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COMMUTATIVE BEZOUT DOMAINS IN WHICH ANY NONZERO PRIME IDEAL IS CONTAINED IN A FINITE SET OF MAXIMAL IDEALS

We investigate commutative Bezout domains in which any nonzero prime ideal is contained in a finite set of maximal ideals. In particular, we have described the class of such rings, which are elementary divisor rings. A ring R is called an elementary divisor ring if every matrix over R has a canonical diagonal reduction (we say that a matrix A over R has a canonical diagonal reduction if for the matrix A there exist invertible matrices P and Q of appropriate sizes and a diagonal matrix $D = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r, 0, \dots, 0)$ such that $PAQ = D$ and $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ for every $1 \le i \le r-1$). We proved that a commutative Bezout domain R in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ the ideal aR a decomposed into a product $aR = Q_1 \dots Q_n$, where Q_i $(i = 1, \dots, n)$ are pairwise comaximal ideals and rad $Q_i \in$ spec R, is an elementary divisor ring.

Key words and phrases: Bezout domain, elementary divisor ring, adequate ring, ring of stable range, valuation ring, prime ideal, maximal ideal, comaximal ideal.

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INTRODUCTION

The classical notion of an elementary divisor ring was first introduced by I. Kaplansky [5]. Among the well-known classes of rings, a special place is occupied by adequate rings introduced by Helmer [3]. Henriksen proved that in an adequate ring any nonzero prime ideal is contained in a unique maximal ideal, i.e. an adequate ring is a PM^*-ring [4]. Larsen, Lewis and Shores [6] raised the question: is it true that every commutative Bezout domain, in which any non-zero prime ideal is contained in a unique maximal ideal, is an adequate ring? In [1], an example is given for a commutative PM^* Bezout domain that is not adequate, but when is an elementary divisor ring. Gatalevych and Zabavsky proved that a commutative Bezout domain, in which any nonzero prime ideal is contained in a unique maximal ideal (PM^*-ring), is an elementary divisor ring [9]. While investigating Bezout rings with the Noetherian spectrum [2], the authors encountered examples of commutative Bezout domains, in which any nonzero prime ideal is contained in a finite set of maximal ideals. An obvious example of such a ring is an adequate ring. In this paper, the existence and properties of such rings are established.

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We introduce the necessary definitions and facts.

All rings considered will be commutative with identity. A ring is *a Bezout ring*, if every its finitely generated ideal is principal. Let $\mathrm{GL}_n(R)$ be the group (*the general linear group*) of all invertible $(n \times n)$-matrices over the ring R. We say that matrices A and B over a ring R are *equivalent* if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. The fact that matrices A and B are equivalent is denoted by $A \sim B$. If for a matrix A there exists a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that $A \sim D$ and $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ for every i then we say that the matrix A has *a canonical diagonal reduction*. A ring R is called *an elementary divisor ring* if every matrix over R has a canonical diagonal reduction.

Let I be an ideal of a ring R. The *radical of an ideal* I, denoted by $\mathrm{rad}\, I$ or $\sqrt{I}$, is defined as

$$\mathrm{rad}\, I = \{\, a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N} \,\}.$$

Obviously, $\mathrm{rad}\, I = \bigcap\limits_{P \in \mathrm{spec}\, I} P$ where $\mathrm{spec}\, I$ denotes the set of all the prime ideals of the ring R containing the ideal I (the spectrum of the ideal I). Note that $\mathrm{rad}\, I$ can be defined differently, namely $\mathrm{rad}\, I = \bigcap\limits_{P \in \mathrm{minspec}\, I} P$, where $\mathrm{minspec}\, I$ is the set of minimal ideals of the ideal I, i.e. proper prime ideals of $\mathrm{spec}\, I$, not containing prime ideals from $\mathrm{spec}\, I$.

Two ideals I, J of a ring R are said to be *comaximal* if $x + y = 1$ for some $x \in I$ and $y \in J$.

1 SECTION WITH RESULTS

Let R be a commutative domain, $\mathrm{mspec}\, R$ be a set of all maximal ideals of the ring R, M be any maximal ideal of the ring R ($M \in \mathrm{mspec}\, R$). Let us denote by R_M the localization of the ring R with respect to the multiplicatively closed set $S = R \backslash M$. Note that if R is a commutative Bezout domain, then R_M is a local Bezout domain for any maximal ideal $M \in \mathrm{mspec} R$. And since a local Bezout domain is a valuation ring, i.e. a ring in which the set of ideals is linearly ordered with respect to ideal inclusion, we obtain such a result.

Proposition 1. *Let R be a commutative Bezout domain. For any maximal ideal $M \in \mathrm{mspec}\, R$, the set of the prime ideals of R, contained in M, is linearly ordered with respect to inclusion.*

The Proposition 1 shows that $\mathrm{spec}\, R$ is a tree [1].

Let us consider the case of the commutative Bezout domain R in which the set $\mathrm{minspec}\, R$ is finite for any nonzero element $a \in R$.

Theorem 1. *Let R be a commutative Bezout domain, a be a nonzero element R such that $\mathrm{minspec}\, aR$ is a finite and any prime ideal of $\mathrm{spec}\, aR$ is contained in a unique maximal ideal. Then the factor ring R/aR is the direct sum of valuation rings.*

Proof. Let $P_1, P_2, \ldots, P_n \in \mathrm{minspec}\, aR$. We consider the factor ring $\overline{R} = R/aR$. We denote $\overline{P}_i = P_i/aR$, where $P_i \in \mathrm{minspec}\, aR$, $i = 1, 2, \ldots, n$. Note that $\overline{P}_i \in \mathrm{minspec}\, \overline{R}$ are all minimal prime ideals of the ring $\overline{R}$. Moreover, by Proposition 1, the ideals $\overline{P}_i$ are comaximal in $\overline{R}$. Obviously, $\mathrm{rad}\, \overline{R} = \bigcap\limits_{i=1}^{n} \overline{P}_i$, and by the Chinese remainder theorem we have

$$\overline{R}/\mathrm{rad}\, \overline{R} \cong \overline{R}/\overline{P}_1 \oplus \overline{R}/\overline{P}_2 \oplus \ldots \oplus \overline{R}/\overline{P}_n.$$

Since any prime ideal of $\operatorname{spec} aR$ is contained in a unique maximal ideal, $\overline{R}/\overline{P}_i$ are valuation rings. Moreover, there exist pairwise orthogonal idempotents $\overline{\overline{e}}_1, \ldots, \overline{\overline{e}}_n$, where $\overline{\overline{e}}_i \in \overline{R}/\overline{P}_i$ such that $\overline{\overline{e}}_1 + \ldots + \overline{\overline{e}}_n = \overline{\overline{1}}$. Then, by lifting the idempotent $\overline{\overline{e}}_i$ modulo $\operatorname{rad} \overline{R}$ to pairwise orthogonal idempotents $\overline{e}_1, \ldots, \overline{e}_n \in \overline{R}$ we find that $1 - (e_1 \ldots + e_n)$ is an idempotent and $1 - (e_1 + \ldots + e_n) \in \operatorname{rad} \overline{R}$, which is possible only if it is zero. Therefore,

$$\overline{R} = \overline{e}_1\overline{R} \oplus \overline{e}_2\overline{R} \oplus \cdots \oplus \overline{e}_n\overline{R}$$

and each $\overline{e}_i\overline{R}$ is a homomorphic image of $\overline{R}$, i.e. a commutative Bezout ring. Since any prime ideal of $\overline{R}$ is contained in a unique maximal ideal, $\overline{e}_i\overline{R}$ is a valuation ring. □

A minor modification of the proof of Theorem 1 gives us the following result.

Theorem 2. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set* $\operatorname{minspec} aR$ *is finite, the factor ring $\overline{R} = R/aR$ is a direct sum of semilocal rings.*

Proof. According to the notations from Theorem 1 and its proof, we have

$$\overline{R} = \overline{e}_1\overline{R} \oplus \overline{e}_2\overline{R} \oplus \ldots \oplus \overline{e}_n\overline{R}.$$

Since any prime ideal of the ring $\overline{R}$ is contained in a finite set of maximal ideals, $\overline{e}_i\overline{R}$ is a semilocal ring. □

Obviously, if a commutative ring R is a direct sum of valuation rings R_i, then R is a commutative Bezout ring. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be any elements of R, where $a_i, b_i \in R_i$, $i = 1, 2, \ldots, n$. Since R_i is a valuation ring, $a_i = r_i s_i$, where $r_i R + b_i R = R$ and $s'_i R_i + b_i R_i \neq R_i$ for any non invertible divisor s'_i of the element s_i. If $r = (r_1, \ldots, r_n)$, $s = (s_1, \ldots, s_n)$ then obviously $a = rs$, $rR + bR = R$. For each i such that s'_i is a non invertible divisor of $s_i \in R_i$, we have $s_i R_i + b_i R_i \neq R_i$. Hence $s'R + bR \neq R$, i.e. a is an adequate element.

Recall the definitions.

Definition 1. *An element a of a commutative ring R is called* adequate, *if for every element $b \in R$ one can find elements $r, s \in R$ such that:*

1) $a = rs$;

2) $rR + bR = R$;

3) $s'R + bR \neq R$ for any $s' \in R$ such that $sR \subset s'R \neq R$.

The most trivial examples of adequate elements are units, atoms in a ring, and also square-free elements [8].

A ring R is said to be *everywhere adequate* if any element of R is adequate.

Note that, as shown above, in the case of a commutative ring, which is a direct sum of valuation rings, any element of the ring (in particular zero) is adequate, i.e. this ring is everywhere adequate. Moreover, by [10], this ring is clean, i.e. a ring in which any element is the sum of an idempotent and an invertible element.

Definition 2. *A ring R is called a ring of stable range 1 if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $(a + bt)R = R$.*

Definition 3. *An nonzero element a of a ring R is called an element of almost stable range 1 if the quotient-ring R/aR is a ring of stable range 1.*

Any ring of stable range 1 is a ring of almost stable 1 (see [7]). But not every element of stable range 1 is an element of almost stable range 1. For example, let e be a nonzero idempotent of a commutative ring R and $eR + aR = R$. Then $ex + ay = 1$ for some elements $x, y \in R$ and $(1 - e)ex + (1 - e)ay = 1 - e$, so $e + a(1 - e)y = 1$. And we have that e is an element of stable range 1 for any commutative ring. However if you consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ and the element $e = (1, 0) \in R$ then, as shown above, e is an element of stable range 1, by $R/eR \cong \mathbb{Z}$, and e is not element of almost stable range 1. Moreover, if R is a commutative principal ideal domain (i.e. ring of integers), which is not of stable range 1, then every nonzero element of R is an element of almost stable range 1.

Definition 4. *A commutative ring in which every nonzero element is an element of almost stable range 1 is called a ring of* almost stable range 1.

The first example of a ring of almost stable range 1 is a ring of stable range 1. Also, every commutative principal ideal ring which is not a ring of stable range 1 (for example, the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

We note that the semilocal ring is an example of a ring of stable range 1. Moreover, the direct sum of rings of stable range 1 is a ring of stable range 1. As a result, we obtain the result from the previous theorems.

Theorem 3. *Let R be a commutative Bezout domain, a be a nonzero element R such that the set* $\operatorname{minspec} aR$ *is finite and any prime ideal of* $\operatorname{spec} aR$ *is contained in a unique maximal ideal. Then the factor ring R/aR is everywhere adequate if and only if R/aR is a direct sum of a valuation rings.*

Proof. Since R be a commutative Bezout domain, a be a nonzero element R such that the set $\operatorname{minspec} aR$ is finite and any prime ideal of $\operatorname{spec} aR$ is contained in a unique maximal ideal, factor ring R/aR is a semilocal ring. By [6] proof the semilocal ring R is everywhere adequate if and only if R is a direct sum of a valuation rings. □

Theorem 4. *Let R be a commutative Bezout domain and a be a nonzero element of R such that the set* $\operatorname{minspec} aR$ *is finite, and any nonzero prime ideal* $\operatorname{spec} aR$ *is contained in a finite set of maximal ideals. Then a is an element of almost stable range 1.*

The proof of the Theorem 4 is similar to the proof of the Theorem 3.

Proposition 2 ([2])**.** *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then the following properties are equivalent:*

1) *for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \ldots Q_n$, where $Q_1, \ldots, Q_n$ are pairwise commaximal ideals such that* $\operatorname{rad} Q_i$ *is a prime ideal;*

2) $\operatorname{minspec} aR$ *is finite.*

As a result of Proposition 2 and Theorem 4 we obtain the following results.

Theorem 5. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \dots Q_n$, where $Q_1, \dots, Q_n$ are pairwise comaximal ideals such that* $\operatorname{rad} Q_i \in \operatorname{spec} R$. *Then R is a ring of almost stable range 1.*

Proof. Since R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ there exists a representation $aR = Q_1 \dots Q_n$, where $Q_1, \dots, Q_n$ are pairwise comaximal ideals such that $\operatorname{rad} Q_i \in \operatorname{spec} R$, $\operatorname{minspec} aR$ is finite. By Theorem 4, a is an element of almost stable range 1. Then R is a ring of almost stable range 1. ☐

Since a commutative Bezout ring of almost stable range 1 is an elementary divisor ring [7], as a result, we obtain the following.

Theorem 6. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals and for any nonzero element $a \in R$ let the ideal aR is decomposed into a product $aR = Q_1 \dots Q_n$, where Q_i $(i = 1, \dots, n)$ are pairwise comaximal ideals and* $\operatorname{rad} Q_i \in \operatorname{spec} R$. *Then R is an elementary divisor ring.*

Open Question. Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?

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Забавський Б.В., Романів О.М. *Комутативні області Безу, в яких довільний ненульовий простий ідеал міститься у скінченній множині максимальних ідеалів* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 402–407.

Досліджуються комутативні області Безу, яких довільний ненульовий простий ідеал міститься в скінченній множині максимальних ідеалів. Зокрема описано клас таких кілець, які є кільцями елементарних дільників. Кільце R називається кільцем елементарних дільників, якщо кожна матриця над R володіє канонічною діагональною редукцією (матриця A володіє канонічною діагональною редукцією, якщо існує така діагональна матриця $D = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$, що матриці A та D еквівалентні і $R\varepsilon_i \subseteq R\varepsilon_{i+1}$ для кожного $1 \leq i \leq r-1$). Зокрема, ми довели, що комутативна область Безу R, в якій кожен ненульовий простий ідеал міститься в скінченній множині максимальних ідеалів і для довільного елемента $a \in R$ ідеал aR розкладається в добуток $aR = Q_1 \ldots Q_n$, де Q_i $(i = 1, \ldots, n)$ є попарно комаксимальними ідеалами і $\operatorname{rad} Q_i \in \operatorname{spec} R$, є кільцем елементарних дільників.

Ключові слова і фрази: кільце Безу, кільце елементарних дільників, адекватне кільце, кільце стабільного рангу, кільце нормування, простий ідеал, максимальний ідеал, комаксимальний ідеал.