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MULTIPLICATIVE CONVOLUTION ON THE ALGEBRA OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS

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We introduce and study a multiplicative convolution operator on the spectrum of the algebra of block-symmetric analytic functions of bounded type on an infinite ℓ_1-sum of copies of the Banach space $\mathbb{C}^s$. The case of the algebra of block-symmetric functions on the space $\mathbb{C}^2$ and the action of multiplicative convolution on its spectrum are studied separately.

Introduction

The algebra of analytic functions of bounded type in a complex Banach space X is a standard object of investigations in nonlinear functional analysis. They were studied in [2, 4, 16, 17] and many other publications. In [1, 3, 5, 6–8, 10, 15], the authors studied the spectra of the algebras of symmetric (invariant) analytic functions on the spaces ℓ_p, $1 \le p \le \infty$, L_∞, and $L_\infty[0,+\infty) \bigcap L_1[0,+\infty)$ relative to the groups of isometric mappings of these spaces. The analytic functions on the ℓ_p-sums of finite-dimensional spaces ("blocks") symmetric relative to the permutation of these blocks (block-symmetric) were considered in [11–13].

Thus, in particular, the algebraic basis of block-symmetric polynomials in the space

$$\mathbb{C}^n \otimes \ell_p = \oplus_{\ell_p} \mathbb{C}^n$$

was described in [13]. In the present work, we study an analog of symmetric multiplicative convolution (see [6]) for the case of block-symmetric polynomials.

Note that the block-symmetric polynomials are used in combinatorics and have applications in the field of quantum mechanics, where they are also called the MacMahon symmetric polynomials, diagonal polynomials, or multisymmetric polynomials (see [9, 14]).

1. Preliminarily Data

Let $\mathcal{X}_\infty^s = \oplus_{\ell_1} \mathbb{C}^s$ be an infinite ℓ_1-sum of copies of the Banach space $\mathbb{C}^s$. Then every element from $\bar{x} \in \mathcal{X}_\infty^s$ can be represented in the form of a sequence $\bar{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbb{C}^s$, with the norm

$$\| \bar{x} \| = \sum_{k=1}^{\infty} \sum_{i=1}^{s} \left| x_k^i \right|.$$

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A polynomial P in the space $\mathcal{X}_\infty^s = \oplus_{\ell_1}\mathbb{C}^s$ is called block-symmetric (or vector-symmetric) if

$$P\left(\begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^s \end{pmatrix}_1, \ldots, \begin{pmatrix} x_m^1 \\ x_m^2 \\ \vdots \\ x_m^s \end{pmatrix}_m, \ldots\right) = P\left(\begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^s \end{pmatrix}_{\sigma(1)}, \ldots, \begin{pmatrix} x_m^1 \\ x_m^2 \\ \vdots \\ x_m^s \end{pmatrix}_{\sigma(m)}, \ldots\right),$$

for any substitution $\sigma \in \mathcal{G}$, where $\mathcal{G}$ is the group of substitutions on the set $\mathbb{N}$ and $(x_i^1, x_i^2, \ldots, x_i^s)^\top \in \mathbb{C}^s$. By $\mathcal{P}_{vs}(\mathcal{X}_\infty^s)$ we denote the algebra of block-symmetric polynomials on $\mathcal{X}_\infty^s$.

In [11], it was proved that the algebraic basis of the algebra $\mathcal{P}_{vs}(\mathcal{X}_\infty^s)$ is formed by the polynomials

$$H_n^{k_1,k_2,\ldots,k_s}(x^1, x^2, \ldots, x^s) = \sum_{i=1}^{\infty} (x_i^1)^{k_1}(x_i^2)^{k_2} \ldots (x_i^s)^{k_s},$$

$$k_1 + k_2 + \ldots + k_s = n.$$

By $\mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$ we denote the algebra of block-symmetric analytic functions of bounded type. Let $\mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$ be the spectrum of this algebra.

2. Multiplicative Convolution

In [6], the definition of multiplicative shift was introduced for elements of the space ℓ_1. In a similar way, we introduce the definition of multiplicative shift for elements of the space $\mathcal{X}_\infty^s$.

Definition 1. *Let $(x^1, x^2, \ldots, x^s)$ and $(y^1, y^2, \ldots, y^s) \in \mathcal{X}_\infty^s$. The multiplicative shift of the elements $(x^1, x^2, \ldots, x^s)$ and $(y^1, y^2, \ldots, y^s)$ is introduced as a vector formed by the elements $\left(x_i^1 y_j^1, x_i^2 y_j^2, \ldots, x_i^s y_j^s\right)^\top$ enumerated in any order, $i, j \in \mathbb{N}$, and denoted by $(x^1, x^2, \ldots, x^s) \lozenge (y^1, y^2, \ldots, y^s)$.*

Proposition 1. *For any $\tilde{x} = (x^1, x^2, \ldots, x^s)$, $\tilde{y} = (y^1, y^2, \ldots, y^s) \in \mathcal{X}_\infty^s$, the following assertions are true:*

(1°) $\tilde{x} \lozenge \tilde{y} \in \mathcal{X}_\infty^s$ *and* $\|\tilde{x} \lozenge \tilde{y}\| \le \|\tilde{x}\| \cdot \|\tilde{y}\|$;

(2°) $H_n^{k_1,\ldots,k_s}(\tilde{x} \lozenge \tilde{y}) = H_n^{k_1,\ldots,k_s}(\tilde{x}) H_n^{k_1,\ldots,k_s}(\tilde{y})$;

(3°) if P is an n-homogeneous polynomial on $\mathcal{X}_\infty^s$, and $\tilde{y}$ is a fixed element from $\mathcal{X}_\infty^s$, then the function $\tilde{x} \mapsto P(\tilde{x} \lozenge \tilde{y})$ is an n-homogeneous polynomial.

Proof. It is clear that

$$\|\tilde{x} \lozenge \tilde{y}\| = \sum_{i,j=1}^{\infty} \sum_{k=1}^{s} \left| x_i^k y_j^k \right| \le \sum_{i=1}^{\infty} \sum_{k=1}^{s} \left| x_i^k \right| \cdot \sum_{i=1}^{\infty} \sum_{k=1}^{s} \left| y_j^k \right|$$

$$= \left\|(x^1, x^2, \ldots, x^s)\right\| \cdot \left\|(y^1, y^2, \ldots, y^s)\right\| = \|\tilde{x}\| \cdot \|\tilde{y}\|.$$

Moreover,

$$H_n^{k_1,\ldots,k_s}(\tilde{x} \lozenge \tilde{y}) = \sum_{i,j=1}^{\infty} \prod_{m=1}^{s} (x_i^m y_j^m)^{k_m} = \sum_{i=1}^{\infty} \prod_{m=1}^{s} (x_i^m)^{k_m} \cdot \sum_{j=1}^{\infty} \prod_{m=1}^{s} (y_j^m)^{k_m}.$$

Condition *(3°)* follows from the equality $\lambda(\tilde{x} \lozenge \tilde{y}) = (\lambda \tilde{x} \lozenge \tilde{y})$.

The proposition is proved.

Let $\tilde{y} \in \mathcal{X}_\infty^s$. The mapping

$$\tilde{x} \in \mathcal{X}_\infty^s \overset{\pi_{\tilde{y}}}{\rightarrow} (\tilde{x} \lozenge \tilde{y}) \in \mathcal{X}_\infty^s$$

is linear and continuous, which follows from Proposition 1. If $f \in H_{bvs}(\mathcal{X}_\infty^s)$, then $f \circ \pi_{\tilde{y}} \in H_{bvs}(\mathcal{X}_\infty^s)$ because $f \circ \pi_{\tilde{y}}$ is analytic and bounded on bounded sets, and $f(\sigma(\tilde{x}) \lozenge \tilde{y}) = f(\tilde{x} \lozenge \tilde{y})$ for any permutation $\sigma \in \mathcal{G}$. We call the operator $M_{\tilde{y}} = f \circ \pi_{\tilde{y}}$ *a multiplicative convolution operator*. It is obvious that $M_{\tilde{y}} = M_{\sigma(\tilde{y})}$ for any permutation $\sigma \in \mathcal{G}$ and $M_{\tilde{y}}(H_n^{k_1,\ldots,k_s}) = H_n^{k_1,\ldots,k_s}(\tilde{y}) \cdot H_n^{k_1,\ldots,k_s}$.

Moreover, it is clear that

$$\pi_{\tilde{y}+\tilde{z}}(\tilde{x}) = (\tilde{x} \lozenge (\tilde{y} + \tilde{z})) = (\tilde{x} \lozenge \tilde{y}) + (\tilde{x} \lozenge \tilde{z}) = \pi_{\tilde{y}}(\tilde{x}) + \pi_{\tilde{z}}(\tilde{x}),$$

$$\pi_{\lambda\tilde{y}}(\tilde{x}) = (\tilde{x} \lozenge \lambda\tilde{y}) = \lambda(\tilde{x} \lozenge \tilde{y}) = \lambda\pi_{\tilde{y}}(\tilde{x}).$$

Proposition 2. *For any* $\tilde{y} \in \mathcal{X}_\infty^s$, *the multiplicative convolution operator* $M_{\tilde{y}}$ *is a continuous homomorphism of the algebra* $\mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$ *into itself.*

Proof. Let $\tilde{x} = (x^1, x^2, \ldots, x^s)$, $\tilde{y} = (y^1, y^2, \ldots, y^s) \in \mathcal{X}_\infty^s$, and $f(\tilde{x}) = f(x^1, x^2, \ldots, x^s) \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$. We now show that $f(\tilde{x} \lozenge \tilde{y}) \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$. Since every function $f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$ can be uniformly approximated by the polynomials $P_n \in \mathcal{P}_{vs}(\mathcal{X}_\infty^s)$, we find

$$f(\tilde{x} \lozenge \tilde{y}) = \sum_{n=0}^{\infty} P_n(\tilde{x} \lozenge \tilde{y})$$

$$= \sum_{n=0}^{\infty} G_n\big(H^{1,0,\ldots,0}(\tilde{x} \lozenge \tilde{y}), \ldots, H^{k_1,k_2,\ldots,k_s}(\tilde{x} \lozenge \tilde{y}), \ldots\big)$$

$$= \sum_{n=0}^{\infty} G_n\big(H^{1,0,\ldots,0}(\tilde{x}) \cdot H^{1,0,\ldots,0}(\tilde{y}),$$

$$\ldots, H^{k_1,k_2,\ldots,k_s}(\tilde{x}) \cdot H^{k_1,k_2,\ldots,k_s}(\tilde{y}), \ldots\big) \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s).$$

The fact that $M_{\tilde{y}}$ is a homomorphism follows from the equalities

$$\forall \lambda \in \mathbb{C} \quad M_{\lambda\tilde{y}}(P_n) = P_n \circ \pi_{\lambda\tilde{y}} = P_n \circ \lambda\pi_{\tilde{y}} = \lambda^n P_n \circ \pi_{\tilde{y}} = \lambda^n M_{\tilde{y}}(P_n),$$

$$M_{\tilde{y}+\tilde{z}}(P_n) = P_n \circ \pi_{\tilde{y}+\tilde{z}} = P_n \circ (\pi_{\tilde{y}} + \pi_{\tilde{z}})$$

$$= P_n \circ \pi_{\tilde{y}} + P_n \circ \pi_{\tilde{z}} = M_{\tilde{y}}(P_n) + M_{\tilde{z}}(P_n).$$

The continuity of the operator $M_{\tilde{y}}$ is a consequence of the inequality

$$\big\| M_{\tilde{y}}(P_n) \big\| = \big\| P_n \circ \pi_{\tilde{y}} \big\| \le \| P_n \| \cdot \| y \|^n.$$

Thus, $M_{\tilde{y}}$ is a continuous homomorphism on $\mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$.

The proposition is proved.

In [12], the authors introduced the notion of radius function $R(\varphi)$ of a complex homomorphism $\varphi \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$ as the infimum of all r such that φ is continuous on $A_{uvs}(rB_{\mathcal{X}_\infty^s})$, where $A_{uvs}(rB_{\mathcal{X}_\infty^s})$ is the algebra of all uniformly continuous block-symmetric analytic functions on the sphere $rB_{\mathcal{X}_\infty^s} \subset \mathcal{X}_\infty^s$ of radius r. It was proved that the quantity $R(\varphi)$ can be found by using the following formula:

$$R(\varphi) = \limsup_{n \to \infty} \| \varphi_n \|^{\frac{1}{n}},$$

where φ_n is the restriction of the functional φ to the subspace of n-homogeneous block-symmetric polynomials.

Proposition 3. *For all* $\theta \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)'$ *and any* $\tilde{y} \in \mathcal{X}_\infty^s$, *the radius function of the continuous homomorphism* $\theta \circ M_{\tilde{y}}$ *can be estimated as follows:*

$$R(\theta \circ M_{\tilde{y}}) \le R(\theta) \cdot \big\| \tilde{y} \big\|. \tag{1}$$

Proof. We perform reasoning similar to that used in [6]. Let $\tilde{y} \in \mathcal{X}_\infty^s$. We denote by $(\theta \circ M_{\tilde{y}})_n$ (resp., θ_n), the restriction of $\theta \circ M_{\tilde{y}}$ (resp., θ) to the subspace of n-homogeneous block-symmetric polynomials. We get

$$\left\| (\theta \circ M_{\tilde{y}})_n \right\| = \sup_{\|f\| \le 1} \left| \theta_n \left(\frac{M_{\tilde{y}}(f_n)}{\|\tilde{y}\|^n} \right) \right| \cdot \|\tilde{y}\|^n \le \|\theta_n\| \cdot \|\tilde{y}\|^n .$$

Thus,

$$R(\theta \circ M_{\tilde{y}}) \le \limsup_{n\to\infty} \left(\|\theta_n\| \cdot \|\tilde{y}\|^n \right)^{\frac{1}{n}} = R(\theta) \cdot \|\tilde{y}\| .$$

The proposition is proved.

We now introduce the multiplicative convolution on $\mathcal{H}_{bvs}(\mathcal{X}_\infty^s)'$ by analogy with [6].

Definition 2. *For any functions $f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$ and $\theta \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)'$, the multiplicative convolution is defined by the formula*

$$(\theta \lozenge f)(\tilde{x}) = \theta\left[M_{\tilde{x}}(f) \right] \quad \textit{for every} \quad \tilde{x} \in \mathcal{X}_\infty^s .$$

Definition 3. *For any $\phi, \theta \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)'$, the multiplicative convolution is defined by the formula*

$$(\phi \lozenge \theta)(f) = \phi(\theta \lozenge f) \quad \textit{for any} \quad f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s) .$$

Proposition 4. *If $\phi, \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$, then $\phi \lozenge \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$.*

Proof. It follows from the multiplicativity of $M_{\tilde{y}}$ that $\phi \lozenge \theta$ is a character.

The fact that $\phi \lozenge \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$ follows from (1), i.e., from the inequality

$$R(\phi \lozenge \theta) \le R(\phi) \cdot R(\theta) .$$

Hence, $\phi \lozenge \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$.

The proposition is proved.

Theorem 1. *For any $\phi, \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$, the multiplicative convolution is commutative and associative. Moreover, this convolution satisfies the equality*

$$(\phi \lozenge \theta)(H^{k_1,\ldots,k_s}) = \phi(H^{k_1,\ldots,k_s}) \cdot \theta(H^{k_1,\ldots,k_s}) . \tag{2}$$

Proof. Since every function $f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$ can be uniformly approximated on bounded sets by block-symmetric polynomials that can be represented in the form of the algebraic combination of polynomials $H^{k_1,\ldots,k_s}$, it suffices to check the associativity and commutativity of multiplicative convolution for the polynomials $H^{k_1,\ldots,k_s}$.

First, we check equality (2). Indeed,

$$\begin{aligned}\big(\theta \lozenge H^{k_1,\ldots,k_s}\big)(\tilde{x}) &= \theta\big(M_{\tilde{x}}(H^{k_1,\ldots,k_s})\big) \\ &= \theta\big(H^{k_1,\ldots,k_s}(\tilde{x}) \cdot H^{k_1,\ldots,k_s}\big) = H^{k_1,\ldots,k_s}(\tilde{x}) \cdot \theta\big(H^{k_1,\ldots,k_s}\big).\end{aligned}$$

This is why, we get

$$\begin{aligned}(\phi \lozenge \theta)(H^{k_1,\ldots,k_s}) &= \phi(\theta \lozenge H^{k_1,\ldots,k_s}) \\ &= \phi\big(H^{k_1,\ldots,k_s}(\tilde{x}) \cdot \theta(H^{k_1,\ldots,k_s})\big) = \phi\big(H^{k_1,\ldots,k_s}\big) \cdot \theta\big(H^{k_1,\ldots,k_s}\big).\end{aligned}$$

The last inequality implies the associativity and commutativity of multiplicative convolution on the polynomials $H^{k_1,\ldots,k_s}$ and, hence, for any function $f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$.

The theorem is proved.

For the elements $\tilde{x}, \tilde{y} \in \mathcal{X}_\infty^s$, the notion of symmetric shift $\tilde{x} \bullet \tilde{y}$ was introduced in [12] by the formula

$$\tilde{x} \bullet \tilde{y} = \left(\begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^s \end{pmatrix}, \begin{pmatrix} y_1^1 \\ y_1^2 \\ \vdots \\ y_1^s \end{pmatrix}, \ldots, \begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^s \end{pmatrix}, \begin{pmatrix} y_i^1 \\ y_i^2 \\ \vdots \\ y_i^s \end{pmatrix}, \ldots \right).$$

Moreover, it was shown that $\tilde{x} \bullet \tilde{y} \in \mathcal{X}_\infty^s$.

In [12], for any $\phi, \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$ and $f \in \mathcal{H}_{bvs}(\mathcal{X}_\infty^s)$, the operation of symmetric convolution $\phi \star \theta$ was defined as follows:

$$(\phi \star \theta)(f) = \phi(\theta \star f) = \phi\big(\tilde{y} \mapsto \theta(\mathcal{T}_{\tilde{y}}^s(f))\big),$$

where

$$\mathcal{T}_{\tilde{y}}^s(f)(\tilde{x}) = f(\tilde{x} \cdot \tilde{y}).$$

Proposition 5. *For any* $\varphi, \phi, \theta \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^s)$, *the following equality is true:*

$$\theta \lozenge (\varphi \star \phi) = (\theta \lozenge \varphi) \star (\theta \lozenge \phi).$$

Proof. By using equality (2) and Theorem 3 [12], we obtain

$$\begin{aligned}((\theta \Diamond \varphi) \star (\theta \Diamond \phi))(H^{k_1,\ldots,k_s}) &= (\theta \Diamond \varphi)(H^{k_1,\ldots,k_s}) + (\theta \Diamond \phi)(H^{k_1,\ldots,k_s}) \\ &= \theta(H^{k_1,\ldots,k_s}) \cdot \varphi(H^{k_1,\ldots,k_s}) + \theta(H^{k_1,\ldots,k_s}) \cdot \phi(H^{k_1,\ldots,k_s}) \\ &= \theta(H^{k_1,\ldots,k_s}) \cdot \big(\varphi(H^{k_1,\ldots,k_s}) + \phi(H^{k_1,\ldots,k_s})\big) \\ &= \theta(H^{k_1,\ldots,k_s}) \cdot (\varphi \star \phi)(H^{k_1,\ldots,k_s}) \\ &= (\theta \Diamond (\varphi \star \phi))(H^{k_1,\ldots,k_s}) .\end{aligned}$$

The proposition is proved.

3. The Case of the Space $\mathcal{X}_\infty^2 = \oplus_{\ell_1} \mathbb{C}^2$

Another algebraic basis of the algebra $\mathcal{P}_{vs}(\mathcal{X}_\infty^2)$ is formed by the polynomials

$$R^{k_1,k_2}(x^1,x^2) = \sum_{\substack{i_1<\ldots<i_{k_1} \\ j_1<\ldots<j_{k_2} \\ i_k \neq j_l}}^{\infty} x^1_{i_1} \ldots x^1_{i_{k_1}} x^2_{j_1} \ldots x^2_{j_{k_2}} ,$$

where k_1 and k_2 are the numbers of elements $x^1_{i_k}$ and $x^2_{j_\ell}$, respectively [12].

Let $\mathbb{C}\{t_1,t_2\}$ be the space of all power series over $\mathbb{C}^2$. The representations

$$\mathcal{R}(\varphi)(t_1,t_2) = \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} \varphi\big(R^{k_1,k_2}\big) ,$$

$$\mathcal{H}(\varphi)(t_1,t_2) = \sum_{\substack{n=1 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} \varphi\big(H^{k_1,k_2}\big)$$

acting from $\mathcal{M}_{bvs}(\mathcal{X}_\infty^2)$ into $\mathbb{C}\{t_1,t_2\}$ were considered in [12]. In particular, it was shown that the set

$$\big\{R(\varphi)(t_1,t_2) : \varphi \in \mathcal{M}_{bvs}(\mathcal{X}_\infty^2)\big\}$$

is the set of functions of exponential type.

Note that, for any $(a_1, a_2) \in \mathbb{C}^2$ and the vector

$$(a^1, a^2) = \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right),$$

we get

$$\left(\delta_{(a^1,a^2)} \Diamond \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} R^{k_1,k_2} \right)(x^1, x^2)$$

$$= M_{(x^1,x^2)} \left(\sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} R^{k_1,k_2} \right)(a^1, a^2)$$

$$= \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} R^{k_1,k_2} \big((x^1, x^2) \Diamond (a^1, a^2) \big)$$

$$= \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} R^{k_1,k_2} \left(\begin{pmatrix} x_1^1 a_1 \\ x_1^2 a_2 \end{pmatrix}, \begin{pmatrix} x_2^1 a_1 \\ x_2^2 a_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_i^1 a_1 \\ x_i^2 a_2 \end{pmatrix}, \ldots \right)$$

$$= \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} a_1^{k_1} a_2^{k_2} R^{k_1,k_2}(x^1, x^2).$$

Therefore,

$$\mathcal{R}\Big(\varphi \Diamond \delta_{(a^1,a^2)} \Big) = \varphi \left(\sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} a_1^{k_1} a_2^{k_2} R^{k_1,k_2} \right)$$

$$= \sum_{\substack{n=0 \\ k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} a_1^{k_1} a_2^{k_2} \varphi(R^{k_1,k_2}).$$

It follows from Theorem 3 [12] that

$$\delta_{(a^1,a^2)} \star \delta_{(b^1,b^2)} = \delta_{\left(\binom{a_1}{a_2}, \binom{b_1}{b_2}, \binom{0}{0}, \ldots, \binom{0}{0}, \ldots \right)}.$$

By using Proposition 5 and Theorem 4 [12], we obtain

$$\mathcal{R}\left(\varphi \Diamond \delta_{\left(\binom{a_1}{a_2},\binom{b_1}{b_2},\binom{0}{0},\ldots,\binom{0}{0},\ldots\right)}\right)(t_1,t_2)$$

$$= \mathcal{R}\left(\left(\varphi \Diamond \delta_{(a^1,a^2)}\right) \star \left(\varphi \Diamond \delta_{(b^1,b^2)}\right)\right)(t_1,t_2)$$

$$= \mathcal{R}\left(\varphi \Diamond \delta_{(a^1,a^2)}\right)(t_1,t_2) \cdot \mathcal{R}\left(\varphi \Diamond \delta_{(b^1,b^2)}\right)(t_1,t_2)$$

$$= \sum_{\substack{n=0\\k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} a_1^{k_1} a_2^{k_2} \varphi(R^{k_1,k_2}) \cdot \sum_{\substack{n=0\\k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} b_1^{k_1} b_2^{k_2} \varphi\left(R^{k_1,k_2}\right).$$

In a more general form, we can write

$$\mathcal{R}\left(\varphi \Diamond \delta_{\left(\binom{x_1^1}{x_1^2},\ldots,\binom{x_m^1}{x_m^2},\binom{0}{0},\ldots\right)}\right)(t_1,t_2) = \prod_{\ell=1}^{m} \sum_{\substack{n=0\\k_1+k_2=n}}^{\infty} t_1^{k_1} t_2^{k_2} (x_\ell^1)^{k_1} (x_\ell^2)^{k_2} \varphi\left(R^{k_1,k_2}\right).$$

Since the sequence

$$\left(\delta_{\left(\binom{x_1^1}{x_1^2},\ldots,\binom{x_m^1}{x_m^2},\binom{0}{0},\ldots\right)}\right)_m$$

is pointwise convergent to

$$\delta_{\left(\binom{x_1^1}{x_1^2},\ldots,\binom{x_m^1}{x_m^2},\ldots\right)}$$

in $\mathcal{M}_{bvs}(\mathcal{X}_\infty^2)$, the sequence

$$\left(\varphi \Diamond \delta_{\left(\binom{x_1^1}{x_1^2},\ldots,\binom{x_m^1}{x_m^2},\binom{0}{0},\ldots\right)}\right)_m$$

is pointwise convergent to

$$\varphi \Diamond \delta_{\left(\binom{x_1^1}{x_1^2},\ldots,\binom{x_m^1}{x_m^2},\ldots\right)}.$$

Therefore,

$$\mathcal{R}\Big(\varphi\lozenge\delta_{(x^1,x^2)}\Big)(t_1,t_2) = \prod_{\ell=1}^{\infty}\sum_{\substack{n=0\\ k_1+k_2=n}}^{\infty} t_1^{k_1}t_2^{k_2}(x_\ell^1)^{k_1}(x_\ell^2)^{k_2}\varphi\big(R^{k_1,k_2}\big)$$

for any

$$(x^1,x^2) = \left(\begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix},\ldots,\begin{pmatrix} x_m^1 \\ x_m^2 \end{pmatrix},\ldots w\right)\in\mathcal{X}_\infty^2 .$$

In [12], we constructed a family $(\phi_{(k,\ell)}:\ (k,\ell)\in\mathbb{C}^2)$ of elements of the set $\mathcal{M}_{bvs}(\mathcal{X}_\infty^2)$ such that

$$\phi_{(k,\ell)}(H^{1,0}) = k, \quad \phi_{(k,\ell)}(H^{0,1}) = \ell, \quad \text{and} \quad \phi_{(k,\ell)}(H^{k_1,k_2}) = 0 \quad \forall k_1,k_2 > 1 .$$

It was also shown that

$$\mathcal{R}(\phi_{(k,\ell)})(t_1,t_2) = e^{kt_1+\ell t_2} .$$

It is easy to see that

$$(\phi_{(k,\ell)}\lozenge\varphi)(H^{1,0}) = k\varphi(H^{1,0}) ,$$

$$(\phi_{(k,\ell)}\lozenge\varphi)(H^{0,1}) = \ell\varphi(H^{0,1}) ,$$

$$(\phi_{(k,\ell)}\lozenge\varphi)(H^{k_1,k_2}) = 0 \quad \forall k_1,k_2 > 1 ,$$

$$\mathcal{R}(\phi_{(k,\ell)}\lozenge\varphi)(t_1,t_2) = e^{k\varphi(H^{1,0})t_1+\ell\varphi(H^{0,1})t_2} .$$

Thus, the operation of multiplicative convolution with functionals $\phi_{(k,\ell)}$ acts as a shift on the elements $\mathcal{R}(\phi_{(k,\ell)})(t_1,t_2)$.

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