Arch. Math. 89 (2007), 157–166

0003/889X/020157-10, *published online* 2007-07-16
DOI 10.1007/s00013-007-2043-4

Archiv der Mathematik

Polynomial automorphisms and hypercyclic operators on spaces of analytic functions

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Abstract. We consider hypercyclic composition operators on $H(\mathbb{C}^n)$ which can be obtained from the translation operator using polynomial automorphisms of $\mathbb{C}^n$. In particular we show that if C_S is a hypercyclic operator for an affine automorphism S on $H(\mathbb{C}^n)$, then $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$ for some polynomial automorphism Θ and vectors a and b, where I is the identity operator. Finally, we prove the hypercyclicity of "symmetric translations" on a space of symmetric analytic functions on ℓ_1.

Mathematics Subject Classification (2000). Primary: 47A16; Secondary: 46E10, 46E50.

Keywords. Hypercyclic operators, functional spaces, polynomial automorphisms, symmetric functions.

Introduction. Let X be a Fréchet linear space. A continuous linear operator $T : X \to X$ is called *hypercyclic* if there is a vector $x_0 \in X$ for which the orbit under T, $\mathrm{Orb}(T, x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$ is dense in X. Every such vector x_0 is called a *hypercyclic vector* of T. The classical Birkhoff theorem [5] asserts that any operator of composition with translation $x \mapsto x + a$, $T_a \colon f(x) \mapsto f(x + a)$ is hypercyclic on the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$ if $a \neq 0$. The Birkhoff translation T_a has also been regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

A generalization of the Birkhoff theorem was proved by Godefroy and Shapiro in [8]. They showed that if $\varphi(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \mapsto \sum_{|\alpha|\geq 0} c_\alpha D^\alpha f, \qquad f \in H(\mathbb{C}^n) \tag{1}$$

is hypercyclic. Moreover, in [8] it is proved that any continuous linear operator T on $H(\mathbb{C}^n)$ which commutes with translations and is not a scalar multiple of the identity, can be expressed by (1) and so is hypercyclic as well.

Let us recall that an operator C_Φ on $H(\mathbb{C}^n)$ is said to be a *composition operator* if $C_\Phi f(x) = f(\Phi(x))$ for some analytic map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$. It is known that only translation operator T_a for some $a \neq 0$ is a hypercyclic composition operator on $H(\mathbb{C})$ [4]. However, if $n > 1$, $H(\mathbb{C}^n)$ supports more hypercyclic composition operators. In [3] Bernal-González established some necessary and sufficient conditions for a composition operator by an affine map to be hypercyclic. In particular, in [3] it is proved that a given affine automorphism $S = A + b$ on $\mathbb{C}^n$, the composition operator $C_S : f(x) \mapsto f(S(x))$ is hypercyclic if and only if the linear operator A is bijective and the vector b is not in the range of $A - I$.

In this paper we consider hypercyclic composition operators on $H(\mathbb{C}^n)$ which can not be described by formula (1) but can be obtained from the translation operator using polynomial automorphisms of $\mathbb{C}^n$. To do it we developed a method which involves the theory of symmetric analytic functions on Banach spaces.

In Section 1 we discuss some relationship between polynomial automorphisms on $\mathbb{C}^n$ and the operation of changing of polynomial bases in an algebra of symmetric analytic functions on the Banach space of summing sequences, ℓ_1. In Section 2 we consider operators of the form $C_\Theta{}^{-1} T_b C_\Theta$ for a polynomial automorphism Θ and show that if C_S is a hypercyclic operator for some affine automorphism S on $\mathbb{C}^n$, then there exists a representation of the form $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$ that is we can write $C_S = C_\Theta{}^{-1} T_b C_\Theta T_a$. To do it we use the method of symmetric polynomials on ℓ_1 as an important tool for constructing and computations. Finally, in the third section we prove the hypercyclicity of a special operator on an algebra of symmetric analytic functions on ℓ_1 which plays the role of translation in this algebra.

For details of the theory of analytic functions on Banach spaces we refer the reader to Dineen's book [7]. Note that an analog of the Godefroy-Shapiro Theorem for a special class of entire functions on Banach space with separable dual was proved by Aron and Bés in [2]. Current state of theory of symmetric analytic functions on Banach spaces can be found in [1, 9]. A detailed survey of hypercyclic operators is given by Grosse-Erdmann in [10].

1. Polynomial automorphisms and symmetric functions. A polynomial map $\Phi = (\Phi_1, \dots, \Phi_n)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ is said to be a *polynomial automorphism* if it is invertible and the inverse map is also a polynomial.

Let X be a Banach space with a symmetric basis $(e_i)_{i=1}^{\infty}$. A function g on X is called *symmetric* if for every $x = \sum_{i=1}^{\infty} x_i e_i \in X$,

$$g(x) = g\Big(\sum_{i=1}^{\infty} x_i e_i\Big) = g\Big(\sum_{i=1}^{\infty} x_i e_{\sigma(i)}\Big)$$

for an arbitrary permutation σ on the set $\{1, ..., m\}$ for any positive integer m. The sequence of homogeneous polynomials $(P_j)_{j=1}^{\infty}$, $\deg P_k = k$ is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials if for every symmetric polynomial P of degree n on X there exists a polynomial q on $\mathbb{C}^n$ such that

$$P(x) = q(P_1(x), \dots , P_n(x)).$$

Throughout this paper we consider the case when $X = \ell_1$. We denote by $\mathcal{P}_s(\ell_1)$ the algebra of all symmetric polynomials on ℓ_1. The next two algebraic bases of $\mathcal{P}_s(\ell_1)$ are useful for us: $(F_k)_{k=1}^{\infty}$ (see [9]) and $(G_k)_{k=1}^{\infty}$, where

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k \quad \text{and} \quad G_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

By the Newton formula $G_1 = F_1$ and for every $k > 1$,

$$G_{k+1} = \frac{1}{k+1}((-1)^k F_{k+1} - F_k G_1 + \cdots + F_1 G_k).$$

Denote by $H_s^n(\ell_1)$ the algebra of entire symmetric functions on ℓ_1 which is topologically generated by polynomials $F_1, \dots , F_n$. It means that $H_s^n(\ell_1)$ is the completion of the algebraic span of $F_1, \dots , F_n$ in the uniform topology on bounded subsets. We say that polynomials $P_1, \dots , P_n$ (not necessary homogeneous) form an *algebraic basis* in $H_s^n(\ell_1)$ if they topologically generate $H_s^n(\ell_1)$. Evidently, if $(P_j)_{j=1}^{\infty}$ is a homogeneous algebraic basis in $\mathcal{P}_s(\ell_1)$, then $(P_1, \dots , P_n)$ is an algebraic basis in $H_s^n(\ell_1)$. We will use notations $\mathbf{F} := (F_k)_{k=1}^n$ and $\mathbf{G} := (G_k)_{k=1}^n$.

Proposition 1.1. *Let $\Phi = (\Phi_1, \dots , \Phi_n)$ be a polynomial automorphism on $\mathbb{C}^n$. Then $(\Phi_1(\mathbf{P}), \dots , \Phi_n(\mathbf{P}))$ is an algebraic basis in $H_s^n(\ell_1)$ for an arbitrary algebraic basis $\mathbf{P} = (P_1, \dots , P_n)$.*

Conversely, if $(\Phi_1(\mathbf{P}), \dots , \Phi_n(\mathbf{P}))$ is an algebraic basis for some algebraic basis $\mathbf{P} = (P_1, \dots , P_n)$ in $H_s^n(\ell_1)$ and a polynomial map Φ on $\mathbb{C}^n$, then Φ is a polynomial automorphism.

Proof. Suppose that Φ is a polynomial automorphism and

$$\Phi^{-1} = ((\Phi^{-1})_1, \dots , (\Phi^{-1})_n)$$

is its inverse. Then $P_k = (\Phi^{-1})_k(\Phi_1(\mathbf{P}), \dots , \Phi_n(\mathbf{P}))$, $1 \le k \le n$. Hence polynomials $\Phi_1(\mathbf{P}), \dots , \Phi_n(\mathbf{P})$ topologically generate $H_s^n(\ell_1)$ and so they form an algebraic basis.

Let now $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ be an algebraic basis in $H_s^n(\ell_1)$ for some algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$. Then for each P_k, $1 \leq k \leq n$, there exists a polynomial q_k on $\mathbb{C}^n$ such that $P_k = q_k(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$. Put $(\Phi^{-1})_k(t) := q_k(t)$, $t \in \mathbb{C}^n$. Since $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis, the map

$$(x_1, \ldots, x_n) \mapsto (\Phi_1(\mathbf{P}(x)), \ldots, \Phi_n(\mathbf{P}(x)))$$

is onto by [1, Lemma 1.1]. Thus $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a bijection and so the mapping $((\Phi^{-1})_1, \ldots, (\Phi^{-1})_n)$ is the inverse polynomial map for Φ. $\square$

2. Similar translations. We start with an evident statement, which actually is a very special case of the Universal Comparison Principle (see e.g. [10, Proposition 4]).

Proposition 2.1. *Let T be a hypercyclic operator on X and A be an isomorphism of X. Then $A^{-1}TA$ is hypercyclic.*

We will say that $A^{-1}TA$ is a *similar* operator to T. If $T = C_R$ is a composition operator on $H(\mathbb{C}^n)$ and $A = C_\Phi$ is a composition by an analytic automorphism Φ of $\mathbb{C}^n$, then $A^{-1}TA = C_{\Phi\circ R\circ\Phi^{-1}}$ is a composition operator too. If A is a composition with a polynomial automorphism, we will say that $A^{-1}TA$ is *polynomially similar* to T. Now we consider operators which are similar to the translation composition $T_a\colon f(x) \mapsto f(x+a)$ on $H(\mathbb{C}^n)$.

Example 2.2. Let $\Phi(t_1, t_2) = (t_1, t_2 - t_1^m)$ for some positive integer m. Clearly, Φ is a polynomial automorphism and $\Phi^{-1}(z_1, z_2) = (z_1, z_2 + z_1^m)$. So

$$\begin{aligned}\Phi(t+a) &= (t_1 + a_1, t_2 + a_2 - (t_1 + a_1)^m) \\ &= \left(t_1 + a_1, t_2 + a_2 - t_1^m - a_1^m - \sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j\right).\end{aligned}$$

Thus we have

$$\Phi\circ(I+a)\circ\Phi^{-1}(t) = \Phi(\Phi^{-1}(t)+a) = \left(t_1+a_1, t_2+a_2-a_1^m-\sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j\right).$$

Hence the composition operator with the $(m-1)$-degree polynomial $\Phi\circ(I+a)\circ\Phi^{-1}$ is similar to the translation operator $T_a = C_{(I+a)}$ and so it must be hypercyclic. Here I is the identity operator.

It is known (see [1]) that the map

$$\mathcal{F}_n^{\mathbf{F}}\colon f(t_1, \ldots, t_n) \mapsto f(F_1(x), \ldots, F_n(x))$$

is a topological isomorphism from the algebra $H(\mathbb{C}^n)$ to the algebra $H_s^n(\ell_1)$. Now we will prove more general statement.

Lemma 2.3. *Let $\mathbf{P} = (P_k)_{k=1}^n$ be an algebraic basis in $H_s^n(\ell_1)$. Then the map*

$$\mathcal{F}_n^{\mathbf{P}} : f(t_1, \dots, t_n) \mapsto f(P_1(x), \dots, P_n(x))$$

is a topological isomorphism from $H(\mathbb{C}^n)$ onto $H_s^n(\ell_1)$.

Proof. Evidently, $\mathcal{F}_n^{\mathbf{P}}$ is a homomorphism. It is known [1] that for every vector $(t_1, \dots, t_n) \in \mathbb{C}^n$ there exists an element $x \in \ell_1$ such that $P_1(x) = t_1, \dots, P_n(x) = t_n$. Therefore the map $\mathcal{F}_n^{\mathbf{P}}$ is injective. Let us show that $\mathcal{F}_n^{\mathbf{P}}$ is surjective. Let $u \in H_s^n(\ell_1)$ and $u = \sum u_k$ be the Taylor series expansion of u at zero. For every homogeneous polynomial u_k there exists a polynomial q_k on $\mathbb{C}^n$ such that $u_k = q_k(P_1, \dots, P_n)$. Put $f(t_1, \dots, t_n) = \sum_{k=1}^{\infty} q_k(t_1, \dots, t_n)$. Since f is a power series which converges for every vector $(t_1, \dots, t_n)$, f is an entire analytic function on $\mathbb{C}^n$. Evidently, $\mathcal{F}_n^{\mathbf{P}}(f) = u$. From the known theorem about automatic continuity of an isomorphism between commutative finitely generated Fréchet algebras [11, p. 43] it follows that $\mathcal{F}_n^{\mathbf{P}}$ is continuous. □

Let $x, y \in \ell_1$, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. We put

$$x \bullet y := (x_1, y_1, x_2, y_2, \dots)$$

and define

$$\mathcal{T}_y(f)(x) := f(x \bullet y).$$

We will say that $x \mapsto x \bullet y$ is the *symmetric translation* and the operator $\mathcal{T}_y$ is the *symmetric translation operator*. It is clear that if f is a symmetric function, then $f(x \bullet y)$ is a symmetric function for any fixed y.

In [6] is proved that $\mathcal{T}_y$ is a topological isomorphism from the algebra of symmetric analytic functions to itself. Evidently, we have that

$$F_k(x \bullet y) = F_k(x) + F_k(y) \tag{2}$$

for every k.

Let $g \in H_s^n(\ell_1)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. Set

$$\mathcal{D}^\alpha g := \mathcal{F}_n^{\mathbf{F}} D^\alpha (\mathcal{F}_n^{\mathbf{F}})^{-1} g = \left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}} f \right)(F_1(\cdot), \dots, F_n(\cdot)),$$

where $f = (\mathcal{F}_n^{\mathbf{F}})^{-1} g$.

Theorem 2.4. *Let $y \in \ell_1$ such that $(F_1(y), \dots, F_n(y))$ is a nonzero vector in $\mathbb{C}^n$. Then the symmetric translation operator $\mathcal{T}_y$ is hypercyclic on $H_s^n(\ell_1)$. Moreover, every operator $\mathcal{A}$ on $H_s^n(\ell_1)$ which commutes with $\mathcal{T}_y$ and is not a scalar multiple of the identity is hypercyclic and can be represented by*

$$\mathcal{A}(g) = \sum_{|\alpha| \geq 0} c_\alpha \mathcal{D}^\alpha g, \tag{3}$$

where c_α are coefficients of a non-constant entire function of exponential type on $\mathbb{C}^n$.

Proof. Let $a = (F_1(y), \dots, F_n(y)) \in \mathbb{C}^n$. If $g \in H_s^n(\ell_1)$, then

$$g(x) = \mathcal{F}_n^{\mathbf{F}}(f)(x) = f(F_1(x), \dots, F_n(x))$$

for some $f \in H_s^n(\ell_1)$ and property (2) implies that

$$\mathcal{T}_y(g)(x) = \mathcal{F}_n^{\mathbf{F}} T_a (\mathcal{F}_n^{\mathbf{F}})^{-1}(g)(x).$$

So the proof follows from Proposition 2.1 and the Godefroy-Shapiro Theorem. □

A given algebraic basis $\mathbf{P}$ on $H_s^n(\ell_1)$ we set

$$T_{\mathbf{P},y} := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}} \quad \text{and} \quad D_{\mathbf{P}}^{\alpha} := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{D}^{\alpha} \mathcal{F}_n^{\mathbf{P}}.$$

Corollary 2.5. *Let $\mathbf{P}$ be an algebraic basis on $H_s^n(\ell_1)$ and let $y \in \ell_1$ such that $(F_1(y), \dots, F_n(y)) \neq 0$. Then the operator $T_{\mathbf{P},y}$ is hypercyclic on $H(\mathbb{C}^n)$. Moreover, every operator A on $H(\mathbb{C}^n)$ which commutes with $T_{\mathbf{P},y}$ and is not a scalar multiple of the identity is hypercyclic and can be represented by the form*

$$A(f) = \sum_{|\alpha| \geq 0} c_\alpha D_{\mathbf{P}}^{\alpha} f, \tag{4}$$

where c_α as in (1).

Note that due to Proposition 1.1 the transformation $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ is nothing else than a composition with $\Phi \circ (I + a) \circ \Phi^{-1}$, where $\Phi(F_1, \dots, F_n) = (P_1, \dots, P_n)$ and $a = (F_1(y), \dots, F_n(y))$. Conversely, every polynomially similar operator to the translation can be represented by the form $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ for some algebraic basis of symmetric polynomials $\mathbf{P}$. This observation can be helpful in order to construct some examples of such operators.

Example 2.6. Let us compute how looks the operator $T_{\mathbf{P},y}$ for $\mathbf{P} = \mathbf{G}$. We observe first that $G_k(x \bullet y) = \sum_{i=0}^k G_i(x) G_{k-i}(y)$, where for the sake of convenience we take $G_0 \equiv 1$. Thus

$$\begin{aligned} \mathcal{T}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \dots, t_n) &= \mathcal{T}_y f(G_1(x), \dots, G_n(x)) = f(G_1(x \bullet y), \dots, G_n(x \bullet y)) \\ &= f\Big(G_1(x) + G_1(y), \dots, \sum_{i=0}^{n} G_i(x) G_{n-i}(y)\Big). \end{aligned}$$

Therefore

$$T_{\mathbf{G},y} f(t_1, \dots, t_n) = f\Big(t_1 + b_1, \dots, \sum_{i=0}^{k} t_i b_{k-i}, \dots, \sum_{i=0}^{n} t_i b_{n-i}\Big), \tag{5}$$

where $t_0 = 1$, $b_0 = 1$ and $b_k = G_k(y)$ for $1 \leq k \leq n$.

According to the Newton formula and Proposition 1.1 the corresponding polynomial automorphism Φ can be given of recurrence form $\Phi_1(t) = t_1$, $\Phi_{k+1}(t) = 1/(k+1)((-1)^k t_{k+1} - t_k \Phi_1(t) + \cdots + t_1 \Phi_k(t))$ which is not so good for computations.

The hypercyclic operator in Example 2.2 is a composition with an $m-1$ degree polynomial and so does not commute with the translation because it can not be generated by formula (1). However, the composition with an affine map in Example 2.6 still does not commute with T_a. Indeed, by (5),

$$T_a \circ T_{\mathbf{G},y} f(t_1, \dots, t_n) = f\Big(t_1 + b_1 + a_1, \dots, \sum_{i=0}^{n} t_i b_{n-i} + a_n\Big);$$

$$T_{\mathbf{G},y} \circ T_a f(t_1, \dots, t_n) = f\Big(t_1 + b_1 + a_1, \dots, \sum_{i=0}^{n} (t_i + a_i) b_{n-i}\Big),$$

where $a_0 = 1$. Evidently, $T_a \circ T_{\mathbf{G},y} \neq T_{\mathbf{G},y} \circ T_a$ for some $a \neq 0$ whenever $b \neq (0, \dots, 0, b_n)$.

Corollary 2.7. *There exists a nonzero vector $b \in \mathbb{C}^n$ and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $\Theta \circ (I+b)\Theta^{-1}(t) = A(t) + c$ where A is a linear operator with the matrix of the form*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{6}$$

and $c \neq 0$.

Proof. We choose $b \in \mathbb{C}^n$ such that all coordinates b_k, $1 \leq k \leq n$ are positive numbers. Let Φ be a polynomial automorphism associated with $T_{\mathbf{G},y}$ in Example 2.6, where $y \in \ell_1$ is such that $G_k(y) = b_k$, $1 \leq k \leq n$. Then, according to (5), we can write $\Phi \circ (I+b)\Phi^{-1}(t) = R(t) + b$, where

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & 1 & 0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 & 1 \end{pmatrix}.$$

We recall that the *index* of an eigenvalue λ of a matrix M is the smallest nonnegative integer k such that $\operatorname{rank}((M-\lambda I)^k) = \operatorname{rank}((M-\lambda I)^{k+1})$. The matrix R has a unique eigenvalue 1 and since all coordinates b_k of b are positive, the index of this eigenvalue is equal to n. Indeed, for each $k < n$, $(R-\lambda I)^k$ contains an $(n-k) \times (n-k)$ triangular matrix with only positive numbers in the main diagonal and $(R-\lambda I)^n = 0$. Therefore, from the Linear Algebra we know that the largest Jordan block A associated with the eigenvalue 1 is $n \times n$ and so it can be represented by (6). Thus there is a linear isomorphism L on $\mathbb{C}^n$ such that $A = LRL^{-1}$. Hence

$$(L \circ \Phi) \circ (I+b) \circ (L \circ \Phi)^{-1}(t) = L \circ (R+b) \circ L^{-1}(t) = A(t) + L(b).$$

So it is enough to set $\Theta := L \circ \Phi$ and $c := L(b)$. $\square$

Theorem 2.8. *Let S be an affine automorphism on $\mathbb{C}^n$ such that C_S is hypercyclic. Then there are vectors a, b and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$.*

Proof. Let $S(t) = A(t) + c$ be an affine map on $\mathbb{C}^n$ such that C_S is hypercyclic. Without loss of the generality we can assume that A is a direct sum of Jordan blocks $A_1, \dots, A_m$ and each block A_j acts on a subspace V_j of $\mathbb{C}^n$. In the proof of Theorem 3.1 of [3] is shown that the spectrum of each block A_j is the singleton $\{1\}$. So each A_j is of the form as in (6). Let $\Theta_{(j)}$ be a polynomial automorphism of V_j as in Corollary 2.7, that is,

$$\Theta_{(j)} \circ (I + b_{(j)}) \circ \Theta_{(j)}^{-1} = A_j + b_{(j)},$$

for some $b_{(j)} \in V_j$. Put $\Theta = \Theta_{(1)} + \cdots + \Theta_{(m)}$ and $b = b_{(1)} + \cdots + b_{(m)}$. Then $\Theta \circ (I + b) \circ \Theta^{-1} = A + b$. Let $a = c - b$. Hence

$$S = A + c = A + b + a = \Theta \circ (I + b) \circ \Theta^{-1} + a.$$

□

Of course, the converse of Theorem 2.8 (with $b \neq 0$) also holds.

We do not know whether it is always possible to choose Θ so that $a = 0$. In other words: *Is every hypercyclic operator which is a composition by an affine automorphism polynomially similar to a translation?* Moreover, we do not know any example of a hypercyclic composition operator on $H(\mathbb{C}^n)$ which is not similar to a translation.

3. The infinity-dimensional case. Let us recall a well known Kitai-Gethner-Shapiro theorem which is also known as the Hypercyclicity Criterion.

Theorem 3.1. *Let X be a separable Fréchet space and $T \colon X \to X$ be a linear and continuous operator. Suppose there exist X_0, Y_0 dense subsets of X, a sequence (n_k) of positive integers and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n \colon Y_0 \to X$ so that*

(1) $T^{n_k}(x) \to 0$ *for every* $x \in X_0$ *as* $k \to \infty$.
(2) $S_{n_k}(y) \to 0$ *for every* $y \in Y_0$ *as* $k \to \infty$.
(3) $T^{n_k} \circ S_{n_k}(y) = y$ *for every* $y \in Y_0$.

Then T is hypercyclic.

The operator T is said to satisfy the *Hypercyclicity Criterion for full sequence* if we can chose $n_k = k$. Note that T_a satisfies the Hypercyclicity Criterion for full sequence [8] and so the symmetric shift $\mathcal{T}_y$ on $H_s^n(\ell_1)$ satisfies the Hypercyclicity Criterion for full sequence provided $(F_1(y), \dots, F_n(y)) \neq 0$.

Finally, we establish our result about hypercyclic operators on the space of symmetric entire functions on ℓ_1. But before this, we need the following general auxiliary statement, which might be of some interest by itself.

Lemma 3.2. *Let X be a Fréchet space and $X_1 \subset X_2 \subset \cdots \subset X_m \subset \cdots$ be a sequence of closed subspaces such that $\bigcup_{m=1}^{\infty} X_m$ is dense in X. Let T be an operator on X such that $T(X_m) \subset X_m$ for each m and each restriction $T|_{X_m}$ satisfies the Hypercyclicity Criterion for full sequence on X_m. Then T satisfies the Hypercyclicity Criterion for full sequence on X.*

Proof. Let $Y_0^{(m)}$ and $X_0^{(m)}$ be dense subsets in X_m, and $S_k^{(m)}$ corresponding sequence of mappings associated with $T|_{X_m}$ as in Theorem 3.1. Put $X_0 = \bigcup_{m=1}^{\infty} X_0^{(m)}$ and $Y_0 = \bigcup_{m=1}^{\infty} Y_0^{(m)}$. It is clear that both X_0 and Y_0 are dense in X. For a given $y \in Y_0$, we denote by $m(y)$ the minimal number m such that $y \in Y_0^{(m)}$. We set $S_k(y) := S_k^{(m(y))}(y)$. Then

$$T^k \circ S_k(y) = T^k|_{X_{m(y)}} \circ S_k^{(m(y))}(y) = y, \quad \forall y \in Y_0$$

and $S_k(y) = S_k^{(m(y))}(y) \to 0$ as $k \to \infty$ for every $y \in Y_0$. Similarly, if $x \in X_0$, then $x \in X_0^{(m)}$ for some m and $T^k(x) = T^k|_{X_m}(x) \to 0$ as $k \to \infty$. So T satisfies the Hypercyclicity Criterion for full sequence on X. In particular, T is hypercyclic. $\square$

We denote by $H_{bs}(\ell_1)$ the Fréchet algebra of symmetric entire functions on ℓ_1 which are bounded on bounded subsets. This algebra is the completion of the space of symmetric polynomials on ℓ_1 endowed with the uniform topology on bounded subsets.

Theorem 3.3. *The symmetric operator $\mathcal{T}_y$ is hypercyclic on $H_{bs}(\ell_1)$ for every $y \neq 0$.*

Proof. Since $y \neq 0$, $F_{m_0}(y) \neq 0$ for some m_0 [1]. So, $\mathcal{T}_y$ is hypercyclic (and satisfies the Hypercyclicity Criterion for full sequence) on $H_s^m(\ell_1)$ whenever $m \geq m_0$. The set $\bigcup_{m=m_0}^{\infty} H_s^m(\ell_1)$ contains the space of all symmetric polynomials on ℓ_1 and so it is dense in $H_{bs}(\ell_1)$. Also $H_s^m(\ell_1) \subset H_s^n(\ell_1)$ if $n > m$. Hence by Lemma 3.2, $\mathcal{T}_y$ is hypercyclic. $\square$

Acknowledgement. The second author wishes to thank the Nipissing University for the invitation for Visiting Professor position in 2005-06 academic year and financial support.

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Received: 8 June 2006

Revised: 26 September 2006