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Spectra of Algebras of Analytic Functions and Polynomials on Banach Spaces

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Abstract. The paper contains a survey of basic facts about spectra of algebras of analytic functions of bounded type on Banach spaces and some new results in this area. In particular we prove an analogue of Hilbert Nullstellensatz Theorem for polynomials on Banach spaces and investigate some derivations on algebras of analytic functions of bounded type which are associated with points of their spectra.

Introduction

Let A be a complex commutative topological algebra. We denote by $M(A)$ the spectrum of A that is the set of continuous complex-valued homomorphisms of A. $M(A)$ can be naturally identified with the set of closed maximal ideals of A. Recall that A is *semisimple* if the complex homomorphisms from $M(A)$ separate points of A. It is well known that every semisimple commutative Fréchet algebra A is isomorphic to some subalgebra of continuous functions on $M(A)$ endowed with a natural topology. More exactly, for every $a \in A$ there exists a function $\widehat{a} : M(A) \to \mathbb{C}$ defined by $\widehat{a}(\phi) := \phi(a)$. The weakest topology on $M(A)$ such that all functions $\widehat{a}$, $a \in A$, are continuous is called the *Gelfand topology.* The Gelfand topology coincides with the weak-star topology of the strong dual space A' restricted to $M(A)$. If A is a Banach algebra, $M(A)$ is a weak-star compact subset of the unit ball of A'.

If A is a uniform algebra of continuous functions on a metric space G, then for any $x \in G$ the *point evaluation functional* $\delta(x) : f \mapsto f(x)$ belongs to $M(A)$. So we can see that the underlying set G may be identified with a subset of $M(A)$. Moreover, we can consider $M(A)$ as the most natural domain for elements of A. For example, every C^* commutative Banach algebra A is isomorphic to the algebra of all continuous (with respect to the Gelfand topology) functions on $M(A)$. On the other hand, if A consists of all continuous and bounded functions on a metric space G, then $M(A)$ is homeomorphic to the Čech-Stone compactification of G. If A is a uniform algebra of holomorphic functions on an analytic manifold, we can

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ask about analytic structure on the set of maximal ideals, $M(A)$. For instance, the spectrum of the algebra of all analytic functions on an open subset $\Omega \subset \mathbb{C}^n$ which are continuous on the closure $\overline{\Omega}$ coincides with the polynomially convex hull of Ω in $\mathbb{C}^n$. In particular, $M(H(\mathbb{C}^n)) = \mathbb{C}^n$, where $H(\mathbb{C}^n)$ is the algebra of entire functions on $\mathbb{C}^n$.

This paper is a survey of basic results about spectra of algebras of analytic functions of bounded type on Banach spaces and a presentation of some new results in this area.

In the first section we consider basic definitions and preliminary results about analytic functions on Banach spaces. Section 2 is devoted to maximal ideals of algebras of polynomials on Banach spaces. Using results of Aron, Cole and Gamelin we prove an analogue of the Hilbert Nullstellensatz Theorem for polynomials on Banach spaces. In the third section we consider some applications of the Nullstellensatz for symmetric polynomials. In Section 4 we refer recent results from [**Z**] about the description of ideals of the algebra $H_b(X)$ of analytic functions of bounded type on a Banach space X. Some examples and applications for homomorphisms on $H_b(X)$ are given in Section 5 and Section 6. In Section 7 we describe some new derivations on $H_b(X)$ which are associated with points of M_b. Section 8 is a short survey of related topics on spectra of algebras of analytic functions on Banach spaces.

1. Definitions and Preliminaries

Let X and Y be complex Banach spaces. A mapping P from X to Y is called a continuous n-homogeneous polynomial if there exists a symmetric continuous n-linear map A^P from n-Cartesian product of X to Y such that $P(x) = A^P(x, \dots, x)$. The Banach space of all n-homogeneous polynomials from the X to Y endowed with the sup-norm topology on the unit ball of X is denoted by $\mathcal{P}(^nX, Y)$. A map $P : X \to Y$ is said to be a polynomial of degree n if $P = P_0 + P_1 + \cdots + P_n$, where $P_0 \in Y$, $P_k \in \mathcal{P}(^kX, Y)$ and $P_n \neq 0$. The space of all polynomials from X to Y will be denoted by $\mathcal{P}(X, Y)$. We denote the spaces $\mathcal{P}(^kX, \mathbb{C})$ and $\mathcal{P}(X, \mathbb{C})$ by $\mathcal{P}(^kX)$ and $\mathcal{P}(X)$ respectively. Note that $\mathcal{P}(X)$ is a topological algebra endowed with the locally convex topology of uniform convergence on bounded sets. We will use the notation $\mathcal{P}(^{\leq n}X, Y)$ for the space of $\mathbb{C}$-valued m-degree polynomials on X, $m \leq n$.

$P \in \mathcal{P}(X)$ is called a *polynomial of finite type* if it is a finite sum of finite products of linear functionals. The space of n-homogeneous finite type polynomials is denoted by $\mathcal{P}_f(^nX)$. The closure of $\mathcal{P}_f(^nX)$ in the topology of uniform convergence on bounded sets is called the space of *approximable polynomials* and denoted by $\mathcal{P}_A(^nX)$. Each approximable polynomial is weakly continuous on bounded sets. The converse statement is true if X' has the approximation property [**AP**]. In [**ACG2**] it is constructed an example of a Banach space X without the approximation property such that $\mathcal{P}_A(^nX)$ is a proper subspace in the space of all weakly continuous polynomials on bounded sets.

Ω is a *finitely open subset* of a Banach space X if for any finite dimensional affine subspace E of X, endowed with the Euclidean topology, $E \cap \Omega$ is open in E.

Definition 1.1. We say that a map $f : \Omega \to Y$ is *G-analytic* (*Gâteaux-analytic*), and write $f \in H_G(\Omega, Y)$, if the restriction of f to $E \cap \Omega$ is analytic for any finite dimensional affine subspace E (or, equivalently, for any complex line

$E \in X$). A G-analytic map defined on an open subset $\Omega \subset X$ to Y is called *analytic*, written $f \in H(\Omega, Y)$, if it is continuous.

Every analytic function $f \in H(\Omega, Y)$ can be locally represented by its Taylor's series expansion

$$f(a+x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n f(a)(x, \ldots, x)$$

which converges uniformly on a neighborhood of $a \in \Omega$, where $d^n f(a)(x, \ldots, x) \in \mathcal{P}({}^n X)$ is the nth Fréchet derivative of f at a in the direction $(x, \ldots, x)$.

The proof of Proposition 1.2 and Theorem 1.3 can be found in [**D1**] or [**H**].

Proposition 1.2. *Let f_k be a sequence of continuous k-homogeneous polynomials from X to Y. A necessary and sufficient condition for existence of $f \in H(X, Y)$ such that $f_k = d^k f(0)$ is that for any given $\epsilon > 0$ each $x \in X$ has a neighborhood U such that $\sup_U \|f_k\|^{1/k} \le \epsilon$ for k large enough.*

Let $f \in H(\Omega, Y)$, where Ω is an open subset of X, and $x \in \Omega$. The *radius of uniform convergence* $\varrho_x(f)$ of f at x is defined as supremum of λ, $\lambda \in \mathbb{C}$ such that $x + \lambda B \subset \Omega$ and the Taylor series of f at x converges to f uniformly on $x + \lambda B$, where B is the unit ball of X. The *radius of boundedness* of f at x is defined as supremum of λ, $\lambda \in \mathbb{C}$ such that f is bounded on $x + \lambda B$.

Theorem 1.3. *The radius of uniform convergence of f at x coincides with the radius of boundedness of f at x and if $f \in H(X, Y)$, then*

$$\varrho_0(f) := \left(\limsup_{n \to \infty} \|f_n\|^{1/n} \right)^{-1},$$

where $f_n = d^k(x) f / n!$.

Denote by $H_b(X, Y)$ the space of Y-valued entire functions of *bounded type*, that is, the space of all entire mappings from X to Y which are bounded on bounded subsets (i.e. have the radius of boundedness equal to infinity). Note that if X is an infinite dimensional Banach space, then there exists a $\mathbb{C}$-valued entire function on X, f, such that $\varrho_x(f) < \infty$ for every $x \in X$ (see e.g. [**D1**], p.169). The space $H_b(X) = H_b(X, \mathbb{C})$ is a Fréchet algebra endowed with topology, generated by seminorms

$$\|f\|_r = \sup\{|f(x)| : x \in X, \|x\| < r\},$$

where $r > 0$ varies over the rational numbers.

Each linear functional $\phi \in H_b(X)'$ is continuous with respect to the norm of uniform convergence on some ball in X. The *radius function* $R(\phi)$ of ϕ is defined as the infimum of all $r > 0$ such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB.

Denote by ϕ_n the restriction of ϕ to the subspace of n-homogeneous polynomials $\mathcal{P}({}^n X)$. Then ϕ_n is a continuous linear functional on $\mathcal{P}({}^n X)$ and

$$\|\phi_n\| = \sup\{\phi(P) : P \in \mathcal{P}({}^n X), \|P\| \le 1\}.$$

The following theorems are some kind of dual versions of Proposition 1.2 and Theorem 1.3.

THEOREM 1.4. (Aron, Cole, Gamelin [**ACG1**]). *The radius function R on $H_b(X)'$ is given by*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

THEOREM 1.5. (Aron, Cole, Gamelin [**ACG1**]). *Suppose that $\phi_n \in \mathcal{P}(^nX)'$ for $n \geq 0$, and suppose that the norms of ϕ_n satisfy $\|\phi_n\| \leq cs^n$ for some $c, s > 0$. Then there is a unique $\phi \in H_b(X)'$ whose restriction to $\mathcal{P}(^nX)$ coincides with ϕ_n, $n \geq 0$.*

2. Spectra of Algebras of Polynomials

As we have indicated the spectrum of the space of entire functions on $\mathbb{C}^n$ can be identified with $\mathbb{C}^n$ as a point set. The following results which were proved by Aron, Cole and Gamelin show that in the infinite dimensional case the role of the points of the underlying space play polynomially convergent nets.

THEOREM 2.1. (Aron, Cole, Gamelin [**ACG1**]). *Let Y be a complex vector space. Let A be an algebra of functions on Y such that the restriction of each $f \in A$ to any finite dimensional subspace of Y is an analytic polynomial. Let I be a proper ideal in A. Then there is a net (y_α) in Y such that $f(y_\alpha) \to 0$ for all $f \in I$.*

COROLLARY 2.2. *Let ϕ be any (possibly discontinuous) complex-valued homomorphism of $H_b(X)$. Then there is a net (x_α) in x such that $P(x_\alpha) \to \phi(P)$ for all analytic polynomials P on X.*

For a given uniform algebra A of continuous functions on a Banach space X we define a *A-topology* on X as the weakest topology such that all functions of A are continuous. So the A-topology is just the restriction of the Gelfand topology to X.

PROPOSITION 2.3. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. Then for every bounded $\mathcal{P}_0$-convergent net $(x_\alpha) \in X$ there is a continuous complex-valued homomorphism ϕ on $\mathcal{P}_0(X)$ such that $P(x_\alpha) \to \phi(P)$ for each $P \in \mathcal{P}_0(X)$.*

PROOF. It is easy to see that $\phi(P) := \lim_\alpha P(x_\alpha)$ is a complex-valued homomorphism on $\mathcal{P}_0(X)$. From the boundedness of x_α it follows that ϕ is continuous. □

Here we need a technical lemma.

LEMMA 2.4. (Aron, Cole, Gamelin [**ACG1**]). *Let Y be a complex vector space. Let $F = (f_1, \ldots, f_n)$ be a map from Y to $\mathbb{C}^n$ such that the restriction of each f_j to any finite dimensional space of Y is a polynomial. Then the closure of the range of F, $F(X)^-$ is an algebraic variety. Moreover there exists a finite dimensional subspace $Y_0 \subset X$ such that $F(Y_0)^- = F(X)^-$.*

THEOREM 2.5. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ with unity which contains all finite type polynomials. Let J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite number of polynomials $P_1, \ldots, P_n \in \mathcal{P}_0(X)$. If the polynomials $P_1, \ldots, P_n$ have no common zeros, then J is not proper.*

PROOF. According to Lemma 2.4 there exists a finite dimensional subspace $Y_0 = \mathbb{C}^m \subset X$ such that $F(Y_0)^- = F(X)^-$, where $F(x) = (P_1(x), \ldots P_n(x))$. Let $e_1, \ldots, e_m$ be a basis in Y_0 and $e_1^*, \ldots, e_m^*$ be the coordinate functionals. Denote by $P_k\mid_{Y_0}$ the restriction of P_k to Y_0. Since $\dim Y_0 = m < \infty$, there exists a continuous projection $T : X \to Y_0$. So any polynomial $Q \in \mathcal{P}(Y_0)$ can be extended to a polynomial $\widehat{Q} \in \mathcal{P}_0(X)$ by formula $\widehat{Q}(x) = Q(T(x))$. $\widehat{Q}$ belongs to $\mathcal{P}_0(X)$ because it is a finite type polynomial. Let us consider the map

$$G(x) = (P_1(x), \dots, P_n(x), \widehat{e_1^*}(x), \dots, \widehat{e_m^*}(x)) : X \to \mathbb{C}^{m+n}.$$

By definition of G, $G(X)^- = G(Y_0)^-$.

Suppose that J is a proper ideal in $\mathcal{P}_0(X)$ and so J is contained in a maximal ideal J_M. Let ϕ be a complex homomorphism such that $J_M = \ker \phi$. By Theorem 2.1 there exists a $\mathcal{P}_0$-convergent net (x_α) such that $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $P \in \mathcal{P}_0(X)$. Since $G(X)^- = G(Y_0)^-$, there is a net $(z_\beta) \subset Y_0$ such that $\lim_\alpha G(x_\alpha) = \lim_\beta G(z_\beta)$. Note that each polynomial $Q \in \mathcal{P}(Y_0)$ is generated by the coordinate functionals. Thus $\lim_\beta Q(z_\beta) = \lim_\alpha \widehat{Q}(x_\alpha) = \phi(Q)$. Also $\lim_\beta P_k \mid_{Y_0} (z_\beta) = \lim_\alpha P_k(x_\alpha) = \phi(P_k)$, $k = 1, \dots, n$. On the other hand, every $\mathcal{P}_0$-convergent net on a finite dimensional subspace is weakly convergent and so it converges to a point $x_0 \in Y_0 \subset X$. Thus $P_k(x_0) = 0$ for $1 \le k \le n$ that contradicts the assumption that $P_1, \dots, P_n$ have no common zeros. $\square$

For an ideal $J \in \mathcal{P}_0(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J. Let G be a subset of X. Then $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ which vanish on G. The set $\mathrm{Rad}J$ is called the *radical* of J if $P^k \in J$ for some positive integer k implies $P \in \mathrm{Rad}J$. P is called a *radical* polynomial if it can be represented by a product of mutually different irreducible polynomials. In this case $(P) = \mathrm{Rad}(P)$.

A subalgebra A_0 of an algebra A is called *factorial* if for every $f \in A_0$ the equality $f = f_1 f_2$ implies that $f_1 \in A_0$ and $f_2 \in A_0$.

Using a standard idea from Algebraic geometry, now we can prove the next theorem which is a generalization of the well known Hilbert Nullstellensatz for algebras of polynomials of infinitely many variables.

THEOREM 2.6. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and let J be an ideal of $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \dots, P_n$. Then* $\mathrm{Rad}J \subset \mathcal{P}_0(X)$ *and*

$$I[V(J)] = \mathrm{Rad}J$$

in $\mathcal{P}_0(X)$.

PROOF. Since $\mathcal{P}_0(X)$ is factorial, $\mathrm{Rad}J \subset \mathcal{P}_0(X)$ for every ideal $J \subset \mathcal{P}_0(X)$. Evidently, $I[V(J)] \supset \mathrm{Rad}J$. Let $P \in \mathcal{P}_0(X)$ and $P(x) = 0$ for every $x \in V(J)$. Let $y \in \mathbb{C}$ be an additional "independent variable" which is associated with a basis vector e of an extra dimension. Consider a Banach space $X \oplus \mathbb{C}e = \{x + ye \colon x \in X, y \in \mathbb{C}\}$. We denote by $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ the algebra of polynomials on $X \oplus \mathbb{C}e$ such that every polynomial in $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ belongs to $\mathcal{P}_0(X)$ for arbitrary $y \in \mathbb{C}$. The polynomials $P_1, \dots, P_n, Py - 1$ have no common zeros. By Theorem 2.5 there are polynomials $Q_1, \dots, Q_{n+1} \in \mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ such that

$$\sum_{i=1}^{n} P_i Q_i + (Py - 1) Q_{n+1} \equiv 1.$$

Since it is an identity it will be still true for all vectors x such that $P(x) \neq 0$ if we substitute $y = 1/P(x)$. Thus

$$\sum_{i=1}^{n} P_i(x)Q_i(x, 1/P(x)) = 1.$$

Taking a common denominator, we find that for some positive integer N,

$$\sum_{i=1}^{n} P_i(x)Q_i'(x)P^{-N}(x) = 1$$

or

$$\sum_{i=1}^{n} P_i(x)Q_i'(x) = P^N(x), \tag{2.1}$$

where $Q'(x) = Q(x, P^{-1})P^N(x) \in \mathcal{P}_0(X)$. The equality (2.1) holds on an open subset $X \setminus \ker P$, so it holds for every $x \in X$. But it means that P^N belongs to J. So $P \in \mathrm{Rad}J$. $\square$

3. Applications for Symmetric Polynomials

Let $\mathcal{G}$ be a group of linear isometries of X. A subset V of X is said to be $\mathcal{G}$*-symmetric* if it is invariant under the action of $\mathcal{G}$ on X. A function with a $\mathcal{G}$-symmetric domain is $\mathcal{G}$*-symmetric* if $f(\sigma(x)) = f(x)$ for every $\sigma \in \mathcal{G}$. It is clear that the kernel of a $\mathcal{G}$-symmetric polynomial is $\mathcal{G}$-symmetric. We consider the question: under which conditions a polynomial with a $\mathcal{G}$-symmetric set of zeros is $\mathcal{G}$-symmetric?

First we observe that if $P(x)$ is an irreducible polynomial then $P(\sigma(x))$ is irreducible for every $\sigma \in \mathcal{G}$. Indeed, if $P(\sigma(x)) = P_1(x)P_2(x)$, then

$$P(x) = P_1(\sigma^{-1}(x))P_2(\sigma^{-1}(x)).$$

Recall that a group homomorphism of $\mathcal{G}$ to $S^1 = \{e^{i\vartheta} : 0 \leq \vartheta < 2\pi\}$ is called a *character* of $\mathcal{G}$.

Proposition 3.1. *Suppose $\mathcal{G}$ has no nontrivial characters. If P is radical and $\ker P$ is a $\mathcal{G}$-symmetric set, then P is a $\mathcal{G}$-symmetric polynomial.*

Proof. Since $\ker P = \ker P \circ \sigma$ for every $\sigma \in \mathcal{G}$, then, by Theorem 2.6, $P = cP \circ \sigma$ for some constant c. Because σ is an isometry, $|c| = 1$. If $c \neq 1$, then $c = c(\sigma)$ is a nontrivial character of $\mathcal{G}$. So $c = 1$. $\square$

Suppose, for example that $\mathcal{G} = S^1$, that is, the group of actions $x \rightsquigarrow e^{i\vartheta}x$. Then a homogeneous polynomial is $\mathcal{G}$-symmetric only if it is a constant. However, the zero set of any homogeneous polynomial is S^1-symmetric.

Note that the subset of all $\mathcal{G}$-symmetric polynomials is a subalgebra in $\mathcal{P}(X)$.

Theorem 3.2. *Suppose that the algebra of $\mathcal{G}$-symmetric polynomials on X is factorial and that $\mathcal{G}$ has no nontrivial characters. Then the kernel of a polynomial P is $\mathcal{G}$-symmetric if and only if P is $\mathcal{G}$-symmetric.*

Proof. Let $P = P_1^{k_1} \dots P_n^{k_n}$, where $P_1, \dots, P_n$ are mutually different irreducible polynomials. Then $P_1 \dots P_n$ has the same zero set as P. So if $\ker P$ is $\mathcal{G}$-symmetric, then by Proposition 3.1, $P_1 \dots P_n$ is $\mathcal{G}$-symmetric. By the assumption of the theorem, all polynomials $P_1, \dots, P_n$ must be $\mathcal{G}$-symmetric. So P is $\mathcal{G}$-symmetric as well. $\square$

Note that if there exist a $\mathcal{G}$-symmetric polynomial $P = P_1P_2$ such that P_1 is not $\mathcal{G}$-symmetric, then $P_1^2P_2$ is not a $\mathcal{G}$-symmetric polynomial with a $\mathcal{G}$-symmetric kernel.

If X is an infinite dimensional space ℓ_p, $1 \leq p < \infty$ and $\mathcal{G}$ is the group of permutations of the canonical basis elements, then it is not difficult to see that the algebra of $\mathcal{G}$-symmetric polynomials is factorial and $\mathcal{G}$ has no nontrivial characters. For any n-dimensional space, $1 < n < \infty$ there exists a nonsymmetric polynomial which has a symmetric kernel. For example $P(x) = x_1^2x_2 \dots x_n$ has a symmetric kernel in $\mathbb{C}^n$ but is not symmetric if $n > 1$.

4. The Spectrum of $H_b(X)$

All results of this section were proved in [**Z**].

Let us denote by $A_n(X)$ the closure of the algebra, generated by all polynomials in $\mathcal{P}(^{\leq n}X)$ with respect to the topology of uniform convergence on bounded sets. It is clear that $A_1(X) \cap \mathcal{P}(^nX) = \mathcal{P}_A(^nX)$ and $A_n(X)$ is a Fréchet algebra of entire analytic functions on X for every n. The closure of the algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets coincides with $H_b(X)$. We will use the short notation M_b for the spectrum $M(H_b(X))$.

Lemma 4.1. *Let $\phi \in H_b(X)'$ such that $\phi(P) = 0$ for every $P \in \mathcal{P}(^mX) \cap A_{m-1}(X)$, where m is a fixed positive integer and $\phi_m \neq 0$. Then there is $\psi \in M_b$ such that $\psi_k = 0$ for $k < m$ and $\psi_m = \phi_m$. The radius function $R(\psi) = \|\phi_m\|^{1/m}$.*

Idea of proof. For every polynomial $P \in \mathcal{P}(^{mk}X)$ we denote by $P_{(m)}(u)$ the polynomial from $\mathcal{P}(^k\otimes^m_{s,\pi}X)$ such that $P_{(m)}(x^{\otimes m}) = P(x)$, where $x^{\otimes m} = \underbrace{x \otimes \cdots \otimes x}_{m}$.

Since $\phi_m \neq 0$, there is an element $w \in (\otimes^m_{s,\pi}X)''$, $w \neq 0$ such that for any m-homogeneous polynomial P, $\phi(P) = \phi_m(P) = \widetilde{P}_{(m)}(w)$, where $\widetilde{P}_{(m)}$ is the Aron-Berner extension of linear functional $P_{(m)}$ from $\otimes^m_{s,\pi}X$ to $(\otimes^m_{s,\pi}X)''$ and $\|w\| = \|\phi_m\|$. For an arbitrary n-homogeneous polynomial Q we set

$$\psi(Q) = \begin{cases} \widetilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\widetilde{Q}_{(m)}$ is the Aron-Berner extension of the k-homogeneous polynomial $Q_{(m)}$ from $\otimes^m_{s,\pi}X$ to $(\otimes^m_{s,\pi}X)''$. Next we extend ψ to a continuous complex homomorphism on $H_b(X)$ by linearity, distributivity and continuity.

The verification that ψ is well defined and multiplicative is a little bit technical (see [**Z**] for details). □

For each fixed $x \in X$, the translation operator T_x is defined on $H_b(X)$ by

$$(T_xf)(y) = f(y + x), \qquad f \in H_b(X).$$

It is not complicated to check that $T_xf \in H_b(X)$ and for fixed $\phi \in H_b(X)'$ the function $x \mapsto \phi(T_xf)$, $x \in X$, belongs to $H_b(X)$ (see [**ACG1**]).

For fixed $\phi, \theta \in H_b(X)'$ the convolution product $\phi * \theta$ in $H_b(X)$ is defined by

$$(\phi * \theta)(f) = \phi(\theta(T_xf)), \qquad f \in H_b(X).$$

Let $\phi, \theta \in M_b$. By Corollary 2.2, there exist nets $(x_\alpha), (y_\beta) \subset X$ such that

$$\phi(P) = \lim_\alpha P(x_\alpha), \qquad \theta(P) = \lim_\beta P(y_\beta) \tag{4.2}$$

for every polynomial P. Thus for every polynomial P we have:

$$(\phi * \theta)(P) = \lim_{\beta} \lim_{\alpha} P(x_\alpha + y_\beta).$$

Note that M_b is a semigroup with respect to the convolution product and $\phi*\theta \neq \theta*\phi$ in general (see [**AGGM**, Remark 3.5]). We denote $\phi_1 * \cdots * \phi_n$ briefly by $\mathop{\ast}\limits_{k=1}^{n} \phi_k$.

Let I_k be the minimal closed ideal in $H_b(X)$, generated by all m-homogeneous polynomials, $0 < m \leq k$. Evidently, I_k is a proper ideal (it contains no unit) so it is contained in a closed maximal ideal (see [**M1**, p. 228]). Let

$$\Phi_k := \{\phi \in M_b \ : \ \ker \phi \supset I_k\}.$$

We set $\Phi_0 := M_b$. The functional $\delta(0)$, that is the point evaluation at zero, belongs to Φ_k for every $k > 0$.

Corollary 4.2. *If $A_m(X) \neq A_{m-1}(X)$ for some $m > 1$, then there exists $\psi \in \Phi_{m-1}$ such that $\psi \notin \Phi_m$.*

Note that $A_1(c_0) = A_n(c_0)$ for every n, but $A_k(\ell_p) = A_m(\ell_p)$ for $k \neq m$ if and only if $k < p$ and $m < p$. Moreover, if X admits a polynomial which is not weakly sequentially continuous, then the chain of algebras $\{A_k(X)\}$ does not stabilize and if X contains ℓ_1, then $A_k(X) \neq A_m(X)$ for $k \neq m$ [**Gon, DiGo**].

Lemma 4.3. *If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-1}$, then $\phi * \psi(P) = \phi(P) + \psi(P)$ for every $P \in \mathcal{P}(^kX)$.*

Given a sequence $(\phi_n)_{n=1}^\infty \subset M_b$, $\phi_n \in \Phi_{n-1}$, the infinite convolution $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n(P) = \mathop{\ast}\limits_{n=1}^{k} \phi_n(P)$ if $P \in \mathcal{P}(^kX)$ for an arbitrary k. This multiplicative functional uniquely determines a functional in M_b (which we denote by the same symbol $\mathop{\ast}\limits_{n=1}^{\infty} \phi_n$) if it is continuous.

The point evaluation operator δ maps X into M_b by $x \mapsto \delta(x)$, $\delta(x)(f) = f(x)$. The operator $\widetilde{\delta}$ is the extension of δ onto X'', i.e. $\widetilde{\delta}(x'')(f) = \widetilde{f}(x'')$ for every $x'' \in X''$.

Theorem 4.4. *There exists a sequence of dual Banach spaces $(E_n)_{n=1}^\infty$ and a sequence of maps $\delta^{(n)} : E_n \to M_b$ such that $E_1 = X''$, $E_n = \mathcal{P}(^nX)' \cap I_{n-1}^\perp$, $\delta^{(1)} = \widetilde{\delta}$ and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation*

$$\phi = \mathop{\ast}\limits_{n=1}^{\infty} \delta^{(n)}(u_n) \tag{4.3}$$

for some $u_n \in E_n$, $n = 1, 2, \ldots$.

Furthermore, every complex homomorphism ϕ on $\mathcal{P}(X)$ which is defined by 4.3 is continuous and so it can be extended to a continuous complex homomorphism on $H_b(X)$ if and only if

$$R(\phi) = \limsup_{m\to\infty} \left\| \mathop{\ast}\limits_{n=1}^{m} \delta^{(n)}(u_n) \right\|^{1/m} < \infty.$$

Let us denote by $\mathbb{E}^\infty$ the space of all finite sequences $(u_1, \ldots, u_m, 0, \ldots)$, $u_k \in E_k$. According to Theorem 4.4, every finite sequence $\mathfrak{u} = (u_1, \ldots, u_m, 0, \ldots)$ defines a character $\phi_{\mathfrak{u}} = \mathop{\ast}\limits_{k=1}^{m} \delta^{(k)}(u_k) \in M_b$. Thus $\mathbb{E}^\infty \subset M_b$ and for every $\mathfrak{u}, \mathfrak{v} \in \mathbb{E}^\infty$, $\phi_{\mathfrak{u}+\mathfrak{v}} \in M_b$. Moreover, from the density of the polynomials in $H_b(X)$ it follows that $\mathbb{E}^\infty$ is dense in M_b with respect to the Gelfand topology. So we have proved the following theorem.

THEOREM 4.5. *M_b contains the dense linear subspace $\mathbb{E}^\infty$ of all finite sequences $(u_1, \dots, u_m, 0, \dots)$, $u_k \in E_k$.*

5. Examples

Let X be a Banach space such that $\mathcal{P}(X) = \mathcal{P}_A(X)$. Then $A_1 = A_n$ for every n and from Theorem 4.4 it follows that $M_b = X''$. This holds, for example, if $X = c_0$. Tsirelson [**Ts**] constructed a reflexive Banach space T, with an unconditional basis which contains no isomorphic copy of any ℓ_p. Alencar, Aron and Dineen [**AAD**] proved that T is *polynomially reflexive* i.e. $\mathcal{P}(^nT)$ is reflexive for every n. Since T has the approximation property, it follows from [**Al**] that $\mathcal{P}(T) = \mathcal{P}_A(T)$ and so $M_b = T$ (see [**ACG1**], [**M2**] for details). In general, if $\mathcal{P}(X) \neq \mathcal{P}_A(X)$, then $A_1 \neq A_n$ for some n and Theorem 4.4 implies that there exists a complex homomorphism ϕ (e.g. $\phi = \delta^{(n)}(u_n)$, $u_n \in E_n$, $u_n \neq 0$) which does not belong to X''.

PROPOSITION 5.1. *Suppose that there exists $P \in \mathcal{P}(^nX)$, $n > 1$ such that $|P(e_k)| = 1$, $k = 1, 2, \dots$ for some normed weakly null sequence $(e_k) \subset X$. Then there exists $\phi \in M_b$ such that $\phi(P) \neq 0$ and $\phi(f) = 0$ for every $f \in X' = \mathcal{P}(^1X)$.*

PROOF. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Put $\phi(f) := \lim_{\mathcal{U}} f(e_k)$. It is clear that $\phi(P) = 1$ and

$$\phi(f) = \lim_{\mathcal{U}} f(e_k) = \lim_{k \to \infty} f(e_k) = 0$$

if $f \in X'$. $\square$

Notice that if $X = \ell_p$ and (e_k) is the standard basis, then $P(x) = \sum_{k=1}^\infty x_k^n$, $n \geq p$, where $x = \sum_{k=1}^\infty x_k e_k$ satisfies conditions of Proposition 5.1.

In [**ACG1**] Aron, Cole and Gamelin proposed some description of the spectrum of $H_b(\ell_1)$ in terms of chains of measures (see [**G**] for an L_1-version of this construction).

According to [**ACG1**], $\phi \in H_b(\ell_1)'$ if and only if for every $m = 1, 2, \dots$ there exists a symmetric measure on $\beta(\mathbb{N}^m)$, ν_m and a constant $c > 0$ such that $\|\nu_m\| \leq c^m$ and for each $P_m \in \mathcal{P}(^m\ell_1)$,

$$\phi(P_m) = \int_{\beta(\mathbb{N}^m)} \overline{P}_m d\nu_m,$$

where $\overline{P}_m$ is just P_m regarded as a vector from $\ell^\infty(\mathbb{N}^m)$. By Theorem 4.4, $\phi \in M_b(\ell_1)$ if and only there is a sequence of symmetric measures (μ_m) which are orthogonal to $\beta(\mathbb{N}^j) \times \beta(\mathbb{N}^k) \subset \beta(\mathbb{N}^m)$, for $m > 1$, $k + j = m$, $k, j > 0$ and functionals

$$u_m(P_m) = \int_{\beta(\mathbb{N}^m)} \overline{P}_m d\mu_m$$

determine ϕ by formula (4.3).

6. Homomorphisms on $H_b(X)$

We will use notations $\mathbb{E}^n \subset \mathbb{E}^\infty \subset M_b$,

$$\mathbb{E}^n := E_1 \times \cdots \times E_n = \{(u_1, \dots, u_n)\colon u_k \in E_k, 1 \le k \le n\}.$$

It is clear that $\mathbb{E}^n$ is a Banach space.

PROPOSITION 6.1. *Let Θ be a continuous homomorphism from $H_b(X)$ to itself. Then for every positive integer n there exists a map $F_n\colon \mathbb{E}^n \to \mathbb{E}^n$ such that for every $f \in A_n(X)$, $\Theta(f) = \widehat{f} \circ F_n$.*

PROOF. If $\mathfrak{u} = (u_1, \dots, u_n) \in \mathbb{E}^n$. Then $\phi_{\mathfrak{u}} \circ \Theta = \mathop{\ast}\limits_{k=1}^{m} \delta^{(k)}(u_k) \circ \Theta \in M_b$. By Theorem 4.4 there exists a point $\mathfrak{v} = (v_1, v_2, \dots) \in M_b$ such that $\phi_{\mathfrak{u}} \circ \Theta(f) = \widehat{f}(\mathfrak{v})$. If $f \in A_n(X)$, $\widehat{f}(\mathfrak{v}) = \widehat{f}((v_1, \dots, v_n))$. So we can assume that $\mathfrak{v} \in \mathbb{E}^n$. Put $F_n(\mathfrak{u}) := \mathfrak{v}$. Thus we have constructed the required mapping $\mathfrak{u} \mapsto F_n(\mathfrak{u})$ with the property that $\Theta(f) = \widehat{f} \circ F_n$.

□

A homomorphism Θ from $H_b(X)$ to itself is called *AB-composition homomorphism* [**CGM**] if there exists $F \in H_b(X'', X'')$ such that $\widetilde{\Theta(f)}(x'') = \widetilde{f}(F(x''))$, where $\widetilde{f}$ is the Aron-Berner extension of f.

THEOREM 6.2. *Every polynomial on X is approximable if and only if every homomorphism on $H_b(X)$ is an AB-composition homomorphism.*

PROOF. Suppose that every polynomial on X is approximable. Then $H_b(X) = A_1(X)$. By Proposition 6.1 for every homomorphism $\Theta\colon H_b(X) \to H_b(X)$ there exists a mapping $F\colon X'' \to X''$ such that $\Theta(f) = \widehat{f} \circ F = \widetilde{f} \circ F$. In particular, for every $f \in X'$, $\widetilde{f} \circ F \in H_b(X)$. So we can say that F is weak-star analytic map on X''. By a classical result of Dunford [**Du**] and Grothendieck [**Gr**] on weak-star analytic mappings, F is analytic on X''. Since $\widetilde{f} \circ F$ is bounded on bounded sets of X'' for every $f \in X'$ and weak-star boundedness implies boundedness, $F \in H_b(X'', X'')$.

Suppose now that $A_n(X) \ne A_1(X)$ for some n. Let us choose $u_n \in E_n$ $u_n \ne 0$ and $l \in X'$, $l \ne 0$. Put $F(x) := l(x)u_n$ and $\Theta(f)(x) := \widehat{f}(F(x))$. Since $F \in H_b(X, \mathbb{E}^n)$, $\Theta(f)(x) \in H_b(X)$. But Θ is not an *AB*-composition homomorphism because $\Theta \not\equiv 0$ and $\Theta(f) = 0$ for every $f \in A_1$. □

Since the approximation property of X' implies that every weakly continuous on bounded sets polynomial is approximable [**AP**], we have the following corollary.

COROLLARY 6.3. (c.f. [**CGM**]). *Let X' have the approximation property. Then every polynomial on X is weakly continuous on bounded sets if and only if every homomorphism on $H_b(X)$ is an AB-composition homomorphism.*

The result of Theorem 6.2 can be improved for a reflexive Banach space.

THEOREM 6.4. (Mujica [**M2**]). *If $\mathcal{P}(X) = \mathcal{P}_A(X)$ for a reflexive Banach space X, then for every continuous homomorphism $\Theta\colon H_b(X) \to H_b(X)$ there is a unique map $F \in H_b(X, X)$ such that $\Theta(f) = f \circ F$.*

COROLLARY 6.5. *Let X be a reflexive Banach space with $\mathcal{P}(X) = \mathcal{P}_A(X)$ and $F \in H_b(X, X)$. Suppose that $\Theta(f) = f \circ F$ is an isomorphism of $H_b(X)$. Then F is invertible and $F^{-1} \in H_b(X, X)$.*

PROOF. By Theorem 6.4 there exists a map $G \in H_b(X,X)$ such that $\Theta^{-1}(f) = f \circ G$. It is easy to see that $G = F^{-1}$. $\square$

7. Derivations on $H_b(X)$

Let $u_k \in E_k$. According to Theorem 4.4 we can define a complex homomorphism $\phi \in M_b = \delta^{(k)}(u_k)$ and $\phi(f) = \widehat{f}(u_k)$ for every $f \in H_b(X)$. However, u_k belongs to $(\otimes^k_{s,\pi} X)''$ and so there is an another natural way to define a linear functional on $H_b(X)$, associated with u_k. Let $\theta = \theta(u_k) = \sum \theta_m \in H_b(X)'$ such that $\theta_k(P) = \widehat{P}(u_k)$ if $P \in \mathcal{P}(^k X)$ and $\theta_m = 0$ if $m \neq k$. Recall that here θ_m is the restriction of θ to $\mathcal{P}(^k X)$. It is easy to see that θ is not a homomorphism if $u_k \neq 0$. We define a linear operator on $H_b(X)$, $\partial_{(k)}(u_k)$ by

$$\partial_{(k)}(u_k)(f)(x) := \theta(u_k) \circ T_x(f).$$

For the multilinear form A^P associated with an n-homogeneous polynomial P we denote by $\widehat{A^P}(x^{n-k}, u_k)$ the value of the Gelfand transform at $u_k \in E_k$ of the k-homogeneous polynomial $A^P(x^{n-k}, \cdot)$, where x is fixed.

THEOREM 7.1. *Let $u_k \in E_k$. Then the operator $\partial_{(k)}(u_k)$ is a continuous derivation on $H_b(X)$,*

$$\partial_{(k)}(u_k)(P)(x) = \binom{n}{k} \widehat{A^P}(x^{n-k}, u_k) \tag{7.1}$$

for every $P \in \mathcal{P}(^n X)$ and

$$\delta^{(k)}(u_k)(f)(x) = \sum_{m=0}^{\infty} \frac{(k!)^m}{(mk)!} \partial^m_{(k)}(u_k)(f)(x) \tag{7.2}$$

for every $f \in H_b(X)$.

PROOF. To prove formula (7.1) we observe that

$$P(z+x) = \sum_{m=0}^{n} \binom{n}{m} A^P(x^{n-m}, z^m).$$

So for a fixed x,

$$\partial_{(k)}(u_k)(P)(x) = \theta(u_k)(P(z+x)) = \binom{n}{k} \widehat{A^P}(x^{n-k}, u_k).$$

Note that if $\deg P \leq k$, then $\partial_{(k)}(u_k)(P)(x) = 0$ for every x by the definition of $\partial_{(k)}(u_k)$.

Let $P \in \mathcal{P}(^n X)$ and $Q \in \mathcal{P}(^m X)$. The multilinear form $A^{PQ}(x^{nm-k}, z^k)$ associated with PQ can be represented by

$$A^{PQ}(x^{nm-k}, z^k) = A_1^{PQ}(x^{nm-k}, z^k) + A_2^{PQ}(x^{nm-k}, z^k) + A_3^{PQ}(x^{nm-k}, z^k),$$

where

$$A_1^{PQ}(x^{n-k}, z^k) = A^P(x^{n-k}, z^k) A^Q(x^m);$$
$$A_2^{PQ}(x^{n-k}, z^k) = A^P(x^n) A^Q(z^k, x^{m-k})$$

and

$$A_3^{PQ}(x^{n-k}, z^k) = \frac{1}{k-1} \sum_{s=1}^{k-1} A^P(x^{n-s} z^s) A^Q(z^{k-s}, x^{m-k+s}).$$

If $n \le k$ (resp. $m \le k$), then A_1^{PQ} (resp. A_2^{PQ}) is equal to zero. By definitions of $\theta(u_k)$ and u_k,

$$\theta(u_k)A_3^{PQ}(x^{n-k}, z^k) = 0$$

for any fixed x. So

$$\partial_{(k)}(u_k)(PQ)(x) = \partial_{(k)}(u_k)(P)(x)Q(x) + P(x)\partial_{(k)}(u_k)(Q)(x).$$

Since $\partial_{(k)}(u_k)$ is linear, it is a differentiation on the algebra $H_b(X)$. The continuity of $\partial_{(k)}(u_k)$ follows from the continuity of $\theta(u_k)$ and the translation T_x.

Let $P \in \mathcal{P}(^nX)$ and $n = km$. From (7.1) we have that

$$\partial_{(k)}^m(u_k)(P) = \left(\begin{array}{c} km \\ k \end{array}\right)\left(\begin{array}{c} k(m-1) \\ k \end{array}\right)\cdots\left(\begin{array}{c} k \\ k \end{array}\right)\widehat{P}(u_k) = \frac{(mk)!}{(k!)^m}\delta^{(k)}(u_k)(P).$$

Thus

$$\delta^{(k)}(u_k) = \sum_{m=0}^{\infty}\frac{(k!)^m}{(mk)!}\partial_{(k)}^m(u_k).$$

$\square$

Aron, Cole and Gamelin in [**ACG1**] considered the operation $\partial_{(k)}(u_k)$ for the case when $k = 1$ and so $u_k = u_1 = z$ for some $z \in X''$. They used notation $(z)T_xf = (*z)f(x)$ instead $\partial_{(1)}(z)f(x)$. For this special case and using this notation formula 7.2 can be rewritten as

$$\delta^{(1)}(z)f = \widetilde{\delta}(z)f = \sum_{m=}^{\infty}\frac{1}{m!}z^{*m} = \exp(*z).$$

8. Related Topics

The problem of the description of the spectrum of $H_b(X)$ is related to questions about spectra of various algebras of analytic functions on the unit ball B of X. Carne, Cole and Gamelin in [**CCG**] investigated the Banach algebra $H^\infty(B)$ of bounded analytic functions on $B \subset X$ if X is a dual Banach space and its subalgebra generated by the weak-star continuous linear functionals.

In [**ACG1**] Aron, Cole and Gamelin introduced a Banach algebra $H_{uc}^\infty(B)$ of uniformly continuous analytic functions on the unit ball B and proved that the spectrum of $H_{uc}^\infty(B)$ consists of elements $\phi \in M_b$ such that $R(\phi) \le 1$. Combining this result with Theorem 4.4, we can see that every element ϕ in the spectrum $M(H_{uc}^\infty(B))$ can be represented by formula (4.3) with $R(\phi) \le 1$. Moreover, The spectrum of $H_{uc}^\infty(B)$ contains unit balls of E_k for every k.

Let H be a uniform algebra such that $H_{uc}^\infty \subseteq H \subseteq H^\infty(B)$. Then there is a natural projection of the spectrum $M(H)$ onto $\{\phi \in M_b \colon R(\phi) \le 1\}$ which is one-to-one over $\{\phi \in M_b \colon R(\phi) < 1\}$. However, the projection onto $\{\phi \in M_b \colon R(\phi) \le 1\}$ is one-to-one if and only if $H = H_{uc}^\infty(B)$ [**ACG1**, 12.1 Theorem]. Note that if X is infinite dimensional, then the algebra of bounded analytic functions on B which are continuous on the closure $\overline{B}$, $H_c^\infty(B)$ does not coincides with $H_{uc}^\infty(B)$.

The next estimations for $R(\phi)$ in terms of norms of u_k were obtained in [**Z**].

Proposition 8.1. *Let $\phi \in M_b$ and $\phi = \mathop{\circledast}\limits_{k=1}^{\infty} \delta^{(k)}(u_k)$, $u_k \in E_k$ be its representation. Then*

$$\limsup_{k\to\infty}\|u_k\|_k^{1/k} \le R(\phi) \le \sum_{k=1}^{\infty}\|u_k\|_k^{1/k}.$$

Dixon [**Dix**] has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. Galindo and al. give in [**GGMM**] a construction of a discontinuous scalar-valued homomorphism of the algebra of polynomials on arbitrary infinite dimensional Banach space. The next corollary shows that the restriction of a discontinuous complex homomorphism on $A_n(X) \cap \mathcal{P}(X)$ can be continuous for every n. Note that the problem of the existence of discontinuous complex homomorphisms on $H_b(X)$ for an infinite dimensional Banach space X is still open and equivalent to the famous Michael Problem [**Mi**], [**M1**, p. 240].

Corollary 8.2. ([**Z**]). *If the sequence of algebras $A_n(X)$ does not stabilize, then there is a discontinuous complex homomorphism ζ on $\mathcal{P}(X)$ such that the restriction of ζ onto $A_n(X) \cap \mathcal{P}(X)$ is a continuous complex homomorphism for every n.*

Note also that the technique of investigation of $H_b(X)$ developed by Aron, Cole and Gamelin in [**ACG1**] and [**ACG2**] was successfully applied for algebras $H_b(U)$ of analytic functions of bounded type on open sets [**AGGM, M2**] and for algebras of vector valued analytic functions of bounded type [**BM, GLMM, GLMP**]. Maximal ideals of *symmetric* analytic functions on ℓ_p were considered in [**AAGZ**].

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