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MULYAVA O.[1], TRUKHAN YU.[2]

ON MEROMORPHICALLY STARLIKE FUNCTIONS OF ORDER α AND TYPE β, WHICH SATISFY SHAH'S DIFFERENTIAL EQUATION

According to M.L. Mogra, T.R. Reddy and O.P. Juneja an analytic in $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$ function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ is said to be meromorphically starlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if $|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|$, $z \in \mathbb{D}_0$. Here we investigate conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2, under which the differential equation of S. Shah $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$ has meromorphically starlike solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. Beside the main case $n + \gamma_2 \neq 0$, $n \geq 1$, cases $\gamma_2 = -1$ and $\gamma_2 = -2$ are considered. Also the possibility of the existence of the solutions of the form $f(z) = \frac{1}{z} + \sum_{n=1}^{m} f_n z^n$, $m \geq 2$, is studied. In addition we call an analytic in $\mathbb{D}_0$ function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ meromorphically convex of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if $|zf''(z) + 2f'(z)| < \beta|zf''(z) + 2\alpha f'(z)|$, $z \in \mathbb{D}_0$ and investigate sufficient conditions on parameters β_0, β_1, γ_0, γ_1, γ_2 under which the differential equation of S. Shah has meromorphically convex solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. The same cases as for the meromorphically starlike solutions are considered.

Key words and phrases: meromorphically starlike function of order α and type β, meromorphically convex function of order α and type β, Shah's differential equation.

[1] National University of Food Technologies, 68 Volodymyrska str., 01601, Kyiv, Ukraine
[2] Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: info@nuft.edu.ua (Mulyava O.), yurkotrukhan@gmail.com (Trukhan Yu.)

INTRODUCTION AND PRELIMINARY LEMMAS

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [2, p. 203] that the condition $\text{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [4] a function f is said to be close-to-convex in $\mathbb{D}$ (see also [2, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\text{Re}\,(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. A close-to-convex function f has the characteristic property that the complement G to the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that a function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \tag{2}$$

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is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark also, that a function defined by (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is a starlike domain with respect to the origin. It is clear that every starlike function is close-to-convex.

S.M. Shah [8] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution given by (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he obtained the following result: if $\beta_1 + \gamma_2 = 0$, $-1 \le \beta_0 < 0$, $\beta_1 > 0$, $\gamma_0 = 0$ and $-\beta_1/2 < \gamma_1 \le 0$, then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)| : |z| = r\}$.

The investigations are continued in papers [9–14]. In particular in the case of complex parameters β_0, β_1, γ_0, γ_1, γ_2 in [13] it is proved that if $\gamma_0 = 0$, $\beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $\dfrac{2(|\beta_1| + |\gamma_1|)}{2 - |\beta_1|} < \ln 2$ then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are starlike and, thus, close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$. An analog of this assertion for convex functions is obtained in [14], where it is proved that if $\gamma_0 = 0$, $\beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $\dfrac{2(|\beta_1| + |\gamma_1|)}{2 - |\beta_1|} < \dfrac{\ln 2}{2}$ then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are convex in $\mathbb{D}$.

Let Σ be the class of functions defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n, \tag{4}$$

analytic in $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said ([3,5]) to be meromorphically starlike of order $\alpha \in [0, 1)$ if $\mathrm{Re}\,\{-zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D}_0)$, and is said to be meromorphically convex of order $\alpha \in [0, 1)$ if $\mathrm{Re}\{-(1 + zf''(z))/f'(z)\} > \alpha$ $(z \in \mathbb{D}_0)$.

Conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2 under which Shah's differential equation has meromorphically starlike and meromorphically convex solutions of order $\alpha \in [0, 1)$ are investigated in [1] . It is known ([1,7]) that if

$$|zf'(z) + f(z)| < |zf'(z) + (2\alpha - 1)f(z)| \tag{5}$$

for all $z \in \mathbb{D}_0$ then the function f is meromorphically starlike of order $\alpha \in [0, 1)$.

By B. Uralegaddi [15] a function $f \in \Sigma$ is meromorphically starlike of order $\beta \in (0, 1]$ if

$$|zf'(z) + f(z)| < \beta|zf'(z) - f(z)|, \quad z \in \mathbb{D}_0. \tag{6}$$

Finally, combining (5) and (6), M.L. Mogra, T.R. Reddy and O.P. Juneja [6] called a function $f \in \Sigma$ meromorphically starlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if

$$|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|, \quad z \in \mathbb{D}_0,$$

and proved the following lemma.

Lemma 1. *If*

$$\sum_{n=1}^{\infty} ((1 + \beta)n + \beta(2\alpha - 1) + 1)|f_n| \le 2\beta(1 - \alpha), \tag{7}$$

then the function defined by (4) *is meromorphically starlike of order* $\alpha \in [0, 1)$ *and type* $\beta \in (0, 1]$.

Here we investigate conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2 such that equation (3) has meromorphically starlike solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$.

We need also the following lemma [1].

Lemma 2. *A function defined by* (4) *is a solution of equation* (3) *if and only if*

$$2-\beta_1+\gamma_2=0, \quad -\beta_0+\gamma_1=0, \gamma_0+2(1+\gamma_2)f_1=0, \quad 3(2+\gamma_2)f_2+2\gamma_1 f_1=0 \tag{8}$$

and for $n \geq 3$

$$(n+1)(n+\gamma_2)f_n+n\gamma_1 f_{n-1}+\gamma_0 f_{n-2}=0. \tag{9}$$

1 MEROMORPHICALLY STARLIKE SOLUTIONS

We assume that

$$n+\gamma_2 \neq 0, \quad n \geq 1. \tag{10}$$

Then equalities (8) and (9) yields that if $\gamma_0 = 0$ then all $f_n = 0$, that is, the condition (7) is equivalent to the condition $0 \leq \beta(1-\alpha)$. Therefore, the following statement is true ([1]).

Proposition 1. *If* $\beta_1 = 2+\gamma_2$, $\beta_0 = \gamma_1$, $\gamma_0 = 0$ *and condition* (10) *holds then differential equation* (3) *has the solution* $f(z) = 1/z$, *which is meromorphically starlike of order* α *and type* β *for each* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1]$.

Now we assume that $\gamma_0 \neq 0$. Then $f_1 = -\dfrac{\gamma_0}{2(1+\gamma_2)}$, $f_2 = -\dfrac{2\gamma_1}{3(2+\gamma_2)}f_1$ and $f_n = -\dfrac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1} - \dfrac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}$. Using these equalities and Lemma 1 we prove the following theorem.

Theorem 1. *Let* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1]$. *If* $\beta_1 = 2+\gamma_2$, $|\gamma_2| < 1$, $\beta_0 = \gamma_1$ *then differential equation* (3) *has a solution given by* (4), *which by the condition*

$$\frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|} \leq 2\beta(1-\alpha)\left(1-\frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)(3-|\gamma_2|)}\right) \tag{11}$$

is meromorphically starlike of order α *and type* β.

Proof. Since $|\gamma_2| < 1$ then (10) holds and from the indicated above equalities for f_j we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\left|\frac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1} + \frac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}\right| \\
&\quad \le 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=3}^{\infty}\frac{n((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_1|}{(n+1)(n-|\gamma_2|)}|f_{n-1}| + \sum_{n=3}^{\infty}\frac{((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_0|}{(n+1)(n-|\gamma_2|)}|f_{n-2}| \\
&\quad = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=2}^{\infty}\frac{(n+1)((1+\beta)(n+1)+\beta(2\alpha-1)+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| \\
&\quad + \sum_{n=1}^{\infty}\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n| \\
&\quad = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| - \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| \\
&\quad + \sum_{n=1}^{\infty}\frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| + \sum_{n=1}^{\infty}\frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n|,
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(1 - \frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+1+\beta(n-1+2\alpha))(n+2)(n+1-|\gamma_2|)}\right. \\
&\quad \left. - \frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+1+\beta(n-1+2\alpha))(n+2-|\gamma_2|)}\right)((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \\
&\quad \le 2(1+\beta\alpha)|f_1| + \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| - \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| \\
&\quad = 2(1+\beta\alpha)|f_1| \le \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}.
\end{aligned} \tag{12}$$

Since the sequences $\left(\frac{(n+2+(n+2\alpha)\beta)}{n+1+\beta(n-1+2\alpha)}\right)$ and $\left(\frac{n+1}{(n+2)(n+1-|\gamma_2|)}\right)$ are decreasing then

$$\frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+1+\beta(n-1+2\alpha))(n+2)(n+1-|\gamma_2|)} \le \frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}, \tag{13}$$

and by analogy

$$\frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+1+\beta(n-1+2\alpha))(n+2-|\gamma_2|)} \le \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}. \tag{14}$$

Condition (11) implies the inequality

$$\frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)} + \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)} < 1.$$

Therefore, from (12) in view of (13) and (14) we have

$$\begin{aligned}
\left(1 - \frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)} - \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}\right) &\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \\
&\le \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|},
\end{aligned}$$

whence in view of (11) we obtain inequality (7). By Lemma 1 function (4) is meromorphically starlike of order α and type β. □

Now we consider the cases where the condition (10) does not hold. At first, we assume that $1+\gamma_2=0$. Then in view of (8) $\gamma_0=0$ and we can choose $f_1\neq 0$, because if $f_1=0$ then in view of (8) $f_2=0$, and in view of (9) all $f_n=0$ and we come to the case $f(z)=1/z$, which we considered above.

We assume that $f_1=a^2\neq 0$ and $\gamma_1=0$. Since $2+\gamma_2\neq 0$, we have $f_2=0$ and in view of the equality $\gamma_0=0$, all $f_n=0$ for $n\geq 2$. Thus, the solution has the form $f(z)=1/z+a^2z=a(1/(az)+az)=2aJ(az)$, where J is the function of Joukowsky. Therefore, using Lemma 1, we get the following statement.

Proposition 2. *If $\beta_1=1$, $\gamma_2=-1$ and $\beta_0=\gamma_1=\gamma_0=0$ then differential equation* (3) *has the solution $f(z)=J(az)$, which by the condition $(1+\beta\alpha)|a|^2\leq\beta(1-\alpha)$ is meromorphically starlike of order α and type β.*

If $\gamma_1\neq 0$ then in view of the equality $\gamma_2=-1$ from (8) we have $f_2=-2\gamma_1 f_1/3$ and since $\gamma_0=0$, we obtain $f_n=-\frac{n\gamma_1}{n^2-1}f_{n-1}$ for $n\geq 3$. Using the recurrent formula we prove the following theorem.

Theorem 2. *If $\beta_1=1$, $\gamma_2=-1$, $\gamma_0=0$, $\beta_0=\gamma_1\neq 0$ then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1|<1 \tag{15}$$

is meromorphically starlike of order α and type β.

Proof. Since, as above,

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|=2(1+\beta\alpha)|f_1|$$
$$+\sum_{n=1}^{\infty}\frac{((1+\beta)(n+1)+\beta(2\alpha-1)+1)}{(1+\beta)n+\beta(2\alpha-1)+1}\frac{(n+1)|\gamma_1|}{((n+1)^2-1)}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|$$
$$\leq 2(1+\beta\alpha)|f_1|+\sum_{n=1}^{\infty}\frac{3+\beta+2\alpha\beta}{2(1+\alpha\beta)}\frac{2|\gamma_1|}{3}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|,$$

then by the condition (15) we have

$$\left(1-\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1|\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|\leq 2(1+\beta\alpha)|f_1|.$$

Therefore, if

$$2(1+\beta\alpha)|f_1|\leq 2\beta(1-\alpha)\left(1-\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1|\right), \tag{16}$$

then by Lemma 1 the function given by (4) is meromorphically starlike of order α and type β. In view of the arbitrariness of f_1 and the condition (15) we can choose f_1 such that the condition (16) holds. □

Now, let $2+\gamma_2=0$. Then $\beta_1=0$ and from (8) and (9) we obtain $f_1=\gamma_0/2$, $\gamma_1 f_1=0$ and $f_n=-\frac{n\gamma_1}{(n+1)(n-2)}f_{n-1}-\frac{\gamma_0}{(n+1)(n-2)}f_{n-2}$ for $n\geq 3$. Hence it follows that either $f_1=0$ or $\gamma_1=0$, and f_2 may be arbitrary number.

At first we suppose that $f_1=0$. Then $\gamma_0=0$ and for $n\geq 3$

$$|f_n|=\frac{n|\gamma_1|}{(n+1)(n-2)}|f_{n-1}|\leq\frac{|\gamma_1|}{n-2}|f_{n-1}|\leq\frac{|\gamma_1|^2}{(n-2)(n-3)}|f_{n-2}|\leq\cdots\leq\frac{|\gamma_1|^{n-2}}{(n-2)!}|f_2|.$$

Hence it follows that

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|\leq(3+\beta+2\alpha\beta)|f_2|$$
$$+\sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\frac{|\gamma_1|^{n-2}}{(n-2)!}|f_2|=K_1(\alpha,\beta,|\gamma_1|)|f_2|$$

where $K(\alpha,\beta,|\gamma_1|)=\text{const}>0$. Since f_2 may be arbitrary the following proposition holds.

Proposition 3. *If $\gamma_2=-2$, $\beta_1=\gamma_0=0$, $\beta_0=\gamma_1\neq 0$ then for each $\alpha\in[0,1)$ and $\beta\in(0,1]$ there exists a solution given by* (4) *of differential equation* (3)*, which is meromorphically starlike of order α and type β.*

Now, we assume that $\gamma_1=0$. Then $f_1=\gamma_0/2$, f_2 may be arbitrary and

$$|f_n|=\frac{|\gamma_0|}{(n+1)(n-2)}|f_{n-2}|\qquad\text{for}\qquad n\geq 3.$$

Using these relations, we prove the following theorem.

Theorem 3. *Let $\alpha\in[0,1)$ and $\beta\in(0,1]$. If $\gamma_2=-2,\beta_1=\beta_0=\gamma_1=0$ then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$(1+\beta\alpha)|\gamma_0|\leq 2\beta(1-\alpha)\left(1-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\beta\alpha)}\right),\tag{17}$$

is meromorphically starlike of order α and type β.

Proof. Since f_2 may be arbitrary, we set $f_2=0$. Then

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|=2(1+\alpha\beta)|f_1|+\sum_{n=3}^{\infty}\frac{((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_0|}{(n+1)(n-2)}|f_{n-2}|$$
$$=2(1+\alpha\beta)|f_1|+\sum_{n=1}^{\infty}\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{n(n+3)}|f_n|$$

and, thus,

$$\sum_{n=1}^{\infty}\left(1-\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{((1+\beta)n+\beta(2\alpha-1)+1)n(n+3)}\right)((1+\beta)n+\beta(2\alpha-1)+1)|f_n|$$
$$\leq 2(1+\alpha\beta)|f_1|.$$

But
$$\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{((1+\beta)n+\beta(2\alpha-1)+1)n(n+3)}\leq\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)}.$$

Therefore,

$$\left(1-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)}\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|\leq(1+\alpha\beta)|\gamma_0|,$$

whence in view of (17) we obtain (7), and by Lemma 1 function (4) is meromorphically starlike of order α and type β. □

Finally, we consider the case, where equation (3) has a solution of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{m}f_nz^n,\quad m\geq 2,\tag{18}$$

where $f_m\neq 0$. Equality (9) yields that

$$(m+2)(m+1+\gamma_2)f_{m+1}+m\gamma_1 f_m+\gamma_0 f_{m-1}=0\quad\text{and}$$
$$(m+3)(m+2+\gamma_2)f_{m+2}+(m+1)\gamma_1 f_{m+1}+\gamma_0 f_m=0.$$

Since $f_{m+2}=f_{m+1}=0$ and $f_m\neq 0$, the second equality implies the equality $\gamma_0=0$ and consequently the first equality implies the equality $\gamma_1=0$. Therefore, in view of (8) and (9) $(n+1)(n+\gamma_2)f_n=0$ for all $n\geq 1$. Since $f_m\neq 0$, so $m+\gamma_2=0$. Thus $n+\gamma_2\neq 0$ for all $n\neq m$ and, therefore, $f_n=0$, except f_m, which may be arbitrary. Hence it follows that the solution given by (18) is possible only if $m+\gamma_2=0$ and is of the form

$$f(z)=\frac{1}{z}+f_mz^m,\tag{19}$$

where f_m is an arbitrary number. (It is easy to verify directly that the function (19) is a solution of equation (3) if and only if $\beta_0=\gamma_0=\gamma_1=2-\beta_1+\gamma_2=m+\gamma_2=0$.)

For each $\alpha\in[0,1)$ and $\beta\in(0,1]$ we choose f_m such that $((1+\beta)m+\beta(2\alpha-1)+1)|f_m|\leq 2\beta(1-\alpha)$. Then the function (19) is meromorphically starlike of order α and type β.

2 MEROMORPHICALLY CONVEX SOLUTIONS

We call a function $f\in\Sigma$ meromorphically convex of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$ if

$$|zf''(z)+2f'(z)|<\beta|zf''(z)+2\alpha f'(z)|,\quad z\in\mathbb{D}_0.$$

Clearly, f is meromorphically convex of order α and type β if and only if $\varphi(z)=-zf'(z)$ is meromorphically starlike of order α and type β. Since $\varphi(z)=\frac{1}{z}-\sum\limits_{n=1}^{\infty}nf_nz^n$, by Lemma 1 the condition

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n|\leq 2\beta(1-\alpha),\tag{20}$$

is sufficient in order that f is meromorphically convex of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$. Therefore, using Lemma 2 we can prove analogues of Theorems 1 - 3.

Theorem 4. *Let $\alpha\in[0,1)$ and $\beta\in(0,1]$. If $\beta_1=2+\gamma_2$, $|\gamma_2|<1$, $\beta_0=\gamma_1$ then differential equation (3) has a solution given by (4), which by the condition*

$$\frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}\leq 2\beta(1-\alpha)\left(1-\frac{2(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta))(2-|\gamma_2|)}-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}\right)\tag{21}$$

is meromorphically convex of order α and type β.

Proof. As in the proof of Theorem 1 we have

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n| = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))2|f_2|$$
$$+\sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\left|\frac{n^2\gamma_1(n-1)f_{n-1}}{(n-1)(n+1)(n+\gamma_2)}+\frac{n\gamma_0(n-2)f_{n-2}}{(n-2)(n+1)(n+\gamma_2)}\right|$$
$$\le 2(1+\beta\alpha)|f_1| + 2(3+\beta(1+2\alpha))|f_2|$$
$$+\sum_{n=2}^{\infty}\frac{(n+1)^2((1+\beta)(n+1)+\beta(2\alpha-1)+1)|\gamma_1|}{n(n+2)(n+1-|\gamma_2|)}n|f_n|$$
$$+\sum_{n=1}^{\infty}\frac{(n+2)((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{n(n+3)(n+2-|\gamma_2|)}n|f_n|$$
$$=2(1+\beta\alpha)|f_1| + 2(3+\beta(1+2\alpha))|f_2| - \frac{4(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1|$$
$$+\sum_{n=1}^{\infty}\frac{(n+1)^2(n+2+(n+2\alpha)\beta)|\gamma_1|}{n(n+2)(n+1-|\gamma_2|)}n|f_n| + \sum_{n=1}^{\infty}\frac{(n+2)(n+3+\beta(n+1+2\alpha))|\gamma_0|}{n(n+3)(n+2-|\gamma_2|)}n|f_n|,$$

whence as above

$$\left(1-\frac{2(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)(3-|\gamma_2|)}\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n|$$
$$\le \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}$$

and in view of (21) we obtain (20). Therefore, the function defined by (4) is meromorphically convex of order α and type β. □

The following theorems can be proved by analogy.

Theorem 5. *If* $\beta_1 = 1$, $\gamma_2 = -1$, $\gamma_0 = 0$, $\beta_0 = \gamma_1 \ne 0$ *then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$\frac{2(3+\beta+2\alpha\beta)}{3(1+\alpha\beta)}|\gamma_1| < 1$$

is meromorphically convex of order α *and type* β*.*

Theorem 6. *Let* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1]$*. If* $\gamma_2 = -2$ *and* $\beta_1 = \beta_0 = \gamma_1 = 0$ *then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$(1+\beta\alpha)|\gamma_0| \le 2\beta(1-\alpha)\left(1-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\beta\alpha)}\right)$$

is meromorphically convex of order α *and type* β*.*

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Мулява О., Трухан Ю. *Про мероморфно зіркові функції порядку α і типу β, що задовольняють диференційне рівняння Шаха* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 154–162.

Згідно з М.Л. Могра, Т.Р. Редді та О.П. Жюнея аналітична в $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$ функція $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ називається мероморфно зірковою порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$, якщо $|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|$, $z \in \mathbb{D}_0$. Тут досліджено умови на комплексні параметри β_0, β_1, γ_0, γ_1, γ_2, за яких диференційне рівняння С. Шаха $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$ має мероморфно зіркові розв'язки порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$. Окрім основного випадку $n + \gamma_2 \neq 0$, $n \geq 1$, розглядаються випадки $\gamma_2 = -1$ і $\gamma_2 = -2$. Також вивчено можливість існування розв'язків вигляду $f(z) = \frac{1}{z} + \sum_{n=1}^{m} f_n z^n$, $m \geq 2$. Крім того, ми називаємо аналітичну в $\mathbb{D}_0$ функцію $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ мероморфно опуклою порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$, якщо $|zf''(z) + 2f'(z)| < \beta|zf''(z) + 2\alpha f'(z)|$, $z \in \mathbb{D}_0$, і досліджуємо достатні умови на параметри β_0, β_1, γ_0, γ_1, γ_2, за яких диференційне рівняння С. Шаха має мероморфно опуклі розв'язки порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$. Розглядаються ті ж випадки, що і для мероморфно зіркових розв'язків.

Ключові слова і фрази: мероморфно зіркова функція порядку α та типу β, мероморфно опукла функція порядку α та типу β, диференційне рівняння Шаха.