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# CONVERGENCE OF SOME BRANCHED CONTINUED FRACTIONS WITH INDEPENDENT VARIABLES

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In this paper, we investigate a convergence of associated multidimensional fractions and multidimensional $J$-fractions with independent variables that are closely related to each other; the coefficients of its partial numerators are positive constants or are non-zero complex constants from parabolic regions. We have established the uniform convergence of the sequences of odd and even approximants of the above mentioned fractions to holomorphic functions on compact subsets of certain domains of $\mathbb{C}^N$. And also, we have proved that a condition of convergence for the considered branched continued fractions in certain subsets of $\mathbb{C}^N$ is the divergence of the series composed of its coefficients. Moreover, we have established that the convergence is uniform to a holomorphic function on all compact subsets of domains of $\mathbb{C}^N$, which are interior of the above mentioned subsets.

**1. Introduction.** It is known that the branched continued fractions are multidimensional generalization of continued fractions [4]. Perhaps the most important subclass of such fractions is the branched continued fractions with independent variables. These fractions are an efficient tool for the approximation of multivariable functions, which are represented by multiple power series [3, 5, 8, 9]. By structure the branched continued fractions with independent variables are their analogues.

A convergence criteria have been given in [1, 3] for multidimensional $C$-fractions with independent variables

$$1+\sum_{i_1=1}^{N}\frac{c_{i(1)}z_{i_1}}{1}\underset{+}{}\sum_{i_2=1}^{i_1}\frac{c_{i(2)}z_{i_2}}{1}\underset{+}{}\sum_{i_3=1}^{i_2}\frac{c_{i(3)}z_{i_3}}{1}\underset{+}{}\cdots,$$

where the $c_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\geq 1$, are complex constants such that $c_{i(k)}\neq 0$, $i(k)\in\mathcal{I}_k$, $k\geq 1$,

$$\mathcal{I}_k=\big\{i(k):\ i(k)=(i_1,i_2,\ldots,i_k),\ 1\leq i_p\leq i_{p-1},\ 1\leq p\leq k,\ i_0=N\big\},\quad k\geq 1,$$

denote the sets of multiindices, and where $\mathbf{z}=(z_1,z_2,\ldots,z_N)\in\mathbb{C}^N$, in [7] for multidimensional $g$-fractions with independent variables

$$\frac{s_0}{1}\underset{+}{}\sum_{i_1=1}^{N}\frac{g_{i(1)}z_{i_1}}{1}\underset{+}{}\sum_{i_2=1}^{i_1}\frac{g_{i(2)}(1-g_{i(1)})z_{i_2}}{1}\underset{+}{}\sum_{i_3=1}^{i_2}\frac{g_{i(3)}(1-g_{i(2)})z_{i_2}}{1}\underset{+}{}\cdots,$$

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where the $s_0$ is a positive constant and the $g_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real constants such that $0 < g_{i(k)} < 1$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$, and in [6] for multidimensional $J$-fractions with independent variables

$$\sum_{i_1=1}^{N} \frac{-a_{i(1)}^2}{b_{i(1)} + z_{i_1} +} \sum_{i_2=1}^{i_1} \frac{-a_{i(2)}^2}{b_{i(2)} + z_{i_2} +} \sum_{i_3=1}^{i_2} \frac{-a_{i(3)}^2}{b_{i(3)} + z_{i_3} +} \cdots,$$

where the $a_{i(k)}$ and $b_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are complex constants such that $a_{i(k)} \neq 0$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$.

In this paper, we investigate the associated multidimensional fractions with independent variables of the form

$$1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{(-1)^{\delta_{i_1,i_2}} a_{i(2)} z_{i_1} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{(-1)^{\delta_{i_2,i_3}} a_{i(3)} z_{i_2} z_{i_3}}{1} + \cdots, \tag{1}$$

where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are complex constants such that $a_{i(k)} \neq 0$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\delta_{k,p}$ is the Kronecker delta, $1 \leq k, p \leq N$, $\mathbf{z} \in \mathbb{C}^N$, which are the expansions of Taylor series for multiple variables [5]. We use the parabolic convergence regions in establishing convergence criteria. They form the basis of Theorems 2 and 3. These theorems give us the intersection of open disk and angular openings for the sets of convergence of associated multidimensional fractions with independent variables. In Theorem 4, the angular openings are obtained for the domain of convergence of the above mentioned fractions. As an application of these theorems, we derive in Section 3 three new convergence criteria for the multidimensional $J$-fractions with independent variables of the form

$$\sum_{i_1=1}^{N} \frac{a_{i(1)}}{z_{i_1}} + \sum_{i_2=1}^{i_1} \frac{(-1)^{\delta_{i_1,i_2}} a_{i(2)}}{z_{i_2}} + \sum_{i_3=1}^{i_2} \frac{(-1)^{\delta_{i_2,i_3}} a_{i(3)}}{z_{i_3}} + \cdots, \tag{2}$$

where the $a_{i(k)} \in \mathbb{C} \backslash \{0\}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, and $\delta_{k,p}$ is the Kronecker delta, $1 \leq k, p \leq N$, $\mathbf{z} \in \mathbb{C}^N$, which are the expansions of Laurent series for multiple variables ([5]).

For use in the proof of our results, we include the following theorem, which follows from Theorem 2 ([2]).

**Theorem 1.** *Let the elements $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of branched continued fraction of the special form*

$$\sum_{i_1=1}^{N} \frac{c_{i(1)}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)}}{1} + \cdots \tag{3}$$

*satisfy the following conditions*

$$\sum_{i_k=1}^{i_{k-1}} \frac{|c_{i(k)}| - \operatorname{Re}(c_{i(k)} e^{-i(\varphi_{i(k-1)} + \varphi_{i(k)})})}{\cos \varphi_{i(k)} - p_{i(k)}} \leq 2(1-\varepsilon) p_{i(k-1)}, \quad i(k) \in \mathcal{I}_k,\ k \geq 1,$$

*where $\varphi_{i(0)}$, $p_{i(0)}$, $\varphi_{i(k)}$ and $p_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real numbers such that*

$$|\varphi_{i(0)}| \leq \varphi, \quad p_{i(0)} \geq 0, \quad |\varphi_{i(k)}| \leq \varphi, \quad 0 \leq p_{i(k)} \leq (1-\varepsilon) \cos \varphi_{i(k)}, \quad i(k) \in \mathcal{I}_k,\ k \geq 1,$$

*where $\varepsilon$ and $\varphi$ are constants such that $0 < \varepsilon < 1$ and $0 < \varphi < \pi/(2(1+\varepsilon))$. Let*

$$f_n = \sum_{i_1=1}^{N} \frac{c_{i(1)}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)}}{1}$$

*be the $n$-th approximant of (3), $n \geq 1$. Then:*

*(A) The approximants of branched continued fraction of special form (3) are all finite and lie in the half-plane*

$$\mathcal{V} = \left\{ \varpi : \ \mathrm{Re}(\varpi e^{-i\varphi_{i(0)}}) \geq -p_{i(0)} \right\}.$$

*(B) Both the sequences of even and odd approximants $\{f_{2n}\}$ and $\{f_{2n-1}\}$ of branched continued fraction of the special form (3) converge.*

*(C) A branched continued fraction of special form (3) converges if the series*

$$\sum_{k=1}^{\infty} \left( \max_{i(k)\in\mathcal{I}_k} |c_{i(k)}| \right)^{-1}$$

*diverges.*

**2. Associated multidimensional fractions with independent variables.** In this section we shall give three new convergence criteria for the associated multidimensional fractions with independent variables (1).

For use in the following theorem we introduce the notation for the tails of (1): $F^{(n)}_{i(n)}(\mathbf{z}) = 1$, $i(n) \in \mathcal{I}_n$, $n \geq 1$,

$$F^{(n)}_{i(k)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{(-1)^{\delta_{i_k,i_{k+1}}} a_{i(k+1)} z_{i_k} z_{i_{k+1}}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{(-1)^{\delta_{i_{n-1},i_n}} a_{i(n)} z_{i_{n-1}} z_{i_n}}{1},$$

where $i(k) \in \mathcal{I}_k$, $1 \leq k \leq n-1$, $n \geq 2$. Let

$$f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{F^{(n)}_{i(1)}(\mathbf{z})}$$

be the $n$-th approximant of (1), $n \geq 1$.

Now we shall prove the following two results.

**Theorem 2.** *Let the coefficients $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of the associated multidimensional fraction with independent variables (1) satisfy the conditions*

$$\sum_{i_1=1}^{N} \frac{|a_{i(1)}| - \mathrm{Re}(a_{i(1)})}{l_{i_1}(1 - g_{i(1)})} \leq (1-\varepsilon) g_{i(0)}, \tag{4}$$

$$\sum_{i_{k+1}=1}^{i_k} \frac{|a_{i(k+1)}| - \mathrm{Re}\big((-1)^{\delta_{i_k,i_{k+1}}} a_{i(k+1)}\big)}{l_{i_{k+1}}(1 - g_{i(k+1)})} \leq \frac{(1-\varepsilon) l_{i_k} g_{i(k)}}{2}, \quad i(k) \in \mathcal{I}_k, \ k \geq 1, \tag{5}$$

*where the $l_k$, $1 \leq k \leq N$, are positive numbers and the $g_{i(0)}$, $g_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real numbers such that*

$$g_{i(0)} \geq 0, \quad 0 \leq g_{i(k)} \leq 1 - \varepsilon, \tag{6}$$

*where $0 < \varepsilon < 1$. Then:*

*(A) For all $\mathbf{z}$ in the set*

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ |z_k| \le \frac{2\cos(\arg(z_k))}{l_k}, \ |\arg(z_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\}, \tag{7}$$

*the even and odd approximants of the associated multidimensional fraction with independent variables (1) converge to finite values $p(\mathbf{z})$ and $q(\mathbf{z})$, respectively. Both even and odd approximants converge uniformly on every compact subset of $\mathrm{Int}\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$, and $p(\mathbf{z})$, $q(\mathbf{z})$ are holomorphic on $\mathrm{Int}\ \mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$.*

*(B) For each $\mathbf{z} \in \mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$, the associated multidimensional fraction with independent variables (1) converges to a finite value $f(\mathbf{z})$ if the series*

$$\sum_{k=1}^{\infty} \left( \max_{i(k)\in\mathcal{I}_k} |a_{i(k)}| \right)^{-1} \tag{8}$$

*diverges. The convergence is uniform in every compact subset of $\mathrm{Int}\ \mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$, and $f(\mathbf{z})$ is holomorphic on $\mathrm{Int}\ \mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$.*

*Proof.* We set $z_k = r_k e^{i\varphi_k}$, $1 \le k \le N$, and choose

$$p_{i(0)} = g_{i(0)}, \quad p_{i(k)} = g_{i(k)} \cos(\varphi_{i_k}), \quad i(k) \in \mathcal{I}_k, \ k \ge 1. \tag{9}$$

Then from (4) it follows that

$$\sum_{i_1=1}^{N} \frac{|a_{i(1)} z_{i_1}| - \mathrm{Re}(a_{i(1)} z_{i_1} e^{-i\varphi_{i_1}})}{r_{i_1} l_{i_1} (1 - g_{i(1)})} \le (1-\varepsilon) g_{i(0)}$$

and for all $i(k) \in \mathcal{I}_k, \ k \ge 1$, inequality (5) implies that

$$\sum_{i_{k+1}=1}^{i_k} \frac{|a_{i(k+1)} z_{i_k} z_{i_{k+1}}| - \mathrm{Re}\big((-1)^{\delta_{i_k,i_{k+1}}} a_{i(k+1)} z_{i_k} z_{i_{k+1}} e^{-i(\varphi_{i_k}+\varphi_{i_{k+1}})}\big)}{r_{i_{k+1}} l_{i_{k+1}} (1 - g_{i(k+1)})} \le \frac{(1-\varepsilon) r_{i_k} l_{i_k} g_{i(k)}}{2}$$

that is

$$\sum_{i_1=1}^{N} \frac{|a_{i(1)} z_{i_1}| - \mathrm{Re}(a_{i(1)} z_{i_1} e^{-i\varphi_{i_1}})}{\dfrac{r_{i_1} l_{i_1}}{\cos(\varphi_{i_1})} (\cos(\varphi_{i_1}) - p_{i(1)})} \le (1-\varepsilon) p_{i(0)} \tag{10}$$

and for all $i(k) \in \mathcal{I}_k, \ k \ge 1$,

$$\sum_{i_{k+1}=1}^{i_k} \frac{|a_{i(k+1)} z_{i_k} z_{i_{k+1}}| - \mathrm{Re}\big((-1)^{\delta_{i_k,i_{k+1}}} a_{i(k+1)} z_{i_k} z_{i_{k+1}} e^{-i(\varphi_{i_k}+\varphi_{i_{k+1}})}\big)}{\dfrac{r_{i_{k+1}} l_{i_{k+1}}}{\cos(\varphi_{i_{k+1}})} (\cos(\varphi_{i_{k+1}}) - p_{i(k+1)})} \le \frac{(1-\varepsilon) r_{i_k} l_{i_k} p_{i(k)}}{2\cos(\varphi_{i_k})}. \tag{11}$$

It follows from (7) and (9) that

$$r_k l_k < 2\cos(\varphi_k), \quad 1 \le k \le N. \tag{12}$$

Using (12) we may write (10) as

$$\sum_{i_1=1}^{N}\frac{|a_{i(1)}z_{i_1}|-\mathrm{Re}(a_{i(1)}z_{i_1}e^{-i\varphi_{i_1}})}{\cos(\varphi_{i_1})-p_{i(1)}}\le 2(1-\varepsilon)p_{i(0)} \tag{13}$$

and for all $i(k)\in\mathcal{I}_k,\ k\ge 1$, (11) as

$$\sum_{i_{k+1}=1}^{i_k}\frac{|a_{i(k+1)}z_{i_k}z_{i_{k+1}}|-\mathrm{Re}\big((-1)^{\delta_{i_k,i_{k+1}}}a_{i(k+1)}z_{i_k}z_{i_{k+1}}e^{-i(\varphi_{i_k}+\varphi_{i_{k+1}})}\big)}{\cos(\varphi_{i_{k+1}})-p_{i(k+1)}}\le 2(1-\varepsilon)p_{i(k)}. \tag{14}$$

Moreover, it follows from (6) that

$$0\le p_{i(k)}\le(1-\varepsilon)\cos(\varphi_{i_k}),\quad i(k)\in\mathcal{I}_k,\quad k\ge 1. \tag{15}$$

Thus, the elements of (1) satisfy the conditions of Theorem 1, with $\varphi_{i(0)}=0$, iff $\mathbf{z}\in\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$. It follows from part (B) of Theorem 1 that the even and odd approximants of (1) converge to finite values for all $\mathbf{z}\in\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Moreover, part (A) of this theorem implies that for every index $i_1$, $1\le i_1\le N$, the values of all tails $F^{(n)}_{i(1)}(\mathbf{z})$, $n\ge 2$, of (1) are finite and lie in the half-plane

$$\mathcal{V}_{i(1)}(\varphi_{i_1},p_{i(1)})=\big\{\varpi:\ \mathrm{Re}(\varpi e^{-i\varphi_{i_1}})\ge\cos(\varphi_{i_1})-p_{i(1)}\big\}. \tag{16}$$

It follows from (15) that $F^{(n)}_{i(1)}(\mathbf{z})\not\equiv 0$ for all indices. Thus, the approximants $f_n(\mathbf{z})$, $n\ge 1$, of (1) form a sequence of holomorphic functions in $\mathrm{Int}\,\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

Let

$$\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}=\left\{\mathbf{z}\in\mathbb{C}^N:\ |z_k|<\frac{2\cos(\arg(z_k))}{(1+\sigma)l_k},\ |\arg(z_k)|<\frac{\sigma\pi}{2(1+\varepsilon)},\ 1\le k\le N\right\}, \tag{17}$$

where $0<\sigma<1$, be a domain contained in $\mathrm{Int}\,\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Set

$$C=\max_{1\le i_1\le N}|a_{i(1)}|.$$

Then for the arbitrary $\mathbf{z}\in\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}\subseteq\mathrm{Int}\ \mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$, we obtain for $n\ge 1$

$$\begin{aligned}|f_n(\mathbf{z})|&\le 1+\sum_{i_1=1}^{N}\frac{|a_{i(1)}||z_{i_1}|}{\mathrm{Re}\big(F^{(n)}_{i(1)}(\mathbf{z})e^{-i\varphi_{i_1}}\big)}<1+\sum_{i_1=1}^{N}\frac{2C\cos(\varphi_{i_1})}{(1+\sigma)l_{i_1}(\cos(\varphi_{i_1})-p_{i(1)})}\le\\&\le 1+\sum_{i_1=1}^{N}\frac{2C}{(1+\sigma)l_{i_1}(1-g_{i(1)})}\le 1+\sum_{i_1=1}^{N}\frac{2C}{(1+\sigma)\varepsilon l_{i_1}}=M(\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}),\end{aligned}$$

where the constant $M(\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon})$ depends only on the domain (17), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{D}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathrm{Int}\,\mathcal{D}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Let us cover $\mathcal{K}$ with domains of form (17). From this cover we choose the finite subcover

$$\mathcal{D}_{l_1^{(1)},l_2^{(1)},\ldots,l_N^{(1)},\sigma^{(1)},\varepsilon^{(1)}},\ \mathcal{D}_{l_1^{(2)},l_2^{(2)},\ldots,l_N^{(2)},\sigma^{(2)},\varepsilon^{(2)}},\ \ldots,\ \mathcal{D}_{l_1^{(k)},l_2^{(k)},\ldots,l_N^{(k)},\sigma^{(k)},\varepsilon^{(k)}}.$$

Set

$$M(\mathcal{K})=\max_{1\le r\le k} M\big(\mathcal{D}_{l_1^{(r)},l_2^{(r)},\dots,l_N^{(r)},\sigma^{(r)},\varepsilon^{(r)}}\big).$$

Then for arbitrary $\mathbf{z}\in\mathcal{K}$ we obtain $|f_n(\mathbf{z})|\le M(\mathcal{K})$, for $n\ge 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of $\operatorname{Int}\mathcal{D}_{l_1,l_2,\dots,l_N,\varepsilon}$. An application of Theorem 24.2 [10, p. 108–109] (see also Theorem 2.17 [4, p. 66]) yields the uniform convergence of the even and odd approximants of (1) to holomorphic functions on all compact subsets of $\operatorname{Int}\mathcal{D}_{l_1,l_2,\dots,l_N,\varepsilon}$. This proves part (A). The proof of part (B) follows readily from part (C) of Theorem 1. □

**Theorem 3.** *Let the coefficients $a_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\ge 1$, of the associated multidimensional fraction with independent variables (1) satisfy the conditions*

$$|a_{i(1)}|-\operatorname{Re}(a_{i(1)})\le(1-\varepsilon)l_{i_1}g_{i(0)}(1-g_{i(1)}),\quad 1\le i_1\le N, \tag{18}$$

$$|a_{i(k)}|-\operatorname{Re}\big((-1)^{\delta_{i_{k-1},i_k}}a_{i(k)}\big)\le(1-\varepsilon)l_{i_{k-1}}l_{i_k}g_{i(k-1)}(1-g_{i(k)})/2,\quad i(k)\in\mathcal{I}_k,\ k\ge 2, \tag{19}$$

*where $l_k$, $1\le k\le N$, are positive numbers and $g_{i(0)}$, $g_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\ge 1$, are real numbers satisfying (6), where $0<\varepsilon<1$. Then:*

*(A) For all $\mathbf{z}$ in the set*

$$\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}=\left\{\mathbf{z}\in\mathbb{C}^N:\ \sum_{k=1}^{N}\frac{l_k|z_k|}{2\cos(\arg(z_k))}\le 1,\ |\arg(z_k)|<\frac{\pi}{2(1+\varepsilon)},\ 1\le k\le N\right\}, \tag{20}$$

*the even and odd approximants of the associated multidimensional fraction with independent variables (1) converge to finite values $p(\mathbf{z})$ and $q(\mathbf{z})$, respectively. Both even and odd approximants converge uniformly on every compact subset of $\operatorname{Int}\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}$, and $p(\mathbf{z})$, $q(\mathbf{z})$ are holomorphic on $\operatorname{Int}\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}$.*

*(B) For each $\mathbf{z}\in\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}$, the associated multidimensional fraction with independent variables (1) converges to a finite value $f(\mathbf{z})$ if the series (8) diverges. The convergence is uniform in every compact subset of $\operatorname{Int}\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}$, and $f(\mathbf{z})$ is holomorphic on $\operatorname{Int}\mathcal{O}_{l_1,l_2,\dots,l_N,\varepsilon}$.*

*Proof.* As in the proof of Theorem 2 we set $z_k=r_ke^{i\varphi_k}$, $1\le k\le N$, and choose the $p_{i(0)}$ and $p_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\ge 1$, as well as in (9). Then from (18) and (20) it follows that

$$\sum_{i_1=1}^{N}\frac{|a_{i(1)}z_{i_1}|-\operatorname{Re}(a_{i(1)}z_{i_1}e^{-i\varphi_{i_1}})}{(1-g_{i(1)})\cos(\varphi_{i_1})}\le 2(1-\varepsilon)g_{i(0)}$$

that is (13). For the arbitrary of multiindex $i(k)\in\mathcal{I}_k$, $k\ge 1$, (19) and (20) imply

$$\sum_{i_{k+1}=1}^{i_k}\frac{|a_{i(k+1)}z_{i_k}z_{i_{k+1}}|-\operatorname{Re}\big((-1)^{\delta_{i_k,i_{k+1}}}a_{i(k+1)}z_{i_k}z_{i_{k+1}}e^{-i(\varphi_{i_k}+\varphi_{i_{k+1}})}\big)}{(1-g_{i(k+1)})\cos(\varphi_{i_{k+1}})}\le(1-\varepsilon)r_{i_k}l_{i_k}g_{i(k)}$$

that is

$$\sum_{i_{k+1}=1}^{i_k}\frac{|a_{i(k+1)}z_{i_k}z_{i_{k+1}}|-\operatorname{Re}\big((-1)^{\delta_{i_k,i_{k+1}}}a_{i(k+1)}z_{i_k}z_{i_{k+1}}e^{-i(\varphi_{i_k}+\varphi_{i_{k+1}})}\big)}{\cos(\varphi_{i_{k+1}})-p_{i(k+1)}}\le\frac{(1-\varepsilon)r_{i_k}l_{i_k}p_{i(k)}}{\cos(\varphi_{i_k})}. \tag{21}$$

It follows from (9) and (20) that inequalities (12) hold. Using (12) we may write (21) as (14). Moreover, from (6) implies that inequalities (15) are valid. Thus, the elements of (1) satisfy the assumptions of Theorem 1, with $\varphi_{i(0)} = 0$, iff $\mathbf{z} \in \mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. It follows from part (B) of Theorem 1 that the even and odd approximants of (1) converge to finite values for all $\mathbf{z} \in \mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Moreover, part (A) of this theorem implies that for every index $i_1$, $1 \le i_1 \le N$, the values of all tails $F^{(n)}_{i(1)}(\mathbf{z})$, $n \ge 2$, of (1) are finite and lie in (16). Inequality (15) yields that $F^{(n)}_{i(1)}(\mathbf{z}) \not\equiv 0$ for all indices. Thus, the approximants $f_n(\mathbf{z})$, $n \ge 1$, of (1) form a sequence of holomorphic functions in $\mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

Let

$$\mathcal{O}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon} =$$
$$= \left\{ \mathbf{z} \in \mathbb{C}^N : \sum_{k=1}^{N} \frac{|z_k|}{l_k \cos(\arg(z_k))} < 2\sigma, \ |\arg(z_k)| < \frac{\sigma\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\}, \tag{22}$$

where $0 < \sigma < 1$, be a domain contained in $\mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Set

$$C = \max_{1 \le i_1 \le N} |a_{i(1)}|, \quad l = \max_{1 \le k \le N} l_k.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{O}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon} \subseteq \mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$, we obtain for $n \ge 1$

$$|f_n(\mathbf{z})| \le 1 + \sum_{i_1=1}^{N} \frac{|a_{i(1)}||z_{i_1}|}{\mathrm{Re}\big(F^{(n)}_{i(1)}(\mathbf{z}) e^{-i\varphi_{i_1}}\big)} \le 1 + \sum_{i_1=1}^{N} \frac{C|z_{i_1}|}{\cos(\varphi_{i_1}) - p_{i(1)}} \le 1 + \frac{2\sigma l C}{\varepsilon} = M(\mathcal{O}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}),$$

where the constant $M(\mathcal{O}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon})$ depends only on the domain (22), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{O}_{l_1,l_2,\ldots,l_N,\sigma,\varepsilon}$.

Let $\mathcal{K}$ be an arbitrary compact subset of $\mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. Let us cover $\mathcal{K}$ with domains of form (22). From this cover we choose the finite subcover

$$\mathcal{O}_{l_1^{(1)},l_2^{(1)},\ldots,l_N^{(1)},\sigma^{(1)},\varepsilon^{(1)}}, \ \mathcal{O}_{l_1^{(2)},l_2^{(2)},\ldots,l_N^{(2)},\sigma^{(2)},\varepsilon^{(2)}}, \ \ldots, \ \mathcal{O}_{l_1^{(k)},l_2^{(k)},\ldots,l_N^{(k)},\sigma^{(k)},\varepsilon^{(k)}}.$$

Set

$$M(\mathcal{K}) = \max_{1 \le r \le k} M\big(\mathcal{O}_{l_1^{(r)},l_2^{(r)},\ldots,l_N^{(r)},\sigma^{(r)},\varepsilon^{(r)}}\big).$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \le M(\mathcal{K})$, for $n \ge 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of $\mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. An application of Theorem 24.2 [10, pp. 108–109] yields the uniform convergence of the even and odd approximants of (1) to holomorphic functions on all compact subsets of $\mathrm{Int}\mathcal{O}_{l_1,l_2,\ldots,l_N,\varepsilon}$. This proves part (A). The proof of part (B) follows readily from part (C) of Theorem 1. □

The following theorem is the application of Theorem 2.

**Theorem 4.** *Let the coefficients $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, of the associated multidimensional fraction with independent variables (1) satisfy the conditions*

$$a_{i(1)} > 0, \quad 1 \le i_1 \le N, \quad (-1)^{\delta_{i_{k-1},i_k}} a_{i(k)} > 0, \quad i(k) \in \mathcal{I}_k, \ k \ge 2. \tag{23}$$

*Then:*

*(A) The even and odd approximants of the associated multidimensional fraction with independent variables (1) converge to holomorphic function in the domain*

$$\mathcal{D}_\varepsilon = \left\{ \mathbf{z} \in \mathbb{C}^N : \ |\arg z_k| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\}, \tag{24}$$

*where $0 < \varepsilon < 1$. Both even and odd approximants converge uniformly on every compact subset of $\mathcal{D}_\varepsilon$.*

*(B) The associated multidimensional fraction with independent variables (1) converges to a holomorphic function in $\mathcal{D}_\varepsilon$ if series (8) diverges. The convergence is uniform in every compact subset of $\mathcal{D}_\varepsilon$.*

*Proof.* If $g_{i(0)} = 1/2$, $g_{i(k)} = 1/2$ and $a_{i(1)} > 0$, $(-1)^{\delta_{i_k, i_{k+1}}} a_{i(k+1)} > 0$ for all $i(k) \in \mathcal{I}_k$, $k \ge 1$, then conditions (4) and (5) holds for all $l_k > 0$, $1 \le k \le N$.

Let $\mathcal{K}$ be an arbitrary compact set contained in $\mathcal{D}_\varepsilon$. Then $\mathcal{K} \subseteq \operatorname{Int} \mathcal{D}_{l_1, l_2, \ldots, l_N, \varepsilon} \subseteq \mathcal{D}_\varepsilon$ for some $l_1, l_2, \ldots, l_N$ sufficiently small, for which $\operatorname{Int} \mathcal{D}_{l_1, l_2, \ldots, l_N, \varepsilon}$ is an interior of set (7).

Thus, Theorem 4 is an immediate consequence of Theorem 2. □

**3. Multidimensional *J*-fractions with independent variables.** Various criteria for the convergence of multidimensional $J$-fractions with independent variables (2) can be obtained by considering the equivalent associated multidimensional fractions with independent variables of the form

$$\sum_{i_1=1}^{N} \frac{a_{i(1)} \xi_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{(-1)^{\delta_{i_1,i_2}} a_{i(2)} \xi_{i_1} \xi_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{(-1)^{\delta_{i_2,i_3}} a_{i(3)} \xi_{i_2} \xi_{i_3}}{1} + \cdots, \tag{25}$$

where $\xi_k = 1/z_k$, $1 \le k \le N$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{C}^N$. For more details on equivalent transformations, see [4, p. 29–33].

From Theorem 2 it follows that for all $\boldsymbol{\xi}$ in the set

$$\mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon} = \left\{ \boldsymbol{\xi} \in \mathbb{C}^N : \ |\xi_k| \le \frac{2\cos(\arg(\xi_k))}{l_k}, \ |\arg(\xi_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\}, \tag{26}$$

where $0 < \varepsilon < 1$ and the $l_k$, $1 \le k \le N$, are positive numbers, the even and odd approximants of (25) converge to finite values $p(\boldsymbol{\xi})$ and $q(\boldsymbol{\xi})$, respectively, if conditions (4)–(6) hold. Both even and odd approximants converge uniformly on every compact subset of $\operatorname{Int} \mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon}$, and $p(\boldsymbol{\xi})$, $q(\boldsymbol{\xi})$ are holomorphic on $\operatorname{Int} \mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon}$. Moreover, for each $\boldsymbol{\xi} \in \mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon}$, the associated multidimensional fraction with independent variables (25) converges to a finite value $f(\boldsymbol{\xi})$ if series (8) diverges. The convergence is uniform in every compact subset of $\operatorname{Int} \mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon}$, and $f(\boldsymbol{\xi})$ is holomorphic on $\operatorname{Int} \mathcal{D}'_{l_1,l_2,\ldots,l_N,\varepsilon}$.

It follows from (26) that

$$|1/z_k| \le (2/l_k)\cos(\arg(1/z_k)), \quad |\arg(1/z_k)| < \pi/(2(1+\varepsilon)), \quad 1 \le k \le N,$$

that is

$$\operatorname{Re}(z_k) \ge l_k/2, \quad |\arg(z_k)| < \pi/(2(1+\varepsilon)), \quad 1 \le k \le N.$$

Therefore, the following corollary is a simple application of Theorem 2.

**Corollary 1.** *Let the coefficients $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of the multidimensional J-fraction with independent variables (2) satisfy the conditions (4) and (5), where the $l_k$, $1 \leq k \leq N$, are positive numbers and $g_{i(0)}$, $g_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real numbers satisfying (6), where $0 < \varepsilon < 1$. Then:*

*(A) For all $\mathbf{z}$ in the set*

$$\mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ \mathrm{Re}(z_k) \geq \frac{l_k}{2}, \ |\arg(z_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \leq k \leq N \right\},$$

*the even and odd approximants of the multidimensional J-fraction with independent variables (2) converge to finite values $p(\mathbf{z})$ and $q(\mathbf{z})$, respectively. Both even and odd approximants converge uniformly on every compact subset of* $\mathrm{Int}\mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$, *and $p(\mathbf{z})$, $q(\mathbf{z})$ are holomorphic on* $\mathrm{Int}\mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

*(B) For each $\mathbf{z} \in \mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$, the multidimensional J-fraction with independent variables (2) converges to a finite value $f(\mathbf{z})$ if the series (8) diverges. The convergence is uniform in every compact subset of* $\mathrm{Int}\ \mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$, *and $f(\mathbf{z})$ is holomorphic on* $\mathrm{Int}\ \mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

From Theorem 3 we obtain the following corollary.

**Corollary 2.** *Let the coefficients $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of the multidimensional J-fraction with independent variables (2) satisfy the conditions (18) and (19), where the $l_k$, $1 \leq k \leq N$, are positive numbers and $g_{i(0)}$, $g_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are real numbers satisfying (6), where $0 < \varepsilon < 1$. Then:*

*(A) For all $\mathbf{z}$ in the set*

$$\mathcal{Q}_{l_1,l_2,\ldots,l_N,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ \sum_{k=1}^{N} \frac{l_k}{\mathrm{Re}(z_k)} \geq 2, \ |\arg(z_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \leq k \leq N \right\},$$

*the even and odd approximants of the multidimensional J-fraction with independent variables (2) converge to finite values $p(\mathbf{z})$ and $q(\mathbf{z})$, respectively. Both even and odd approximants converge uniformly on every compact subset of* $\mathrm{Int}\ \mathcal{Q}_{l_1,l_2,\ldots,l_N,\varepsilon}$, *and $p(\mathbf{z})$, $q(\mathbf{z})$ are holomorphic on* $\mathrm{Int}\ \mathcal{Q}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

*(B) For each $\mathbf{z} \in \mathcal{P}_{l_1,l_2,\ldots,l_N,\varepsilon}$, the multidimensional J-fraction with independent variables (2) converges to a finite value $f(\mathbf{z})$ if the series (8) diverges. The convergence is uniform in every compact subset of* $\mathrm{Int}\ \mathcal{Q}_{l_1,l_2,\ldots,l_N,\varepsilon}$, *and $f(\mathbf{z})$ is holomorphic on* $\mathrm{Int}\ \mathcal{Q}_{l_1,l_2,\ldots,l_N,\varepsilon}$.

Finally, from Theorem 4 we deduce the following result.

**Corollary 3.** *Let the coefficients $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of the multidimensional J-fraction with independent variables (2) satisfy the condition (23). Then:*

*(A) The even and odd approximants of the multidimensional J-fraction with independent variables (2) converge to holomorphic function in domain (24), where $0 < \varepsilon < 1$. The convergence is uniform in every compact subset of $\mathcal{D}_\varepsilon$.*

*(B) The multidimensional J-fraction with independent variables (2) converges to a holomorphic function in $\mathcal{D}_\varepsilon$ if series (8) diverges.*

We remark that from a convergence criteria that have been given in [6], we have the following domains of convergence for (2)

$$\mathcal{R}_{l_1,l_2,\ldots,l_N,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{C}^N : \ \mathrm{Re}(-iz_k) > l_k, \ |\arg(-iz_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \leq k \leq N \right\}$$

and

$$\mathcal{R}_\varepsilon = \left\{ \mathbf{z} \in \mathbb{C}^N : \ |\arg(-iz_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\},$$

where the $l_k$, $1 \le k \le N$, are positive numbers and where $\varepsilon$ is a constant such that $0 < \varepsilon < 1$. In view of this, we conclude that Corollaries 1–3 give us another sets of convergence for the multidimensional $J$-fraction with independent variables (2), in certain conditions, on the coefficients of its partial numerators.

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