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ALGEBRAS OF SYMMETRIC HOLOMORPHIC FUNCTIONS ON ℓ_p

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Abstract

The authors study the algebra of uniformly continuous holomorphic symmetric functions on the ball of ℓ_p, investigating in particular the spectrum of such algebras. To do so, they examine the algebra of symmetric polynomials on ℓ_p-spaces, as well as finitely generated symmetric algebras of holomorphic functions. Such symmetric polynomials determine the points in ℓ_p up to a permutation.

In recent years, algebras of holomorphic functions on the unit ball of standard complex Banach spaces have been considered by a number of authors, and the spectrum of such algebras was studied in [**1**], [**2**] and [**7**]. For example, properties of $A_u(B_X)$, the algebra of uniformly continuous holomorphic functions on the ball of a complex Banach space X have been studied by Gamelin *et al.* Unfortunately, this analogue of the classical disc algebra $A(D)$ has a very complicated, ill-understood, spectrum. If X^* has the approximation property, the spectrum of $A_u(B_X)$ coincides with the closed unit ball of the bidual if, and only if, X^* generates a dense subalgebra in $A_u(B_X)$ (see [**5**]). In a very real sense, however, the problem is that $A_u(B_{\ell_p})$ is usually too large, admitting far too many functions. For instance, $\ell_\infty \subset A_u(B_{\ell_2})$ isometrically via the mapping $a = (a_j) \rightsquigarrow P_a$, where $P_a(x) \equiv \sum_{j=1}^{\infty} a_j x_j^2$.

This paper addresses this problem by severely restricting the functions that we admit. Specifically, we limit our attention here to uniformly continuous *symmetric* holomorphic functions on B_{ℓ_p}. By 'a symmetric function on ℓ_p', we mean a function that is invariant under any reordering of the sequence in ℓ_p. Information on symmetric polynomials in finite-dimensional spaces can be found in [**9**] or [**12**]; in the infinite-dimensional Hilbert space they already appear in [**11**]. Throughout this note, $\mathscr{P}_s(\ell_p)$ is the space of symmetric polynomials on a complex space ℓ_p, $1 \leqslant p < \infty$. Such polynomials determine, as we prove, the points in ℓ_p up to a permutation. We shall use the notation $A_{us}(B_{\ell_p})$ for the uniform algebra of symmetric holomorphic functions that are uniformly continuous on the open unit ball B_{ℓ_p} of ℓ_p, and we also study some particular finitely generated subalgebras. The purpose of this paper is to describe such algebras and their spectra (which we identify with certain subsets of ℓ_∞ and $\mathbb{C}^m$, respectively), and as a result of this we show that $A_{us}(B_{\ell_p})$ is algebraically and topologically isomorphic to a uniform Banach algebra generated by coordinate

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projections in ℓ_∞. This is done in Section 3, following algebraic preliminaries and a brief examination of the finite-dimensional situation in Sections 1 and 2.

We denote by τ_{pw} the topology of pointwise convergence in ℓ_∞. We follow the usual conventions, denoting by $\mathscr{H}_b(X)$ the Fréchet algebra of $\mathbb{C}$-valued holomorphic functions on a complex Banach space X that are bounded on bounded subsets of X, endowed with the topology of uniform convergence on bounded sets. The subalgebra of *symmetric* functions will be denoted $\mathscr{H}_{bs}(X)$. For any Banach or Fréchet algebra A, we put $\mathscr{M}(A)$ for its spectrum: that is, the set of all continuous scalar-valued homomorphisms. For background on analytic functions on infinite-dimensional Banach spaces, we refer the reader to [**3**].

1. *The algebra of symmetric polynomials*

Let X be a Banach space, and let $\mathscr{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathscr{P}_0(X)$ be a subalgebra of $\mathscr{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an *algebraic basis* of $\mathscr{P}_0(X)$ if for every $P \in \mathscr{P}_0(X)$ there is $q \in \mathscr{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(G_1(x), \ldots, G_n(x))$; in other words, if G is the mapping $x \in X \rightsquigarrow G(x) := (G_1(x), \ldots, G_n(x)) \in \mathbb{C}^n$, then $P = q \circ G$.

Let $\langle p \rangle$ be the smallest integer that is greater than or equal to p. In [**8**], it is proved that the polynomials $F_k(\sum a_i e_i) = \sum a_i^k$ for $k = \langle p \rangle, \langle p \rangle + 1, \ldots$ form an algebraic basis in $\mathscr{P}_s(\ell_p)$. So there are no symmetric polynomials of degree less than $\langle p \rangle$ in $\mathscr{P}_s(\ell_p)$ and if $\langle p_1 \rangle = \langle p_2 \rangle$, then $\mathscr{P}_s(\ell_{p_1}) = \mathscr{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathscr{P}_s(\ell_p)$ only for integer values of p. Throughout, we shall assume that p is an integer, $1 \leqslant p < \infty$.

It is well known [**9**, XI, §52] that for $n < \infty$ any polynomial in $\mathscr{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the *elementary symmetric polynomials* $(R_i)_{i=1}^n$, $R_i(x) = \sum_{k_1 < \ldots < k_i} x_{k_1} \ldots x_{k_i}$.

LEMMA 1.1. *Let $\{G_1, \ldots, G_n\}$ be an algebraic basis of $\mathscr{P}_s(\mathbb{C}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$, $G_i(y) = \xi_i$, $i = 1, \ldots, n$, then $x = y$ up to a permutation.*

Proof. First, we suppose that $G_i = R_i$. Then, according to the Vieta formulae [**9**], the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \ldots + (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x) = \xi_i$, and so $x = (x_1, \ldots, x_n)$ as required. Now let G_i be an arbitrary algebraic basis of $\mathscr{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials v_i on $\mathbb{C}^n$. Setting v as the polynomial mapping $x \in \mathbb{C}^n \rightsquigarrow v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $G = w \circ R$; hence $R = (v \circ w) \circ R$, so $v \circ w = \mathrm{id}$. Then v and w are inverse to each other, since $w \circ v$ coincides with the identity on the open set, $\mathrm{Im}(w)$. In particular, v is one-to-one.

Now, the solutions $x_1, \ldots, x_n$ of the equation

$$x^n - v_1(\xi_1, \ldots, \xi_n) x^{n-1} + \ldots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$

satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, and hence $\xi = G(x)$. □

COROLLARY 1.2. *Given* $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, *there is* $x \in \ell_p^{n+p-1}$ *such that*

$$F_p(x) = \xi_1, \ldots, F_{p+n-1}(x) = \xi_n.$$

This result shows that any $P \in \mathscr{P}_s(\ell_p)$ has a *'unique' representation* in terms of $\{F_k\}$, in the sense that if $q \in \mathscr{P}(\mathbb{C}^n)$ for some n is such that $P(x) = q(F_p(x), \ldots, F_{n+p}(x))$ and if $q' \in \mathscr{P}(\mathbb{C}^m)$ for some m is such that $P(x) = q'(F_p(x), \ldots, F_{m+p}(x))$, with, say, $n \leqslant m$, then $q'(\xi_1, \ldots, \xi_m) = q(\xi_1, \ldots, \xi_n)$.

For $x, y \in \ell_p$, we shall write $x \sim y$, whenever there is a permutation T of the basis in ℓ_p such that $x = T(y)$. For any point $x \in \ell_p$, the linear multiplicative functional on $\mathscr{P}_s(\ell_p)$ of evaluation at x will be denoted by δ_x. It is clear that if $x \sim y$, then $\delta_x = \delta_y$.

THEOREM 1.3. *Let* $x, y \in \ell_p$ *and* $F_i(x) = F_i(y)$ *for every* $i > p$. *Then* $x \sim y$.

Proof. Define $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Without loss of generality, we can assume that

$$1 = |x_1| = \ldots = |x_k| > |x_{k+1}| \geqslant \ldots \quad \text{and} \quad 1 \geqslant |y_1| \geqslant |y_2| \geqslant \ldots .$$

If $|y_1| < 1$, then for many big values of j, $|F_j(x)|$ will be close to k, while for all big j, $F_j(y)$ will be close to 0. Thus $|y_1| = 1$. Suppose that $1 = |y_1| = \ldots = |y_m| > |y_{m+1}| \geqslant \ldots$. We claim that $m = k$. Suppose, for a contradiction, that $m < k$. Then, for many big j, $|F_j(x)|$ is close to k, while for all big j, $|F_j(y)| < m + 1/2 < k$. This contradiction shows that $m < k$ is false; similarly, $k < m$ is false, and so $m = k$.

Let $\tilde{x} = (x_1, \ldots, x_k)$ and $\tilde{y} = (y_1, \ldots, y_k)$. Also, for $z = (z_i) \in \ell_p$, let z^j denote the point $(z_1^j, z_2^j, \ldots)$. We claim that $\tilde{x} \sim \tilde{y}$, where we associate $\tilde{x} = (x_1, \ldots, x_k) \in \mathbb{C}^k$, for example, with $(x_1, \ldots, x_k, 0, 0, \ldots)$. Consider the function $f : (S^1)^{2k} \longrightarrow \mathbb{C}$ given by

$$f(\tilde{u}, \tilde{v}) = f(u_1, \ldots, u_k, v_1, \ldots, v_k) = [u_1 + \ldots + u_k] - [v_1 + \ldots + v_k].$$

Since both $F_j(x - \tilde{x}) \to 0$ and $F_j(y - \tilde{y}) \to 0$ as $j \to \infty$, and since we are assuming that $F_j(x) = F_j(y)$ for all $j \geqslant p$, it follows that $f(\tilde{x}^j, \tilde{y}^j) \to 0$ as $j \to \infty$. Now, f is obviously a continuous function, and so it follows that $f(u, v) = 0$ for any point $(u, v) \in (S^1)^{2k}$ that is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geqslant p\}$.

Next, the point $(1, \ldots, 1) \in (S^1)^{2k}$ is a limit point of $\{(\tilde{x}^j, \tilde{y}^j) : j \geqslant p\}$. If the net $(\tilde{x}^{j_t}, \tilde{y}^{j_t})_t \longrightarrow (1, \ldots, 1)$, then $(\tilde{x}^{j_t+1}, \tilde{y}^{j_t+1})_t \longrightarrow (\tilde{x}, \tilde{y})$. Consequently, $f(\tilde{x}, \tilde{y}) = 0$ or, in other words, $F_1(\tilde{x}) = F_1(\tilde{y})$. Similarly, $F_j(\tilde{x}) = F_j(\tilde{y})$ for all j. From Lemma 1.1 it follows that $\tilde{x} \sim \tilde{y}$. So $F_j(x - \tilde{x}) = F_j(y - \tilde{y})$ for every $j \geqslant p$; that is,

$$F_j(0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots) = F_j(0, \ldots, 0, y_{k+1}, y_{k+2}, \ldots)$$

for every $j \geqslant p$. If $|x_{k+1}| = 0$ and $|y_{k+1}| = 0$, then $x_i = 0$ and $y_i = 0$ for $i > k$. Let $|x_{k+1}| = a \neq 0$; then we can repeat the above argument for vectors $x' = (x_{k+1}/a, x_{k+2}/a, \ldots)$ and $y' = (y_{k+1}/a, y_{k+2}/a, \ldots)$, and by induction we shall see that $x \sim y$. □

COROLLARY 1.4. *Let* $x, y \in \ell_p$. *If for some integer* $m \geqslant p$, $F_i(x) = F_i(y)$ *for each* $i \geqslant m$, *then* $x \sim y$.

Proof. Since $m \geqslant p$, then $x, y \in \ell_m$, and from Theorem 1.3 it follows that $x \sim y$ in ℓ_m. So $x \sim y$ in ℓ_p. □

PROPOSITION 1.5 (Nullstellensatz). *Let* $P_1, \ldots, P_m \in \mathscr{P}_s(\ell_p)$ *be such that* $\ker P_1 \cap \ldots \cap \ker P_m = \emptyset$. *Then there are* $Q_1, \ldots, Q_m \in \mathscr{P}_s(\ell_p)$ *such that*

$$\sum_{i=1}^{m} P_i Q_i \equiv 1.$$

Proof. Let $n = \max_i(\deg P_i)$. We may assume that $P_i(x) = g_i(F_p(x), \ldots, F_n(x))$ for some $g_i \in \mathscr{P}(\mathbb{C}^{n-p+1})$. Let us suppose that at some point $\xi \in \mathbb{C}^{n-p+1}$, $\xi = (\xi_1, \ldots, \xi_{n-p+1})$ and $g_i(\xi) = 0$. Then by Corollary 1.2 there is $x_0 \in \ell_p$ such that $F_i(x_0) = \xi_i$. So the common set of zeros of all g_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $q_1, \ldots, q_m$ such that $\sum_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(F_p(x), \ldots, F_n(x))$. □

2. *Finitely generated symmetric algebras*

Let us denote by $\mathscr{P}_s^n(\ell_p)$, $n \geqslant p$, the subalgebra of $\mathscr{P}_s(\ell_p)$ generated by $\{F_p, \ldots, F_n\}$. By appealing to Corollary 1.2, one easily verifies that $\mathscr{P}_s^n(\ell_p) \cap \mathscr{P}(^k\ell_p)$ is a *sup-norm* closed subspace of $\mathscr{P}(^k\ell_p)$ for every $k \in \mathbb{N}$.

Let $A_{us}^n(B_{\ell_p})$ and $\mathscr{H}_{bs}^n(\ell_p)$ be the closed subalgebras of $A_{us}(B_{\ell_p})$ and $\mathscr{H}_{bs}(\ell_p)$ generated by $\{F_p, \ldots, F_n\}$, that is, the closure of $\mathscr{P}_s^n(\ell_p)$ in each of the corresponding algebras. Note that for any $f \in \mathscr{H}_{bs}^n(\ell_p)$, with f having Taylor series $f = \sum P_k$ about 0, we have $P_k \in \mathscr{P}_s^n(\ell_p)$. Indeed, if $f \in \mathscr{P}_s^n(\ell_p)$, it is immediate that $P_k \in \mathscr{P}_s^n(\ell_p) \cap \mathscr{P}(^k\ell_p)$ for all k. Then the same holds for any $f \in \mathscr{H}_{bs}^n(\ell_p)$ if one recalls the continuity of the map that assigns to a holomorphic function its kth Taylor polynomial.

By [**6**, III. 1.4], we may identify the spectrum of $A_{us}^n(B_{\ell_p})$ with the joint spectrum of $\{F_p, \ldots, F_n\}$, denoted by $\sigma(F_p, \ldots, F_n)$. It is well known that $\mathscr{M}(\mathscr{H}(\mathbb{C}^n)) = \mathbb{C}^n$ in the sense that all continuous homomorphisms are evaluations at some point in $\mathbb{C}^n$.

Let us denote by $\mathscr{F}_p^n$ the mapping from ℓ_p to $\mathbb{C}^{n-p+1}$ given by

$$\mathscr{F}_p^n : x \longmapsto (F_p(x), \ldots, F_n(x)).$$

Then $D_p^n := \mathscr{F}_p^n(\overline{B_{\ell_p}})$ is a subset of the closed unit ball of $\mathbb{C}^{n-p+1}$ with the max-norm.

Let K be a bounded set in $\mathbb{C}^n$. Recall that a point x belongs to the *polynomial convex hull* of K, denoted $[K]$, if $|f(x)| \leqslant \sup_{z \in K} |f(z)|$ for every polynomial f. A set is *polynomially convex* if it coincides with its polynomial convex hull. Recall that the sup norm on K of a polynomial coincides with the sup norm on $[K]$. It is well known (see, for example, [**6**]) that the spectrum of the uniform Banach algebra $P(K)$ generated by polynomials on the compact set K coincides with the polynomially convex hull of this set. Thus, $[D_p^n]$ denotes the polynomial convex hull of D_p^n.

THEOREM 2.1.
- (i) *The composition operator* $C_{\mathscr{F}_p^n} : \mathscr{H}(\mathbb{C}^{n+1-p}) \longrightarrow \mathscr{H}_{bs}^n(\ell_p)$ *given by* $C_{\mathscr{F}_p^n}(g) = g \circ \mathscr{F}_p^n$ *is a topological isomorphism.*
- (i′) *The composition operator* $C_{\mathscr{F}_p^n} : P([D_p^n]) \longrightarrow A_{us}^n(B_{\ell_p})$ *given by* $C_{\mathscr{F}_p^n}(g) = g \circ \mathscr{F}_p^n$ *is a topological isomorphism.*
- (ii) $\mathscr{M}(\mathscr{H}_{bs}^n(\ell_p)) = \mathbb{C}^{n+1-p}$.
- (ii′) $\mathscr{M}(A_{us}^n(B_{\ell_p})) = [D_p^n]$.

Proof. Clearly, the composition operators are well defined and one-to-one, so it remains to prove that they are onto.

In (i), let $f \in \mathscr{H}^n_{bs}(\ell_p)$, and let $f = \sum P_k$ be the Taylor series expansion of f at 0. Since $P_k \in \mathscr{P}^n_s(\ell_p)$, there is a homogeneous polynomial $g_k \in \mathscr{P}(\mathbb{C}^{n+1-p})$ such that $P_k(x) = g_k(F_p(x), \dots, F_n(x))$. Put $g(\xi_1, \dots, \xi_{n-p+1}) = \sum_{k=1}^{\infty} g_k(\xi_1, \dots, \xi_{n-p+1})$; since g is a convergent power series in each variable, it is separately holomorphic, and hence holomorphic. Note that $f = g \circ \mathscr{F}^n_p$.

In (i′), observe that for any $g \in P([D^n_p])$,

$$\|C_{\mathscr{F}^n_p}(g)\| = \sup_{x \in \overline{B_{\ell_p}}} |g \circ \mathscr{F}^n_p(x)| = \|g\|_{D^n_p} = \|g\|_{[D^n_p]}.$$

Thus $C_{\mathscr{F}^n_p}$ is an isometry, and hence its range is a closed subspace, which moreover contains $\mathscr{P}^n_s(\ell_p)$; therefore, $C_{\mathscr{F}^n_p}$ is onto $A^n_{us}(B_{\ell_p})$.

Statements (ii) and (ii′) follow from (i) and (i′). □

To conclude, we record the following elementary result, which will be needed in Section 3.

Lemma 2.2. *If* $(\xi^0_1, \dots, \xi^0_m) \in [D^m_p]$ *and* $n < m$, *then* $(\xi^0_1, \dots, \xi^0_n) \in [D^n_p]$.

Proof. If $(\xi^0_1, \dots, \xi^0_n) \notin [D^n_p]$, there is a polynomial q in n variables such that

$$|q(\xi^0_1, \dots, \xi^0_n)| > \sup_{(\xi_1,\dots,\xi_n) \in D^n_p} |q(\xi_1, \dots, \xi_n)|.$$

Consider the polynomial $\tilde{q}$ in m variables, given by $\tilde{q}(\xi_1, \dots, \xi_m) = q(\xi_1, \dots, \xi_n)$. Then

$$\begin{aligned} \sup_{(\xi_1,\dots,\xi_m) \in D^m_p} |\tilde{q}(\xi_1, \dots, \xi_m)| &= \sup_{x \in B_{\ell_p}} |\tilde{q}(F_p(x), \dots, F_{p+m-1}(x))| \\ &= \sup_{x \in B_{\ell_p}} |q(F_p(x), \dots, F_{p+n-1}(x))| \\ &< |q(\xi^0_1, \dots, \xi^0_n)| \\ &= |\tilde{q}(\xi^0_1, \dots, \xi^0_m)|. \end{aligned}$$

But this means that $(\xi^0_1, \dots, \xi^0_m) \notin [D^m_p]$, a contradiction. □

3. *Spectrum of* $A_{us}(B_{\ell_p})$

In the study of the spectrum of $A_{us}(B_{\ell_p})$, the most decisive feature is that the polynomials $\{F^n_p\}_{n=p}^{\infty}$ generate a dense subalgebra. Actually, for every $f \in A_{us}(B_{\ell_p})$, its Taylor polynomials are easily seen to be symmetric, using the fact that each such polynomial can be calculated by integrating f (see, for example, [**3**]).

Note that there are symmetric holomorphic functions on B_{ℓ_p} that are not in $A_{us}(B_{\ell_p})$. One such example is $f = \sum_{k=p}^{\infty} F_k$. To see that f is holomorphic on the *open* ball B_{ℓ_p}, let $x \in B_{\ell_p}$ be arbitrary, and choose $\rho < 1$ such that $\|x\| < \rho$. Then $\sum_{k=p}^{\infty} |F_k(x)|$ converges, since the sequence $(F_k(x/\rho)) = (F_k(x)/\rho^k)$ is null. On the other hand, $f \notin A_{us}(B_{\ell_p})$, since $f(te_1) = t^p/(1 - t^p) \to \infty$ as $t \uparrow 1$.

First, we shall show that the spectrum of the uniform algebra of symmetric holomorphic functions on B_{ℓ_p} does not coincide with equivalence classes of point evaluation functionals. The example also shows that D^n_p is not closed.

Example 3.1. For every n, put

$$v_n = \frac{1}{n^{1/p}}(e_1 + \dots + e_n) \in \overline{B_{\ell_p}}.$$

Then $\delta_{v_n}(F_p) = 1$ and $\delta_{v_n}(F_j) \to 0$ as $n \to \infty$ for every $j > p$. By the compactness of $\mathscr{M}(A_{us}(B_{\ell_p}))$ there is an accumulation point ϕ of the sequence $\{\delta_{v_n}\}$. Then $\phi(F_p) = 1$ and $\phi(F_j) = 0$ for all $j > p$. From Corollary 1.4 it follows that there is no point z in ℓ_p such that $\delta_z = \phi$. Another, more geometric, way of looking at this example is to fix $k \in \mathbb{N}$ and consider $D_p^{p+k} \subset \mathbb{C}^{k+1}$. It is straightforward that $(1, 0, \ldots, 0) \notin D_p^{p+k}$ for $k \geqslant p$, although this point is a limit of the sequence

$$(\mathscr{F}_p^{p+k}(v_n)) = \left(1, \frac{1}{n^{1/p}}, \ldots, \frac{1}{n^{(k-1)/p}}\right).$$

Intuitively, the accumulation point ϕ corresponds to the point $(1, 0, \ldots, 0, \ldots) \in \overline{B_{\ell_\infty}}$.

Let us denote $\Sigma_p := \{(a_i)_{i=p}^{\infty} \in \ell_\infty : (a_i)_{i=p}^{n} \in [D_p^n]$ for every $n\}$. As a consequence of Lemma 2.2, Σ_p is the *limit of the inverse sequence* $\{[D_p^n], \pi_n^m, \mathbb{N}\}$, where $\pi_n^m : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is the projection onto the first n coordinates; see [**4**, 2.5]. When Σ_p is endowed with the product topology (that is, the topology of coordinatewise convergence), it is a non-empty compact Hausdorff space by [**4**, 3.2.13]. Also, Σ_p is a weak-star compact subset of the closed unit ball ℓ_∞, since the weak star topology and the pointwise convergence topology coincide on the closed unit ball of ℓ_∞.

Now we describe the spectrum of $A_{us}(B_{\ell_p})$. It is immediate that it is a connected set; it suffices to recall Shilov's idempotent theorem [**6**, III.6.5], and to notice that there are no idempotent elements in $A_{us}(B_{\ell_p})$.

Theorem 3.2. *Σ_p is homeomorphic to the spectrum of $A_{us}(B_{\ell_p})$.*

Proof (See [**10**, 8.3]). First of all, observe that any $\Psi \in \mathscr{M}(A_{us}(B_{\ell_p}))$ is completely determined by the sequence of values $\{\Psi(F_n)\}$, since Ψ is determined by its behaviour on $\mathscr{P}_s(\ell_p)$, the algebra generated by $\{F_n\}$, which is in turn dense in $A_{us}(B_{\ell_p})$.

We construct an embedding

$$j : (a_i)_{i=p}^{\infty} \in \Sigma_p \rightsquigarrow \Phi \in \mathscr{M}(A_{us}(B_{\ell_p})),$$

and prove that it is a homeomorphism. Given $(a_i)_{i=p}^{\infty} \in \Sigma_p$, a homeomorphism $j[(a_i)_{i=p}^{\infty}] := \Phi$ on $A_{us}(B_{\ell_p})$ is defined in the following way. Every polynomial $P \in \mathscr{P}_s(\ell_p)$ may be written as $g \circ \mathscr{F}_p^n$ for some $n \in \mathbb{N}$ and some polynomial g in $n - p + 1$ variables. Thus we may define $\Phi(P) := g(a_p, \ldots, a_n)$. Certainly, $\Phi(P)$ is well defined, since if $P = h \circ \mathscr{F}_p^m$ for some other polynomial h, and, say, $m > n$, then (by Corollary 1.2) $h = \tilde{g}$, where $\tilde{g}$ has the same meaning as in Lemma 2.2. Hence $g(a_p, \ldots, a_n) = \tilde{g}(a_p, \ldots, a_n, \ldots, a_m) = h(a_p, \ldots, a_n, \ldots, a_m)$. It is easy now to see that Φ is linear and multiplicative on the subalgebra of symmetric polynomials. Also, $|\Phi(P)| = |g(a_p, \ldots, a_n)| \leqslant \|g\|_{[D_p^n]} = \|g\|_{D_p^n} \leqslant \|P\|$. Therefore Φ is uniformly continuous on $\mathscr{P}_s(\ell_p)$, and hence it has a continuous linear and multiplicative extension to the closure of $\mathscr{P}_s(\ell_p)$, that is, to $A_{us}(B_{\ell_p})$. We still denote this extension by Φ.

Obviously, j is one-to-one. Moreover, j is also an onto mapping; indeed, for any $\Psi \in \mathscr{M}(A_{us}(B_{\ell_p}))$, the sequence $\{\Psi(F_n)\} \in \Sigma_p$ because $\{\Psi(F_n)_{n=p}^{m}\}$ is an element of the joint spectrum of $\mathscr{M}(A_{us}^m(B_{\ell_p}))$ (obtained just by taking the restriction of Ψ to $A_{us}^n(B_{\ell_p})$), which we know to be $[D_p^m]$. Of course, $j[\{\Psi(F_n)\}] = \Psi$, since they coincide on each F_n.

Next, this embedding is continuous. To see this, observe first that the spectrum $\mathscr{M}(A_{us}(B_{\ell_p}))$ is an equicontinuous subset of the dual space $(A_{us}(B_{\ell_p}))^*$. Therefore, the weak-star topology coincides on it with the topology of pointwise convergence on the elements of the dense set of all symmetric polynomials, and hence on the generating system $\{F_n\}_{n=p}^{\infty}$.

Finally, j is a homeomorphism as the continuous bijection between two compact Hausdorff spaces. □

We can view Σ_p as the '*joint spectrum*' of the sequence $\{F_n\}_{n=p}^{\infty}$, since $\Phi(F_n) = a_n$. We denote by $\mathscr{F}_p$ the mapping $x \in \overline{B_{\ell_p}} \rightsquigarrow (F_p^n(x)) \in \mathbb{C}^{\mathbb{N}}$. Note that $\mathscr{F}_p(\overline{B_{\ell_p}}) \subset \Sigma_p$. So we may remark that the set $D_p = \mathscr{F}_p(\overline{B_{\ell_p}}) \subset \Sigma_p$ corresponds to the set of point evaluation multiplicative functionals on $A_{us}(B_{\ell_p})$. Actually, we have

$$D_p \subset B_{c_0} \cup \{(e^{pi\theta}, \dots, e^{ni\theta}, \dots) |\ \theta \in [0, 2\pi]\}.$$

To see this, we first let $x \in \overline{B_{\ell_p}}$ be such that $|x_m| < 1$ for all $m \in \mathbb{N}$. Then, as we observed in the proof of Theorem 1.3, the sequence $(F_n(x))_{n=p}^{\infty}$ converges to 0. In the case where $x \in \overline{B_{\ell_p}}$ is such that $|x_{m'}| = 1$ for some $m' \in \mathbb{N}$, then m' is unique, $x_{m'} = e^{i\theta}$ and, further, $x_m = 0$ if $m \neq m'$. Thus $F_n(x) = e^{ni\theta}$.

It is clear that $\overline{D_p^n} \subset [D_p^n]$, but we do not know whether this embedding is proper. This is related to a corona-type theorem for $A_{us}(B_{\ell_p})$, since D_p is dense in Σ_p if $\overline{D_p^n} = [D_p^n]$ for all $n \in \mathbb{N}$.

Note that if $q > p$, then $D_p \subset D_q$ and the inclusion is strict. Indeed, let $x \in B_{\ell_q}$ so that $x \notin \ell_p$. If $\mathscr{F}_q(y) = \mathscr{F}_p(x)$ for some $y \in \ell_q$, then $x \sim y$ in ℓ_q and so $x \sim y$ in ℓ_p, which is a contradiction.

PROPOSITION 3.3. *$\Sigma_p \subset \ell_\infty$ is polynomially convex and coincides with the polynomial convex hull of $D_p \subset (\ell_\infty, \tau_{pw})$.*

Proof. Let $(a_i)_{i=p}^{\infty} \in \ell_\infty$ be such that $|P((a_i))| \leqslant \|P\|_{\Sigma_p}$ for all polynomials $P \in \mathscr{P}(\ell_\infty)$. For any $n \geqslant p$ and any $g \in \mathscr{P}(\mathbb{C}^{n+1-p})$, the mapping Q given by $(x_i)_{i=p}^{\infty} \in \ell_\infty \rightsquigarrow g(x_p, \dots, x_n)$ is a polynomial on ℓ_∞. Hence

$$|g(a_p, \dots, a_n)| = |Q((a_i))| \leqslant \|Q\|_{\Sigma_p} \leqslant \|g\|_{[D_p^n]}.$$

Therefore $(a_p, \dots, a_n) \in [D_p^n]$, as we want, and Σ_p is polynomially convex. So, to finish, it is enough to check that Σ_p is contained in the polynomial convex hull of D_p. To do this, let $(a_i)_{i=p}^{\infty} \in \Sigma_p$, and let $P \in \mathscr{P}((\ell_\infty, \tau_{pw}))$. As P is pointwise continuous, it depends on a finite number of variables, say $x_p, \dots, x_n$. Thus the mapping q given by $(x_p, \dots, x_n) \rightsquigarrow P(x_p, \dots, x_n, 0, \dots, 0, \dots)$ is a polynomial on $\mathbb{C}^{n+1-p}$. Since $(a_p, \dots, a_n) \in [D_p^n]$,

$$\begin{aligned}|P((a_i))| &= |P(a_p, \dots, a_n, 0, \dots, 0, \dots)| \\ &= |q(a_p, \dots, a_n)| \leqslant \|q\|_{[D_p^n]} \\ &= \|q\|_{D_p^n} \leqslant \|P\|_{D_p},\end{aligned}$$

and it follows that $(a_i)_{i=p}^{\infty}$ belongs to the polynomial convex hull of D_p. □

THEOREM 3.4. *There is an algebraic and topological isomorphism between $A_{us}(B_{\ell_p})$ and the uniform Banach algebra on Σ_p generated by the $w^*(\ell_\infty, \ell_1)$ continuous coordinate functionals $\{\pi_k\}_{k=p}^{\infty}$.*

Proof. For every $f \in A_{us}(B_{\ell_p})$ and $\Phi \in \mathscr{M}(A_{us}(B_{\ell_p}))$, denote by $\hat{f}(\Phi) = \Phi(f)$ the standard Gelfand transform, which is known to be an algebraic isometry into $C(\Sigma_p)$. Recall that the range of the Gelfand transform is a closed subalgebra which, as we are going to see, will coincide with A_p, the uniform Banach subalgebra of $C(\Sigma_p)$ generated by the coordinate functionals $\{\pi_k\}_{k=p}^{\infty}$.

Since $\hat{F}_k(\xi) = \xi_k$ for $\xi = (\xi_i)_i \in \Sigma_p$, it follows that the Gelfand transform of F_k is the kth coordinate functional on ℓ_∞. As $A_{us}(B_{\ell_p})$ is the closure of the algebra generated by $\{F_k : k \geqslant p\}$, it follows that $\hat{f} \in A_p$ for every $f \in A_{us}(B_{\ell_p})$. Therefore A_p is precisely the range of the Gelfand transform. □

PROPOSITION 3.5. *The mapping* $S : f \in A(D) \longrightarrow F \in A_{us}(B_{\ell_p})$ *defined by* $F((x_i)) = \sum_{i=1}^{\infty} x_i^p f(x_i)$ *is an isometry onto the closed subspace* $\mathscr{F}$ *of* $A_{us}(B_{\ell_p})$ *generated by* $\{F_{k+p}\}_{k=0}^{\infty}$.

Proof. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be the Taylor series expansion. For each $(x_i) \in B_{\ell_p}$, put

$$F((x_i)) := \sum_{k=0}^{\infty} c_k F_{k+p}((x_i)) = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} c_k x_i^{p+k}.$$

Since $|F_{k+p}((x_i))| \leqslant \|(x_i)\|^{p+k}$ and the series $\sum_{k=0}^{\infty} c_k t^k$ is absolutely convergent in the open unit disc,

$$\sum_{k=0}^{\infty}\sum_{i=1}^{\infty} \left|c_k x_i^{p+k}\right| = \sum_{k=0}^{\infty} |c_k| \sum_{i=1}^{\infty} \left|x_i^{p+k}\right| = \sum_{k=0}^{\infty} |c_k| F_{k+p}((|x_i|))$$
$$\leqslant \sum_{k=0}^{\infty} |c_k| (\|(x_i)\|^{p+k}) = \|(x_i)\|^p \sum_{k=0}^{\infty} |c_k| (\|(x_i)\|^k) < \infty.$$

So $F((x_i))$ is well defined, and $F((x_i)) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} c_k x_i^{p+k} = \sum_{i=1}^{\infty} x_i^p f(x_i)$.

Also

$$|F((x_i))| = \left|\sum_{i=1}^{\infty} x_i^p f(x_i)\right| \leqslant \sum_{i=1}^{\infty} \left|x_i^p\right| |f(x_i)| \leqslant \|f\|_D \|(x_i)\|^p,$$

and hence $\|F\|_{B_{\ell_p}} \leqslant \|f\|_D$. On the other hand, if $a \in D$ and $x_0 = (a, 0, \ldots, 0, \ldots)$, we have $x_0 \in B_{\ell_p}$ and $|F(x_0)| = |a|^p |f(a)|$. By the maximum principle, it follows that $\|F\|_{B_{\ell_p}} \geqslant \|f\|_D$. Consequently, $\|F\|_{B_{\ell_p}} = \|f\|_D$.

Now we check that $F \in A_{us}(B_{\ell_p})$ and then that, actually, $F \in \mathscr{F}$. To do this, let $s_m(t) = \sum_{k=0}^{m} c_k t^k$ be the partial sums of the Taylor series of f, and let $\psi_n = (1/n)(s_0 + s_1 + \ldots + s_n)$ be the Cesàro means. Put

$$S_m((x_i)) = \sum_{k=0}^{m} c_k F_{k+p}((x_i)) = \sum_{i=1}^{\infty} x_i^p s_m(x_i).$$

Then

$$\Psi_n((x_i)) = \frac{1}{n}(S_0((x_i)) + S_1((x_i)) + \ldots + S_n((x_i)))$$
$$= \frac{1}{n}\sum_{i=1}^{\infty} x_i^p (s_0(x_i) + s_1(x_i) + \ldots + s_n(x_i)) = \sum_{i=1}^{\infty} x_i^p \psi_n(x_i)$$

are the Cesáro means partial sums of $\sum_{k=0} c_k F_{k+p}$.

Since

$$|\Psi_n((x_i)) - F((x_i))| = \left|\sum_{i=1}^{\infty} x_i^p(\psi_n(x_i) - f(x_i))\right| \leqslant \|\psi_n - f\| \cdot \|(x_i)\|,$$

the uniform convergence of ψ_n to f on D implies the uniform convergence of Ψ_n to F on B_{ℓ_p}. So $F \in A_{us}(B_{\ell_p})$ and, moreover, $F \in \mathscr{F}$, since every Ψ_n is obviously in $\mathscr{F}$.

The mapping S being an isometry, its range is a closed subspace of $A_{us}(B_{\ell_p})$. Therefore, its range is onto $\mathscr{F}$ since F_{k+p} is the image of z^k. □

PROPOSITION 3.6. *$\Sigma_p \neq \bar{B}_{\ell_\infty}$ for every positive integer p.*

Proof. We show that no point of the form $(e^{i\theta}, \pm 1, 0, \ldots, 0, \ldots)$ is in Σ_p. This will follow from Proposition 3.5, applied to every linear fractional transformation $f(z) = (z - a)/(1 - \bar{a}z)$, $|a| < 1$, for which the Taylor series

$$f(z) = -a + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1 - |a|^2)z^n$$

has radius of convergence bigger than 1. Its image F by the mapping S in Proposition 3.5 is $F = -aF_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1 - |a|^2)F_{n+p}$. Moreover, the convergence of this series is uniform on B_{ℓ_p}, and therefore the Gelfand transform of F is

$$\hat{F} = -a\pi_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1 - |a|^2)\pi_{n+p}.$$

Choose θ such that $-ae^{i\theta} = |a|$, and assume that the point $(e^{i\theta}, 1, 0, \ldots, 0, \ldots)$ is in Σ_p. Then $|\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| \leqslant \|F\| = \|f\| = 1$. However,

$$\begin{aligned} |\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| &= \left|\left(-a\pi_p + \sum_{n=1}^{\infty} \bar{a}^{n-1}(1 - |a|^2)\pi_{n+p}\right)(e^{i\theta}, 1, 0, \ldots, 0, \ldots)\right| \\ &= \left| -ae^{i\theta} + 1 - |a|^2 \right| = |a| + 1 - |a|^2 > 1, \end{aligned}$$

which is a contradiction. □

We remark that arguments similar to those in Theorem 1.3 enable us to show that no point of the form $(1, -1, -1, z_4, z_5, \ldots) \in \overline{B_{\ell_\infty}}$ can be in Σ_p.

Our final result describes the class of functionals on ℓ_∞ that belong to the range of $A_{us}(B_{\ell_p})$ under the Gelfand transform, thereby completing a circle of connections between $A_{us}(B_{\ell_p})$, $A(D), C(\Sigma_p)$, and certain functionals on ℓ_∞. Recall that such Gelfand transforms are weak-star continuous on Σ_p.

PROPOSITION 3.7. *Let ϕ be a linear functional on ℓ_∞ that is weak-star continuous on Σ_p. Then ϕ is the Gelfand transform of some $F \in A_{us}(B_{\ell_p})$ and, furthermore, there is $f \in A(D)$ with $\|\phi\|_{\Sigma_p} = \|f\|_D$ and such that*

$$\phi(\mathscr{F}_p(x)) = \sum_{i=1}^{\infty} a_i^p f(a_i), \quad x = (a_i) \in B_{\ell_p}.$$

Proof. Every $(a_i)_{i=p}^{\infty} \in \Sigma_p$ is the $w(\ell_\infty, \ell_1)$ convergent series $\sum_{i=p}^{\infty} a_i e_i$. Therefore $\phi((a_i)) = \sum_{i=p}^{\infty} a_i \phi(e_i)$ and, setting $c_i = \phi(e_i)$, we find that the series $\sum_{i=p}^{\infty} c_i \pi_i$ is pointwise convergent in Σ_p to ϕ. Moreover, the partial sums of this series are uniformly bounded on Σ_p, since

$$\left|\sum_{j=p}^{l} c_j\pi_j((a_i))\right| = \left|\sum_{j=p}^{l} c_j a_j\right| = \left|\sum_{j=p}^{l} \phi(e_j)a_j\right|$$
$$= |\phi(a_p,\ldots,a_l,0,\ldots,0,\ldots)| \leqslant \|\phi\|_{\ell_\infty}.$$

Thus ϕ is the weak limit in $C(\Sigma_p)$ of the series $\sum_{i=p}^{\infty} c_i\pi_i$. Since each of the terms in the series belongs to the range of the Gelfand transform, it follows that there is $F \in A_{us}(B_{\ell_p})$ such that $\hat{F} = \phi$, and also that the series $F = \sum_{i=p}^{\infty} c_iF_i$ converges weakly in $A_{us}(B_{\ell_p})$.

Note that $\|\phi\|_{\Sigma_p} = \|F\|_{B_{\ell_p}}$, and also that F belongs to the weakly closed subspace $\mathscr{F}$ generated by $\{F_{k+p}\}_{k=0}^{\infty}$. Thus by Proposition 3.5 there is $f \in A(D)$ such that $F(x) = F(\sum_{i=1}^{\infty} x_ie_i) = \sum_{i=1}^{\infty} x_i^p f(x_i)$. Therefore, $\phi(\mathscr{F}_p(x)) = \hat{F}(\mathscr{F}_p(x)) = F(x)$, as we wanted. □

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