Hindawi
Journal of Function Spaces
Volume 2017, Article ID 4947925, 8 pages
https://doi.org/10.1155/2017/4947925

Research Article

On Algebraic Basis of the Algebra of Symmetric Polynomials on $\ell_p(\mathbb{C}^n)$

Victoriia Kravtsiv, Taras Vasylyshyn, and Andriy Zagorodnyuk

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka Str., Ivano-Frankivsk 76000, Ukraine

Correspondence should be addressed to Andriy Zagorodnyuk; andriyzag@yahoo.com

Received 28 April 2017; Accepted 28 September 2017; Published 30 October 2017

Academic Editor: Alberto Fiorenza

We consider polynomials on spaces $\ell_p(\mathbb{C}^n)$, $1 \le p < +\infty$, of p-summing sequences of n-dimensional complex vectors, which are symmetric with respect to permutations of elements of the sequences, and describe algebraic bases of algebras of continuous symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

1. Introduction

Algebras of polynomials and analytic functions on a Banach space X which are invariant (symmetric) with respect to a group of linear operators $G(X)$ acting on X were studied by a number of authors [1–10] (see also a survey [11]). If X has a symmetric structure, then it is natural to consider the case when $G(X)$ is a group of operators which preserve this structure. In particular, if X is a rearrangement-invariant sequence space, then $G(X)$ is used to be the group of permutations of positive integers. In [8] Nemirovskii and Semenov described algebraic bases of algebras of continuous symmetric polynomials on real spaces ℓ_p, where $1 \le p < +\infty$. Their results were generalized by González et al. [7] to real separable rearrangement-invariant sequence spaces.

Algebraic basis plays a crucial role in the problem of description of spectra of algebras generated by polynomials [1–4]. For example, each complex homomorphism on the algebra of symmetric polynomials on ℓ_p is completely defined by its values on the basis elements.

Note that an algebra of symmetric functions essentially depends on a representation of a given group G on X. In particular, in [12–14] the group of permutations of positive integers was considered which acts on the complex space ℓ_1 permutating "blocks" of coordinates. Polynomials which are invariant with respect to the action are called block-symmetric. It is natural to consider such polynomials as symmetric polynomials on $\ell_1(\mathbb{C}^n)$.

In this work we get an explicit description of algebraic bases of algebras of symmetric polynomials on $\ell_p(\mathbb{C}^n)$, where $1 \le p < +\infty$.

2. Materials and Methods

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

A mapping $P : X \to \mathbb{C}$, where X is a complex Banach space, is called an N-homogeneous polynomial if there exists an N-linear form $A_P : X^N \to \mathbb{C}$ such that P is the restriction to the diagonal of A_P, that is, $P(x) = A_P(\underbrace{x, \ldots, x}_{N})$ for every $x \in X$. By [15, Corollary 2.3], N-homogeneous polynomial P is continuous if and only if its norm $\|P\| = \sup_{\|x\|\le 1}|P(x)|$ is finite. Definition of N-homogeneous polynomial implies the inequality $|P(x)| \le \|P\|\|x\|^N$ for every $x \in X$. A mapping $P = P_0 + P_1 + \cdots + P_m$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, m\}$, is called a polynomial of degree at most m.

Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let us denote $\ell_p(\mathbb{C}^n)$ the vector space of all sequences

$$x = (x_1, x_2, \ldots), \tag{1}$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(n)}) \in \mathbb{C}^n$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{\infty}\sum_{s=1}^{n} |x_j^{(s)}|^p$ is convergent. The space $\ell_p(\mathbb{C}^n)$ with norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty}\sum_{s=1}^{n}\left|x_j^{(s)}\right|^p\right)^{1/p} \tag{2}$$

is a Banach space.

Definition 1. A function $f : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ is called symmetric if $f(x \circ \sigma) = f(x)$ for every $x \in \ell_p(\mathbb{C}^n)$ and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, where $x \circ \sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$.

Let us denote $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$ the algebra of all symmetric continuous polynomials on $\ell_p(\mathbb{C}^n)$.

3. Results and Discussion

3.1. Power Sum Symmetric Polynomials on $\ell_p(\mathbb{C}^n)$. For a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ let $|k| = k_1 + \cdots + k_n$. For every $k \in \mathbb{Z}_+^n$ such that $|k| \ge \lceil p \rceil$, where $\lceil p \rceil$ is a ceiling of p, let us define a mapping $H_k : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ by

$$H_k(x) = \sum_{j=1}^{\infty}\prod_{\substack{s=1\\k_s>0}}^{n}\left(x_j^{(s)}\right)^{k_s}. \tag{3}$$

Also we set $H_{(0,\ldots,0)}(x) \equiv 1$. Note that H_k is a symmetric $|k|$-homogeneous polynomial. Polynomials H_k are generalizations of so-called *power sum symmetric polynomials* on finite-dimensional spaces (see, e.g., [16, page 23] or [17, page 297]).

Proposition 2. *For $p \in [1, +\infty)$ and for every $k \in \mathbb{Z}_+^n$ such that $|k| \ge \lceil p \rceil$, polynomial H_k on $\ell_p(\mathbb{C}^n)$ is continuous and $\|H_k\| \le 1$.*

Proof. Let $x \in \ell_p(\mathbb{C}^n)$ such that $\|x\|_p \le 1$. Note that

$$\left|H_k(x)\right| \le \sum_{j=1}^{\infty}\prod_{\substack{s=1\\k_s>0}}^{n}\left|x_j^{(s)}\right|^{k_s}. \tag{4}$$

Since $|x_j^{(s)}| \le \max_{1\le m\le n}|x_j^{(m)}|$ for every $s \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$, it follows that

$$\prod_{\substack{s=1\\k_s>0}}^{n}\left|x_j^{(s)}\right|^{k_s} \le \left(\max_{1\le m\le n}\left|x_j^{(m)}\right|\right)^{|k|} \tag{5}$$

for every $j \in \mathbb{N}$. Note that

$$\left(\max_{1\le m\le n}\left|x_j^{(m)}\right|\right)^{|k|} = \max_{1\le m\le n}\left|x_j^{(m)}\right|^{|k|} \le \sum_{m=1}^{n}\left|x_j^{(m)}\right|^{|k|}. \tag{6}$$

Therefore,

$$\left|H_k(x)\right| \le \sum_{j=1}^{\infty}\sum_{m=1}^{n}\left|x_j^{(m)}\right|^{|k|}. \tag{7}$$

Since $\|x\|_p \le 1$, it follows that $|x_j^{(m)}| \le 1$ for every $m \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$. Therefore, $|x_j^{(m)}|^{|k|} \le |x_j^{(m)}|^p$. Thus,

$$\left|H_k(x)\right| \le \sum_{j=1}^{\infty}\sum_{m=1}^{n}\left|x_j^{(m)}\right|^p = \|x\|_p^p \le 1. \tag{8}$$

Therefore, $\|H_k\| = \sup_{\|x\|_p\le 1}|H_k(x)| \le 1$. Hence, H_k is bounded and, consequently, it is continuous. □

For $m \in \mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the space of all sequences $x = (x_1, \ldots, x_m, 0, \ldots)$, where $x_1, \ldots, x_m \in \mathbb{C}^n$ and $0 = (0, \ldots, 0) \in \mathbb{C}^n$. Note that $c_{00}^{(m)}(\mathbb{C}^n)$ is isomorphic to $(\mathbb{C}^n)^m$. Let $c_{00}(\mathbb{C}^n) = \bigcup_{m=1}^{\infty} c_{00}^{(m)}(\mathbb{C}^n)$. Note that $c_{00}(\mathbb{C}^n)$ is a dense subspace in $\ell_p(\mathbb{C}^n)$. Also note that H_k is well-defined on $c_{00}(\mathbb{C}^n)$ for every $k \in \mathbb{Z}_+^n$.

For arbitrary $x = (x_1, \ldots, x_m, 0, \ldots)$, $y = (y_1, \ldots, y_s, 0, \ldots) \in c_{00}(\mathbb{C}^n)$, we set

$$x \oplus y = (x_1, \ldots, x_m, y_1, \ldots, y_s, 0, \ldots). \tag{9}$$

For $x^{(1)}, \ldots, x^{(r)} \in c_{00}(\mathbb{C}^n)$, let

$$\bigoplus_{j=1}^{r} x^{(j)} = x^{(1)} \oplus \cdots \oplus x^{(r)}. \tag{10}$$

Note that

$$\left\|\bigoplus_{j=1}^{r} x^{(j)}\right\|_p^p = \sum_{j=1}^{r}\left\|x^{(j)}\right\|_p^p. \tag{11}$$

Also note that for every $k \in \mathbb{Z}_+^n$, such that $|k| \ge 1$,

$$H_k\left(\bigoplus_{j=1}^{r} x^{(j)}\right) = \sum_{j=1}^{r} H_k\left(x^{(j)}\right). \tag{12}$$

For every $m \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, we set

$$\alpha_{mj} = \frac{1}{m^{1/m}}\exp\left(\frac{2\pi i j}{m}\right). \tag{13}$$

Also we set $\alpha_{01} = 0$. For $l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$, let

$$a_l = \bigoplus_{j_1=1}^{\widehat{l_1}} \cdots \bigoplus_{j_n=1}^{\widehat{l_n}} \left(\left(\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}\right), (0, \ldots, 0), \ldots\right), \tag{14}$$

where $\widehat{l_j} = \max\{1, l_j\}$ for $j \in \{1, \ldots, n\}$.

Let us define a partial order on $\mathbb{Z}_+^n$ by the following way. For $k, l \in \mathbb{Z}_+^n$ we set $k \succeq l$ if and only if there exists $m \in \mathbb{Z}_+^n$ such that $k_s = m_s l_s$ for every $s \in \{1, \ldots, n\}$. We write $k \succ l$, if $k \succeq l$ and $k \ne l$.

Proposition 3. *For $k \in \mathbb{Z}_+^n$ such that $|k| \ge 1$ and for arbitrary $l \in \mathbb{Z}_+^n$*

$$H_k(a_l) = \begin{cases} \displaystyle\prod_{\substack{s=1\\k_s>0}}^{n}\frac{1}{l_s^{k_s/l_s-1}}\prod_{\substack{s=1\\k_s=0}}^{n}\widehat{l_s}, & \text{if } k \succeq l \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

where, by the definition, product of an empty set of multipliers is equal to 1. In particular, $H_k(a_k) = 1$.

Proof. By (12) and (14),

$$H_k(a_l) = \sum_{j_1=1}^{\widehat{l_1}} \cdots \sum_{j_n=1}^{\widehat{l_n}} H_k\left(\left(\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}\right), (0, \ldots, 0), \ldots\right). \quad (16)$$

By the definition of H_k,

$$H_k\left(\left(\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}\right), (0, \ldots, 0), \ldots\right) = \prod_{\substack{s=1 \\ k_s>0}}^{n} \left(\alpha_{l_s j_s}\right)^{k_s}. \quad (17)$$

Therefore,

$$\begin{aligned} H_k(a_l) &= \sum_{j_1=1}^{\widehat{l_1}} \cdots \sum_{j_n=1}^{\widehat{l_n}} \prod_{\substack{s=1 \\ k_s>0}}^{n} \left(\alpha_{l_s j_s}\right)^{k_s} \\ &= \prod_{\substack{s=1 \\ k_s>0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} \left(\alpha_{l_s j_s}\right)^{k_s} \prod_{\substack{s=1 \\ k_s=0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} 1 \\ &= \prod_{\substack{s=1 \\ k_s>0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} \left(\alpha_{l_s j_s}\right)^{k_s} \prod_{\substack{s=1 \\ k_s=0}}^{n} \widehat{l_s}. \end{aligned} \quad (18)$$

Let $k \geq l$. Then there exists $m \in \mathbb{Z}_+^n$ such that $k_s = m_s l_s$ for every $s \in \{1, \ldots, n\}$. For $s \in \{1, \ldots, n\}$ such that $k_s > 0$, we have that $l_s > 0$ too. Consequently, for such s we have $\widehat{l_s} = l_s$, and, by (13),

$$\begin{aligned} \sum_{j_s=1}^{\widehat{l_s}} \left(\alpha_{l_s j_s}\right)^{k_s} &= \sum_{j_s=1}^{l_s} \left(\frac{1}{l_s^{1/l_s}} \exp\left(\frac{2\pi i j_s}{l_s}\right)\right)^{m_s l_s} \\ &= \frac{1}{l_s^{m_s}} \sum_{j_s=1}^{l_s} \exp\left(2\pi i j_s m_s\right) = \frac{1}{l_s^{m_s}} \sum_{j_s=1}^{l_s} 1 \\ &= \frac{1}{l_s^{m_s-1}} = \frac{1}{l_s^{k_s/l_s-1}}. \end{aligned} \quad (19)$$

Therefore, by (18),

$$H_k(a_l) = \prod_{\substack{s=1 \\ k_s>0}}^{n} \frac{1}{l_s^{k_s/l_s-1}} \prod_{\substack{s=1 \\ k_s=0}}^{n} \widehat{l_s}. \quad (20)$$

In the case $k = l$ we have

$$H_k(a_k) = \prod_{\substack{s=1 \\ k_s>0}}^{n} \frac{1}{k_s^{k_s/k_s-1}} \prod_{\substack{s=1 \\ k_s=0}}^{n} \widehat{k_s} = 1. \quad (21)$$

Let $k \ngeq l$. Then we have two cases. *Case 1.* There exists $s \in \{1, \ldots, n\}$ such that $k_s > l_s = 0$. Then

$$\sum_{j_s=1}^{\widehat{l_s}} \left(\alpha_{l_s j_s}\right)^{k_s} = \left(\alpha_{01}\right)^{k_s} = 0; \quad (22)$$

therefore, $H_k(a_l) = 0$. *Case 2.* There exists $s \in \{1, \ldots, n\}$ such that $l_s > k_s > 0$. Then

$$\begin{aligned} \sum_{j_s=1}^{\widehat{l_s}} \left(\alpha_{l_s j_s}\right)^{k_s} &= \sum_{j_s=1}^{l_s} \left(\frac{1}{l_s^{1/l_s}} \exp\left(\frac{2\pi i j_s}{l_s}\right)\right)^{k_s} \\ &= \frac{1}{l_s^{k_s/l_s}} \sum_{j_s=1}^{l_s} \exp\left(\frac{2\pi i j_s}{l_s}\right)^{k_s}. \end{aligned} \quad (23)$$

It is known that

$$\sum_{j=1}^{q} \exp\left(\frac{2\pi i j}{q}\right)^{r} = 0 \quad (24)$$

for every $q \in \{2, 3, \ldots\}$ and $r \in \{1, \ldots, q-1\}$. Therefore,

$$\sum_{j_s=1}^{l_s} \exp\left(\frac{2\pi i j_s}{l_s}\right)^{k_s} = 0 \quad (25)$$

and, consequently, $H_k(a_l) = 0$. □

Let us prove the following auxiliary proposition.

Proposition 4. *A function $g : (0, +\infty) \to \mathbb{R}$, $g(x) = (c_1^x + \cdots + c_m^x)^{1/x}$, where $m \in \mathbb{N}$ and $c_1, \ldots, c_m > 0$, is strictly decreasing.*

Proof. Let us prove that $g'(x) < 0$ for every $x \in (0, +\infty)$. Note that $g(x) = ((1/x)\ln(c_1^x + \cdots + c_m^x))$. Therefore,

$$\begin{aligned} g'(x) &= g(x)\left(-\frac{1}{x^2}\ln\left(c_1^x + \cdots + c_m^x\right) + \frac{1}{x}\right. \\ &\quad \left.\cdot \frac{c_1^x \ln c_1 + \cdots + c_m^x \ln c_m}{c_1^x + \cdots + c_m^x}\right) \\ &= -\frac{g(x)}{x^2\left(c_1^x + \cdots + c_m^x\right)}\left(\left(c_1^x + \cdots + c_m^x\right)\right. \\ &\quad \left.\cdot \ln\left(c_1^x + \cdots + c_m^x\right) - x\left(c_1^x \ln c_1 + \cdots + c_m^x \ln c_m\right)\right) \\ &= -\frac{g(x)}{x^2\left(c_1^x + \cdots + c_m^x\right)}\left(c_1^x\left(\ln\left(c_1^x + \cdots + c_m^x\right)\right.\right. \\ &\quad \left.\left.- \ln c_1^x\right) + \cdots + c_m^x\left(\ln\left(c_1^x + \cdots + c_m^x\right) - \ln c_m^x\right)\right). \end{aligned} \quad (26)$$

Since $g(x)/x^2(c_1^x + \cdots + c_m^x) > 0$ and $\ln(c_1^x + \cdots + c_m^x) > \ln c_j^x$ for every $j \in \{1, \ldots, m\}$, it follows that $g'(x) < 0$. □

Corollary 5. *For every $x \in \ell_p(\mathbb{C}^n)$ and for every $q \geq p$*

$$\|x\|_p \geq \|x\|_q. \quad (27)$$

For an arbitrary nonempty finite set $M \subset \mathbb{Z}_+^n$ let us define a mapping $\pi_M : c_{00}(\mathbb{C}^n) \to \mathbb{C}^{|M|}$, where $|M|$ is the cardinality of M, by

$$\pi_M(x) = \left(H_k(x)\right)_{k \in M}, \quad (28)$$

where $(H_k(x))_{k\in M}$ is an $|M|$-dimensional vector of values of H_k on x, indexed by $k \in M$. We endow the space $\mathbb{C}^{|M|}$ with norm $\|\xi\|_\infty = \max_{k\in M}|\xi_k|$, where $\xi = (\xi_k)_{k\in M} \in \mathbb{C}^{|M|}$.

Theorem 6. *Let M be a finite nonempty subset of $\mathbb{Z}_+^n$ such that $|k| \geq 1$ for every $k \in M$. Then*

(i) *there exists $m \in \mathbb{N}$ such that for every $\xi = (\xi_k)_{k\in M} \in \mathbb{C}^{|M|}$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi) = \xi$;*

(ii) *there exists a constant $\rho_M > 0$ such that if $\|\xi\|_\infty < 1$, then $\|x_\xi\|_p < \rho_M$ for every $p \in [1, +\infty)$.*

Proof. (i) Let $\xi = (\xi_k)_{k\in M} \in \mathbb{C}^{|M|}$. For every $k \in M$, let us define $\eta_k \in \mathbb{C}$ and $b_k \in c_{00}(\mathbb{C}^n)$ by the following way. For minimal elements k of the partially ordered set $(M, \preceq)$, let $\eta_k = \xi_k$ and $b_k = \sqrt[|k|]{\eta_k}a_k$, where a_k is defined by (14) and

$$\sqrt[|k|]{\eta_k} = \begin{cases} \sqrt[|k|]{|\eta_k|}e^{i\arg\eta_k/|k|}, & \text{if } \eta_k \neq 0 \\ 0, & \text{if } \eta_k = 0. \end{cases} \tag{29}$$

For $k \in M$, which are not minimal elements of $(M, \preceq)$, we define η_k and b_k inductively by

$$\eta_k = \xi_k - \sum_{\substack{l\in M\\ l\prec k}} H_k(b_l), \tag{30}$$

$$b_k = \sqrt[|k|]{\eta_k}a_k. \tag{31}$$

We set $x_\xi = \bigoplus_{l\in M} b_l$. Note that $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$, where

$$m = \sum_{k\in M} \min\left\{j \in \mathbb{N} : a_k \in c_{00}^{(j)}(\mathbb{C}^n)\right\}. \tag{32}$$

For $k \in M$, by (12), $H_k(x_\xi) = \sum_{l\in M} H_k(b_l)$. Since H_k is a $|k|$-homogeneous polynomial,

$$H_k(b_l) = \left(\sqrt[|l|]{\eta_l}\right)^{|k|} H_k(a_l). \tag{33}$$

By Proposition 3, $H_k(a_l)$ is not equal to zero only for $l \in M$ such that $l \preceq k$. Therefore,

$$H_k(x_\xi) = H_k(b_k) + \sum_{\substack{l\in M\\ l\prec k}} H_k(b_l). \tag{34}$$

By Proposition 3, $H_k(a_k) = 1$, and therefore, by (33), $H_k(b_k) = \eta_k$. Hence,

$$H_k(x_\xi) = \eta_k + \sum_{\substack{l\in M\\ l\prec k}} H_k(b_l). \tag{35}$$

Taking into account (30), we have $H_k(x_\xi) = \xi_k$. Hence, $\pi_M(x_\xi) = \xi$.

(ii) Let $\xi = (\xi_k)_{k\in M} \in \mathbb{C}^{|M|}$ be such that $\|\xi\|_\infty < 1$. For $k \in M$ let

$$\langle k\rangle = \max\left\{s \in \mathbb{N} : \exists l^{(1)}, \ldots, l^{(s)} \in M \text{ such that } l^{(1)} \prec \cdots \prec l^{(s)} = k\right\}. \tag{36}$$

Note that for minimal elements $k \in M$ we have $\langle k\rangle = 1$.

Let

$$C = \max\left\{1, \max_{k\in M}\|a_k\|_1\right\}. \tag{37}$$

Let

$$r = \max_{k\in M}\langle k\rangle, \tag{38}$$

and for every $j \in \{1, \ldots, r\}$ let

$$\mu_j = \prod_{s=1}^{j}(1 + m_s), \tag{39}$$

where

$$m_s = |\{k \in M : \langle k\rangle = s\}|. \tag{40}$$

Also we set $\mu_0 = 1$.

Note that for every $j \in \{1, \ldots, r\}$

$$\begin{aligned} \mu_j &= \mu_{j-1}(1 + m_j) = \mu_{j-1} + \mu_{j-1}m_j \\ &= \mu_{j-2} + \mu_{j-2}m_{j-1} + \mu_{j-1}m_j = \cdots \\ &= \mu_0 + \mu_0 m_1 + \mu_1 m_2 + \cdots + \mu_{j-1}m_j. \end{aligned} \tag{41}$$

Let us prove that for every $k \in M$

$$\|b_k\|_1 < \mu_{\langle k\rangle-1}C^{\langle k\rangle}. \tag{42}$$

We proceed by induction on $\langle k\rangle$. In the case $\langle k\rangle = 1$, we have $\eta_k = \xi_k$, and therefore, $\|b_k\|_1 = \sqrt[|k|]{|\xi_k|}\|a_k\|_1$. Since $|\xi_k| < 1$, it follows that $\|b_k\|_1 < \|a_k\|_1 \leq C = \mu_0 C$. If $r = 1$, then (42) is proved. Let $r \geq 2$ and $j \in \{2, \ldots, r\}$. Suppose that inequality (42) holds for every $k \in M$ such that $\langle k\rangle \in \{1, \ldots, j-1\}$. Let us prove (42) for $k \in M$ such that $\langle k\rangle = j$. By (31) and (37),

$$\|b_k\|_1 \leq \sqrt[|k|]{|\eta_k|}\,\|a_k\|_1 \leq \sqrt[|k|]{|\eta_k|}C. \tag{43}$$

By (30),

$$|\eta_k| \leq |\xi_k| + \sum_{\substack{l\in M\\ l\prec k}} |H_k(b_l)|. \tag{44}$$

Since H_k is a $|k|$-homogeneous polynomial on the space $\ell_1(\mathbb{C}^n)$ and $\|H_k\| \leq 1$,

$$|H_k(b_l)| \leq \|H_k\|\,\|b_l\|_1^{|k|} \leq \|b_l\|_1^{|k|}. \tag{45}$$

Therefore, taking into account $|\xi_k| < 1$, we have

$$|\xi_k| + \sum_{\substack{l\in M\\ l\prec k}} |H_k(b_l)| < 1 + \sum_{\substack{l\in M\\ l\prec k}} \|b_l\|_1^{|k|}. \tag{46}$$

Therefore,

$$\sqrt[|k|]{|\eta_k|} < \left(1 + \sum_{\substack{l\in M\\ l\prec k}} \|b_l\|_1^{|k|}\right)^{1/|k|}. \tag{47}$$

By Proposition 4,

$$\left(1+\sum_{\substack{l\in M\\ l\prec k}}\|b_l\|_1^{|k|}\right)^{1/|k|}\leq 1+\sum_{\substack{l\in M\\ l\prec k}}\|b_l\|_1. \tag{48}$$

Note that if $l\prec k$, then $\langle l\rangle<\langle k\rangle$. Therefore,

$$\sum_{\substack{l\in M\\ l\prec k}}\|b_l\|_1\leq\sum_{\substack{l\in M\\ \langle l\rangle<\langle k\rangle}}\|b_l\|_1. \tag{49}$$

Since $\langle k\rangle=j$,

$$\sum_{\substack{l\in M\\ \langle l\rangle<\langle k\rangle}}\|b_l\|_1=\sum_{s=1}^{j-1}\sum_{\substack{l\in M\\ \langle l\rangle=s}}\|b_l\|_1. \tag{50}$$

By the induction hypothesis, if $\langle l\rangle=s$, where $s\in\{1,\dots,j-1\}$, then $\|b_l\|_1<\mu_{s-1}C^s$. Therefore,

$$\sum_{\substack{l\in M\\ \langle l\rangle=s}}\|b_l\|_1<\sum_{\substack{l\in M\\ \langle l\rangle=s}}\mu_{s-1}C^s=\mu_{s-1}C^s\sum_{\substack{l\in M\\ \langle l\rangle=s}}1=\mu_{s-1}m_sC^s. \tag{51}$$

Since $C\geq 1$, it follows that $C^s\leq C^{j-1}$ for every $s\in\{1,\dots,j-1\}$, and therefore,

$$\begin{aligned}1+\sum_{s=1}^{j-1}\mu_{s-1}m_sC^s&\leq 1+C^{j-1}\sum_{s=1}^{j-1}\mu_{s-1}m_s\\&\leq\left(1+\sum_{s=1}^{j-1}\mu_{s-1}m_s\right)C^{j-1}.\end{aligned} \tag{52}$$

Since $\mu_0=1$, by (41),

$$1+\sum_{s=1}^{j-1}\mu_{s-1}m_s=\mu_{j-1}. \tag{53}$$

By (47)–(53),

$$\sqrt[|k|]{|\eta_k|}<\mu_{j-1}C^{j-1}. \tag{54}$$

By (43) and (54), $\|b_k\|_1\leq\mu_{j-1}C^j$. Hence, inequality (42) holds for every $k\in M$.

By (11) and by Proposition 4,

$$\|x_\xi\|_1\leq\sum_{l\in M}\|b_l\|_1. \tag{55}$$

By (42),

$$\begin{aligned}\sum_{l\in M}\|b_l\|_1&=\sum_{j=1}^{r}\sum_{\substack{l\in M\\ \langle l\rangle=j}}\|b_l\|_1<\sum_{j=1}^{r}\sum_{\substack{l\in M\\ \langle l\rangle=j}}\mu_{j-1}C^j\\&=\sum_{j=1}^{r}\mu_{j-1}C^j\sum_{\substack{l\in M\\ \langle l\rangle=j}}1=\sum_{j=1}^{r}\mu_{j-1}m_jC^j\\&\leq\left(\sum_{j=1}^{r}\mu_{j-1}m_j\right)C^r<\left(\mu_0+\sum_{j=1}^{r}\mu_{j-1}m_j\right)C^r\\&=\mu_rC^r.\end{aligned} \tag{56}$$

Set $\rho_M=\mu_rC^r$. We have that $\|x_\xi\|_1<\rho_M$ if $\|\xi\|_\infty<1$. By Corollary 5, $\|x_\xi\|_p\leq\|x_\xi\|_1\leq\rho_M$ for every $p\in[1,+\infty)$. □

Corollary 7. *Let $M=\{k^{(1)},\dots,k^{(s)}\}\subset\mathbb{Z}_+^n$ such that $|k^{(j)}|\geq 1$ for every $j\in\{1,\dots,s\}$. Then there exists $m\in\mathbb{N}$ such that for every $m'\geq m$ polynomials $H_{k^{(1)}},\dots,H_{k^{(s)}}$ are algebraically independent on $c_{00}^{(m')}(\mathbb{C}^n)$.*

Proof. By Theorem 6, there exists $m\in\mathbb{N}$ such that for every $\xi=(\xi_1,\dots,\xi_s)\in\mathbb{C}^s$ there exists $x_\xi\in c_{00}^{(m)}(\mathbb{C}^n)$ such that

$$H_{k^{(j)}}(x_\xi)=\xi_j \tag{57}$$

for every $j\in\{1,\dots,s\}$. Let us show that $H_{k^{(1)}},\dots,H_{k^{(s)}}$ are algebraically independent on $c_{00}^{(m')}(\mathbb{C}^n)$ for every $m'\geq m$. Let $Q:\mathbb{C}^s\to\mathbb{C}$ be a polynomial such that

$$Q\left(H_{k^{(1)}}(x),\dots,H_{k^{(s)}}(x)\right)=0 \tag{58}$$

for every $x\in c_{00}^{(m')}(\mathbb{C}^n)$. Set $x=x_\xi$. Taking into account (57), we have $Q(\xi_1,\dots,\xi_s)=0$ for arbitrary $\xi_1,\dots,\xi_s\in\mathbb{C}$, that is, $Q\equiv 0$. Hence, $H_{k^{(1)}},\dots,H_{k^{(s)}}$ are algebraically independent. □

3.2. Algebraic Basis of the Algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$

Theorem 8. *Every N-homogeneous polynomial $P\in\mathcal{P}_s(c_{00}^{(m)}(\mathbb{C}^n))$, where m is an arbitrary positive integer, can be represented as an algebraic combination of polynomials H_k, where $k\in\mathbb{Z}_+^n$ such that $1\leq|k|\leq N$.*

Proof. We proceed by induction on m. In the case $m=1$ for $x=(x_1,0,\dots)\in c_{00}^{(1)}(\mathbb{C}^n)$, we have

$$\begin{aligned}P(x)&=\sum_{\substack{k\in\mathbb{Z}_+^n\\ |k|=N}}\alpha_k\left(x_1^{(1)}\right)^{k_1}\cdots\left(x_1^{(n)}\right)^{k_n}\\&=\sum_{\substack{k\in\mathbb{Z}_+^n\\ |k|=N}}\alpha_kH_k(x),\end{aligned} \tag{59}$$

where $\alpha_k\in\mathbb{C}$. Suppose the statement holds for $m-1$ and prove it for m. Let $P\in\mathcal{P}_s(c_{00}^{(m)}(\mathbb{C}^n))$ and $x=(x_1,\dots,x_m,0,\dots)\in c_{00}^{(m)}(\mathbb{C}^n)$. Then $P(x)$ can be represented as a sum of terms

$$\beta_k\left(x_m^{(1)}\right)^{k_1}\cdots\left(x_m^{(n)}\right)^{k_n}f_k\left((x_1,\dots,x_{m-1},0,\dots)\right), \tag{60}$$

where $\beta_k\in\mathbb{C}$, $k=(k_1,\dots,k_n)\in\mathbb{Z}_+^n$ such that $1\leq|k|\leq N$, and f_k is an $(N-|k|)$-homogeneous polynomial. Note that $f_k\in\mathcal{P}_s(c_{00}^{(m-1)}(\mathbb{C}^n))$, and therefore, by the induction hypothesis, $f_k((x_1,\dots,x_{m-1},0,\dots))$ can be represented as an algebraic combination of $H_l((x_1,\dots,x_{m-1},0,\dots))$, where $l\in\mathbb{Z}_+^n$ such that $1\leq|l|\leq N-|k|$. Note that

$$\begin{aligned}&H_l\left((x_1,\dots,x_{m-1},0,\dots)\right)\\&\quad=H_l(x)-\left(x_m^{(1)}\right)^{l_1}\cdots\left(x_m^{(n)}\right)^{l_n}.\end{aligned} \tag{61}$$

Therefore, $P(x)$ can be represented as an algebraic combination of $H_l(x)$ and $x_m^{(1)}, \ldots, x_m^{(n)}$. Since P and H_l are symmetric, it follows that together with term

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \left(x_m^{(1)}\right)^{r_1} \cdots \left(x_m^{(n)}\right)^{r_n} H_{l_1}^{t_1}(x) \cdots H_{l_s}^{t_s}(x), \quad (62)$$

where $\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \in \mathbb{C}$, $l_1, \ldots, l_s \in \mathbb{Z}_+^n$ and $r_1, \ldots, r_n, t_1, \ldots, t_s \in \mathbb{Z}_+$, the sum must contain terms

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \left(x_j^{(1)}\right)^{r_1} \cdots \left(x_j^{(n)}\right)^{r_n} H_{l_1}^{t_1}(x) \cdots H_{l_s}^{t_s}(x), \quad (63)$$

where $j \in \{1, \ldots, m-1\}$. Therefore, $P(x)$ can be represented as a sum of terms

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \left(\frac{1}{m} \sum_{j=1}^{m} \left(x_j^{(1)}\right)^{r_1} \cdots \left(x_j^{(n)}\right)^{r_n} \right) \cdot H_{l_1}^{t_1}(x) \cdots H_{l_s}^{t_s}(x). \quad (64)$$

Since $\sum_{j=1}^{m} (x_j^{(1)})^{r_1} \cdots (x_j^{(n)})^{r_n} = H_r(x)$, where $r = (r_1, \ldots, r_n)$, it follows that P is an algebraic combination of polynomials H_k, where $k \in \mathbb{Z}_+^n$ such that $1 \le |k| \le N$. □

Theorem 9. *Let $P : c_{00}(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric N-homogeneous polynomial. Let $M_N = \{k \in \mathbb{Z}_+^n : 1 \le |k| \le N\}$. There exists a polynomial $q : \mathbb{C}^{|M_N|} \to \mathbb{C}$ such that $P = q \circ \pi_{M_N}$, where the mapping π_{M_N} is defined by (28).*

Proof. By Corollary 7, there exists $m \in \mathbb{N}$ such that for every $m' \ge m$ polynomials H_k, where $k \in M$, are algebraically independent. Therefore, the representation, given by Theorem 8 for the restriction of P to $c_{00}^{(m')}(\mathbb{C}^n)$, is unique. Thus, for every $m' \ge m$ there exists a unique polynomial $q_{m'} : \mathbb{C}^{|M_N|} \to \mathbb{C}$ such that $P(x) = (q_{m'} \circ \pi_{M_N})(x)$ for every $x \in c_{00}^{(m')}(\mathbb{C}^n)$. Since $c_{00}^{(m')}(\mathbb{C}^n) \supset c_{00}^{(m)}(\mathbb{C}^n)$, it follows that q_m is the restriction of $q_{m'}$ to $\pi_{M_N}(c_{00}^{(m)}(\mathbb{C}^n))$. By Theorem 6, $\pi_{M_N}(c_{00}^{(m)}(\mathbb{C}^n)) = \mathbb{C}^{|M_N|}$, and therefore, $q_{m'} \equiv q_m$. Let $q = q_m$. Then $P(x) = (q \circ \pi_{M_N})(x)$ for every $x \in c_{00}(\mathbb{C}^n)$. □

Theorem 10. *Polynomials H_k, where $k \in \mathbb{Z}_+^n$, form an algebraic basis of the algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$.*

Proof. Let us prove that every symmetric continuous polynomial on $\ell_1(\mathbb{C}^n)$ can be uniquely represented as an algebraic combination of polynomials H_k. It suffices to prove the statement only for homogeneous polynomials. Let $P : \ell_1(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric continuous N-homogeneous polynomial. By Theorem 9, the restriction of P to $c_{00}(\mathbb{C}^n)$ can be uniquely represented as an algebraic combination of polynomials H_k, where $k \in \mathbb{Z}_+^n$ such that $1 \le |k| \le N$. Since $c_{00}(\mathbb{C}^n)$ is dense in $\ell_1(\mathbb{C}^n)$ and polynomials H_k are well-defined and continuous on $\ell_1(\mathbb{C}^n)$, it follows that given representation can be extended to $\ell_1(\mathbb{C}^n)$. □

3.3. Algebraic Basis of the Algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$. Let $p \in (1, +\infty)$. In this section, we describe an algebraic basis of the algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$.

Let us prove a complex analog of [8, Lemma 2].

Lemma 11. *Let $K \subset \mathbb{C}^m$ and $\varkappa : K \to \mathbb{C}^{m-1}$ be an orthogonal projection: $\varkappa((x_1, x_2, \ldots, x_m)) = (x_2, \ldots, x_m)$. Let $K_1 = \varkappa(K)$, $\operatorname{int} K_1 \ne \emptyset$ and for every open set $U \subset K_1$ a set $\varkappa^{-1}(U)$ is unbounded. If polynomial $Q(x_1, \ldots, x_m)$ is bounded on K, then Q does not depend on x_1.*

Proof. Suppose that Q depends on x_1. Then

$$Q(x_1, \ldots, x_m) = \sum_{j=0}^{k} q_j(x_2, \ldots, x_m) x_1^j, \quad (65)$$

where $1 \le k \le \deg Q$ and $q_k \not\equiv 0$. Note that $q_k \not\equiv 0$ on $\operatorname{int} K_1$, and therefore, there exists point $a \in \operatorname{int} K_1$ such that $q_k(a) \ne 0$. Since $\operatorname{int} K_1$ is open and q_k is continuous, there exists $r > 0$ such that $B(a, r) \subset \operatorname{int} K_1$ and $\inf_{b \in B(a,r)} |q_k(b)| > 0$, where $B(a, r)$ is an open ball with center a and radius r in the space $\mathbb{C}^{m-1}$. Note that, for $(x_1, \ldots, x_m) \in \varkappa^{-1}(B(a, r))$,

$$\begin{aligned} |Q(x_1, \ldots, x_m)| &\ge |q_k(x_2, \ldots, x_m)| |x_1|^k - \sum_{j=0}^{k-1} |q_j(x_2, \ldots, x_m)| |x_1|^j \\ &\ge c |x_1|^k - \sum_{j=0}^{k-1} d_j |x_1|^j, \end{aligned} \quad (66)$$

where $c = \inf_{b \in B(a,r)} |q_k(b)|$ and $d_j = \sup_{b \in B(a,r)} |q_j(b)|$ for $j \in \{0, \ldots, k-1\}$. Note that for the polynomial $c x_1^k + \sum_{j=0}^{k-1} d_j x_1^j$ there exists $R > 0$ such that if $|x_1| > R$, then $c|x_1|^k > 2 \sum_{j=0}^{k-1} d_j |x_1|^j$, that is, $\sum_{j=0}^{k-1} d_j |x_1|^j < (1/2) c |x_1|^k$. Therefore, if $|x_1| > R$, then

$$c |x_1|^k - \sum_{j=0}^{k-1} d_j |x_1|^j > c |x_1|^k - \frac{1}{2} c |x_1|^k = \frac{1}{2} c |x_1|^k. \quad (67)$$

Since $\varkappa^{-1}(B(a, r))$ is unbounded, there exists a sequence $((x_1^{(n)}, \ldots, x_m^{(n)}))_{n \in \mathbb{N}} \subset \varkappa^{-1}(B(a, r))$ such that $x_1^{(n)} \to \infty$ as $n \to +\infty$. Taking into account (66) and (67), we have

$$\left| Q\left(x_1^{(n)}, \ldots, x_m^{(n)}\right) \right| > \frac{1}{2} c \left| x_1^{(n)} \right|^k \longrightarrow +\infty \quad (68)$$

as $n \to +\infty$, which contradicts the boundedness of Q on K. □

For $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, let $\mathcal{V}(k) = \{s \in \{1, \ldots, n\} : k_s \ne 0\}$ and $\nu(k) = |\mathcal{V}(k)|$.

Lemma 12. *For $k, l \in \mathbb{Z}_+^n$ if $l \succ k$ and $\nu(l) \ge \nu(k)$, then $|l| > |k|$.*

Proof. Since $l \succ k$, there exists $m \in \mathbb{Z}_+^n$ such that $(l_1, \ldots, l_n) = (m_1 k_1, \ldots, m_n k_n)$, and $l \ne k$. Therefore, if $k_s = 0$ for some $s \in \{1, \ldots, n\}$, then $l_s = 0$ too. It means that $\mathcal{V}(l) \subset \mathcal{V}(k)$. On the other hand, $\nu(l) \ge \nu(k)$. Therefore, $\mathcal{V}(l) = \mathcal{V}(k)$; that is, for $s \in \{1, \ldots, n\}$, we have that $l_s \ne 0$ if and only if $k_s \ne 0$. Therefore, for every $s \in \mathcal{V}(l)$ we have that $m_s \ge 1$. Since $l \ne k$, there exists $s_0 \in \mathcal{V}(l)$ such that $m_{s_0} \ge 2$. Therefore,

$$|l| = m_1 k_1 + \cdots + m_n k_n > k_1 + \cdots + k_n = |k|. \quad (69)$$

□

For $N \in \mathbb{N}$ and $J \in \{1, \ldots, n\}$ let

$$M_N^{(J)} = \{l \in \mathbb{Z}_+^n : 1 \le |l| < \lceil p \rceil,\ \nu(l) \ge J\} \cup \{l \in \mathbb{Z}_+^n : \lceil p \rceil \le |l| \le N\}. \tag{70}$$

By Theorem 6, for $M = M_N^{(1)}$ there exists $\rho = \rho_M > 0$ such that $\pi_M(V_\rho)$ contains the open unit ball of the space $\mathbb{C}^{|M|}$ with norm $\|\cdot\|_\infty$, where

$$V_\rho = \left\{x \in c_{00}(\mathbb{C}^n) : \|x\|_p < \rho\right\}. \tag{71}$$

Proposition 13. *For $J \in \{1, \ldots, N\}$, let $q((\xi_l)_{l\in M_N^{(J)}})$ be a polynomial on $\mathbb{C}^{|M_N^{(J)}|}$. If q is bounded on $\pi_{M_N^{(J)}}(V_\rho)$, then q does not depend on ξ_k such that $\nu(k) = J$ and $1 \le |k| < \lceil p \rceil$.*

Proof. Let $k \in \mathbb{Z}_+^n$ such that $\nu(k) = J$ and $1 \le |k| < \lceil p \rceil$. Let $K = \pi_{M_N^{(J)}}(V_\rho)$, $K_1 = \pi_{M_N^{(J)}\setminus\{k\}}(V_\rho)$ and $\varkappa : K \to K_1$ be an orthogonal projection, defined by

$$\varkappa : (\xi_l)_{l\in M_N^{(J)}} \longmapsto (\xi_l)_{l\in M_N^{(J)}\setminus\{k\}}. \tag{72}$$

Let us show that, for every ball

$$B(u,r) = \left\{\xi \in \mathbb{C}^{|M_N^{(J)}\setminus\{k\}|} : \|\xi - u\|_\infty < r\right\} \tag{73}$$

with center $u = (u_l)_{l\in M_N^{(J)}\setminus\{k\}} \in \mathbb{C}^{|M_N^{(J)}\setminus\{k\}|}$ and radius $r > 0$ such that $B(u,r) \subset \pi_{M_N^{(J)}\setminus\{k\}}(V_\rho)$, a set $\varkappa^{-1}(B(u,r))$ is unbounded. Since $u \in \pi_{M_N^{(J)}\setminus\{k\}}(V_\rho)$, there exists $x_u \in V_\rho$ such that $\pi_{M_N^{(J)}\setminus\{k\}}(x_u) = u$. For $m \in \mathbb{N}$, we set $x_m = \bigoplus_{j=1}^m (1/j^{1/|k|})a_k$, where a_k is defined by (14). Choose ε such that

$$0 < \varepsilon < \min\left\{1, \frac{\rho - \|x_u\|_p}{\|a_k\|_p \zeta(p/|k|)^{1/p}}, \frac{r}{\|a_k\|_1^N \zeta(1 + 1/|k|)}\right\}, \tag{74}$$

where $\zeta(\cdot)$ is a Riemann zeta-function. Let $x_{m,\varepsilon} = (\varepsilon x_m) \oplus x_u$. Let us show that $x_{m,\varepsilon} \in V_\rho$. By (11),

$$\|x_m\|_p^p = \sum_{j=1}^m \left\|\frac{1}{j^{1/|k|}} a_k\right\|_p^p = \sum_{j=1}^m \frac{1}{j^{p/|k|}} \|a_k\|_p^p = \|a_k\|_p^p \sum_{j=1}^m \frac{1}{j^{p/|k|}} < \|a_k\|_p^p \zeta\left(\frac{p}{|k|}\right). \tag{75}$$

Therefore, $\|x_m\|_p < \|a_k\|_p \zeta(p/|k|)^{1/p}$. By the triangle inequality,

$$\|x_{m,\varepsilon}\|_p \le \varepsilon \|x_m\|_p + \|x_u\|_p < \varepsilon \|a_k\|_p \zeta\left(\frac{p}{|k|}\right)^{1/p} + \|x_u\|_p. \tag{76}$$

Since $\varepsilon < (\rho - \|x_u\|_p)/\|a_k\|_p \zeta(p/|k|)^{1/p}$, it follows that $\|x_{m,\varepsilon}\|_p < \rho$. Hence, $x_{m,\varepsilon} \in V_\rho$.

Note that for arbitrary $l \in \mathbb{Z}_+^n$ such that $|l| \ge 1$, by (12),

$$H_l(x_m) = \sum_{j=1}^m \frac{1}{j^{|l|/|k|}} H_l(a_k) = H_l(a_k) \sum_{j=1}^m \frac{1}{j^{|l|/|k|}}, \tag{77}$$

$$H_l(x_{m,\varepsilon}) = \varepsilon^{|l|} H_l(x_m) + H_l(x_u) = \varepsilon^{|l|} H_l(a_k) \sum_{j=1}^m \frac{1}{j^{|l|/|k|}} + H_l(x_u). \tag{78}$$

Let us show that $\pi_{M_N^{(J)}\setminus\{k\}}(x_{m,\varepsilon}) \in B(u,r)$. For $l \in M_N^{(J)} \setminus \{k\}$ such that $l \not\succ k$, by Proposition 3, $H_l(a_k) = 0$, and therefore, by (78),

$$H_l(x_{m,\varepsilon}) = H_l(x_u) = u_l. \tag{79}$$

Let $l \in M_N^{(J)} \setminus \{k\}$ be such that $l \succ k$. If $\lceil p \rceil \le |l| \le N$, then $|l| > |k|$, since $|k| < \lceil p \rceil$. If $1 \le |l| < \lceil p \rceil$ and $\nu(l) \ge J$, then $|l| > |k|$ by Lemma 12. Hence, $|l| > |k|$ in both cases. By (78),

$$\left|H_l(x_{m,\varepsilon}) - u_l\right| \le \varepsilon^{|l|} \left|H_l(a_k)\right| \sum_{j=1}^m \frac{1}{j^{|l|/|k|}}. \tag{80}$$

Since $\varepsilon < 1$, it follows that $\varepsilon^{|l|} \le \varepsilon$. Since $\|H_l\| \le 1$, it follows that $|H_l(a_k)| \le \|a_k\|_1^{|l|}$. Taking into account $\|a_k\|_p \ge 1$ and $|l| \le N$, we have that $|H_l(a_k)| \le \|a_k\|_1^N$. Since $|l|$ and $|k|$ are integer numbers and $|l| > |k|$, it follows that $|l| \ge |k| + 1$, and therefore,

$$\sum_{j=1}^m \frac{1}{j^{|l|/|k|}} \le \sum_{j=1}^m \frac{1}{j^{1+1/|k|}} < \zeta\left(1 + \frac{1}{|k|}\right). \tag{81}$$

Hence,

$$\left|H_l(x_{m,\varepsilon}) - u_l\right| < \varepsilon \|a_k\|_1^N \zeta\left(1 + \frac{1}{|k|}\right). \tag{82}$$

Since $\varepsilon < r/\|a_k\|_1^N \zeta(1+1/|k|)$, it follows that $|H_l(x_{m,\varepsilon}) - u_l| < r$, and therefore, $\pi_{M_N^{(J)}\setminus\{k\}}(x_{m,\varepsilon}) \in B(u,r)$.

By Proposition 3, $H_k(a_k) = 1$, and therefore, by (78),

$$H_k(x_{m,\varepsilon}) = \varepsilon^{|l|} \sum_{j=1}^m \frac{1}{j} + H_k(x_u) \longrightarrow \infty \tag{83}$$

as $m \to +\infty$. Hence, $\varkappa^{-1}(B(u,r))$ is unbounded. By Lemma 11, q does not depend on ξ_k. □

Theorem 14. *Let $P \in \mathcal{P}_s(\ell_p(\mathbb{C}^n))$ be an N-homogeneous polynomial. If $N < \lceil p \rceil$, then $P \equiv 0$. Otherwise, there exists a unique polynomial $q : \mathbb{C}^{|M_{p,N}|} \to \mathbb{C}$ such that $P = q \circ \pi_{M_{p,N}}^{(p)}$, where $M_{p,N} = \{k \in \mathbb{Z}_+^n : \lceil p \rceil \le |k| \le N\}$ and $\pi_{M_{p,N}}^{(p)} : \ell_p(\mathbb{C}^n) \to \mathbb{C}^{|M_{p,N}|}$ is defined by $\pi_{M_{p,N}}^{(p)}(x) = (H_k(x))_{k\in M_{p,N}}$.*

Proof. Let P_0 be the restriction of P to $c_{00}(\mathbb{C}^n)$. Note that P_0 is a continuous symmetric N-homogeneous polynomial. By Theorem 9, there exists a unique polynomial $q : \mathbb{C}^{|M_N|} \to \mathbb{C}$, where $M_N = M_N^{(1)}$ such that $P_0 = q \circ \pi_{M_N}$. Since P_0 is continuous, P_0 is bounded on V_ρ, defined by (71). Therefore, q is bounded on $\pi_{M_N}(V_\rho)$.

Let us prove that q does not depend on arguments ξ_k such that $1 \le |k| < \lceil p \rceil$ by induction on $\nu(k)$. By Proposition 13, for $J = 1$ we have that $q((\xi_k)_{k \in M_N})$ does not depend on arguments ξ_k such that $\nu(k) = 1$ and $1 \le |k| < \lceil p \rceil$. Suppose that the statement holds for $\nu(k) \in \{1, \ldots, J-1\}$, where $J \in \{2, \ldots, n\}$, that is, $q((\xi_k)_{k \in M_N})$ does not depend on arguments ξ_k such that $1 \le \nu(k) \le J-1$ and $1 \le |k| < \lceil p \rceil$. Then the restriction of q to $\mathbb{C}^{|M_N^{(J)}|}$, by Proposition 13, does not depend on ξ_k such that $\nu(k) = J$ and $1 \le |k| < \lceil p \rceil$. Hence, q does not depend on ξ_k such that $1 \le |k| < \lceil p \rceil$.

Since polynomials H_k, where $k \in M_{p,N}$, are well-defined and continuous on $\ell_p(\mathbb{C}^n)$ and $c_{00}(\mathbb{C}^n)$ is dense in $\ell_p(\mathbb{C}^n)$, it follows that $P = q \circ \pi_{M_{p,N}}^{(p)}$. Note that in the case $N < \lceil p \rceil$ we have $M_{p,N} = \emptyset$ and, therefore, $P \equiv 0$. □

Corollary 15. *Polynomials H_k, where $k \in \{l \in \mathbb{Z}_+^n : |l| \ge \lceil p \rceil\} \cup \{0\}$, form an algebraic basis of the algebra $\mathscr{P}_s(\ell_p(\mathbb{C}^n))$.*

4. Conclusions

Power sum symmetric polynomials on $\ell_p(\mathbb{C}^n)$ are algebraically independent and form an algebraic basis of the algebra of all continuous symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

Results of this work generalize results of works [7, 8, 14].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] R. Alencar, R. Aron, P. Galindo, and A. Zagorodnyuk, "Algebras of symmetric holomorphic functions on l_p," *Bulletin of the London Mathematical Society*, vol. 35, no. 1, pp. 55–64, 2003.

[2] R. Aron, P. Galindo, D. Pinasco, and I. Zalduendo, "Group-symmetric holomorphic functions on a Banach space," *Bulletin of the London Mathematical Society*, vol. 48, no. 5, pp. 779–796, 2016.

[3] I. Chernega, P. Galindo, and A. Zagorodnyuk, "Some algebras of symmetric analytic functions and their spectra," *Proceedings of the Edinburgh Mathematical Society*, vol. 55, no. 1, pp. 125–142, 2012.

[4] I. Chernega, P. Galindo, and A. Zagorodnyuk, "The convolution operation on the spectra of algebras of symmetric analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 395, no. 2, pp. 569–577, 2012.

[5] I. Chernega, P. Galindo, and A. Zagorodnyuk, "A multiplicative convolution on the spectra of algebras of symmetric analytic functions," *Revista Matemática Complutense*, vol. 27, no. 2, pp. 575–585, 2014.

[6] P. Galindo, T. Vasylyshyn, and A. Zagorodnyuk, "The algebra of symmetric analytic functions on L∞," *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, vol. 147, no. 4, pp. 743–761, 2017.

[7] M. González, R. Gonzalo, and J. Jaramillo, "Symmetric polynomials on rearrangement-invariant function spaces," *Journal Of The London Mathematical Society-Second Series*, vol. 59, no. 2, pp. 681–697, 1999.

[8] A. S. Nemirovskii and S. M. Semenov, "On polynomial approximation of functions on hilbert space," *Matematicheskii Sbornik*, vol. 21, no. 2, pp. 255–277, 1973.

[9] T. V. Vasylyshyn, "Continuous block-symmetric polynomials of degree at most two on the space (L∞)2," *Carpathian Mathematical Publications*, vol. 8, no. 1, pp. 38–43, 2016.

[10] T. Vasylyshyn, "Symmetric continuous linear functionals on complex space L8," *Carpathian Mathematical Publications*, vol. 6, , no. 1, pp. 10-10, 2014.

[11] I. Chernega, "Symmetric polynomials and holomorphic functions on infinite dimensional spaces," *Journal of Vasyl Stefanyk Precarpathian National University*, vol. 2, no. 4, 2015.

[12] V. Kravtsiv, "The analogue of Newton's formula for block-symmetric polynomials," *International Journal of Mathematical Analysis*, vol. 10, no. 5-8, pp. 323–327, 2016.

[13] V. V. Kravtsiv and A. V. Zagorodnyuk, "Representation of spectra of algebras of block-symmetric analytic functions of bounded type," *Carpathian Mathematical Publications*, vol. 8, no. 2, pp. 263–271, 2016.

[14] V. V. Kravtsiv and A. V. Zagorodnyuk, "On algebraic bases of algebras of block-symmetric polynomials on Banach spaces," *Matematychni Studii*, vol. 37, no. 1, pp. 109–112, 2012.

[15] J. Mujica, *Complex Analysis in Banach Spaces*, vol. 120 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1986.

[16] I. G. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press, Oxford University Press, New York, 1979.

[17] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, vol. 62 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1999.