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RESEARCH ARTICLE

Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces

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Abstract
We consider some conditions for continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions of bounded type on ℓ_1, $L_\infty[0, 1)$, and $L_\infty[0, \infty) \cap L_1[0, \infty)$. We establish hypercyclicity of some special composition operators, namely of compositions with translations on algebras of symmetric analytic functions and some other algebras generated by countable sequences of homogeneous polynomials.

Keywords Hypercyclic operators · Functional spaces · Composition operators · Symmetric analytic functions

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1 Introduction

Let X be a Fréchet linear space. We recall that a continuous linear operator $T: X \to X$ is *hypercyclic* if there is a vector $x_0 \in X$ for which the orbit under T, $\mathrm{Orb}(T, x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, is dense in X. Every such vector x_0 is called a *hypercyclic vector* of T. The classical Birkhoff theorem [6] asserts that any composition with translation $x \mapsto x + a$, $T_a: f(x) \mapsto f(x + a)$, $x, a \in \mathbb{C}$, is hypercyclic on the space of entire functions $H(\mathbb{C})$, if $a \neq 0$. The Birkhoff translation T_a can also be regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

From this perspective Godefroy and Shapiro generalized the Birkhoff theorem in [14]. They showed that if $\varphi(z) = \sum_{|\alpha|\geqslant 0} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \mapsto \sum_{|\alpha|\geqslant 0} c_\alpha D^\alpha f, \quad f \in H(\mathbb{C}^n), \tag{1}$$

is hypercyclic. Moreover, they proved that any continuous linear operator T on $H(\mathbb{C}^n)$ which commutes with translations and is not a scalar multiple of the identity can be expressed in the form (1) and so is hypercyclic as well. These results were generalized to separable spaces of analytic functions of bounded type on Banach spaces in [2]. Note that in the general case, the most interesting algebras of entire functions on a Banach space are non-separable and so they do not admit any hypercyclic operators. In this paper we consider some special subalgebras of analytic functions of bounded type on a Banach space such as algebras of symmetric analytic functions and construct for them hypercyclic composition operators.

Let us recall that a continuous operator C_Φ on the space of analytic functions on a Banach space X is said to be a *composition operator* if $C_\Phi f(x) = f(\Phi(x))$ for some analytic map $\Phi: X \to X$. It is known that only translation operator T_a for some $a \neq 0$ is a hypercyclic composition operator on $H(\mathbb{C})$ [5]. However, in $H(\mathbb{C}^n)$ with $n > 1$ the situation is quite different.

In [19], there was proposed a method, using symmetric analytic functions on ℓ_1, to construct hypercyclic composition operators on $H(\mathbb{C}^n)$, which cannot be described by formula (1). Some generalizations of these results to spaces ℓ_p, $1 < p < \infty$, were obtained in [18].

Let us recall a well-known Kitai–Gethner–Shapiro theorem which is also known as the Hypercyclicity Criterion.

Theorem 1.1 (Hypercyclicity Criterion) *Let X be a separable Fréchet space and $T: X \to X$ be a linear, continuous operator. Suppose there exist X_0, Y_0, dense subsets of X, a sequence (n_k) of positive integers, and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n: Y_0 \to X$ such that*

(i) $T^{n_k}(x) \to 0$ *for every $x \in X_0$ as $k \to \infty$.*

(ii) $S_{n_k}(y) \to 0$ *for every* $y \in Y_0$ *as* $k \to \infty$.
(iii) $T^{n_k} \circ S_{n_k}(y) = y$ *for every* $y \in Y_0$.

Then T *is hypercyclic.*

The operator T is said to satisfy the *Hypercyclicity Criterion for the entire sequence* if we can choose $n_k = k$. It is known that there are hypercyclic operators which do not satisfy the Hypercyclicity Criterion (see [4]).

Let X be a complex Banach space and $\mathcal{G}$ be a group or semigroup of bounded linear operators on X. A function f on X is called $\mathcal{G}$-*symmetric* (or just *symmetric*) if for every $\sigma \in \mathcal{G}$,

$$f(x) = f(\sigma(x)).$$

A sequence of homogeneous polynomials $(P_j)_{j=1}^{\infty}$, $\deg P_k = k$, is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials if for every symmetric polynomial P of degree n on X there exists a unique polynomial q on $\mathbb{C}^n$ such that

$$P(x) = q(P_1(x), \ldots, P_n(x)).$$

For a fixed group $\mathcal{G}$ we denote by $\mathcal{P}_s(X)$ the space of all continuous symmetric polynomials on X and by $H_{bs}(X)$ its closure in the algebra of all analytic functions of bounded type on X. In other words, $H_{bs}(X)$ consists of all symmetric entire functions which are bounded on bounded subsets of X. $H_{bs}(X)$ is a Fréchet algebra over $\mathbb{C}$ with respect to the metrizable topology of the uniform convergence on bounded subsets.

In this paper we consider the following three basic examples of algebras of $\mathcal{G}$-symmetric functions which admit algebraic bases.

***Example* 1.2** Let $X = \ell_1$ and $\mathcal{G}$ be the group of permutations of basis vectors. There are two bases in the corresponding algebra of symmetric polynomials which are interesting for us: the basis of *power series*

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k$$

and the basis of *elementary symmetric polynomials*

$$G_k(x) = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1} \ldots x_{n_k},$$

$x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_1$.

***Example* 1.3** Let $X = L_\infty[0, 1]$ and $\mathcal{G}$ be the group of operators of compositions with measurable transformations of $[0, 1]$ preserving the Lebesgue measure. In [12], it is proved that the functions

$$R_n(x) = \int_{[0,1]} (x(t))^n \, dt, \quad x(t) \in L_\infty[0, 1], \quad n \in \mathbb{N},$$

form an algebraic basis in the algebra of all symmetric polynomials $\mathcal{P}_s(L_\infty[0, 1]) \subset H_{bs}(L_\infty[0, 1])$ and that $\mathcal{P}_s(L_\infty[0, 1])$ is a dense subspace of $H_{bs}(L_\infty[0, 1])$. Similar results for L_p spaces, $1 \leqslant p < \infty$, were obtained in [15]. In [13] it is proved that ℓ_∞ admits no symmetric polynomials.

***Example* 1.4** Let $X = L_\infty[0;\infty) \cap L_1[0;\infty)$ and $\mathcal{G}$ be the group of operators of compositions with measurable transformations of $[0;\infty)$ preserving the Lebesgue measure. We denote by $\mathcal{P}_s(L_\infty[0;\infty) \cap L_1[0;\infty))$ the algebra of all symmetric polynomials on $L_\infty[0;\infty)\cap L_1[0;\infty)$. Like in the case of $L_\infty[0,1]$, the sequence $(\mathcal{R}_k)_{k=1}^\infty$, where

$$\mathcal{R}_k(x) = \int_{[0;\infty]} (x(t))^k\,dt, \quad k \in \mathbb{N},$$

forms an algebraic basis in $\mathcal{P}_s(L_\infty[0;\infty) \cap L_1[0;\infty))$ (see [20]).

For our purpose it is important to have some information about the spectrum $M_{bs}(X)$ of a given algebra $H_{bs}(X)$. $M_{bs}(X)$ consists of all continuous characters, that is, complex-valued homomorphisms of $H_{bs}(X)$. Clearly, for every $x \in X$, the point evaluation functional δ_x, $\delta_x(f) = f(x)$, $f \in H_{bs}(X)$, belongs to $M_{bs}(X)$. It is known [12] that $M_{bs}(L_\infty[0,1])$ consists only of point evaluation functionals but $M_{bs}(\ell_1)$ does not [1]. If $\phi \in M_{bs}(X)$, then ϕ is continuous as a linear functional on the normed algebra $(H_{bs}(X), \|\cdot\|_r)$ for some $r > 0$, where

$$\|f\|_r = \sup_{\|x\|\leqslant r} |f(x)|.$$

The infimum of such r is called the radius function of ϕ and denoted by $R(\phi)$ [3].

In Sect. 2 we consider algebras generated by countable sequences of polynomials and composition operators on these algebras. In Sect. 3 we consider continuity and hypercyclicity of such operators, especially for algebras of symmetric analytic functions. Also, we prove the hypercyclicity of related operators of differentiation on $H_{bs}(\ell_1)$ and $H_{bs}(L_\infty[0;\infty) \cap L_1[0;\infty))$. For details on the theory of analytic functions on Banach spaces we refer the reader to [11]. The state of art of theory of symmetric analytic functions on Banach spaces can be found in [1,7–10,12,15,20]. A detailed survey of hypercyclic operators is given in [4,16,17].

2 Composition operators

Let $\mathbf{P} = \{P_n\}_{n=1}^\infty$ be a sequence of homogeneous polynomials on a complex Banach space X such that P_n are algebraically independent and $\deg P_n = n$. We denote by $\mathcal{P}_{\mathbf{P}}$ the unital algebra of polynomials generated by $\{P_n\}_{n=1}^\infty$ and by $H_{b\mathbf{P}}$ its completion with respect to the metrizable topology of uniform convergence on bounded subsets of X. We will use also notations $\mathcal{P}_{\mathbf{P}}^n$ for the subalgebra of $\mathcal{P}_{\mathbf{P}}$ generated by $\{P_1,\ldots,P_n\}$ and $H_{\mathbf{P}}^n$ for its closure in $H_{b\mathbf{P}}$. Since $\{P_1,\ldots,P_n\}$ are algebraically independent, $H_{\mathbf{P}}^n$ is isomorphic to $H(\mathbb{C}^n)$. More exactly, the map

$$I_n\colon g(t_1,\ldots,t_n) \mapsto g(P_1(x),\ldots,P_n(x))$$

is a topological and algebraic isomorphism from $H(\mathbb{C}^n)$ onto $H^n_{\mathbf{P}}$. Every function $f \in H_{b\mathbf{P}}$ can be represented as

$$f(x) = \sum_{n=0}^{\infty} \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_j \geqslant 0}} a_{k_1k_2\ldots k_n} P_1^{k_1}(x) P_2^{k_2}(x) \ldots P_n^{k_n}(x), \tag{2}$$

where $a_{k_1k_2\ldots k_n} \in \mathbb{C}$. Using the sequence of homogeneous polynomials $\mathbf{P}$ we can define a relation of equivalence on X by the following way: $x \sim y$ if $P_n(x) = P_n(y)$ for every $n \in \mathbb{N}$. Let us denote by $S_{\mathbf{P}}$ the semigroup of all bounded linear operators A on X to itself such that $A(x) \sim x$ for every $x \in X$. Let us call a function f on X, $S_{\mathbf{P}}$-*symmetric* if $f(x) = f(A(x))$ for every $A \in S_{\mathbf{P}}$.

Definition 2.1 We say that the sequence $\mathbf{P}$ is *symmetrically complete* if $H_{b\mathbf{P}}$ consists of all $S_{\mathbf{P}}$-symmetric analytic functions of bounded type on X.

Let $X = \ell_1$, $P_1(x) = x_1$, and $P_n(x) = F_n$ for $n > 1$. It is easy to see that $S_{\mathbf{P}}$ consists of all permutations σ of the basis vectors such that $\sigma(1, 0, 0, \ldots) = (1, 0, 0, \ldots)$. The sequence $\mathbf{P} = \{P_n\}_{n=1}^{\infty}$ is not symmetrically complete because $F_1 \notin H_{b\mathbf{P}}$ but F_1 is $S_{\mathbf{P}}$-symmetric.

Algebras of symmetric analytic functions of bounded type on ℓ_1, $L_\infty[0, 1]$, and $L_\infty[0, 1] \cap L_1[0, 1]$ are typical examples of $H_{b\mathbf{P}}$ for various algebraic bases $\mathbf{P}$, and by definitions of these algebras their algebraic bases are symmetrically complete. Conversely, if $\mathbf{P}$ is symmetrically complete, then $H_{b\mathbf{P}} = H_{bs}(X)$ for the semigroup of operators $G = S_{\mathbf{P}}$.

Theorem 2.2 *Let $\mathbf{P}$ be a symmetrically complete sequence of polynomials on a Banach space X and $F\colon X \to X$ be an analytic map of bounded type such that $x \sim y$ implies $F(x) \sim F(y)$, $x, y \in X$. Then the composition $C_F\colon f \mapsto f \circ F$ is a continuous homomorphism of $H_{b\mathbf{P}}$ to itself.*

Proof Since F is of bounded type, C_F is a continuous homomorphism from $H_{b\mathbf{P}}$ to $H_b(X)$. From the property

$$x \sim y \implies F(x) \sim F(y)$$

it follows that $f \circ F$ is $S_{\mathbf{P}}$-symmetric for every $f \in H_{b\mathbf{P}}$. Since $H_{b\mathbf{P}}$ contains all $S_{\mathbf{P}}$-symmetric analytic functions, C_F is a continuous homomorphism from $H_{b\mathbf{P}}$ to itself. □

Symmetric translation operators on spaces of symmetric analytic functions on ℓ_1 provide some examples of composition operators as in Theorem 2.2. More precisely, for a given $y \in \ell_1$, the operator

$$x \mapsto x \bullet y := (x_1, y_1, x_2, y_2, \ldots)$$

is a continuous 1-degree polynomial which satisfies Theorem 2.2 for any algebraic basis in $H_{bs}(\ell_1)$. Moreover, the operation '$\bullet$' can be extended to a continuous operation '$\star$' on the spectrum $M_{bs}(\ell_1)$ of $H_{bs}(\ell_1)$, and

$$\phi \star \psi(F_n) = \phi(F_n) + \psi(F_n), \quad n \in \mathbb{N}$$

(see [7,8]).

Another operation on the spectrum of $H_{\mathrm{bs}}(\ell_1)$ can be defined on elements of $\ell_1/\sim$ in the following way:

$$x \diamond y = \{x_i y_j\}_{i,j=1}^{\infty}$$

and naturally extended to $M_{\mathrm{bs}}(\ell_1)$ (see [9]). It is easy to check that $F_n(x \diamond y) = F_n(x)F_n(y)$, $n \in \mathbb{N}$. Combining $x \mapsto x \bullet y$ and $x \mapsto x \diamond y$ we can get a lot of continuous composition operators.

It is clear that the map $F_n \mapsto 1/F_n$ cannot be extended to a continuous homomorphism on $H_{\mathrm{bs}}(\ell_1)$. Also, it is known [8] that $F_n \mapsto -F_n$ cannot be extended to a continuous homomorphism on $H_{\mathrm{bs}}(\ell_1)$ (we will prove this fact for more general situation in Corollary 3.6). However, combining these operators we can get a continuous homomorphism. Let us denote by $H_{\mathrm{bs}}(B_1)$ the Fréchet algebra of analytic functions on the unit ball B_1 of ℓ_1 which are bounded on all balls B_r centered at zero of radius $r < 1$. The Fréchet topology on $H_{\mathrm{bs}}(B_1)$ can be generated by countable family of norms $\|f\| = \sup_{\|x\| \leqslant r} |f(x)|$, $r \in \mathbb{Q}$, $0 < r < 1$.

***Example* 2.3** Let $x \in \ell_1$ and $\|x\| < 1$. Set

$$w_m(x) = \Big(\cdots \big(((1, 0, 0, \ldots) \bullet x) \bullet (x \diamond x) \big) \bullet \cdots \Big) \bullet (\underbrace{x \diamond \cdots \diamond x}_{m}).$$

Since

$$\|w_n(x)\| \leqslant \sum_{k=0}^{m} \|x\|^k \leqslant \frac{1}{1 - \|x\|} \quad \text{and} \quad \|w_{n+1}(x) - w_n(x)\| \leqslant \|x\|^n,$$

$w_n(x)$ converges to a vector $W(x) \in \ell_1$. Moreover,

$$F_n(W(x)) = \frac{1}{1 - F_n(x)}, \quad n \in \mathbb{N}, \tag{3}$$

and since the map $x \mapsto W(x)$ is analytic and bounded on all balls B_r, $r < 1$, C_W is a continuous homomorphism from $H_{\mathrm{bs}}(\ell_1)$ to $H_{\mathrm{bs}}(B_1)$ which can be defined on the basis polynomials F_n by formula (3).

It is possible to construct an analogue of '$\bullet$' on $L_\infty[0, \infty) \cap L_1[0, \infty)$ as follows. Let $x = x(t)$ and $y = y(t)$ belong to $L_\infty[0, \infty) \cap L_1[0, \infty)$. Let $z = x \bullet y$, where

$$z(t) = \begin{cases} x(t - n), & \text{if } 2n \leqslant t < 2n + 1, \\ y(t - n - 1), & \text{if } 2n + 1 \leqslant t < 2n + 2, \end{cases}$$

$n = 0, 1, 2, \ldots$ Clearly, the map $x \mapsto x \bullet y$ satisfies conditions of Theorem 2.2 and

$$\mathcal{R}_n(x \bullet y) = \mathcal{R}_n(x) + \mathcal{R}_n(y), \quad n \in \mathbb{N}.$$

3 Hypercyclicity of translation operators

Apart from the composition operators, we can consider operators generated by some mappings on the set of basis polynomials. Such approach is helpful for constructing hypercyclic operators. The following lemma is proved in [19].

Lemma 3.1 *Let Y be a Fréchet space and $Y_1 \subset Y_2 \subset \cdots \subset Y_m \subset \cdots$ be a sequence of closed subspaces such that $\bigcup_{m=1}^{\infty} Y_m$ is dense in Y. Let T be a continuous operator on Y such that $T(Y_m) \subset Y_m$ for each m and each restriction $T|_{Y_m}$ satisfies the Hypercyclicity Criterion for the entire sequence on Y_m. Then T satisfies the Hypercyclicity Criterion for the entire sequence on Y.*

Theorem 3.2 *Let $T\colon H_{b\mathbf{P}} \to H_{b\mathbf{P}}$ be a continuous homomorphism such that $T(P_n) = P_n + a_n$ for some non-zero sequence of complex numbers $\{a_n\}$. Then T is hypercyclic and satisfies the Hypercyclicity Criterion for the entire sequence.*

Proof Since $\{a_n\}$ is a non-zero sequence, there is n_0 such that $\mathbb{C}^n \ni (a_1, \ldots, a_n) \neq 0$ for every $n > n_0$ and so

$$T_n\colon g(t_1, \ldots, t_n) \mapsto g(t_1 + a_1, \ldots, t_n + a_n)$$

satisfies the Hypercyclicity Criterion for the entire sequence on $H(\mathbb{C}^n)$. Thus

$$T|_{H^n_{\mathbf{P}}} = I_n \circ T_n \circ I_n^{-1}$$

satisfies the Hypercyclicity Criterion for the entire sequence on $H^n_{\mathbf{P}}$. Now, the proof follows from Lemma 3.1. □

In other words, a homomorphism on $\mathcal{P}_{\mathbf{P}}$ of the form $P_n \mapsto P_n + a_n$ can be extended to a hypercyclic operator on $H_{b\mathbf{P}}$ if and only if it is continuous. Note that the question about continuity of a such operator is not trivial in the general case.

Proposition 3.3 *Let T be a continuous homomorphism as in Theorem 3.2. Then there is a character ϕ on $H_{b\mathbf{P}}$ such that $a_n = \phi(P_n)$ for every n.*

Proof From the definition of T we have

$$\delta_0 \circ T(P_n) = P_n(0) + a_n = a_n,$$

where δ_0 is the zero-evaluation functional. So we can set $\phi = \delta_0 \circ T$. □

For every continuous linear functional ϕ on $H_{b\mathbf{P}}$ the *radius function* $R(\phi)$ is the infimum of all $r > 0$ such that ϕ is continuous with respect to the norm of uniform convergence on the ball B_r of the radius r in X [3]. We say that a subset Ω of all characters $M(H_{b\mathbf{P}})$ is *additive with respect to* $\mathbf{P}$ if for for every $\phi \in M(H_{b\mathbf{P}})$ and $\psi \in \Omega$ there is $\theta \in M(H_{b\mathbf{P}})$ such that

$$\phi(P_n) + \psi(P_n) = \theta(P_n), \quad n = 1, 2, \ldots,$$

and

$$R(\theta) \leqslant C(R(\phi), R(\psi)), \tag{4}$$

where $C(R(\phi), R(\psi))$ is a positive constant which depends only on $R(\phi)$ and $R(\psi)$. The following proposition is a kind of converse to Proposition 3.3.

Proposition 3.4 *Let Ω be additive with respect to a symmetrically complete sequence $\mathbf{P}$ and T be a homomorphism on $\mathcal{P}_{\mathbf{P}}$ with $T(P_n) = P_n + a_n$ for all n. If there is a character $\psi \in \Omega$ such that $a_n = \psi(P_n)$ for every n, then T is continuous.*

Proof For every $\phi \in M(H_{b\mathbf{P}})$ we have

$$\phi \circ T(P_n) = \phi(P_n) + a_n = \phi(P_n) + \psi(P_n) = \theta(P_n).$$

So $\phi \circ T = \theta$ is continuous for every $\phi \in M(H_{b\mathbf{P}})$. By continuity it can be extended to the whole space $H_{b\mathbf{P}}$ and

$$\phi \circ T(f) = \theta(f), \quad f \in H_{b\mathbf{P}}.$$

So, for every $x \in X$,

$$T(f)(x) = \delta_x \circ T(f) = \theta_x(f),$$

where θ_x is such that $\delta_0(P_n) + \psi(P_n) = \theta_x(P_n)$, $n \in \mathbb{N}$. Thus $T(f)$ is a well-defined map on $H_{b\mathbf{P}}$. From representation (2) it follows that $T(f)$ is G-analytic, that is, analytic on every finite dimension subspace. Inequality (4) implies that $T(f)$ is bounded on bounded subsets of X. So $T(f)$ is a homomorphism from $H_{b\mathbf{P}}$ to itself. Since $H_{b\mathbf{P}}$ is semi-simple and $\phi \circ T$ is continuous for every $\phi \in M(H_{b\mathbf{P}})$, T must be continuous.□

Note that the spectrum $M_{bs}(\ell_1)$ of $H_{bs}(\ell_1)$ is additive with respect to the basis $\mathbf{F} = \{F_n\}_{n=1}^{\infty}$ because

$$\phi \star \psi(F_n) = \phi(F_n) + \psi(F_n), \quad n \in \mathbb{N}.$$

However, sequences $\{\phi(F_n)\}_{n=1}^{\infty}$, $\phi \in M_{bs}$, do not form a linear space. On the other hand, according to [12], the spectrum $M_{bs}(L_\infty[0,1])$ of $H_{bs}(L_\infty[0,1])$ can be identified with the set of all sequences

$$\left\{(c_n)_{n=1}^{\infty} : c_n \in \mathbb{C},\ \limsup_{n\to\infty} |c_n|^{1/n} < \infty\right\}$$

by $(c_n)_{n=1}^{\infty} = (R_n(x))_{n=1}^{\infty}$, $x \in L_\infty[0,1]$, and such a set is a sequence linear space. But we do not know whether inequality (4) holds for this case.

Theorem 3.5 *Let ϕ be a character on $\mathcal{P}_s(\ell_1)$ such that $\phi(F_n) = a$ for all $n \in \mathbb{N}$. Then ϕ is continuous if and only if $a \in \mathbb{Z}_+$.*

Proof Let $a \in \mathbb{Z}_+$. We set

$$x_a = (\underbrace{1, \ldots, 1}_{a}, 0, \ldots).$$

Then for $\phi = \delta_{x_a}$ we have $\phi(F_n) = F_n(x_a) = a$ for all n and so ϕ is continuous.

For the general case, by the Newton formula,

$$nG_n = F_1 G_{n-1} - F_2 G_{n-2} + \cdots + (-1)^{n-1} F_n,$$

and induction with the equality $\phi(F_n) = a$ implies

$$\phi(G_n) = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

If $a \notin \mathbb{Z}_+$, then $\phi(G_n) \neq 0$ for all $n \in \mathbb{N}$ and it grows like a^n/n. Hence

$$\sum_{n=0}^{\infty} \frac{\lambda^n a^n}{n}$$

is not a function of exponential type of $\lambda \in \mathbb{C}$. On the other hand, in [8] it is proved that if ϕ is a continuous character, then

$$\sum_{n=0}^{\infty} \lambda^n \phi(G_n)$$

is a function of exponential type. A contradiction. □

From similar reasoning it follows that the map $G_n \mapsto a$ cannot be extended to a continuous complex homomorphism of $H_{\mathrm{bs}}(\ell_1)$ if $a \neq 0$ because

$$\sum_{n=0}^{\infty} a\lambda^n$$

is not a function of exponential type.

Corollary 3.6 (i) *Each homomorphism T_a and M_a on $\mathcal{P}_{\mathrm{s}}(\ell_1)$ defined on $\mathbf{F}$ by*

$$T_a \colon F_n \mapsto F_n + a \quad \textit{and} \quad M_a \colon F_n \mapsto aF_n$$

is continuous if and only if $a \in \mathbb{Z}_+$. In this case T_a satisfies the Hypercyclicity Criterion for the entire sequence on $H_{\mathrm{bs}}(\ell_1)$.

(ii) *Let $y \in L_\infty[0;\infty) \cap L_1[0;\infty)$ and $\mathcal{R}_n(y) \neq 0$ for some n, then the composition operator*

$$f(x) \mapsto f(x \bullet y), \quad f \in H_{\mathrm{bs}}(L_\infty[0;\infty) \cap L_1[0;\infty)),$$

satisfies the Hypercyclicity Criterion for the entire sequence on $H_{\mathrm{bs}}(L_\infty[0;\infty)\cap L_1[0;\infty))$.

Proof We need to prove statement (i) only for the operator M_a. Let M_a be continuous and δ_1 be the point evaluation functional at $(1,0,0,\ldots)$. Then $\phi = \delta_1\circ M_a \in M_{\mathrm{bs}}$ and $\phi(F_n) = a$ for all $n\in\mathbb{N}$. Hence $a\in\mathbb{Z}_+$. Conversely, if $a\in\mathbb{Z}_+$, then M_a can be defined by the following formula:

$$M_a(f)(x) = f(\underbrace{x\bullet\cdots\bullet x}_{a}),\quad f\in H_{\mathrm{bs}},$$

and it is continuous.

The proof of (ii) follows directly from Theorems 2.2 and 3.2. □

Let us consider some differential operators related to '$\bullet$' on algebras $H_{\mathrm{bs}}(\ell_1)$ and $H_{\mathrm{bs}}(L_\infty[0;\infty)\cap L_1[0;\infty))$. Let X be ℓ_1 or $L_\infty[0;\infty)\cap L_1[0;\infty)$ and $f\in H_{\mathrm{bs}}(X)$. We denote by $e\in X$ the element $(1,0,0,\ldots)$ if $X=\ell_1$ and

$$e(t) = \chi_{[0,1]}(t) = \begin{cases} 1, & \text{if } 0\leqslant t\leqslant 1,\\ 0, & \text{otherwise,}\end{cases}$$

if $X = L_\infty[0;\infty)\cap L_1[0;\infty)$. Set

$$\partial_1(f) = \lim_{\lambda\to 0}\frac{f(x\bullet\lambda e) - f(x)}{\lambda}.$$

From the continuity of '$\bullet$' on $H_{\mathrm{bs}}(X)$ it follows that ∂_1 is well defined on $H_{\mathrm{bs}}(X)$ and continuous. Also, direct calculations show that ∂_1 is linear and

$$\partial_1(fg) = \partial_1(f)g + f\partial_1(g),\quad f,g\in H_{\mathrm{bs}}(X),$$

that is, ∂_1 is a differentiation. Moreover, $\partial_1(F_1) = 1$ and $\partial_1(F_n) = 0$ if $n>1$ for the case $X=\ell_1$. Also $\partial_1(\mathcal{R}_1) = 1$ and $\partial_1(\mathcal{R}_n) = 0$ if $n>1$ for the case $X = L_\infty[0;\infty)\cap L_1[0;\infty)$. So, if we consider subalgebras $H^n_{\mathrm{bs}}(X)\subset H_{\mathrm{bs}}(X)$ generated by $\{F_1,\ldots,F_n\}$ (resp. $\{\mathcal{R}_1,\ldots,\mathcal{R}_n\}$) and the isomorphism $I_n\colon H(\mathbb{C}^n)\to H^n_{\mathrm{bs}}(X)$, then

$$\partial_1 = I_n\circ\frac{\partial}{\partial t_1}\circ I_n^{-1}.$$

Since $\frac{\partial}{\partial t_1}$ satisfies the Hypercyclicity Criterion for the entire sequence on $H(\mathbb{C}^n)$ for every n, then from Lemma 3.1 the next result follows.

Proposition 3.7 *Let X be ℓ_1 or $L_\infty[0;\infty)\cap L_1[0;\infty)$. Then ∂_1, defined as above, satisfies the Hypercyclicity Criterion for the entire sequence on $H_{\mathrm{bs}}(X)$.*

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