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Hardy type spaces associated with compact unitary groups

Oleh Lopushansky [a], Andriy Zagorodnyuk [b,c,*]

[a] Institute of Mathematics, Rzeszów University, 16A Rejtana Str., 35-310 Rzeszów, Poland

[b] Department of Mathematics and Computer Sciences, Precarpathian National University, 57 Shevchenka Str., 76025 Ivano-Frankivsk, Ukraine

[c] Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, 3 b, Naukova Str., Lviv 79060, Ukraine

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ABSTRACT

We investigate Hilbertian Hardy type spaces of complex analytic functions of infinite many variables, associated with compact unitary groups and the corresponding invariant Haar's measures. For such analytic functions we establish a Cauchy type integral formula and describe natural domains. Also we show some relations between constructed spaces of analytic functions and the symmetric Fock space.

1. Introduction

Let Ω be a domain in a complex Banach Space X. The Hardy space $\mathcal{H}^\infty(\Omega)$ which is the uniform algebra of bounded analytic functions on Ω is a standard object of Infinite-Dimensional Complex Analysis and was investigated by many authors (see for example [1–6] and others). However, it is not so clear what are infinite-dimensional analogues of $\mathcal{H}^p(\Omega)$ spaces if $1 \le p < \infty$. In this work we concentrate in an important partial case when $p = 2$ and Ω is a special domain.

Consider the Hardy space $\mathcal{H}^2(\mathsf{B}_n)$ of analytic functions on the open unit Hilbertian ball $\mathsf{B}_n \subset \mathbb{C}^n$ with the unit sphere S_n and the scalar product $\langle \cdot \mid \cdot \rangle_{\mathbb{C}^n}$. The classical Cauchy integral formula

$$f(x) = \int_{\mathsf{S}_n} \frac{f(a)\mathrm{d}a}{(1 - \langle x \mid a \rangle_{\mathbb{C}^n})^n}, \quad x = (x_1, \ldots, x_n) \in \mathsf{B}_n \tag{1}$$

applied for a function $f \in \mathcal{H}^2(\mathsf{B}_n)$ actually describes the following representation

$$f(x) = \left\langle f(\cdot) \,\middle|\, \frac{1}{(1 - \langle x \mid \cdot \rangle_{\mathbb{C}^n})^n} \right\rangle_{\mathcal{H}^2(\mathsf{B}_n)}, \quad x \in \mathsf{B}_n.$$

In other words, the Cauchy integral kernel is a partial case of abstract reproducing kernels and $\mathcal{H}^2(\mathsf{B}_n)$ is a reproducing kernel space (see [7]).

* Corresponding author at: Department of Mathematics and Computer Sciences, Precarpathian National University, 57 Shevchenka Str., 76025 Ivano-Frankivsk, Ukraine. Tel.: +380 955333348.

E-mail addresses: ovlopushansky@hotmail.com (O. Lopushansky), andriyzag@yahoo.com (A. Zagorodnyuk).

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A natural question arise: *What is an analogue of formula* (1) *for infinite-dimensional Banach spaces*? Unfortunately, there is no "canonical" Cauchy formula for the unit ball of infinite-dimensional Hilbert space. Indeed, it is well known in Complex Analysis (see [8]), that polynomials

$$p_n^{(k)}(x) = x_1^{k_1} \cdots x_n^{k_n} \quad \text{with } (k) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$$

form an orthogonal basis in $\mathcal{H}^2(\mathsf{B}_n)$ and

$$\|p_n^{(k)}\|^2_{\mathcal{H}^2(\mathsf{B}_n)} = \frac{(n-1)!k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n - 1)!}.$$

Thus, $\left\|p_n^{(k)}\right\|_{\mathcal{H}^2(\mathsf{B}_n)} \to 0$ if the dimension n approaches infinity.

Consider now the Hardy space $\mathcal{H}^2(\mathsf{B}_n^\infty)$ of analytic functions on the open unit *polydisk*

$$\mathsf{B}_n^\infty = \left\{x = (x_1, \ldots, x_n) \in \mathbb{C}^n \colon |x_j| < 1,\ j = 1, \ldots, n\right\}$$

that is the unit ball in the n-dimensional ℓ^∞-space. The corresponding Cauchy formula

$$f(x) = \int_{\mathsf{S}_n^\infty} f(a) \prod_{i=1}^n \frac{1}{1 - x_i \bar{a}_i}\, \mathrm{d}a_1 \cdots \mathrm{d}a_n, \quad x \in \mathsf{B}_n^\infty$$

with $\mathsf{S}_n^\infty = \left\{z \in \mathbb{C} \colon |z| = 1\right\}$ can be extended to infinite dimensions. In this case polynomials $p_n^{(k)}$ still form an orthogonal basis in $\mathcal{H}^2(\mathsf{B}_n^\infty)$ but $\left\|p_n^{(k)}\right\|_{\mathcal{H}^2(\mathsf{B}_n^\infty)} = 1$ for every $(k) \in \mathbb{Z}_+^n$. Since B_n^∞ is the unit ball of the n-dimensional space ℓ^∞, we can take the unit ball

$$\mathsf{B}^\infty = \left\{x = (x_j) \in \ell^\infty \colon |x_j| < 1,\ j \in \mathbb{N}\right\}$$

as an infinite-dimensional analogue of the polydisk. The condition that polynomials

$$\left\{p_n^{(k)} \colon n \in \mathbb{N},\ (k) \in \mathbb{Z}_+^n\right\}$$

form an *orthonormal* basis uniquely defines a Hilbertian norm on the linear span of this polynomials. However, if we consider the completion of this linear span, we get some analytic functions which are well-defined on a dense subset

$$\ell^1 \bigcap \mathsf{B}^\infty$$

of the unit ball of ℓ^∞. We can still see (cf. [4]) that an appropriated domain for analytic functions belonging to $\mathcal{H}^2(\mathsf{B}^\infty)$ is also

$$\ell^2 \bigcap \mathsf{B}^\infty,$$

where the space ℓ^2 is naturally embedded into ℓ^∞.

These examples suggest us to consider infinite Cartesian products of finite-dimensional balls in a Hilbert space as natural domains for Hardy type classes of analytic functions. Each of these domains has a compact group of unitary operators and we can consider the Haar measure for this group and get an integral representation for analytic functions.

Notice that an another approach to Hilbertian Hardy type classes, being reproducing kernel spaces on infinite-dimensional balls, which generally do not have the form of a polydisk, using the Bishop–De Leeuw theorem about representing measures, has been proposed in [7].

In the given work the case of topological compact groups $\mathfrak{G}$, which look like the countable Cartesian products of full finite-dimensional unitary groups with arbitrary dimensions, acting on a corresponding Hilbert space

$$\mathsf{E} = \ell^2_{\mathfrak{G}},$$

is considered, where $\ell^2_{\mathfrak{G}}$ denotes a Hilbert space naturally associated with all irreducible representations of $\mathfrak{G}$.

In the space $L^2(\mathrm{d}\varsigma)$ of quadratically integrable functions with respect to a $\mathfrak{G}$-invariant probability measure ς a complex closed subspace $\mathcal{H}^2(\mathrm{d}\varsigma)$ generated by the orthogonal basis of homogeneous Hilbert–Schmidt polynomials on E is researched.

We prove that $\mathcal{H}^2(\mathrm{d}\varsigma)$ have a structure of Hardy type space of analytic functions on the open domain

$$\ell^2_{\sqrt{n_r}} \bigcap \mathsf{B}^\infty_{\mathfrak{G}}$$

of the appropriate weighted infinite-dimensional Hilbert space $\ell^2_{\sqrt{n_r}}$, densely embedded in the initial Hilbert space E. A Cauchy type integral formula for analytic extensions on the open domain

$$\ell^2_{\sqrt{n_r}} \bigcap \mathsf{B}^\infty_{\mathfrak{G}}$$

of all function, belonging to $\mathcal{H}^2(\mathrm{d}\varsigma)$, is established in Theorem 7.1.

It is observed also that the Hardy type space $\mathcal{H}^2(\mathrm{d}\varsigma)$ in some sense has the same orthogonal basis, as a Hermitian dual symmetric Fock space F^*, associated with the Hilbert space E. As a consequence, in the case of the infinite-dimensional polydisk group $\mathfrak{G}$, having only 1-dimensional irreducible unitary representations, we obtain that the Hardy type space $\mathcal{H}^2(\mathrm{d}\varsigma)$ is continuously and densely embedded in the dual symmetric Fock space F^*. In the general case we have proved that functions in $\mathcal{H}^2(\mathrm{d}\varsigma)$ agree with functions in F^* on common domains.

2. Preliminaries

Denote by $\mathfrak{U}_r = \mathfrak{U}(n_r)$ the group of all linear unitary operators in an n_r-dimensional complex Hilbert space $\mathbb{C}^{n_r}$ with the scalar product $\langle \cdot \mid \cdot \rangle_{\mathbb{C}^{n_r}}$ and an orthonormal basis

$$\mathcal{E}_r := \left\{ e_{j_r(1)}, \dots, e_{j_r(n_r)} \right\}.$$

A given subsequence $\{n_r : r \in \mathbb{N}\}$ of natural numbers corresponds to the Cartesian product

$$\mathfrak{G} := \mathop{\times}_{r\in\mathbb{N}} \mathfrak{U}_r = \left\{ U = (U_r) : U_r \in \mathfrak{U}_r \right\},$$

endowed with the product's topology, which is an infinite-dimensional compact topological group. As it is well known (see e.g. [9]), the compact group $\mathfrak{G}$ can be unitary represented on the countable orthogonal Hilbertian sum

$$\mathsf{E} := \ell^2_{\mathfrak{G}},$$

$$\ell^2_{\mathfrak{G}} = \bigoplus_{r\in\mathbb{N}} \mathsf{E}_r = \left\{ x = (x_r) : x_r \in \mathbb{C}^{n_r},\ \|x\| := \Big(\sum_{r\in\mathbb{N}} \|x_r\|^2_{\mathbb{C}^{n_r}} \Big)^{1/2} < \infty \right\}$$

with the scalar product $\langle x \mid y \rangle := \sum_r \langle x_r \mid y_r \rangle_{\mathbb{C}^{n_r}}$, where $\left\{ \mathsf{E}_r : r \in \mathbb{N} \right\}$ is a sequence of $\mathfrak{G}$-irreducible subspaces such that

$$\mathsf{E}_r \bigcap \mathsf{E}_s = \{0\} \quad \text{for all } r \neq s$$

and each

$$\mathsf{E}_r \quad \text{is unitary equivalent to } \mathbb{C}^{n_r}$$

for the corresponding $r \in \mathbb{N}$. For simplicity we identify any element $x_r \in \mathbb{C}^{n_r}$ with its range $(0, \dots, 0, x_r, 0, \dots) \in \mathsf{E}$ under the canonical embedding $\mathbb{C}^{n_r} \hookrightarrow \mathsf{E}$. So, we can consider in E the orthonormal basis

$$\mathcal{E} := \bigcup_{r\in\mathbb{N}} \mathcal{E}_r = \left\{ e_j \right\}_{j\in\mathbb{N}}$$

indexed such that $j < i$ for all $e_j \in \mathsf{E}_r$ and $e_i \in \mathsf{E}_{r+1}$. Let

$$\mathsf{B} := \left\{ x \in \mathsf{E} : \|x\| < 1 \right\}$$

denote the open Hilbertian ball in E. We use

$$\mathsf{E}^* = \left\{ x^* := \langle \cdot \mid x \rangle : x \in \mathsf{E} \right\}$$

to denote of the Hermitian dual space.

Let $\otimes^n \mathsf{E}$ be the algebraic tensor product of n copies of E endowed with the scalar product

$$\left\langle x_1 \otimes \cdots \otimes x_n \mid y_1 \otimes \cdots \otimes y_n \right\rangle = \langle x_1 \mid y_1 \rangle \cdots \langle x_n \mid y_n \rangle$$

and $\otimes^n_h \mathsf{E}$ denotes a completion of $\otimes^n \mathsf{E}$ by the Hilbertian norm

$$\|u\|_{\otimes^n_h \mathsf{E}} = \left(\sum_{(j)} |c_{(j)}|^2 \right)^{1/2}, \quad u = \sum_{(j)} c_{(j)}\, e_{j_1} \otimes \cdots \otimes e_{j_n} \in \otimes^n \mathsf{E}$$

with $c_{(j)} \in \mathbb{C}$ and $(j) := (j_1, \dots, j_n) \in \mathbb{N}^n$. It is easy to see that the system

$$\left\{ e_{j_1} \otimes \cdots \otimes e_{j_n} : e_{j_1}, \dots, e_{j_n} \in \mathcal{E},\ (j) \in \mathbb{N}^n \right\}$$

forms an orthonormal basis in $\otimes^n_h \mathsf{E}$. For Hermitian dual Hilbert spaces the natural unitary isometry

$$\left(\otimes^n_h \mathsf{E} \right)^* = \otimes^n_h \mathsf{E}^*$$

is true. Thus every element $u \in \otimes^n_h \mathsf{E}$ uniquely generates a continuous linear form

$$u^* := \langle \cdot \mid u \rangle \in \otimes^n_h \mathsf{E}^*.$$

If $\mathfrak{S}(n) \ni \mathfrak{s} : \{1, \dots, n\} \longmapsto \{\mathfrak{s}(1), \dots, \mathfrak{s}(n)\}$ denotes the group of permutations, then the corresponding symmetrization operator

$$\mathfrak{s}_n : \otimes^n_h \mathsf{E} \ni x_1 \otimes \cdots \otimes x_n \longmapsto x_1 \odot \cdots \odot x_n \in \odot^n_h \mathsf{E}$$

$$x_1 \odot \cdots \odot x_n := \frac{1}{n!} \sum_{\mathfrak{s}\in\mathfrak{S}(n)} x_{\mathfrak{s}(1)} \otimes \cdots \otimes x_{\mathfrak{s}(n)}$$

is a continuous orthogonal projection onto $\odot_h^n \mathsf{E}$. Hence

$$\otimes_h^n \mathsf{E} = \left[\odot_h^n \mathsf{E}\right] \bigoplus \operatorname{Ker} \mathfrak{s}_n,$$

where $\operatorname{Ker} \mathfrak{s}_n$ is the kernel of $\mathfrak{s}_n$. Denote

$$x^{\otimes n} := x \otimes \cdots \otimes x \in \odot_h^n \mathsf{E}, \quad x \in \mathsf{E}.$$

So, for every element $f_n \in \otimes_h^n \mathsf{E}$ there exists a function

$$f_n^*(x) := \left\langle x^{\otimes n} \mid f_n \right\rangle, \quad x \in \mathsf{E},$$

which is usually called an *n-homogeneous Hilbert–Schmidt polynomial* on E. The set of all such polynomials we denote by $\mathscr{P}_h^n(\mathsf{E})$.

Let $[j]$ denote a multi-index

$$(j_1, \ldots, j_n) \in \mathbb{N}^n \quad \text{such that } j_1 \le \cdots \le j_n,$$

and (k) denotes an arbitrary multi-index $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n$. Use the standard notations

$$|(k)| := k_1 + \cdots + k_n \quad \text{and} \quad (k)! := k_1! \cdots k_n!.$$

It is well known (see e.g. [10]) that the elements

$$\mathscr{E}_n := \left\{ e_{[j]}^{\otimes(k)} := e_{j_1}^{\otimes k_1} \odot \cdots \odot e_{j_n}^{\otimes k_n} : \ [j] \in \mathbb{N}^n, (k) \in \mathbb{Z}_+^n, \ |(k)| = n \right\}$$

form an orthogonal basis in $\odot_h^n \mathsf{E}$. If $n = |(k)| = 0$, we set $e_{[j]}^{\otimes(k)} = \mathbf{1}$. So,

$$\mathscr{E}_0 = \{1\} \quad \text{and} \quad \mathscr{E}_1 = \mathcal{E}.$$

Proposition 2.1. *The system of elements*

$$\left\{ \omega_{[j](k)}^{\otimes n}(\varepsilon_1, \ldots, \varepsilon_n) \in \odot_h^n \mathsf{E} \colon |(k)| = n, \ \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \right\}$$

with $[j] = (j_1, \ldots, j_n) \in \mathbb{N}^n$ *and* $(k) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, *where each element*

$$\omega_{[j](k)}(\varepsilon_1, \ldots, \varepsilon_n) := \frac{\varepsilon_1 e_{j_1} + \cdots + \varepsilon_n e_{j_n}}{\sqrt{n}} \in \mathsf{E}$$

with the unit norm is such that any addend e_{j_m} *occurs* k_m *times in the set* $\left\{e_{j_1}, \ldots, e_{j_n}\right\}$, *is total in the space* $\odot_h^n \mathsf{E}$ *for all* $n \in \mathbb{N}$. *As a consequence, the one-to-one adjoint-linear corresponding to*

$$\odot_h^n \mathsf{E} \ni f_n \rightleftarrows f_n^* \in \mathscr{P}_h^n(\mathsf{E})$$

holds.

Proof. By the well-known polarization formula (see e.g. [11]) we obtain

$$e_{[j]}^{\otimes(k)} = \frac{(\sqrt{n})^n}{2^n n!} \sum_{l=1}^{n} \sum_{\varepsilon_l = \pm 1} \varepsilon_1 \cdots \varepsilon_n \, \omega_{[j](k)}^{\otimes n}(\varepsilon_1, \ldots, \varepsilon_n), \quad |(k)| = n.$$

Therefore, if there exists an element $f_n \in \odot_h^n \mathsf{E}$ such that for every multi-index $[j] \in \mathbb{N}^n$, $(k) \in \mathbb{Z}_+^n$

$$\left\langle \omega_{[j](k)}^{\otimes n}(\varepsilon_1, \ldots, \varepsilon_n) \mid f_n \right\rangle_{\otimes_h^n \mathsf{E}} = 0, \quad \text{then } \left\langle e_{[j]}^{\otimes(k)} \mid f_n \right\rangle_{\otimes_h^n \mathsf{E}} = 0.$$

The elements $\left\{e_{[j]}^{\otimes(k)}\right\}$ form a basis in E_h^n, hence $f_n = 0$. So, the set

$$\left\{ \omega_{[j](k)}^{\otimes n}(\varepsilon_1, \ldots, \varepsilon_n) \right\}$$

is total in the space $\odot_h^n \mathsf{E}$. $\square$

Remark 2.2. Using Proposition 2.1 we identify algebraically and topologically the space of Hilbert–Schmidt polynomials $\mathscr{P}_h^n(\mathsf{E})$ with the Hermitian dual Hilbert space

$$\left(\odot_h^n \mathsf{E}\right)^* = \odot_h^n \mathsf{E}^*.$$

In the symmetric Fock space, generated by the Hilbert space E,

$$\mathsf{F} := \bigoplus_{n \in \mathbb{Z}_+} \left[\odot_h^n \mathsf{E}\right], \quad \odot_h^0 \mathsf{E} = \mathbb{C},$$

the system

$$\mathscr{E}_{\mathsf{F}} := \left\{\mathscr{E}_n : n \in \mathbb{Z}_+\right\}$$

forms an orthogonal basis (see e.g. [10, 2.2.2]). We also consider the corresponding basis of Hilbert–Schmidt homogeneous polynomials

$$\mathscr{E}^*_{\mathsf{F}} := \left\{\mathscr{E}^*_n : n \in \mathbb{Z}_+\right\},$$

$$\mathscr{E}^*_n := \left\{e^{*(k)}_{[j]} = e^{*k_1}_{j_1} \cdots e^{*k_n}_{j_n} \in \mathcal{P}^n_h(\mathsf{E}) : e^{\otimes(k)}_{[j]} \in \mathscr{E}_n,\ |(k)| = n\right\},$$

generated by the Riesz involution $x \longmapsto x^*$ on E. Clearly such polynomials form an orthogonal basis in the Hermitian dual symmetric Fock space F^*.

3. Representing invariant measures

Consider the Banach space

$$\ell^\infty_{\mathfrak{G}} = \left\{x = (x_r) \in \bigtimes_{r\in\mathbb{N}} \mathbb{C}^{n_r} : \|x\|_{\ell^\infty_{\mathfrak{G}}} = \sup_{r\in\mathbb{N}} \|x_r\|_{\mathbb{C}^{n_r}} < \infty\right\}$$

and the compact metric spaces

$$\mathsf{S}^\infty_{\mathfrak{G}} := \bigtimes_{r\in\mathbb{N}} \mathsf{S}_r, \qquad \mathsf{S}_r := \left\{x_r \in \mathbb{C}^{n_r} : \|x_r\|_{\mathbb{C}^{n_r}} = 1\right\},$$

$$\mathsf{K}^\infty_{\mathfrak{G}} := \bigtimes_{r\in\mathbb{N}} \mathsf{K}_r, \qquad \mathsf{K}_r := \left\{x_r \in \mathbb{C}^{n_r} : \|x_r\|_{\mathbb{C}^{n_r}} \le 1\right\},$$

endowed with the product topologies. It is easy to see that $\mathsf{K}^\infty_{\mathfrak{G}}$ coincides with a norm closed unit ball of $\ell^\infty_{\mathfrak{G}}$ endowed with the weak-star topology. The contractive embedding

$$\mathsf{E} \hookrightarrow \ell^\infty_{\mathfrak{G}}, \qquad \|x\|_{\ell^\infty_{\mathfrak{G}}} \le \|x\|, \quad x \in \mathsf{E}$$

holds. A norm open unit ball in $\ell^\infty_{\mathfrak{G}}$ we denote by

$$\mathsf{B}^\infty_{\mathfrak{G}} := \left\{x \in \ell^\infty_{\mathfrak{G}} : \|x\|_{\ell^\infty_{\mathfrak{G}}} < 1\right\}.$$

As well we consider the uniform algebra of all continuous complex functions ψ on $\mathsf{K}^\infty_{\mathfrak{G}}$,

$$C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right) \quad \text{endowed with the norm } \|\psi\|_{C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)} = \sup_{x\in\mathsf{K}^\infty_{\mathfrak{G}}} |\psi(x)|.$$

Clearly $C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ contains the unity function $\mathbf{1}_{\mathsf{K}^\infty_{\mathfrak{G}}}$.

Remark 3.1. Note that each linear functional $e^*_j \in \mathscr{E}^*_{\mathsf{F}}$ can be uniquely extended to a weakly-star continuous linear functional on $\ell^\infty_{\mathfrak{G}}$ which we denote by the same symbol. Up to this extension we can write

$$\mathscr{E}^*_{\mathsf{F}} \subset C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right).$$

Let $A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ be a closure in $C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ of the complex linear span of extended Hilbert–Schmidt polynomials $\mathscr{E}^*_{\mathsf{F}}$. Clearly, $A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ is a uniform subalgebra in the algebra $C\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$.

Recall that an element $x \in \mathsf{K}^\infty_{\mathfrak{G}}$ is a peak point if there is a function $f \in A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ such that $f(x) = 1$ and $|f(y)| < 1$ for all $y \in \mathsf{K}^\infty_{\mathfrak{G}}$ if $y \ne x$.

Proposition 3.2. *Let $\mathfrak{P}$ be the set of peak points and ∂A denotes the Choquet boundary of $A(\mathsf{K}^\infty_{\mathfrak{G}})$. Then*

$$\mathsf{S}^\infty_{\mathfrak{G}} = \mathfrak{P} = \partial A.$$

For the uniform algebra

$$A(\mathsf{S}^\infty_{\mathfrak{G}}) := A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)|_{\mathsf{S}^\infty_{\mathfrak{G}}}$$

endowed with the uniform norm $\sup_{x\in\mathsf{S}^\infty_{\mathfrak{G}}} |f(x)|$ the isometry

$$A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right) = A(\mathsf{S}^\infty_{\mathfrak{G}}) \tag{2}$$

holds.

Proof. From [1] (see also [6]) it follows that the algebra $A(\mathsf{K}^\infty_{\mathfrak{G}})$, as a uniform closure of the linear span of finite type continuous polynomials, consists of complex analytic functions in the norm open unit ball $\mathsf{B}^\infty_{\mathfrak{G}}$ having the form

$$\mathsf{B}^\infty_{\mathfrak{G}} = \bigtimes_{r\in\mathbb{N}} \mathsf{B}_r, \qquad \mathsf{B}_r := \{x_r \in \mathbb{C}^{n_r} : \|x_r\|_{\mathbb{C}^{n_r}} < 1\}.$$

Hence if $a \in \mathfrak{P}$, then $a \in \mathsf{S}^\infty_{\mathfrak{G}}$ via the Maximum Principle for analytic functions. Therefore, $\mathfrak{P} \subset \mathsf{S}^\infty_{\mathfrak{G}}$.

In the other hand, for every fixed $a_r \in \mathsf{S}_r$ there exists an analytic in B_r and continuous on B_r function f_r such that

$$f_r(a_r) = 1, \quad \text{and} \quad |f_r(x_r)| < 1 \quad \text{for all } x_r \in \mathsf{K}_r \setminus \{a_r\}$$

(see e.g. [12]). Then for each $a \in \mathsf{S}^\infty_{\mathfrak{G}}$ such that the natural projection of a onto the subspace $\mathsf{E}_r \simeq \mathbb{C}^{n_r}$ is equal to a fixed a_r, the analytic function

$$f = f_r \cdot \prod_{j \in \mathbb{N} \setminus \{r\}} \mathbf{1}_j \in A\big(\mathsf{K}^\infty_{\mathfrak{G}}\big)$$

satisfies the conditions

$$f(a) = 1, \quad \text{and} \quad |f(x)| < 1 \quad \text{for all } x \in \mathsf{K}^\infty_{\mathfrak{G}} \setminus \{a\},$$

where $\mathbf{1}_j$ denotes the identically unit function on K_j. Hence $a \in \mathfrak{P}$ and the embedding $\mathsf{S}^\infty_{\mathfrak{G}} \subset \mathfrak{P}$ are proved. Therefore, $\mathfrak{P} = \mathsf{S}^\infty_{\mathfrak{G}}$ and so we have the isometrical isomorphism (2).

Finally, since $\mathsf{K}_{\ell^\infty_{\mathfrak{G}}}$ is a compact metric separable space, the sets of peak points $\mathfrak{P}$ of $A\big(\mathsf{K}^\infty_{\mathfrak{G}}\big)$ are a G_δ-subset in $\mathsf{K}^\infty_{\mathfrak{G}}$. Hence the equality $\partial A = \mathfrak{P}$ is also true [12, II.11.2]. □

As is well-known there exists a probability Haar measure χ on the group $\mathfrak{G}$ (respectively, there exists a probability Haar measure on $\mathfrak{U}_r$, χ_r) for which

$$\chi(\phi) := \int_{\mathfrak{G}} \phi(U)\mathrm{d}\chi(U) = \int_{\mathfrak{G}} \phi(VU)\mathrm{d}\chi(U) = \int_{\mathfrak{G}} \phi(UV)\mathrm{d}\chi(U)$$

with all $U, V \in \mathfrak{G}$ and $\phi \in C(\mathfrak{G})$ such that $\chi(\mathfrak{G}) = \|\chi\|$, where $C(\mathfrak{G})$ stands for the uniform algebra of continuous complex functions on $\mathfrak{G}$ (similarly for the measure χ_r and the uniform algebra $C(\mathfrak{U}_r)$).

The unitary group $\mathfrak{G}$ on the compact set $\mathsf{S}^\infty_{\mathfrak{G}}$ acts continuously. This group generates a group of linear operators on the algebra $C(\mathsf{S}^\infty_{\mathfrak{G}})$:

$$C(\mathsf{S}^\infty_{\mathfrak{G}}) \ni \varphi \longmapsto \varphi \circ U.$$

For a fixed $a \in \mathsf{S}^\infty_{\mathfrak{G}}$ the mapping $\mathfrak{G} \ni U \longmapsto Ua \in \mathsf{S}^\infty_{\mathfrak{G}}$ is continuous and surjective. Hence, the function $U \longmapsto (\varphi \circ U)(a)$ belongs to $C(\mathfrak{G})$ for all $\varphi \in C(\mathsf{S}^\infty_{\mathfrak{G}})$. Therefore,

$$\sup_{U \in \mathfrak{G}} |(\varphi \circ U)(a)| = \sup_{x \in \mathsf{S}^\infty_{\mathfrak{G}}} |\varphi(x)|.$$

The Riesz representation theorem implies that the Haar measure χ uniquely defines a probability $\mathfrak{G}$-invariant measure ς on the $\mathfrak{G}$-orbit $\mathsf{S}^\infty_{\mathfrak{G}} = \{Ua : U \in \mathfrak{G}\}$ by the formula

$$\varsigma(\varphi) := \int_{\mathsf{S}^\infty_{\mathfrak{G}}} \varphi \mathrm{d}\varsigma = \int_{\mathfrak{G}} \varphi(Ua)\mathrm{d}\chi(U), \quad \varphi \in C(\mathsf{S}^\infty_{\mathfrak{G}}), \tag{3}$$

where ς does not depend on a via the transitivity of $\mathfrak{G}$ on the $\mathfrak{G}$-orbit. Recall that a probability measure ς on $\mathsf{S}^\infty_{\mathfrak{G}}$ is $\mathfrak{G}$-invariant, if ς satisfies the relation $\varsigma = \varsigma \circ U$ for all $U \in \mathfrak{G}$.

For a given E_r let

$$\mathsf{E}_r^\perp := \{x = (x_m) \in \ell^\infty_{\mathfrak{G}} : x_r = 0\}.$$

Then

$$\mathsf{E}_r^\perp \oplus \mathsf{E}_r = \ell^\infty_{\mathfrak{G}}$$

and for every $a \in \ell^\infty_{\mathfrak{G}}$ we have $a = a_r^\perp + a_r$, where $a_r \in \mathsf{E}_r$, $a_r^\perp \in \mathsf{E}_r^\perp$ and $a \longmapsto a_r$ is a continuous projection.

For a given n_r-dimensional subgroup $\mathfrak{U}_r$ the mappings

$$a \longmapsto U_r(a_r) + a_r^\perp \quad \text{with } U_r \in \mathfrak{U}_r$$

generate linear operators

$$T_r\varphi(a) := \varphi(U_r(a_r) + a_r^\perp), \quad \varphi \in C\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$$

acting in $C\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$. We will use the following useful formulas.

Proposition 3.3. *For any $r_1, \ldots, r_m \in N$ the equality*

$$\int_{\mathsf{S}^\infty_{\mathfrak{G}}} \varphi \mathrm{d}\varsigma = \int_{S^\infty_{\mathfrak{G}}} \mathrm{d}\varsigma(a) \prod_{i=1}^m \int_{\mathfrak{U}_{r_i}} T_{r_i}\varphi(a)\mathrm{d}\chi_{r_i}(U_{r_i}), \quad \varphi \in C\big(\mathsf{S}^\infty_{\mathfrak{G}}\big) \tag{4}$$

holds. Thereto, for any compact subgroup $\mathfrak{G}_0 \subset \mathfrak{G}$ with the probability Haar measure ς_0 the equality

$$\int_{\mathsf{S}^\infty_{\mathfrak{G}}} \varphi \mathrm{d}\varsigma = \int_{S^\infty_{\mathfrak{G}}} \mathrm{d}\varsigma(a) \int_{\mathfrak{G}_0} \varphi(Ua)\mathrm{d}\varsigma_0(U), \quad \varphi \in C\left(\mathsf{S}^\infty_{\mathfrak{G}}\right) \tag{5}$$

holds.

Proof. For each $\varphi \in C\left(\mathsf{S}^\infty_{\mathfrak{G}}\right)$ the function

$$(a, U_{r_1}, \dots, U_{r_m}) \longmapsto T_{r_1} \cdots T_{r_m}\varphi(a)$$

is continuous on the Cartesian product $\mathsf{S}^\infty_{\mathfrak{G}} \times \mathfrak{U}_{r_1} \times \cdots \times \mathfrak{U}_{r_m}$. By the Fubini theorem, we have

$$\int_{\mathsf{S}^\infty_{\mathfrak{G}}} \mathrm{d}\varsigma(a) \prod_{i=1}^m \int_{\mathfrak{U}_{r_i}} T_{r_i}\varphi(a)\mathrm{d}\chi_{r_i} = \prod_{i=1}^m \int_{\mathfrak{U}_{r_i}} \mathrm{d}\chi_{r_i} \int_{\mathsf{S}^\infty_{\mathfrak{G}}} T_{r_i}\varphi(a)\mathrm{d}\varsigma(a).$$

However, the internal integral on the right side does not depend on $T_{r_1}, \dots, T_{r_m}$. Therefore, taking into account that

$$\int_{\mathfrak{U}_{r_i}} \mathrm{d}\chi_{r_i} = 1,$$

we obtain (4). The formula (5) can be proved similarly. □

Proposition 3.4. *The $\mathfrak{G}$-invariant measure ς represents the character $\delta_0(f) = f(0)$ of the algebra $A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ i.e. it satisfies the following relation*

$$\delta_0(f) = \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f\mathrm{d}\varsigma, \quad f \in A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right). \tag{6}$$

Proof. By formula (5) for any $e^{*(k)}_{[j]} \in \mathscr{E}^*_n$ we obtain

$$\begin{aligned}\int_{\mathsf{S}^\infty_{\mathfrak{G}}} e^{*(k)}_{[j]}\mathrm{d}\varsigma &= \frac{1}{2\pi} \int_{\mathsf{S}^\infty_{\mathfrak{G}}} \mathrm{d}\varsigma(a) \int_{-\pi}^{\pi} e^{*(k)}_{[j]}(\exp(\mathrm{i}\vartheta)a)\mathrm{d}\vartheta \\ &= \frac{1}{2\pi} \int_{\mathsf{S}^\infty_{\mathfrak{G}}} e^{*(k)}_{[j]}(a)\mathrm{d}\varsigma(a) \int_{-\pi}^{\pi} \exp(\mathrm{i}n\vartheta)a\mathrm{d}\vartheta \\ &= \begin{cases} 0: & n \neq 0 \\ 1: & n = 0. \end{cases}\end{aligned}$$

Uniformly approaching any function $f \in A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ by polynomials $\mathscr{E}^*_{\mathsf{F}}$ and using the linearity and continuity on $A\left(\mathsf{K}^\infty_{\mathfrak{G}}\right)$ of the integral with the measure ς, we come to (6). □

4. A Hardy type space

Let the probability $\mathfrak{G}$-invariant measure ς, defined by the formula (3), be given. The functional

$$\|f\|_{L^2_\varsigma} = \left(\int_{\mathsf{S}^\infty_{\mathfrak{G}}} |f|^2\mathrm{d}\varsigma\right)^{1/2}, \quad f \in C(\mathsf{S}^\infty_{\mathfrak{G}})$$

is a Hilbertian norm on the space $C(\mathsf{S}^\infty_{\mathfrak{G}})$. Indeed, let f be a nonzero function in $C(\mathsf{S}^\infty_{\mathfrak{G}})$ and $\varphi := |f|^2$. If we suppose that $\varsigma(\varphi) = 0$ then (3) implies that for a fixed $a \in \mathsf{S}^\infty_{\mathfrak{G}}$,

$$(\varphi \circ U)(a) \overset{\text{a.e.}}{=} 0, \quad U \in \mathfrak{G}$$

with respect to the Haar measure χ defined on $\mathfrak{G}$. The function

$$U \longmapsto (\varphi \circ U)(a)$$

belongs to $C(\mathfrak{G})$, hence $(\varphi \circ U)(a) \equiv 0$ as a function of $U \in \mathfrak{G}$. Since the mapping

$$\mathfrak{G} \ni U \longmapsto U(a) \in \mathsf{S}^\infty_{\mathfrak{G}}$$

is surjective, we obtain $\varphi \equiv 0$ on $\mathsf{S}^\infty_{\mathfrak{G}}$. Consequently $f \equiv 0$ on $\mathsf{S}^\infty_{\mathfrak{G}}$ which contradicts the assumption.

Consider the Hilbert space $L^2_\varsigma = L^2(\mathrm{d}\varsigma)$ of all quadratically ς-integrable complex functions with the scalar product

$$\langle f \mid g\rangle_{L^2_\varsigma} = \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f\bar{g}\mathrm{d}\varsigma, \quad f, g \in L^2(\mathrm{d}\varsigma).$$

Definition 4.1. *The Hardy type* space $\mathcal{H}^2(\mathrm{d}\varsigma)$ we define as a closure of the algebra $A(\mathsf{K}_{\mathfrak{G}}^{\infty})$ (or $A\big(\mathsf{S}_{\mathfrak{G}}^{\infty}\big)$, that is the same) in the space $L^2(\mathrm{d}\varsigma)$ endowed with the L_{ς}^2-norm.

Note that the embedding $A\big(\mathsf{K}_{\mathfrak{G}}^{\infty}\big) \hookrightarrow \mathcal{H}^2(\mathrm{d}\varsigma)$ is continuous, since

$$\|f\|_{L_{\varsigma}^2} \le \|f\|_{C\left(\mathsf{S}_{\mathfrak{G}}^{\infty}\right)}, \quad f \in A\big(\mathsf{K}_{\mathfrak{G}}^{\infty}\big). \tag{7}$$

For a fixed $n \in \mathbb{Z}_+$ let $\mathcal{H}_n^2$ be a closure in the space $L^2(\mathrm{d}\varsigma)$ of the complex linear span of homogenous Hilbert–Schmidt polynomials $\mathscr{E}_n^*$ (extended to $\mathsf{K}_{\mathfrak{G}}^{\infty}$) and $\mathcal{H}_0^2 := \mathbb{C}$.

Theorem 4.2. *The sequence of homogeneous Hilbert–Schmidt polynomials $\mathscr{E}_{\mathsf{F}}^*$ forms an orthogonal basis in $\mathcal{H}^2(\mathrm{d}\varsigma)$. In particular, the subsequence $\mathscr{E}_n^*$ forms an orthogonal basis in $\mathcal{H}_n^2$ for any $n \in \mathbb{Z}_+$ and $\mathcal{H}_n^2 \perp \mathcal{H}_m^2$ in $\mathcal{H}^2(\mathrm{d}\varsigma)$ whenever $n \neq m$.*

Proof. Every element $a = \sum_{i\in\mathbb{N}} e_i^*(a)e_i \in \ell_{\mathfrak{G}}^{\infty}$ can be written as

$$a = a_s^{\perp} + e_s^*(a)e_s,$$

where $a_s^{\perp}$ denotes a projection of a onto the complementing subspace

$$e_s^{\perp} := \big\{a \in \ell_{\mathfrak{G}}^{\infty} : e_s^*(a) = 0\big\}, \quad s \in \mathbb{N}.$$

Consider the 1-dimensional subgroups in $\mathfrak{G}$ of linear transformations

$$U_s(\vartheta)a := \exp(\mathrm{i}\vartheta)e_s^*(a)e_s + a_s^{\perp}, \qquad U_0(\vartheta)a := \exp(\mathrm{i}\vartheta)a$$

with $a \in \mathsf{S}_{\mathfrak{G}}^{\infty}$ and $\vartheta \in [-\pi, \pi]$. We assign to these transformations 1-parameter groups of linear operators on $C\big(\mathsf{S}_{\mathfrak{G}}^{\infty}\big)$

$$\vartheta \longmapsto T_s(\vartheta)f := f\big(U_s(\vartheta)a\big), \qquad \vartheta \longmapsto T_0(\vartheta)f := f\big(U_0(\vartheta)a\big)$$

with $f \in C\big(\mathsf{S}_{\mathfrak{G}}^{\infty}\big)$ and $a \in \mathsf{S}_{\mathfrak{G}}^{\infty}$. Formulas (4) and (5) imply that

$$\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} f\,\mathrm{d}\varsigma = \frac{1}{2\pi}\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} \mathrm{d}\varsigma(a) \int_{-\pi}^{\pi} T(\vartheta)f(a)\,\mathrm{d}\vartheta \tag{8}$$

for any $T \in \big\{T_0, T_s : s \in \mathbb{N}\big\}$. If $|(k)| \neq |(l)|$, then from (8) it follows that

$$\begin{aligned}\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} e_{[j]}^{*(k)} \cdot \bar{e}_{[i]}^{*(l)}\,\mathrm{d}\varsigma &= \int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} e_{[j]}^{*(k)}\big(\exp(\mathrm{i}\vartheta)a\big)\bar{e}_{[i]}^{*(l)}\big(\exp(\mathrm{i}\vartheta)a\big)\mathrm{d}\varsigma(a)\\ &= \frac{1}{2\pi}\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} e_{[j]}^{*(k)} \cdot \bar{e}_{[i]}^{*(l)}\,\mathrm{d}\varsigma \int_{-\pi}^{\pi} \exp\big(\mathrm{i}(|(k)| - |(l)|)\vartheta\big)\mathrm{d}\vartheta = 0.\end{aligned}$$

So, $e_{[j]}^{*(k)} \perp e_{[i]}^{*(l)}$ in $L^2(\mathrm{d}\varsigma)$ if $|(k)| \neq |(l)|$ for all $[j], [i] \in \mathbb{N}^n$.

If $|(k)| = |(l)|$ and the corresponding elements $e_{[j]}^{*(k)}$ with $[j] = (j_1, \ldots, j_n)$ and $e_{[i]}^{*(l)}$ with $[i] = (i_1, \ldots, i_m)$ are different, then there exists an index $j_s \in \big\{j_1, \ldots, j_n\big\}$ such that $j_s \notin \big\{i_1, \ldots, i_m\big\}$. Now (8) implies that

$$\begin{aligned}\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} e_{[j]}^{*(k)} \cdot \bar{e}_{[i]}^{*(l)}\,\mathrm{d}\varsigma &= \int T_{j_s}(\vartheta)e_{[j]}^{*(k)} \cdot \overline{T_{j_s}(\vartheta)e_{[i]}^{*(l)}}\,\mathrm{d}\varsigma\\ &= \frac{1}{2\pi}\int_{\mathsf{S}_{\mathfrak{G}}^{\infty}} e_{[j]}^{*(k)} \cdot \bar{e}_{[i]}^{*(l)}\,\mathrm{d}\varsigma \int_{-\pi}^{\pi} \exp\big(\mathrm{i}k_s\vartheta\big)\mathrm{d}\vartheta = 0,\end{aligned}$$

hence, $e_{[i]}^{*(l)} \perp e_{[j]}^{*(k)}$ in $L^2(\mathrm{d}\varsigma)$ too. □

Further we use the following notations.

Let $[j]_r := \big(j_{r(1)}, \ldots, j_{r(n_r)}\big) \in \mathbb{N}^{n_r}$ denote a sub-index of the multi-index $[j] = (j_1, \ldots, j_n) \in \mathbb{N}^n$ such that $j_{r(1)} \le \cdots \le j_{r(n_r)}$.

Let $(k)_r := \big(k_{r(1)}, \ldots, k_{r(n_r)}\big) \in \mathbb{Z}_+^{n_r}$ stand for a sub-index of the index $(k) = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ with $n_r \le n$.

As is usual, $|(k)_r| := k_{r(1)} + \cdots + k_{r(n_r)}$ and $(k)_r! := k_{r(1)}! \cdots k_{r(n_r)}!$.

Theorem 4.3. *If a Hilbert–Schmidt polynomial*

$$e_{[j]}^{*(k)} = e_{j_1}^{*k_1} \cdots e_{j_n}^{*k_n} \in \mathscr{E}_n^*, \quad [j] \in \mathbb{N}^n,\ (k) \in \mathbb{Z}_+^n,\ |(k)| = n$$

is of the form

$$e^{*(k)}_{[j]} = e^{*(k)_{r(1)}}_{[j]_{r(1)}} \cdots e^{*(k)_{r(t)}}_{[j]_{r(t)}}$$

with the block-indices $[j] = \big([j]_{r(1)}, \ldots, [j]_{r(t)}\big)$ *and* $(k) = \big((k)_{r(1)}, \ldots, (k)_{r(t)}\big)$ *such that*

$$n_{r(1)} + \cdots + n_{r(t)} = n,$$

where $\big\{e_{j_{r(1)}}, \ldots, e_{j_{r(n_r)}}\big\} = \big\{e_{j_1}, \ldots, e_{j_n}\big\} \bigcap \mathsf{E}_r$ *and* $t \in \mathbb{N}$ *stands for the number of all such sub-indices in* $[j]$*, then*

$$\Big\| e^{*(k)}_{[j]} \Big\|^2_{L^2_\varsigma} = \prod_{r \in \{r_1, \ldots, r_t\}} \frac{(n_r - 1)!(k)_r!}{(n_r - 1 + |(k)_r|)!}. \tag{9}$$

Proof. Use that

$$T_r \left| e^{*(k)_r}_{[j]_r} \right|^2 (a) = \left| e^{*(k)_r}_{[j]_r} \right|^2 \big(U_r(a_r)\big)$$

for any $a = (a_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$ with $a_r \in \mathsf{S}_r$. As is known [8, 1.4.9],

$$\begin{aligned}\int_{\mathfrak{U}_r} T_r \left| e^{*(k)_r}_{[j]_r} \right|^2 (a) \mathrm{d}\chi_r &= \int_{\mathfrak{U}_r} \left| e^{*(k)_r}_{[j]_r} \right|^2 \big(U_r(a_r)\big) \mathrm{d}\chi_r(U_r) \\ &= \frac{(n_r - 1)!(k)_r!}{(n_r - 1 + |(k)_r|)!}\end{aligned}$$

with the Haar measure χ_r on $\mathfrak{U}_r$. Thus formula (4) immediately implies (9). □

5. A Cauchy type kernel

Let us define the following auxiliary Banach space, associated with $\mathfrak{G}$,

$$\ell^1_{n_r} := \left\{ x = (x_r) \in \mathop{\times}_{r \in \mathbb{N}} \mathbb{C}^{n_r} : \|x\|_{\ell^1_{n_r}} := \sum_{r \in \mathbb{N}} n_r \|x_r\|_{\mathbb{C}^{n_r}} < \infty \right\}.$$

Note that $e_j \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r}$ for all $j \in \mathbb{N}$ and the group

$$\mathfrak{G} \ni U \longrightarrow Ux = \big(U_r x_r\big)_{r \in \mathbb{N}}$$

acts isometrically in both $\ell^\infty_{\mathfrak{G}}$ and $\ell^1_{n_r}$. Since the embedding

$$\ell^1_{n_r} \hookrightarrow \ell^\infty_{\mathfrak{G}}$$

is continuous, the set $\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r}$ is open and the set $\mathsf{K}^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r}$ is closed in $\ell^1_{n_r}$.

Let us examine a Cauchy type kernel

$$\mathfrak{C}(x, a) := \prod_{r \in \mathbb{N}} \frac{1}{(1 - \langle x_r \mid a_r \rangle_{\mathbb{C}^{n_r}})^{n_r}}, \quad a \in \mathsf{S}^\infty_{\mathfrak{G}} \tag{10}$$

which is a priori a Gâteaux analytic mapping of x running over the finitely open set $\bigcup_{r \in \mathbb{N}} \mathsf{B}_1 \times \cdots \times \mathsf{B}_r$ with values in the uniform algebra $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$.

Proposition 5.1. *The Cauchy type kernel* $\mathfrak{C}$ *is a well defined analytic* $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$*-values mapping*

$$\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r} \ni x \longmapsto \mathfrak{C}(x, a), \quad a \in \mathsf{S}^\infty_{\mathfrak{G}}.$$

Proof. For every $\varrho \in (0, 1)$ the series

$$\ln(1 - \zeta)^{-r} = -r \sum_{n \in \mathbb{N}} \frac{\zeta^n}{n} = -r\zeta \sum_{n \in \mathbb{N}} \frac{\zeta^{n-1}}{n}, \quad r \in \mathbb{N}$$

is convergent absolutely for all $|\zeta| \le \varrho$. Therefore the estimation

$$\left| \ln(1 - \zeta)^{-r} \right| \le c_\varrho r |\zeta| \quad \text{for all } |\zeta| \le \varrho \tag{11}$$

with

$$c_{\varrho} := \sum_{n\in\mathbb{N}} \frac{\varrho^{n-1}}{n} < \infty$$

holds. Denote by $\mathsf{B}^1_{n_r}$ and $\mathsf{K}^1_{n_r}$ the open and the closed unit balls in the space $\ell^1_{n_r}$, respectively. Consider the following 1-parametric families of sets

$$\mathsf{K}_\varepsilon := \varepsilon \mathsf{K}^\infty_{\mathfrak{G}} \bigcap \frac{1}{1-\varepsilon}\mathsf{K}^1_{n_r}, \qquad \mathsf{B}_\varepsilon := \varepsilon \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \frac{1}{1-\varepsilon}\mathsf{B}^1_{n_r}$$

with $\varepsilon \in (0, 1)$. Clearly, K_ε and B_ε are closed and open sets in $\ell^1_{n_r}$ respectively because the embedding $\ell^1_{n_r} \hookrightarrow \ell^\infty_{\mathfrak{G}}$ is continuous.

Let $x = (x_r) \in \mathsf{K}_\varepsilon$ and $a = (a_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$. Then we obtain for instance

$$\sup_{\|a_r\|=1} |\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}}| \le \frac{\varepsilon}{n_r} < 1.$$

Hence, the inequality (11) implies

$$\sum_{r\in\mathbb{N}} \left| \ln \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \right| \le \sum_{r\in\mathbb{N}} c_{\varrho(r)} n_r \,|\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}}|$$

with $\varrho(r) := \frac{\varepsilon}{n_r}$. Since $c_{\varrho(r)} \le c_\varepsilon$ for any $r \in \mathbb{N}$, it follows that

$$\begin{aligned}
\sup_{\|a\|_{\ell^\infty_{\mathfrak{G}}}=1} \sum_{r\in\mathbb{N}} \left| \ln \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \right| &\le c_\varepsilon \sup_{\|a\|_{\ell^\infty_{\mathfrak{G}}}=1} \sum_{r\in\mathbb{N}} n_r\,|\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}}| \\
&\le c_\varepsilon \sum_{r\in\mathbb{N}} n_r \sup_{\|a_r\|_{\mathbb{C}^{n_r}}=1} |\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}}| \\
&= c_\varepsilon \sum_{r\in\mathbb{N}} n_r \|x_r\|_{\mathbb{C}^{n_r}} = c_\varepsilon \|x\|_{\ell^1_{n_r}}.
\end{aligned}$$

Consequently, the series of $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-values functions

$$\mathsf{K}_\varepsilon \ni x \longmapsto \sum_{r\in\mathbb{N}} \ln \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \in A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big), \quad a \in \mathsf{S}^\infty_{\mathfrak{G}}, \tag{12}$$

converges absolutely and uniformly on K_ε. Hence, its sum represents a bounded continuous $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-values function on K_ε for any $\varepsilon \in (0, 1)$. Moreover, one is Gâteaux-analytic in the open domain B_ε since its restriction to any 1-dimensional affine subspace is obviously analytic. Thus, the function (12) is analytic in B_ε. By the analyticity of the exponential function, the following map

$$\mathsf{K}_\varepsilon \ni x \longmapsto \mathfrak{C}_\varepsilon(x, a) := \exp \sum_{r\in\mathbb{N}} \ln \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \in A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big) \tag{13}$$

is a bounded continuous function, which is analytic on B_ε. As it is easy to see, for any $a \in \mathsf{S}^\infty_{\mathfrak{G}}$ and $x \in \mathsf{B}_\varepsilon$ we have

$$\mathfrak{C}_\varepsilon(x, a) = \prod_{r\in\mathbb{N}} \frac{1}{(1-\langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}}.$$

If $\varepsilon_1 < \varepsilon_2$ then $\mathsf{B}_{\varepsilon_1} \subset \mathsf{B}_{\varepsilon_2}$ and the function $\mathfrak{C}_{\varepsilon_1}$ defined on $\mathsf{B}_{\varepsilon_1}$ has a unique analytic extension $\mathfrak{C}_{\varepsilon_2}$ on $\mathsf{B}_{\varepsilon_2}$ such that

$$\mathfrak{C}_{\varepsilon_2}|_{\mathsf{B}_{\varepsilon_1}} = \mathfrak{C}_{\varepsilon_1},$$

by virtue of Uniqueness Principle for analytic functions. Therefore, the Cauchy kernel $\mathfrak{C}$ defined by (10), has a unique $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-value analytic extension on the open domain

$$\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r} = \bigcup_{\varepsilon\in(0,1)} \mathsf{B}_\varepsilon,$$

which we also denote by $\mathfrak{C}$. $\square$

6. A Hilbertian extension of the Cauchy type kernel

Now we define a Hilbert space, associated with $\mathfrak{G}$,

$$\ell^2_{\sqrt{n_r}} := \left\{ x = (x_r) \in \mathop{\times}_{r\in\mathbb{N}} \mathbb{C}^{n_r} : \|x\|_{\ell^2_{\sqrt{n_r}}} := \left(\sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} \right)^{1/2} < \infty \right\},$$

where the group $\mathfrak{G}$ acts isometrically. The inequality

$$\sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} \le \left(\sum_{r\in\mathbb{N}} \sqrt{n_r} \|x_r\|_{\mathbb{C}^{n_r}} \right)^2 \le \left(\sum_{r\in\mathbb{N}} n_r \|x_r\|_{\mathbb{C}^{n_r}} \right)^2$$

implies that the following continuous embedding is true

$$\ell^1_{n_r} \hookrightarrow \ell^2_{\sqrt{n_r}}.$$

Proposition 6.1. *The Cauchy kernel $\mathfrak{C}$ possesses a unique analytic $A\big(\mathsf{S}_{\ell^\infty_{\mathfrak{G}}}\big)$-values extension*

$$\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \mathfrak{C}(x, a), \quad a \in \mathsf{S}^\infty_{\mathfrak{G}}.$$

Proof. Let $x = (x_r) \in \ell^\infty_{\mathfrak{G}}$ with $x_r \in \mathsf{E}_r$ and we denote

$$\hat{x} := (\hat{x}_r) \quad \text{with}\, \hat{x}_r := \upsilon_r x_r,\, \upsilon_r = \frac{1}{\sqrt{n_r 2^r}}.$$

Note that if $x = (x_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$ with $x_r \in \mathsf{S}_r$ then $\hat{x} \in \ell^2_{\sqrt{n_r}}$ and

$$\|\hat{x}\|^2_{\ell^2_{\sqrt{n_r}}} = \sum_{r\in\mathbb{N}} \frac{1}{2^r} \|x_r\|^2_{\mathbb{C}^{n_r}} = 1.$$

Consider the linear mapping

$$\hat{\upsilon} \colon \ell^\infty_{\mathfrak{G}} \ni x \longmapsto \hat{x} \in \ell^2_{\sqrt{n_r}}.$$

The mapping $\hat{\upsilon}$ is continuous, since

$$\|\hat{x}\|^2_{\ell^2_{\sqrt{n_r}}} = \sum_{r\in\mathbb{N}} \frac{1}{2^r} \|x_r\|^2_{\mathbb{C}^{n_r}} \le \|x\|^2_{\ell^\infty_{\mathfrak{G}}}.$$

Moreover, from

$$\|x\|^2_{\ell^\infty_{\mathfrak{G}}} = \sup_{r\in\mathbb{N}} \|x_r\|^2_{\mathbb{C}^{n_r}} \le \sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} = \|x\|^2_{\ell^2_{\sqrt{n_r}}}$$

we come to the continuous embedding

$$\ell^2_{\sqrt{n_r}} \hookrightarrow \ell^\infty_{\mathfrak{G}}.$$

Note that the restriction $\hat{\upsilon}\,|_{\ell^2_{\sqrt{n_r}}}$ maps continuously from $\ell^2_{\sqrt{n_r}}$ into $\ell^1_{n_r}$. In fact, from the Cauchy–Schwartz Inequality it follows that

$$\|\hat{x}\|_{\ell^1_{n_r}} = \sum_{r\in\mathbb{N}} n_r \|x_r\|_{\mathbb{C}^{n_r}} \upsilon_r \le \|x\|_{\ell^2_{\sqrt{n_r}}},$$

since $\sum_{r\in\mathbb{N}} n_r \upsilon_r^2 = 1$ and $\|\hat{x}\|_{\ell^\infty_{\mathfrak{G}}} \le \|x\|_{\ell^\infty_{\mathfrak{G}}}$. Hence, the mapping

$$\hat{\upsilon} \colon \ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \hat{x} \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r},$$

is continuous as well. By Proposition 5.1 the mapping

$$\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^1_{n_r} \ni z \longmapsto \mathfrak{C}(z, a) \quad \text{with}\, a \in \mathsf{S}^\infty_{\mathfrak{G}}$$

is an analytic $\mathcal{H}^2(\mathrm{d}\varsigma)$-values function. Hence, putting $z = \hat{x}$ with an element $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$, we obtain that the mapping

$$\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}} \ni x \longmapsto \mathfrak{C}(\hat{x}, a)$$

is also analytic. Note that

$$\frac{1}{\big(1 - \langle \hat{x}_r \mid a_r \rangle_{\mathbb{C}^{n_r}}\big)^{n_r}} = \frac{1}{\big(1 - \langle x_r \mid \hat{a}_r \rangle_{\mathbb{C}^{n_r}}\big)^{n_r}}.$$

Thus we have the following equalities

$$\begin{aligned}\mathfrak{C}(\hat{x}, a) &= \prod_{r\in\mathbb{N}} \frac{1}{\left(1 - \langle \hat{x}_r \mid a_r\rangle_{\mathbb{C}^{n_r}}\right)^{n_r}} \\ &= \prod_{r\in\mathbb{N}} \frac{1}{\left(1 - \langle x_r \mid \hat{a}_r\rangle_{\mathbb{C}^{n_r}}\right)^{n_r}} = \mathfrak{C}(x, \hat{a}),\end{aligned} \tag{14}$$

which are true for all $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ and for a suitable vector a such that the right side product in (14) converges. Let us check that it converges for every $a \in \mathsf{S}^\infty_{\mathfrak{G}}$. Using notations from the proof of Proposition 5.1 and the Cauchy–Schwartz Inequality we obtain

$$\begin{aligned}\sup_{\|a\|_{\ell^\infty_{\mathfrak{G}}}=1} \sum_{r\in\mathbb{N}} \left| \ln \frac{1}{\left(1 - \langle x_r \mid \hat{a}_r\rangle_{\mathbb{C}^{n_r}}\right)^{n_r}} \right| &\le c_\varepsilon \sup_{\|a\|_{\ell^\infty_{\mathfrak{G}}}=1} \sum_{r\in\mathbb{N}} n_r \left|\langle x_r \mid \hat{a}_r\rangle_{\mathbb{C}^{n_r}}\right| \\ &\le c_\varepsilon \Big(\sum_{r\in\mathbb{N}} n_r \|x_r\|^2_{\mathbb{C}^{n_r}} \Big)^{1/2} = c_\varepsilon \|x\|_{\ell^2_{\sqrt{n_r}}}\end{aligned}$$

for all $a = (a_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$ and $x = (x_r) \in \ell^2_{\sqrt{n_r}} \bigcap \varepsilon\mathsf{K}^\infty_{\mathfrak{G}}$ with $\varepsilon \in (0, 1)$.

From the density of $\hat{\upsilon}\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$ in $\mathsf{S}^\infty_{\mathfrak{G}}$ it follows that the previous inequality has a unique continuous extension to $\mathsf{S}^\infty_{\mathfrak{G}}$ i.e.

$$\sup_{\|a\|_{\ell^\infty_{\mathfrak{G}}}=1} \sum_{r\in\mathbb{N}} \left| \ln \frac{1}{(1 - \langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \right| \le c_\varepsilon \|x\|_{\ell^2_{\sqrt{n_r}}}$$

for all $a = (a_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$ and $x = (x_r) \in \ell^2_{\sqrt{n_r}} \bigcap \varepsilon\mathsf{K}^\infty_{\mathfrak{G}}$ with $\varepsilon \in (0, 1)$. Consequently, the following product of $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-values functions

$$\begin{aligned}\ell^2_{\sqrt{n_r}} \bigcap \varepsilon\mathsf{K}^\infty_{\mathfrak{G}} \ni x \longmapsto \exp \sum_{r\in\mathbb{N}} \ln \frac{1}{(1 - \langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \\ = \prod_{r\in\mathbb{N}} \frac{1}{(1 - \langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} \in A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)\end{aligned}$$

with $a \in \mathsf{S}^\infty_{\mathfrak{G}}$ converges absolutely and uniformly. Finally, this product represents a bounded continuous $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-values function on $\ell^2_{\sqrt{n_r}} \bigcap \varepsilon\mathsf{K}^\infty_{\mathfrak{G}}$ for all $\varepsilon \in (0, 1)$ and therefore it has a unique analytic $A\big(\mathsf{S}^\infty_{\mathfrak{G}}\big)$-values extension on the open domain $\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}} = \bigcup_{\varepsilon\in(0,1)} \ell^2_{\sqrt{n_r}} \bigcap \varepsilon\mathsf{K}^\infty_{\mathfrak{G}}$. □

7. A Cauchy type integral formula

Now we can already formulate and prove the first main result.

Theorem 7.1. *Every function*

$$f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2(\mathrm{d}\varsigma) \quad \textit{with } f_n \in \mathcal{H}^2_n$$

has an analytic extension into the open domain $\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$*, which can be represented by the Cauchy type integral formula*

$$\mathfrak{C}[f](x) := \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f(a)\mathfrak{C}(x, a)\mathrm{d}\varsigma(a), \quad x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}. \tag{15}$$

The corresponding Cauchy type kernel $\mathfrak{C}$ *can be represented, in turn, by the series*

$$\mathfrak{C}(x, a) = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}_n(x, a), \quad a \in \mathsf{S}^\infty_{\mathfrak{G}} \textit{ with } \mathfrak{C}_n(x, a) := \sum_{|(k)|=n} \sum_{[j]\in\mathbb{N}^n} \frac{e^{*(k)}_{[j]}(x)\bar{e}^{*(k)}_{[j]}(a)}{\left\| e^{*(k)}_{[j]} \right\|^2_{L^2_\varsigma}},$$

weakly convergent in $\mathcal{H}^2(\mathrm{d}\varsigma)$ *for all* $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$*. The Taylor coefficients at the origin are uniquely defined by the formula*

$$\frac{\mathrm{d}^n_0\mathfrak{C}[f](x)}{n!} = \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f_n(a)\mathfrak{C}_n(x, a)\mathrm{d}\varsigma(a), \quad x \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}. \tag{16}$$

Proof. For any $f \in \mathcal{H}^2(\mathrm{d}\varsigma)$ the linear functional

$$\varsigma_f : \mathcal{H}^2(\mathrm{d}\varsigma) \ni g \longmapsto \int_{\mathsf{S}^\infty_{\mathfrak{G}}} fg\,\mathrm{d}\varsigma$$

is continuous. Since $\mathfrak{C}[f](x) = \varsigma_f \circ \mathfrak{C}(x, \cdot)$, the function $\mathfrak{C}[f]$ defined by formula (15) is analytic in $\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ via Proposition 6.1.

Let $a = (a_r) \in \mathsf{S}^\infty_{\mathfrak{G}}$ with $a_r \in \mathsf{S}_r$ and $x = (x_r) \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ with $x_r \in \mathbb{C}^{n_r}$. Consider a polynomial $e^{*(k)}_{[j]} \in \mathscr{E}^*_n$ of the form

$$e^{*(k)}_{[j]} = e^{*(k)r_1}_{[j]r_1} \cdots e^{*(k)r_t}_{[j]r_t},$$

cited in Theorem 4.3. Since

$$x_r = e^*_{j_{r(1)}}(x_r)e_{j_{r(1)}} + \cdots + e^*_{j_{r(n_r)}}(x_r)e_{j_{r(n_r)}},$$

and $\|x_r\|_{\mathbb{C}^{n_r}} < 1$, $\|a_r\|_{\mathbb{C}^{n_r}} = 1$, we obtain

$$\begin{aligned}\frac{1}{(1 - \langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} &= \sum_{n\in\mathbb{Z}_+} \frac{(n_r - 1 + n)!}{(n_r - 1)!n!} \langle x_r \mid a_r\rangle^n_{\mathbb{C}^{n_r}} \\ &= \sum_{n\in\mathbb{Z}_+} \frac{(n_r - 1 + n)!}{(n_r - 1)!n!} \Big(\sum_{i=1}^{n_r} e^*_{j_{r(i)}}(x_r)\bar{e}^*_{j_{r(i)}}(a_r)\Big)^n \\ &= \sum_{n\in\mathbb{Z}_+} \sum_{|(k)_r|=n} \frac{(n_r - 1 + n)!}{(n_r - 1)!(k)_r!} e^{*(k)_r}_{[j]_r}(x_r)\bar{e}^{*(k)_r}_{[j]_r}(a_r).\end{aligned}$$

Taking into account Theorem 4.3, it follows that

$$\mathfrak{C}(x, a) = \prod_{r\in\mathbb{N}} \frac{1}{(1 - \langle x_r \mid a_r\rangle_{\mathbb{C}^{n_r}})^{n_r}} = \sum_{n\in\mathbb{Z}_+} \mathfrak{C}_n(x, a)$$

with

$$\begin{aligned}\mathfrak{C}_n(x, a) &= \sum_{[j]\in\mathbb{N}^n} \sum_{|(k)|=n} \frac{e^{*(k)}_{[j]}(x)\bar{e}^{*(k)}_{[j]}(a)}{\left\| e^{*(k)}_{[j]} \right\|^2_{L^2_\varsigma}} \\ &= \sum_{\substack{([j]r_1,\dots,[j]r_t)\in\mathbb{N}^n \\ |(k)r_1|+\cdots+|(k)r_t|=n}} \prod_{i=1}^{t} \frac{(n_{r_i} - 1 + n)!}{(n_{r_i} - 1)!(k)_{r_i}!} e^{*(k)r_i}_{[j]r_i}(x_{r_i})\bar{e}^{*(k)r_i}_{[j]r_i}(a_{r_i}).\end{aligned} \tag{17}$$

On the other hand, the equality (17) implies that for any $x \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$

$$\int_{\mathsf{S}^\infty_{\mathfrak{G}}} e^{*(k)}_{[j]}(a)\mathfrak{C}_n(x, a)\mathrm{d}\varsigma(a) = e^{*(k)}_{[j]}(x), \quad e^{*(k)}_{[j]} \in \mathscr{E}^*_n.$$

Since $\mathscr{E}^*_n$ forms an orthogonal basis in $\mathcal{H}^2_n$, the kernel $\mathfrak{C}_n$ realizes the identity mapping in $\mathcal{H}^2_n$. It follows that for any $f_n \in \mathcal{H}^2_n$

$$f_n(x) = \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f_n(a)\mathfrak{C}_n(x, a)\mathrm{d}\varsigma(a), \quad x \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}.$$

Using that $f_n \perp \mathfrak{C}_l(x, \cdot)$ at $n \neq l$, we obtain

$$\mathfrak{C}[f](x) = \int_{\mathsf{S}^\infty_{\mathfrak{G}}} f(a)\mathfrak{C}(x, a)\mathrm{d}\varsigma(a) = \sum_{n\in\mathbb{Z}_+} \varrho^n f_n(y)$$

for all $x = \varepsilon y \in \ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ with $\|y\|_{\ell^{2,\infty}_{\sqrt{n_r}}} = 1$ and $\varepsilon \in [0, 1)$. Now the equality

$$f_n(y) = \frac{1}{n!} \frac{\mathrm{d}^n_0 \mathfrak{C}[f](\varepsilon y)}{\mathrm{d}\varepsilon^n}\bigg|_{\varepsilon=0} \tag{18}$$

implies that f_n is a Taylor coefficient of $\mathfrak{C}[f]$.

Finally, the relation $\mathfrak{C}[f](x) = \varsigma_f \circ \mathfrak{C}(x, \cdot)$ with $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ implies that for any $f \in \mathcal{H}^2(\mathrm{d}\varsigma)$ the series (15) is pointwise by $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ weakly convergent in $\mathcal{H}^2(\mathrm{d}\varsigma)$, as a function of the variable $a \in \mathsf{S}^\infty_{\mathfrak{G}}$. Clearly, $\mathrm{d}^n_0\mathfrak{C}[f]$ in (18) can be extended on $\ell^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$, as a continuous polynomial. Thus, the formula (16) is true.

It remains to note that Taylor coefficients $\frac{1}{n!}\mathrm{d}^n_0\mathfrak{C}[f] = f_n$ uniquely define the analytic function $\mathfrak{C}[f]$ in the open domain $\mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$. □

Remark 7.2. In the finite dimensional case if $\mathsf{E} = \mathbb{C}^n$, $n \in \mathbb{N}$, the Cauchy formula (15) obtains the classical form (see [8])

$$\mathfrak{C}[f](x) = \int_{\mathsf{S}_n} \frac{f(a)\mathrm{d}\varsigma(a)}{(1 - \langle x \mid a\rangle_{\mathbb{C}^n})^n}, \quad x \in \mathsf{B}.$$

Remark 7.3. In the partial case if $n_r = 1$ for all $r \in \mathbb{N}$, the ball $\mathsf{K}^\infty_{\mathfrak{G}}$ has a polydisk form. For this case the Cauchy type formula has been earlier established in [13].

Corollary 7.4. *For every $x \in \mathsf{B}^\infty_{\mathfrak{G}} \bigcap \ell^2_{\sqrt{n_r}}$ the point-evaluation functional*

$$\delta_x(f) : f \mapsto f(x)$$

is continuous on $\mathcal{H}^2(\mathrm{d}\varsigma)$.

Proof. From Theorem 7.1 we have

$$\delta_x(f) = f(x) = \langle \mathfrak{C}(x, \cdot) \mid f(\cdot)\rangle_{\mathcal{H}^2(\mathrm{d}\varsigma)}$$

and from Proposition 6.1 that $\mathfrak{C}(x, \cdot) \in A\big(\mathsf{S}_{\ell^\infty_{\mathfrak{G}}}\big) \subset \mathcal{H}^2(\mathrm{d}\varsigma)$. □

8. Relations between Hardy type spaces and the symmetric Fock space

The Hermitian dual of symmetric Fock space F^* and the Hardy class $\mathcal{H}^2(\mathrm{d}\varsigma)$ possess the same orthogonal basis $\mathcal{E}^*_{\mathsf{F}}$ (see Remark 2.2 and Theorem 4.2). The following proposition is a specification of the statement from [14] (an another interpretation of this statement was given in [15, Theorem 2.6]).

Proposition 8.1. *Every element*

$$f^* = \sum_{n\in\mathbb{Z}_+} f^*_n \in \mathsf{F}^* \quad \text{with } f^*_n \in \odot^n_h \mathsf{E}^*$$

generates an analytic function defined in the Hilbertian open ball B,

$$\mathfrak{F}[f^*](x) = \sum_{n\in\mathbb{Z}_+} f^*_n(x), \quad x \in \mathsf{B} \tag{19}$$

with the Taylor series expansion at the origin

$$f^*_n(x) = \sum_{|(k)|=n} \sum_{[j]\in\mathbb{N}^n} \frac{n!}{(k)!} e^{*(k)}_{[j]}(x) \langle e^{(k)}_{[j]} \mid f_n\rangle_{\otimes^n_h \mathsf{E}}$$

and the point-evaluation functional

$$\delta^{\mathsf{F}}_x : f^* \mapsto \mathfrak{F}[f^*](x)$$

is continuous for every $x \in \mathsf{B}$.

Proof. Since

$$x = \sum_{t\in\mathbb{Z}_+} e^*_t(x) e_t, \quad \text{we have } \|x\|^2 = \sum_{t\in\mathbb{Z}_+} \left|e^*_t(x)\right|^2.$$

The Fourier decomposition of the element $x^{\otimes n}$ is

$$x^{\otimes n} = \Big(\sum_{n\in\mathbb{Z}_+} e^*_t(x) e_t\Big)^{\otimes n} = \sum_{|(k)|=n} \sum_{[j]} \frac{n!}{(k)!} e^{*(k)}_{[j]}(x) e^{\otimes(k)}_{[j]} \tag{20}$$

and the series converges in $\odot_h^n \mathsf{E}$ since

$$\begin{aligned}\left\|x^{\otimes n}\right\|^2_{\otimes_h^n \mathsf{E}} &= \sum_{|(k)|=n}\sum_{[j]} \frac{n!^2}{(k)!^2}|e^{*(k)}_{[j]}(x)|^2 \left\|e^{*\otimes(k)}_{[j]}\right\|^2_{\otimes_h^n \mathsf{E}} \\ &= \sum_{|(k)|=n}\sum_{[j]} \frac{n!}{(k)!}\left|e^{*(k)}_{[j]}(x)\right|^2 = \Big(\sum_{n\in\mathbb{Z}_+}\left|e^*_t(x)\right|^2\Big)^n.\end{aligned}$$

Using the orthogonal property $x^{\otimes n} \perp x^{\otimes m}$ in F for $n \neq m$, we obtain

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}_+} x^{\otimes n}\right\|^2_{\mathsf{F}} &= \sum_{n\in\mathbb{Z}_+}\left\|x^{\otimes n}\right\|^2_{\otimes_h^n \mathsf{E}} \\ &= \sum_{n\in\mathbb{Z}_+}\|x\|^{2n} = \frac{1}{1-\|x\|^2}.\end{aligned}$$

Thus, the series $\sum_{n\in\mathbb{Z}_+} x^{\otimes n}$ is absolutely and uniformly convergent in the Fock space F on any closed subball in B.

As is known [16, Proposition 2.4.2] $\sum_{n\in\mathbb{Z}_+} x^{\otimes n}$ is an analytic map from B into F. For any $f^* = \sum_{n\in\mathbb{Z}_+} f^*_n \in \mathsf{F}^*$ with $f^*_n \in \odot_h^n \mathsf{E}^*$ the orthogonality $f^*_l \perp x^{\otimes n}$ for $n \neq l$ implies

$$\mathfrak{F}[f^*](x) := f^*\Big(\sum_{n\in\mathbb{Z}_+} x^{\otimes n}\Big) = \sum_{n\in\mathbb{Z}_+} f^*_n(x), \quad x \in \mathsf{B}.$$

Hence, the complex function $\mathfrak{F}[f^*]$ is a composition of two analytic maps on B:$f^* \in \mathsf{F}^*$ and $x \mapsto \sum_{n\in\mathbb{Z}_+} x^{\otimes n} \in \mathsf{F}$ and so must be analytic on B (see [16, Proposition 3.1.2]). For any $x = ra \in \mathsf{B}$ with $\|a\| = 1$ and $r \in [0, 1)$ we have

$$f^*_n(a) = \frac{1}{n!}\frac{\mathrm{d}_0^n \mathfrak{F}[f^*](ra)}{\mathrm{d}r^n}\bigg|_{r=0}.$$

Thus each polynomial f^*_n is a Taylor coefficient of $\mathfrak{F}[f^*]$, defined as an orthogonal projection of $f^* \in \mathsf{F}^*$ onto the subspace $\odot_h^n \mathsf{E}^*$. Now it remains to substitute instead of $x^{\otimes n}$ the orthogonal decomposition (20).

Since

$$\mathfrak{F}[f^*](x) = \Big\langle \sum_{n\in\mathbb{Z}_+} x^{\otimes n} \mid f(\cdot)\Big\rangle_{\mathsf{F}}$$

and $\sum_{n\in\mathbb{Z}_+} x^{\otimes n} \in \mathsf{F}$ for every $x \in \mathsf{B}$, the functional δ^{F}_x is continuous on F^*. □

Proposition 8.2. *In the case if*

$$n_r = 1 \quad \text{for all } r \in \mathbb{N},$$

the following contractive dense embeddings

$$\mathcal{H}^2(\mathrm{d}\varsigma) \hookrightarrow \mathsf{F}^* \quad \text{and} \quad \mathcal{H}^2_n \hookrightarrow \mathsf{E}^{*n}_h \quad \text{for all } n \in \mathbb{Z}_+ \tag{21}$$

hold.

Proof. As is well known (see e.g. [10, 2.2.2]), the system $\mathscr{E}_{\mathsf{F}}$ forms orthogonal bases in the symmetric Fock space F and

$$\left\|e^{\otimes(k)}_{[j]}\right\|^2_{\mathsf{F}} = \left\|e^{\otimes(k)}_{[j]}\right\|^2_{\otimes_h^n \mathsf{E}} = \frac{(k)!}{n!}, \quad n = |(k)| \text{ for all } [j] \in \mathbb{N}^n.$$

From Theorem 4.3 it follows, that

$$\left\|e^{\otimes(k)}_{[j]}\right\|^2_{\mathsf{F}} \le 1 = \left\|e^{*(k)}_{[j]}\right\|^2_{L^2_\varsigma}.$$

Via Theorem 4.2 for every function $f_n \in \mathcal{H}^2_n$ there exists a Fourier decomposition

$$f_n = \sum_{|(k)|=n}\sum_{[j]\in\mathbb{N}^n} \alpha^{(k)}_{[j]} e^{*(k)}_{[j]} \quad \text{in } \mathcal{H}^2_n$$

with the coefficients $\alpha^{(k)}_{[j]} \in \mathbb{C}$. It follows, that

$$\|f_n\|^2_{\otimes^n_h \mathsf{E}} = \sum_{|(k)|=n} \sum_{[j]\in\mathbb{N}^n} \left|\alpha^{(k)}_{[j]}\right|^2 \frac{(k)!}{n!} \leq \sum_{|(k)|=n} \sum_{[j]\in\mathbb{N}^n} \left|\alpha^{(k)}_{[j]}\right|^2 = \|f_n\|^2_{L^2_\varsigma}.$$

Hence, the embedding $\mathcal{H}^2_n \hookrightarrow \mathsf{E}^{*n}_h$ is contractive for all n. Therefore

$$\|f\|^2_{\mathsf{F}} = \sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{\otimes^n_h \mathsf{E}} \leq \sum_{n\in\mathbb{Z}_+} \|f_n\|^2_{L^2_\varsigma} = \|f\|^2_{L^2_\varsigma}$$

for all $f = \sum_{n\in\mathbb{Z}_+} f_n \in \mathcal{H}^2(\mathrm{d}\varsigma)$ with $f_n \in \mathcal{H}^2_n$ and the embeddings (21) are proved. Since the system $\mathscr{E}^*_n$ forms an orthogonal basis in $\mathcal{H}^2_n$ for all n, the embeddings (21) are dense. □

For the general case of $\mathsf{B}^\infty_{\mathfrak{G}}$ we have the following theorem.

Theorem 8.3. *Let $x \in \mathsf{B} \bigcap \mathsf{B}^\infty_{\mathfrak{G}}$ and $f \in \mathsf{F}^* \bigcap \mathcal{H}^2(\mathrm{d}\varsigma)$. Then*

$$\mathfrak{C}[f](x) = \mathfrak{F}[f](x).$$

Proof. We observe that $\mathfrak{C}[f](x) = \delta_x(f)$, $\mathfrak{F}[f](x) = \delta^{\mathsf{F}}_x(f)$ and both δ_x and δ^{F}_x are continuous if $x \in \mathsf{B} \bigcap \mathsf{B}^\infty_{\mathfrak{G}}$. So they coincide on the common domain if they are equal each to other on basis functions. But

$$\delta_x\left(e^{*\otimes(k)}_{[j]}\right) = e^{*\otimes k_1}_{j_1}(x) e^{*\otimes k_2}_{j_2}(x) \cdots e^{*\otimes k_n}_{j_n}(x) = \delta^{\mathsf{F}}_x\left(e^{*\otimes(k)}_{[j]}\right)$$

for all multi-indices (k) and $[j]$. So $\mathfrak{C}[f](x) = \mathfrak{F}[f](x)$ for every $f \in \mathsf{F}^* \bigcap \mathcal{H}^2(\mathrm{d}\varsigma)$ and $x \in \mathsf{B} \bigcap \mathsf{B}^\infty_{\mathfrak{G}}$. □

The following proposition gives a natural isomorphism between F^* and $\mathcal{H}^2(\mathrm{d}\varsigma)$ for the general case of $\mathcal{H}^2(\mathrm{d}\varsigma)$.

Proposition 8.4. *Let J be a linear operator from F^* to $\mathcal{H}^2(\mathrm{d}\varsigma)$ defined on the basis functions by the following way*

$$J\left(e^{*\otimes(k)}_{[j]}\right) = \sqrt{\frac{(k)!}{n!}} \frac{e^{*\otimes(k)}_{[j]}}{\sqrt{\left\langle e^{*\otimes(k)}_{[j]} \mid e^{*\otimes(k)}_{[j]} \right\rangle_{\mathcal{H}^2(\mathrm{d}\varsigma)}}}, \quad n = |(k)|.$$

Then J is an isometrical isomorphism.

Proof. From the definition of J we have that

$$J\left(\frac{e^{*\otimes(k)}_{[j]}}{\|e^{*\otimes(k)}_{[j]}\|_{\mathsf{F}^*}}\right) = \frac{e^{*\otimes(k)}_{[j]}}{\|e^{*\otimes(k)}_{[j]}\|_{\mathcal{H}^2(\mathrm{d}\varsigma)}}.$$

That is, J maps one-to-one the orthonormal basis of F^* onto the orthonormal basis of $\mathcal{H}^2(\mathrm{d}\varsigma)$. So J is an isometrical isomorphism. □

Note that if $n_r = 1$ for all $r \in \mathbb{N}$, then

$$J\left(e^{*\otimes(k)}_{[j]}\right) = \sqrt{\frac{(k)!}{n!}} e^{*\otimes(k)}_{[j]}.$$

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