On balancing numbers

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A positive integer n is called a *balancing number* if the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$ holds for some positive integer r which is called balancer [1]. For example, 6, 35, 204 and 1189 are balancing numbers with balancers 2, 14, 84 and 292, respectively.

The balancing numbers $\{B_n\}_{n\geq 0}$ is defined by the recurrence relation $B_{n+1}=6B_n-B_{n-1}$, $n\geq 1$, with initial terms $B_0=0$ and $B_1=1$.

Using Trudi's formula (see, for example, [2]) for determinants of Toeplitz-Hessenberg matrices entries of which are balancing numbers, we obtain identities with multinomial coefficients for the these numbers.

Theorem. *Let $n\geq 1$, except when noted otherwise. Then*

$$\sum(-1)^T p_n(t) B_0^{t_1} B_1^{t_2}\cdots B_{n-1}^{t_n}=\frac{\sqrt{7}}{14}\Big((3-\sqrt{7})^{n-1}-(3+\sqrt{7})^{n-1}\Big),$$

$$\sum p_n(t) B_0^{t_1} B_1^{t_2}\cdots B_{n-1}^{t_n}=6^{n-2},\qquad n\geq 2,$$

$$\sum(-1)^T p_n(t) B_1^{t_1} B_2^{t_2}\cdots B_{n}^{t_n}=\frac{\sqrt{21}}{21}\left(\left(\frac{5-\sqrt{21}}{2}\right)^n-\left(\frac{5+\sqrt{21}}{2}\right)^n\right),$$

$$\sum p_n(t) B_1^{t_1} B_2^{t_2}\cdots B_{n}^{t_n}=\frac{\sqrt{45}}{45}\left(\left(\frac{7+\sqrt{45}}{2}\right)^n-\left(\frac{7-\sqrt{45}}{2}\right)^n\right),$$

$$\sum p_n(t) B_1^{t_1} B_3^{t_2}\cdots B_{2n-1}^{t_n}=36\cdot 35^{n-2},\qquad n\geq 2,$$

$$\sum(-1)^T p_n(t) B_2^{t_1} B_3^{t_2}\cdots B_{n+1}^{t_n}=0,\qquad n\geq 3,$$

$$\sum(-1)^T p_n(t) B_2^{t_1} B_4^{t_2}\cdots B_{2n}^{t_n}=\frac{\sqrt{195}}{65}\Big((14-\sqrt{195})^{n}-(14+\sqrt{195})^{n}\Big),$$

$$\sum(-1)^T p_n(t) B_3^{t_1} B_5^{t_2}\cdots B_{2n+1}^{t_n}=36,\qquad n\geq 2,$$

where the summation is over integers $t_i\geq 0$ satisfying $t_1+2t_2+\cdots+nt_n=n$, $T=t_1+\cdots+t_n$ and $p_n(t)=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ is the multinomial coefficient.

References

1. *Behera A., Panda G.K.* On the square roots of triangular numbers // Fibonacci Quart. 1999. Vol. 1, № 2. P. 98–105.

2. *Goy T.* Some Tribonacci identities using Toeplitz-Hessenberg determinants // Proceedings of 18th International Scientific M. Kravchuk Conference. Vol. 1. Kyiv: NTUU «KPI», 2017. P. 159–161.