6x\sum_{n\geq 0}C_{2n}(x)z^{n+1}$$

or, equivalently,

$$\sum_{n\geq 1}B_{2n}(x)z^n-(18x^2-1)\sum_{n\geq 1}B_{2(n-1)}(x)z^n=6x\sum_{n\geq 1}C_{2(n-1)}(x)z^n.$$

Comparing the coefficients on both sides, we get (7). ☐

For convention, throughout this paper the empty sums are evaluated to 0.

Theorem 3. *For $n\geq 1$, the following formulas hold:*

$$3x\big(B_n(x)-B_{n-1}(x)\big)=C_{2n-1}(x)-(36x^2-6x-2)\sum_{k=1}^{n-1}B_k(x)C_{2(n-k)-1}(x), \tag{8}$$

$$B_n(x)-(18x^2-1)B_{n-1}(x)=C_{2(n-1)}(x)-(36x^2-6x-2)\sum_{k=1}^{n-1}B_k(x)C_{2(n-k-1)}(x),$$

$$B_{2n+1}(x)-3xB_{2n-1}(x)=C_n(x)+C_{n-1}(x)-(36x^2-6x-2)\sum_{k=0}^{n-1}B_{2k+1}(x)C_{n-k-1}(x),$$

$$B_{2n}(x)-3xB_{2(n-1)}(x)=6xC_{n-1}(x)-(36x^2-6x-2)\sum_{k=1}^{n-1}B_{2k}(x)C_{n-k-1}(x).$$

Proof. We prove only (8). Proceeding as in the proof of Theorem 2 above, using (3) and (5), we deduce that

$$zc_1(x,z)-3x(1-z)b(x,z)=(36x^2-6x-2)zb(x,z)c_1(x,z).$$

Expanding both sides of the last equation as a power series in z yields

$$\begin{aligned}\sum_{n\geq 0}C_{2n+1}(x)z^{n+1}-3x\sum_{n\geq 0}B_n(x)z^n+3x\sum_{n\geq 0}B_n(x)z^{n+1}\\ =(36x^2-6x-2)\sum_{n\geq 0}\sum_{k=0}^{n}B_k(x)C_{2(n-k)+1}z^{n+1},\\ \sum_{n\geq 1}C_{2n-1}(x)z^n-3x\sum_{n\geq 1}B_n(x)z^n+3x\sum_{n\geq 1}B_{n-1}(x)z^n\\ =(36x^2-6x-2)\sum_{n\geq 1}\sum_{k=0}^{n-1}B_k(x)C_{2(n-k)-1}z^n.\end{aligned}$$

Comparing the coefficients on both sides implies the stated formula. ☐

3 Balancing relations using exponential generating functions

In this section, we will use the structure of the exponential generating functions to prove our results. In this case, we will use significantly the following lemma from [12].

Lemma 4. *The recurrence of type $u_n(x) = pu_{n-1}(x) + qu_{n-2}(x)$, $n \geq 2$, where $p^2 + 4q \neq 0$ and $u_0(x)$ and $u_1(x)$ are the initial terms, has the exponential generating function*

$$\sum_{n\geq 0} u_n(x)\frac{z^n}{n!} = \frac{e^{\frac{p}{2}z}}{\Delta}\Big((u_1(x) - \beta u_0(x))e^{\frac{\Delta}{2}z} - (u_1(x) - \alpha u_0(x))e^{-\frac{\Delta}{2}z}\Big),$$

while for odd and even indexed sequences

$$\sum_{n\geq 0} u_{2n+1}(x)\frac{z^n}{n!} = \frac{e^{\frac{p^2+2q}{2}z}}{p\Delta}\Big((u_3(x) - \sigma u_1(x))e^{\frac{p\Delta}{2}z} - (u_3(x) - \rho u_0(x))e^{-\frac{p\Delta}{2}z}\Big),$$

$$\sum_{n\geq 0} u_{2n}(x)\frac{z^n}{n!} = \frac{e^{\frac{p^2+2q}{2}z}}{p\Delta}\Big((u_2(x) - \sigma u_0(x))e^{\frac{p\Delta}{2}z} - (u_2(x) - \rho u_0(x))e^{-\frac{p\Delta}{2}z}\Big),$$

where $\Delta = \sqrt{p^2+4q}$, $\alpha = \frac{p+\Delta}{2}$, $\beta = \frac{p-\Delta}{2}$, $\rho = \frac{p^2+2q+p\Delta}{2}$, *and* $\sigma = \frac{p^2+2q-p\Delta}{2}$.

From lemma above it can be shown fairly easily that the exponential generating functions of the balancing polynomials and their odd (even) indexed companions are given by

$$b^*(x,z) = \sum_{n\geq 0} B_n(x)\frac{z^n}{n!} = \frac{e^{3xz}}{\sqrt{9x^2-1}}\sinh\big(\sqrt{9x^2-1}\,z\big), \tag{9}$$

$$b_1^*(x,z) = \sum_{n\geq 0} B_{2n+1}(x)\frac{z^n}{n!}$$

$$= \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}}\left(3x\sinh\big(6x\sqrt{9x^2-1}\,z\big) + \sqrt{9x^2-1}\cosh\big(6x\sqrt{9x^2-1}\,z\big)\right), \tag{10}$$

$$b_2^*(x,z) = \sum_{n\geq 0} B_{2n}(x)\frac{z^n}{n!} = \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}}\sinh\big(6x\sqrt{9x^2-1}\,z\big). \tag{11}$$

In the similar manner, we obtain the exponential generating functions of the Lucas-balancing polynomial sequences:

$$c^*(x,z) = \sum_{n\geq 0} C_n(x)\frac{z^n}{n!} = e^{3xz}\cosh\big(\sqrt{9x^2-1}\,z\big), \tag{12}$$

$$c_1^*(x,z) = \sum_{n\geq 0} C_{2n+1}(x)\frac{z^n}{n!}$$

$$= e^{(18x^2-1)z}\left(3x\cosh\big(6x\sqrt{9x^2-1}\,z\big) + \sqrt{9x^2-1}\sinh\big(6x\sqrt{9x^2-1}\,z\big)\right), \tag{13}$$

$$c_2^*(x,z)=\sum_{n\geq 0} C_{2n}(x)\frac{z^n}{n!}=e^{(18x^2-1)z}\cosh\left(6x\sqrt{9x^2-1}\,z\right). \tag{14}$$

Next, we present some new relations between balancing and Lucas-balancing polynomials involving binomial coefficients.

Theorem 5. *For $n\geq 1$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{1+(-1)^{n-k}}{\left(\sqrt{9x^2-1}\right)^{k-1}}B_k(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{1-(-1)^{n-k}}{\left(\sqrt{9x^2-1}\right)^{k}}C_k(x), \tag{15}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda(x)+(-1)^{n-k}\lambda^{-1}(x)}{\left(6x\sqrt{9x^2-1}\right)^{k-1}}B_{2k+1}(x)=6x\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda(x)-(-1)^{n-k}\lambda^{-1}(x)}{\left(6x\sqrt{9x^2-1}\right)^{k}}C_{2k+1}(x), \tag{16}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{1+(-1)^{n-k}}{\left(6x\sqrt{9x^2-1}\right)^{k-1}}B_{2k}(x)=6x\sum_{k=1}^{n}\binom{n}{k}\frac{1-(-1)^{n-k}}{\left(6x\sqrt{9x^2-1}\right)^{k}}C_{2k}(x), \tag{17}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1+(-1)^{n-k}}{\left(6x\sqrt{9x^2-1}\right)^{k-1}}B_{2k+1}(x)=6x\sum_{k=0}^{n}\binom{n}{k}\frac{\lambda(x)-(-1)^{n-k}\lambda^{-1}(x)}{\left(6x\sqrt{9x^2-1}\right)^{k}}C_{2k}(x),$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{\lambda(x)+(-1)^{n-k}\lambda^{-1}(x)}{\left(6x\sqrt{9x^2-1}\right)^{k-1}}B_{2k}(x)=6x\sum_{k=0}^{n}\binom{n}{k}\frac{1-(-1)^{n-k}}{\left(6x\sqrt{9x^2-1}\right)^{k}}C_{2k+1}(x),$$

where $\lambda(x)=3x+\sqrt{9x^2-1}$.

Proof. We will prove (15). In view of (9) and (12), we have

$$\sqrt{9x^2-1}\cosh\left(\sqrt{9x^2-1}z\right)b^*(x,z)=\sinh\left(\sqrt{9x^2-1}z\right)c^*(x,z)$$

or, equivalently,

$$\begin{aligned}\sqrt{9x^2-1}\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}\sum_{n\geq 0}\left(\sqrt{9x^2-1}\right)^n\left(1+(-1)^n\right)\frac{z^n}{n!}\\ =\sum_{n\geq 0}C_n(x)\frac{z^n}{n!}\sum_{n\geq 0}\left(\sqrt{9x^2-1}\right)^n\left(1-(-1)^n\right)\frac{z^n}{n!},\end{aligned}$$

$$\begin{aligned}\sqrt{9x^2-1}\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}B_k(x)\left(\sqrt{9x^2-1}\right)^{n-k}\left(1+(-1)^{n-k}\right)\frac{z^n}{n!}\\ =\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}C_k(x)\left(\sqrt{9x^2-1}\right)^{n-k}\left(1-(-1)^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides yields (15).

The formulas (16) and (17) follow from the relations

$$\begin{aligned}\sqrt{9x^2-1}\left(3x\cosh\left(6x\sqrt{9x^2-1}\right)+\sqrt{9x^2-1}\sinh\left(6x\sqrt{9x^2-1}\right)\right)b_1^*(x,z)\\ =\left(3x\cosh\left(6x\sqrt{9x^2-1}\right)-\sqrt{9x^2-1}\sinh\left(6x\sqrt{9x^2-1}\right)\right)c_1^*(x,z)\end{aligned}$$

and

$$\sqrt{9x^2-1}\cosh\big(6x\sqrt{9x^2-1}\big)\, b_2^*(x,z) = \sinh\big(6x\sqrt{9x^2-1}\big)\, c_2^*(x,z),$$

that one can obtain from (10), (13) and (11), (14), respectively. The proof of the other formulas is similar. □

In the next theorem we list a range of more advanced further relations for balancing and Lucas-balancing polynomials that can be derived in a similar manner.

Theorem 6. *For $n \geq 1$, we have*

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}\left(\lambda(x)+(-1)^{n-k}\lambda^{-1}(x)\right)\left(\frac{18x^2-1}{18x^2\sqrt{9x^2-1}}\right)^{k-1} B_k(x)\\
&= \left(\frac{18x^2-1}{18x^2}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}\left(1-(-1)^{n-k}\right)\left(\frac{3x}{(18x^2-1)\sqrt{9x^2-1}}\right)^{k} C_{2k+1}(x),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}\left(1+(-1)^{n-k}\right)\left(\frac{18x^2-1}{18x^2\sqrt{9x^2-1}}\right)^{k-1} B_k(x)\\
&= \left(\frac{18x^2-1}{18x^2}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}\left(1-(-1)^{n-k}\right)\left(\frac{3x}{(18x^2-1)\sqrt{9x^2-1}}\right)^{k} C_{2k}(x),
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\left(1+(-1)^{n-k}\right)\left(\frac{3x}{(18x^2-1)\sqrt{9x^2-1}}\right)^{k-1} B_{2k+1}(x)\\
&= \left(\frac{18x^2}{18x^2-1}\right)^{n-1} 6x\sum_{k=0}^{n}\binom{n}{k}\left(\lambda(x)-(-1)^{n-k}\lambda^{-1}(x)\right)\left(\frac{18x^2-1}{18x^2\sqrt{9x^2-1}}\right)^{k} C_k(x),
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{n}\binom{n}{k}\left(1+(-1)^{n-k}\right)\left(\frac{3x}{(18x^2-1)\sqrt{9x^2-1}}\right)^{k-1} B_{2k}(x)\\
&= \left(\frac{18x^2}{18x^2-1}\right)^{n-1} 6x\sum_{k=0}^{n}\binom{n}{k}\left(1-(-1)^{n-k}\right)\left(\frac{18x^2-1}{18x^2\sqrt{9x^2-1}}\right)^{k} C_k(x),
\end{aligned}$$

where $\lambda(x) = 3x + \sqrt{9x^2-1}$.

Proof. Formula (18) follows from the functional relation

$$\begin{aligned}
&\Big(3x\cosh\big(6x\sqrt{9x^2-1}z\big)+\sqrt{9x^2-1}\sinh\big(6x\sqrt{9x^2-1}z\big)\Big)b^*\left(x,\frac{18x^2-1}{3x}z\right)\\
&= \sinh\left(\frac{(18x^2-1)(\sqrt{9x^2-1})}{3x}z\right) c_1^*(x,z),
\end{aligned}$$

writing in terms of power series, and collecting terms.

The proof of the other formulas is similar. We omit the corresponding details. □

4 Chebyshev polynomial relations via balancing and Lucas-balancing polynomials

As usual, Chebyshev polynomials $\big(T_n(x)\big)_{n\geq 0}$ of the first kind and the Chebyshev polynomials $\big(U_n(x)\big)_{n\geq 0}$ of the second kind can be defined by the recurrence [10]

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \quad n \geq 2, \tag{19}$$

but with different initial terms $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$.

Many properties of Chebyshev polynomials can be obtained immediately from the following relations initially observed by the first author in [2].

Lemma 7. *For $n \geq 1$, the following relations hold:*

$$B_n\Big(\frac{x}{3}\Big) = U_{n-1}(x), \quad C_n\Big(\frac{x}{3}\Big) = T_n(x). \tag{20}$$

Applying (20) to Theorems 2 and 3 yields the following Chebyshev polynomial relations.

Corollary 8. *For $n \geq 1$,*

$$T_{2n-1}(x) - 2(2x^2 - x - 1)\sum_{k=1}^{n-1} U_{k-1}(x)T_{2n-2k-1}(x) = x(U_{n-1}(x) - U_{n-2}(x)),$$

$$T_{2n-2}(x) - 2(2x^2 - x - 1)\sum_{k=1}^{n-1} U_{k-1}(x)T_{2(n-k-1)}(x) = U_{n-1}(x) - (2x^2 - 1)U_{n-2}(x),$$

$$T_n(x) + T_{n-1}(x) + 2(2x^2 - x - 1)\sum_{k=1}^{n-1} U_{2k}(x)T_{n-k-1}(x) = U_{2n}(x) - xU_{2(n-1)}(x),$$

$$2xT_{n-1}(x) + 2(2x^2 - x - 1)\sum_{k=1}^{n-1} U_{2k-1}(x)T_{n-k-1}(x) = U_{2n-1}(x) - xU_{2n-3}(x).$$

By virtue of (20), from Theorems 5 and 6 we have the following summation formulas involving Chebyshev polynomials and binomial coefficients.

Corollary 9. *Let $\omega(x) = x + \sqrt{x^2-1}$. Then for $n \geq 1$ the following relations hold:*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{U_{k-1}(x)}{\big(\sqrt{x^2-1}\big)^{k-1}}\big(1 + (-1)^{n-k}\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{T_k(x)}{\big(\sqrt{x^2-1}\big)^{k}}\big(1 - (-1)^{n-k}\big),$$

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}&\frac{U_{2k}(x)}{\big(2x\sqrt{x^2-1}\big)^{k-1}}\big(\omega(x) + (-1)^{n-k}\omega^{-1}(x)\big)\\ &= 2x\sum_{k=0}^{n}\binom{n}{k}\frac{T_{2k+1}(x)}{\big(2x\sqrt{x^2-1}\big)^{k}}\big(\omega(x) - (-1)^{n-k}\omega^{-1}(x)\big),\end{aligned}$$

$$\sqrt{x^2-1}\sum_{k=1}^{n}\binom{n}{k}\frac{U_{2k-1}(x)}{\big(2x\sqrt{x^2-1}\big)^{k}}\big(1 + (-1)^{n-k}\big) = \sum_{k=0}^{n}\binom{n}{k}\frac{T_{2k}(x)}{\big(2x\sqrt{x^2-1}\big)^{k}}\big(1 - (-1)^{n-k}\big),$$

$$\sum_{k=1}^{n}\binom{n}{k}U_{k-1}(x)\left(\frac{2x^2-1}{2x^2\sqrt{x^2-1}}\right)^{k-1}\left(\omega(x)+(-1)^{n-k}\omega^{-1}(x)\right)$$
$$=\left(\frac{2x^2-1}{2x^2}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}T_{2k+1}(x)\left(\frac{x}{(2x^2-1)\sqrt{x^2-1}}\right)^{k}\left(1-(-1)^{n-k}\right),$$
$$\sum_{k=1}^{n}\binom{n}{k}U_{k-1}(x)\left(\frac{2x^2-1}{2x^2\sqrt{x^2-1}}\right)^{k-1}\left(1+(-1)^{n-k}\right)$$
$$=\left(\frac{2x^2-1}{2x^2}\right)^{n-1}\sum_{k=0}^{n}\binom{n}{k}T_{2k}(x)\left(\frac{x}{(2x^2-1)\sqrt{x^2-1}}\right)^{k}\left(1-(-1)^{n-k}\right),$$
$$\sqrt{x^2-1}\sum_{k=0}^{n}\binom{n}{k}U_{2k}(x)\left(\frac{x}{(2x^2-1)\sqrt{x^2-1}}\right)^{k}\left(1+(-1)^{n-k}\right)$$
$$=\left(\frac{2x^2}{2x^2-1}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}T_{k}(x)\left(\frac{2x^2-1}{2x^2\sqrt{x^2-1}}\right)^{k}\left(\omega(x)-(-1)^{n-k}\omega^{-1}(x)\right),$$
$$\sqrt{x^2-1}\sum_{k=0}^{n}\binom{n}{k}\frac{U_{2k}(x)}{\left(2x\sqrt{x^2-1}\right)^{k}}\left(1+(-1)^{n-k}\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{T_{2k}(x)}{\left(2x\sqrt{x^2-1}\right)^{k}}\left(\omega(x)-(-1)^{n-k}\omega^{-1}(x)\right),$$
$$(x^2-1)\sum_{k=1}^{n}\binom{n}{k}U_{2k-1}(x)\left(\frac{x}{(2x^2-1)\sqrt{x^2-1}}\right)^{k}\left(1+(-1)^{n-k}\right)$$
$$=\left(\frac{2x^2}{2x^2-1}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}T_{k}(x)\left(\frac{2x^2-1}{2x^2\sqrt{x^2-1}}\right)^{k}\left(1-(-1)^{n-k}\right),$$
$$\sum_{k=1}^{n}\binom{n}{k}\frac{U_{2k-1}(x)}{\left(2x\sqrt{x^2-1}\right)^{k-1}}\left(\omega(x)+(-1)^{n-k}\omega^{-1}(x)\right)$$
$$=2x\sum_{k=0}^{n}\binom{n}{k}\frac{T_{k}(x)}{\left(2x\sqrt{x^2-1}\right)^{k}}\left(1-(-1)^{n-k}\right).$$

Remark 10. There also exist relations between balancing (Lucas-balancing) polynomials and Chebyshev polynomials of the third and fourth kinds, $V_n(x)$ and $W_n(x)$, respectively [10]. These polynomials satisfy the recurrence (19) with initial terms $V_0(x)=1$, $V_1(x)=2x-1$, and $W_0(x)=1$, $W_1(x)=2x+1$. Well-known relations $V_n(x)=\sqrt{\frac{2}{1+x}}\,T_{2n+1}\Big(\sqrt{\frac{1+x}{2}}\Big)$ and $W_n(x)=U_{2n}\Big(\sqrt{\frac{1+x}{2}}\Big)$ give

$$\sqrt{\frac{1+x}{2}}V_n(x)=C_{2n+1}\left(\sqrt{\frac{1+x}{18}}\right),\qquad W_n(x)=B_{2n+1}\left(\sqrt{\frac{1+x}{18}}\right).$$

5 Fibonacci-Lucas identities via balancing polynomial relations

The balancing and Lucas-balancing polynomials are closely related to the Fibonacci and Lucas numbers. Using this connection, in this section we obtain many Fibonacci-Lucas identities.

Let F_n denote the n-th Fibonacci number and L_n the n-th Lucas number, both satisfying the recurrence $u_n = u_{n-1} + u_{n-2}, n \geq 2$, but with the respective initial terms $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

Lemma 11. *For $n \geq 0$, the following relations hold:*

$$B_n\left(\frac{1}{2}\right) = F_{2n}, \qquad C_n\left(\frac{1}{2}\right) = \frac{L_{2n}}{2}. \tag{21}$$

Proof. The first formula follows from (1) and the fact that sequence $(F_{2n})_{n\geq 0}$ satisfies the recurrence $u_n = 3u_{n-1} - u_{n-2}$, $n \geq 2$. The proof of the second formula is omitted. □

Using (21), from Theorems 2 and 3 we immediately can obtain the following summation Fibonacci-Lucas identities.

Corollary 12. *For $n \geq 1$, we have*

$$3F_{2n-1} = L_{4n-2} - 4\sum_{k=1}^{n-1} F_{2k}L_{4n-4k-2},$$

$$2F_{2n-1} - 5F_{2n-2} = L_{4n-4} - 4\sum_{k=1}^{n-1} F_{2k}L_{4n-4k-4},$$

$$2F_{4n+2} - 3F_{4n-2} = L_{2n} + L_{2n-2} + 4\sum_{k=0}^{n-1} F_{4k+2}L_{2n-2k-2},$$

$$2F_{4n} - 3F_{4n-4} = 3L_{2n-2} + 2\sum_{k=1}^{n-1} F_{4k}L_{2n-2k-2}.$$

Next, by (21), from Theorem 5 and 6 we obtain Fibonacci-Lucas identities involving binomial coefficients.

Corollary 13. *Let α be the golden ratio, $\alpha = (1+\sqrt{5})/2$, and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. For $n \geq 1$, we have*

$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{\sqrt{5}}\right)^k\left(1+(-1)^{n-k}\right)F_{2k} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{\sqrt{5}}\right)^k\left(1-(-1)^{n-k}\right)L_{2k},$$

$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^k\left(\alpha^2+(-1)^{n-k}\beta^2\right)F_{4k+2} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^k\left(\alpha^2-(-1)^{n-k}\beta^2\right)L_{4k+2},$$

$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^k\left(1+(-1)^{n-k}\right)F_{4k} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^k\left(1-(-1)^{n-k}\right)L_{4k},$$

$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{14}{9\sqrt{5}}\right)^{k}\big(\alpha^2+(-1)^{n-k}\beta^2\big)F_{2k}=\left(\frac{7}{9}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{6}{7\sqrt{5}}\right)^{k}\big(1-(-1)^{n-k}\big)L_{4k+2},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{14}{9\sqrt{5}}\right)^{k}\big(1+(-1)^{n-k}\big)F_{2k}=\left(\frac{7}{9}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{6}{7\sqrt{5}}\right)^{k}\big(1-(-1)^{n-k}\big)L_{4k},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{7\sqrt{5}}\right)^{k}\big(1+(-1)^{n-k}\big)F_{4k+2}=\left(\frac{9}{7}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{14}{9\sqrt{5}}\right)^{k}\big(\alpha^2-(-1)^{n-k}\beta^2\big)L_{2k},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^{k}\big(1+(-1)^{n-k}\big)F_{4k+2}=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^{k}\big(\alpha^2-(-1)^{n-k}\beta^2\big)L_{4k},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{7\sqrt{5}}\right)^{k}\big(1+(-1)^{n-k}\big)F_{4k}=\left(\frac{9}{7}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{14}{9\sqrt{5}}\right)^{k}\big(1-(-1)^{n-k}\big)L_{2k},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^{k}\big(\alpha^2+(-1)^{n-k}\beta^2\big)F_{4k}=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\sqrt{5}}\right)^{k}\big(1-(-1)^{n-k}\big)L_{4k+2}.$$

The next result relates balancing and Lucas-balancing polynomials to Fibonacci and Lucas numbers [2].

Lemma 14. *For $n \geq 0$ and $s \geq 1$, the following hold:*

$$B_n\left(\frac{\varepsilon_n}{6}\,L_s\right)=\varepsilon_n^{n-1}\frac{F_{sn}}{F_s},\qquad C_n\left(\frac{\varepsilon_n}{6}\,L_s\right)=\varepsilon_n^{n}\frac{L_{sn}}{2},\tag{22}$$

where $\varepsilon_n=\begin{cases}1, & \text{if } n \text{ is even;}\\ i, & \text{otherwise.}\end{cases}$

The following corollary is an immediate consequence of Theorems 2, 3 and (22).

Corollary 15. *For $n, m \geq 0$, we have*

$$\begin{gathered}
2F_{sn}=F_sL_{s(n-1)}+L_sF_{s(n-1)},\\
L_s\big(F_{s(2n+1)}-(-1)^sF_{s(2n-1)}\big)=F_{2s}\big(L_{s(2n+1)}+(-1)^sL_{s(2n-1)}\big),\\
2F_{2sn}-\big(L_s^2-(-1)^s2\big)F_{2s(n-1)}=F_sL_sL_{2s(n-1)},\\
L_s\big(F_{sn}-(-1)^s\varepsilon_sF_{s(n-1)}\big)\\
=(-1)^s\varepsilon_s^{n+1}F_sL_{s(2n-1)}-(-1)^s\big(\varepsilon_s^2L_{2m}^2-\varepsilon_sL_{2m}-2\big)\sum_{k=1}^{n-1}\varepsilon_s^{n-k}F_{sk}L_{s(2n-2k-1)},\\
2F_{sn}-\varepsilon_s(L_{2m}^2-(-1)^s2)F_{s(n-1)}\\
=(-1)^s\varepsilon_s^{n+1}F_sL_{2s(n-1)}+\big(\varepsilon_s^2L_s^2+\varepsilon_sL_s+2\big)\sum_{k=1}^{n-1}\varepsilon_s^{n-k}F_{sk}L_{2s(n-k-1)},
\end{gathered}$$

$$
\begin{gathered}
\varepsilon_s^n\big(2\varepsilon_s F_{s(2n+1)} - L_s F_{s(2n-1)}\big)\\
= F_s\big(\varepsilon_s L_{sn} + L_{s(n-1)}\big) - \big(\varepsilon_s^2 L_s^2 + \varepsilon_s L_s + 2\big)\sum_{k=0}^{n-1}\varepsilon_s^k F_{s(2k+1)} L_{s(n-k-1)},\\
2F_{s(2n+1)} - (L_s^2 - (-1)^s 2)F_{s(2n-1)} = F_s(L_{2sn} + (-1)^s L_{2s(n-1)}),\\
\varepsilon_s^n(2F_{2sn} - (-1)^s\varepsilon_s L_s F_{2s(n-1)})\\
= \varepsilon_s F_s L_s L_{s(n-1)} - (\varepsilon_s^2 L_s^2 + \varepsilon_s L_s + 2)\sum_{k=1}^{n-1}\varepsilon_s^{k+1} F_{2sk} L_{s(n-k-1)},\\
F_{2sn} - (-1)^s F_{2s(n-1)} = F_s L_{s(2n-1)}.
\end{gathered}
$$

Our last statement follows from Theorems 5, 6 and (22).

Corollary 16. *For $n, s \geq 0$, we have*

$$
\begin{gathered}
\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_s}{2}\right)^{n-k}\big(1+(-1)^{n-k}\big)F_{ks} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_s}{2}\right)^{n-k}\big(1-(-1)^{n-k}\big)L_{ks},\\
\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}\big(\alpha^s + (-1)^{n-k}\beta^s\big)F_{(2k+1)s}\\
= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}\big(\alpha^s - (-1)^{n-k}\beta^s\big)L_{(2k+1)s},\\
\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}\big(1+(-1)^{n-k}\big)F_{2ks} = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}\big(1-(-1)^{n-k}\big)L_{2ks},\\
\sqrt{5}L_{2s}^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}L_s}{2L_{2s}}\right)^{n-k}(\alpha^s + (-1)^{n-k}\beta^s)F_{ks}\\
= L_s^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}L_{2s}F_s}{2L_s}\right)^{n-k}(1-(-1)^{n-k})L_{(2k+1)s},\\
\sqrt{5}L_{2s}^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}L_s}{2L_{2s}}\right)^{n-k}(1+(-1)^{n-k})F_{ks}\\
= L_s^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}L_{2s}F_s}{2L_s}\right)^{n-k}(1-(-1)^{n-k})L_{2ks},\\
\sqrt{5}(2L_s)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}L_{2s}F_s}{2L_s}\right)^{n-k}(1+(-1)^{n-k})F_{(2k+1)s}\\
= (L_{2s})^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}L_s}{L_{2s}}\right)^{n-k}(\alpha^s - (-1)^{n-k}\beta^s)L_{ks},
\end{gathered}
$$

$$\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}(1+(-1)^{n-k})F_{(2k+1)s}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}(\alpha^s-(-1)^{n-k}\beta^s)L_{ks},$$
$$\sqrt{5}L_s^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}L_{2s}F_s}{2L_s}\right)^{n-k}(1+(-1)^{n-k})F_{2ks}$$
$$=L_{2s}^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}L_s}{2L_{2s}}\right)^{n-k}(1-(-1)^{n-k})L_{ks},$$
$$\sqrt{5}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}(\alpha^s+(-1)^{n-k}\beta^s)F_{2ks}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}F_{2s}}{2}\right)^{n-k}(1-(-1)^{n-k})L_{(2k+1)s},$$

where $\alpha=(1+\sqrt{5})/2$ *and* $\beta=(1-\sqrt{5})/2$.

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