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RISING FACTORIAL POWERS AND A NONELEMENTARY FUNCTION OF THE DAWSON'S INTEGRAL TYPE

ВОЗРАСТАЮЩИЕ ФАКТОРИАЛЬНЫЕ СТЕПЕНИ И НЕЭЛЕМЕНТАРНАЯ ФУНКЦИЯ ТИПА ИНТЕГРАЛА ДОУСОНА

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Abstract. A new nonelementary real-valued function of the Dawson's integral type is studied. It is constructed as a power series with the help of rising factorial powers. Its connection with the error function is determined. It is proved that the new function is a solution of the Riccati equation.

Аннотация. Исследована новая неэлементарная функция действительного переменного типа интеграла Доусона, построенная в виде степенного ряда с помощью возрастающих факториальных степеней. Установлена ее связь с функцией ошибок (функцией вероятностей). Показано, что новая функция есть решением уравнения Риккати.

Keywords: rising factorial power, Dawson's integral, error function, Riccati equation.

Ключевые слова: возрастающая факториальная степень, интеграл Доусона, функция ошибок, уравнение Риккати.

Introduction. Duality of rising and falling factorial powers is a common feature in the combinatorial analysis. In other words, if a problem leads to some combinatorial identity constructed with the help of falling factorial powers, then often there is a dual combinatorial problem, which leads to a dual combinatorial identity involving rising factorial powers. One can find some examples of these dual combinatorial identities in [1-3].

The classic exponential function e^x is given by the corresponding power series with factorials, which can be written as the falling factorial power $n^{\underline{n}}$. Replacing the falling factorial powers by the corresponding rising factorial powers

$n^{\overline{n}}$, we get the function $\mathrm{Exp}(x)$ [4]. Now if in the Dawson's integral $F(x)=\exp(-x^2)\int_0^x \exp(t^2)\,dt$ we replace the exponential function by $\mathrm{Exp}(x)$, then we get a new nonelementary function $D(x)=\left(\mathrm{Exp}(-x^2)\right)^{-1}\int_0^x \mathrm{Exp}(t^2)dt$, the basic properties of which are to be studied in this article.

The Dawson's integral and its generalization are widely applied in theory of electric oscillation, astrophysics, spectroscopy, processes of heat conduction, viscosity mechanics, finance and applied mathematics.

Preliminaries and Notations.

Definition 1. For an arbitrary $x\in\mathbf{R}$ and $m\in\mathbf{N}$ the factorial power m with the step of $k\in\mathbf{R}$ is the expression

$$x^{m\{k\}}=x(x+k)\cdot(x+2k)\cdot\ldots\cdot(x+(m-1)k).$$

By definition, $x^{0\{k\}}=1$. If $k=0$, then we have a simple power, i.e. $x^{m\{0\}}=x^m$.

Most often, there are rising factorial powers with the step of 1 and falling factorial powers with the step of (– 1), which we will denote by

$$x^{\overline{m}}=x^{m\{1\}}=x(x+1)(x+2)\cdot\ldots\cdot(x+m-1),\qquad x^{\underline{m}}=x^{m\{-1\}}=x(x-1)(x-2)\cdot\ldots\cdot(x-m+1).$$

respectively. It is obvious that $1^{\overline{m}}=m^{\underline{m}}=m!$

In analogy to the known power series $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, which can be treated as the series constructed with the help of falling factorial powers. The "dual" function $\mathrm{Exp}(x)$, constructed with the help of rising factorial powers, it is studied in [4].

Definition 2. By $\mathrm{Exp}(x)$ we will denote the function defined with the help of the power series

$$\mathrm{Exp}(x)=\sum_{n=0}^{\infty}\frac{x^n}{n^{\overline{n}}}=1+\frac{x}{1}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4\cdot 5}+\ldots+\frac{x^n}{n\cdot(n+1)\cdot\ldots\cdot(2n-1)}+\ldots.$$

It is obvious that

$$\mathrm{Exp}(x)=1+\sum_{n=1}^{\infty}\frac{(n-1)!}{(2n-1)!}x^n \tag{1}$$

and the series in (1) converges on the real axis.

In [4] it is proved that

$$\mathrm{Exp}(x)=1+\sqrt{\pi x}\exp\left(x/4\right)\Phi\left(\sqrt{x}/2\right), \tag{2}$$

where $\Phi(p)=\operatorname{erf}\, p=\frac{2}{\sqrt{\pi}}\int_0^p \exp(-t^2)\,dt$ is the error function [5; p. 405]

Function of the Dawson's integral type. Replacing in the Dawson's integral $F(x)=\exp(-x^2)\int_0^x \exp(t^2)\,dt$ [5; p. 427] the exponentials by $\operatorname{Exp}(x)$, we obtain a nonelementary real-valued function, which we will denote by $D(x)$, i.e. let

$$D(x)=\left(\operatorname{Exp}(-x^2)\right)^{-1}\int_0^x \operatorname{Exp}(t^2)\,dt. \tag{3}$$

From (2), (3) we obtain the formula

$$D(x)=\frac{2\sqrt{\pi}\exp(x^2/4)\Phi(x/2)-x}{\sqrt{\pi}x\exp(x^2/4)\Phi(x/2)+1}.$$

The graphs of the function $y=D(x)$ (solid graph) and the Dawson's integral $y=F(x)$ (dotted graph) are shown in Figure 1.

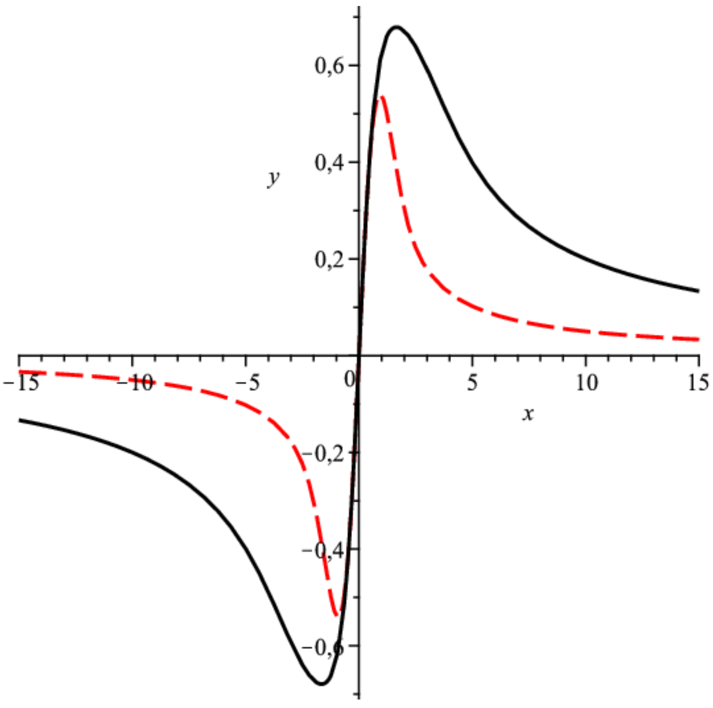

Fig. 1. The graphs of the functions $y=D(x)$ and $y=F(x)$

Differential equation of the function D(x). The Dawson's integral is a solution of the linear nonhomogeneous equation $y'+2xy=1$ [5, p. 429]. Let us prove that the function $D(x)$ is a solution of the Riccati equation.

Theorem. The function $y=D(x)$ is a solution of the Cauchy problem

$$y'=\frac{x^2-2}{2(x^2+2)}y^2-\frac{x(x^2+6)}{2(x^2+2)}y+1, \quad y(0)=0. \tag{4}$$

Proof. It follows from the formula (3) that the integral curve $y = D(x)$ passes through the origin. Let us prove that the function $y = D(x)$ is a solution of the Riccati equation from (4). Indeed, since

$$y' = \frac{1}{2}\left(1+\sqrt{\pi}x\exp\left(x^2/4\right)\Phi\left(x/2\right)\right)^{-2}\times$$

$$\times\left(2\sqrt{\pi}x\exp\left(x^2/4\right)\Phi\left(x/2\right)+\exp\left(x^2/4\right)x^3\,\Phi\left(x/2\right)-4\pi x\exp\left(x^2/2\right)\Phi^2\left(x/2\right)+2x^2+2\right),$$

excluding from this formula and from (3) the expression $\sqrt{\pi}\exp\left(x^2/4\right)\Phi\left(x/2\right)$, we obtain the relation

$$y' = \frac{x^2-2}{2(x^2+2)}y^2 - \frac{x(x^2+6)}{2(x^2+2)}y+1.$$

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