International Journal of Mathematical Analysis
Vol. 10, 2016, no. 7, 323 - 327
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2016.617

The Analogue of Newton's Formula for Block-Symmetric Polynomials

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Abstract

The paper contains a proof of Newton's formula for the block-symmetric polynomials.

Mathematics Subject Classification: 46J15, 46E10, 46E50

Keywords: block-symmetric polynomials, algebra of block-symmetric polynomials, Newton's formula, algebraic basis

1 Introduction

Let X be a Banach spaces, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an algebraic basis of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is a unique $q \in \mathcal{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(G_1(x), \ldots, G_n(x))$; in other words, if G is mapping $x \in X \rightsquigarrow (G_1(x), \ldots, G_n(x)) \in \mathbb{C}^n$, then $P = q \circ G$.

Let $\mathcal{P}_s(\ell_1)$ be the algebra of symmetric polynomials on the space ℓ_1. In [4], it is proved that the polynomials $F_k = \sum\limits_{i=1}^{\infty} x_i^k$, where $k \geq 1$ form an algebraic basis in $\mathcal{P}_s(\ell_1)$. It is well known that any polynomial in $\mathcal{P}_s(\ell_1)$ is uniquely representable as a polynomial in the elementary symmetric polynomials $\{G_i\}_{i=1}^{\infty}$,

$G_k = \sum\limits_{i_1<i_2<\ldots<i_k}^{\infty} x_{i_1} x_{i_2} \ldots x_{i_k}$, where $k \geq 1$. The algebra of symmetric analytic functions $H_{bs}(X)$ were investigated by many authors ([1], [2], [3]).

On the other hand, there are more representations of S_∞ in Banach spaces. For example, if $\mathcal{X}$ is a directs sum of infinite many of "blocks" which are copies of a Banach space X, then S_∞ acts permutations the "blocks". For this case we have invariants — the algebra of block-symmetric analytic functions. Note that this algebra is much more complicated and in the general case has no algebraic basis (see e. g. [6], [9], [5]).

It is well known the Newton's formula for symmetric polynomials [8]:

$$nG_n = F_1 G_{n-1} - F_2 G_{n-2} + F_3 G_{n-3} - \ldots + (-1)^{n-2} F_{n-1} G_1 + (-1)^{n-1} F_n.$$

In this paper we propose a genralization of this formula for block-symmetric polynomials on ℓ_1.

2 Main Result

Let

$$\mathcal{X}^2 = \oplus_{\ell_1} \mathbb{C}^2$$

be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $\overline{x} \in \mathcal{X}^2$ can be represented as a sequence $\overline{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbb{C}^2$, with the norm $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|$.

A polynomial P on the space $\mathcal{X}^2$ is called block-symmetric (or vector-symmetric) if:

$$P\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_1, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}_m, \ldots\right) = P\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_{\sigma(1)}, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}_{\sigma(m)}, \ldots\right),$$

for every permutation σ on the set $\mathbb{N}$, where $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{C}^2$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In paper [7] it was shown that the following vectors form an algebraic bases of $\mathcal{P}_{vs}(\mathcal{X}^2)$:

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p}, \tag{1}$$

where $0 \leq p \leq n$, $(x_i, y_i) \in \mathbb{C}^2$, $i \geq 1$ or "elementary" block-symmetric polynomials:

$$R^{p,n-p}(x,y) = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_{n-p} \\ i_k \neq j_l}}^{\infty} x_{i_1} \ldots x_{i_p} y_{j_1} \ldots y_{j_{n-p}}, \tag{2}$$

where $0 \le p \le n$, $n \ge 1$ and $(x_i, y_i) \in \mathbb{C}^2$.

For equations (1) and (2) of generators we can write an analogue of Newton's formula.

Theorem 2.1 *The following formula is true*

$$\begin{gathered} nR^{p,n-p} = H^{1,0}R^{p-1,n-p} + H^{0,1}R^{p,n-1-p} - \Big(H^{2,0}R^{p-2,n-p} + 2H^{1,1}R^{p-1,n-p-1} + \\ + H^{0,2}R^{p,n-p-2}\Big) + \Big(H^{3,0}R^{p-3,n-p} + 3H^{2,1}R^{p-2,n-p-1} + 3H^{1,2}R^{p-1,n-p-2} + \\ + H^{0,3}R^{p,n-p-3}\Big) - \ldots + (-1)^{s-1}\sum_{k=0}^{s} C_s^k H^{s-k,k}R^{p-(s-k),n-p-k} + \ldots + \\ + (-1)^{n-2}\Big(C_{n-1}^{p-1}H^{p-1,n-p}R^{1,0} + C_{n-1}^{p}H^{p,n-p-1}R^{0,1}\Big) + (-1)^{n-1}C_n^p H^{p,n-p}, \end{gathered} \tag{3}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$, $0 \le p \le n$ *and if* $s-k > p$ *or* $k > n-p$, *then we consider that* $R^{p-(s-k),n-p-s} \equiv 0$.

Proof Let us consider the polynomial $P(x+jy)$, which is symmetric on the space ℓ_1 with respect to simultaneously permutations of $x_i + jy_i$, $i \ge 1$. For the algebraic bases $F_k(x+jy)$ and $G_k(x+jy)$ of this polynomial the Newton formula holds

$$\begin{gathered} nG_n(x+jy) = F_1(x+jy)G_{n-1}(x+jy) - \\ - F_2(x+jy)G_{n-2}(x+jy) + F_3(x+jy)G_{n-3}(x+jy) - \ldots + \\ + (-1)^{n-2}F_{n-1}(x+jy)G_1(x+jy) + (-1)^{n-1}F_n(x+jy). \end{gathered} \tag{4}$$

Each of polynomials $F_k(x+jy)$ and $G_k(x+jy)$ can be represented as a linear combination of polynomials $H^{p,k-p}(x,y)$ and $R^{p,k-p}(x,y)$ respectively. Indeed,

$$\begin{gathered} G_n(x+jy) = \sum_{i_1<\ldots<i_n}^{\infty}(x_{i_1}+jy_{i_1})\ldots(x_{i_n}+jy_{i_n}) = \\ = R^{n,0}(x,y) + jR^{n-1,1}(x,y) + j^2R^{n-2,2}(x,y) + j^3R^{n-3,3}(x,y) + \ldots + \\ + j^kR^{n-k,k}(x,y) + \ldots + j^nR^{0,n}(x,y) \end{gathered} \tag{5}$$

and

$$\begin{gathered} F_n(x+jy) = \sum_{i=1}^{\infty}(x_i+jy_i)^n = \\ = H^{n,0}(x,y) + jC_n^1H^{n-1,1}(x,y) + j^2C_n^2H^{n-2,2}(x,y) + \\ + j^3C_n^3H^{n-3,3}(x,y) + \ldots + j^kC_n^kH^{n-k,k}(x,y) + \ldots + j^nH^{0,n}(x,y), \end{gathered} \tag{6}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

So each term of equality (4) can be represented by over polynomials $H^{p,k-p}(x,y)$ and $R^{p,k-p}(x,y)$. Then we obtain

$$F_1(x+jy)G_{n-1}(x+jy) = H^{1,0}R^{n-1,0} + j(H^{1,0}R^{n-2,1} + H^{0,1}R^{n-1,0}) +$$

$$+j^2(H^{1,0}R^{n-3,2}+H^{0,1}R^{n-2,1})+j^3(H^{1,0}R^{n-4,3}+H^{0,1}R^{n-3,2})+\ldots+$$
$$+j^{n-1}(H^{1,0}R^{0,n-1}+H^{0,1}R^{1,n-2})+j^nH^{0,1}R^{0,n-1},$$
$$F_2(x+jy)G_{n-2}(x+jy)=H^{2,0}R^{n-2,0}+j(H^{2,0}R^{n-3,1}+2H^{1,1}R^{n-2,0})+$$
$$+j^2(H^{2,0}R^{n-4,2}+2H^{1,1}R^{n-3,1}+H^{0,2}R^{n-2,0})+$$
$$+j^3(H^{2,0}R^{n-5,3}+2H^{1,1}R^{n-4,2}+H^{0,2}R^{n-3,1})+\ldots+$$
$$+j^{n-1}(2H^{1,1}R^{0,n-2}+H^{0,2}R^{1,n-3})+j^nH^{0,2}R^{0,n-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_{n-1}(x+jy)G_1(x+jy)=H^{n-1,0}R^{1,0}+j(H^{n-1,0}R^{0,1}+C^1_{n-1}H^{n-2,1}R^{1,0})+$$
$$+j^2(C^1_{n-1}H^{n-2,1}R^{0,1}+C^2_{n-1}H^{n-3,2}R^{1,0})+$$
$$+j^3(C^2_{n-1}H^{n-3,2}R^{0,1}+C^3_{n-1}H^{n-4,3}R^{1,0})+\ldots+$$
$$+j^{n-1}(C^{n-2}_{n-1}H^{1,n-2}R^{0,1}+H^{0,n-1}R^{1,0})+j^nH^{n-1}R^{0,1}.$$

If we substitute this equalities and equalities (5), (6) to (4) and equate multipliers at the same degrees of j, we obtain next equalities:

$$nR^{n,0}=H^{1,0}R^{n-1,0}-H^{2,0}R^{n-2,0}+H^{3,0}R^{n-3,0}-\ldots+(-1)^{n-1}H^{n,0},$$
$$nR^{n-1,1}=H^{1,0}R^{n-2,1}+H^{0,1}R^{n-1,0}-\Big(H^{2,0}R^{n-3,1}+2H^{1,1}R^{n-2,0}\Big)+$$
$$+\Big(H^{3,0}R^{n-4,1}+3H^{2,1}R^{n-3,0}\Big)-\ldots+$$
$$+(-1)^{n-2}\Big(H^{n-1,0}R^{0,1}+C^1_{n-1}H^{n-2,1}R^{1,0}\Big)+(-1)^{n-1}C^1_nH^{n-1,1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$nR^{n-k,k}=H^{1,0}R^{n-k-1,k}+H^{0,1}R^{n-k,k-1}-\Big(H^{2,0}R^{n-k-2,k}+$$
$$+2H^{1,1}R^{n-k-1,k-1}+H^{0,2}R^{n-k,k-2}\Big)+\Big(H^{3,0}R^{n-k-3,k}+3H^{2,1}R^{n-k-2,k-1}+$$
$$+3H^{1,2}R^{n-k-1,k-2}+H^{0,3}R^{n-k,k-3}\Big)-\ldots+(-1)^{n-2}\Big(C^{k-1}_{n-1}H^{n-k,k-1}R^{0,1}+$$
$$+C^k_{n-1}H^{n-k-1,k}R^{1,0}\Big)+(-1)^{n-1}C^k_nH^{n-k,k},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$nR^{0,n}=H^{0,1}R^{0,n-1}-H^{0,2}R^{0,n-2}+H^{0,3}R^{0,n-3}-\ldots+(-1)^{n-1}H^{0,n}.$$

Therefore from these equalities it follows formula (3) for any polynomial $R^{p,n-p}$.

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Received: February 6, 2016; Published: March 5, 2016