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# PROBLEM WITH NONLOCAL CONDITIONS FOR WEAKLY NONLINEAR HYPERBOLIC EQUATIONS

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For weakly nonlinear hyperbolic equations of order $n$, $n \geq 3$, with constant coefficients in the linear part of the operator, we study a problem with nonlocal two-point conditions in time and periodic conditions in the space variable. Generally speaking, the solvability of this problem is connected with the problem of small denominators whose estimation from below is based on the application of the metric approach. For almost all (with respect to the Lebesgue measure) coefficients of the equation and almost all parameters of the domain, we establish conditions for the existence of a unique classical solution of the problem.

The investigation of problems with nonlocal conditions in time (the simplest conditions of this sort are periodic conditions) for hyperbolic equations (both linear and nonlinear) was originated relatively recently (see, e.g., [1–19] and the bibliography in [2, 3]). This can be explained, e.g., by the difficulties encountered in working with the small denominators that appear in constructing solutions of these problems. As far as nonlinear hyperbolic equations are concerned, problems of this sort were studied, as a rule, for equations and systems of the first and second orders.

In the present paper, we study a nonlocal boundary-value problem for weakly nonlinear hyperbolic equations of order $n$, $n \geq 3$. Significant attention is given to the problem of small denominators.

**1.** In a domain $D = \{(t, x) \in \mathbb{R}^2 \colon t \in [0, T], x \in Q\}$, where $Q$ is a unit disk, we consider the problem

$$Lu = \sum_{s=0}^{n} a_s \frac{\partial^n u(t, x)}{\partial t^{n-s} \partial x^s} = \varepsilon f(t, x, u(t, x)) + \Phi(t, x), \tag{1}$$

$$\left.\frac{\partial^j u(t, x)}{\partial t^j}\right|_{t=0} - \mu \left.\frac{\partial^j u(t, x)}{\partial t^j}\right|_{t=T} = 0, \quad j = 0, 1, \ldots, n-1, \tag{2}$$

where $n \geq 3$, $a_s \in \mathbb{R}$, $a_0 = 1$, $\varepsilon, \mu \in \mathbb{C}$, $\mu \neq 0, 1$, the operator $L$ is strictly hyperbolic in the sense of Petrovskii, the function $f(t, x, u)$ is defined continuous in $t$ and sufficiently smooth in $x$ and $u$ in a domain $D_1 = \{(t, x, u) \colon (t, x) \in D, u \in \bar{S}(u^0, r)\}$, where

$$\bar{S}(u^0, r) = \{u(t, x) \in C^n(D) \colon \|u - u_0\|_{C^n(D)} \leq r\}$$

and $u^0 = u^0(t, x)$ is a solution of the nonperturbed problem (1), (2) (with $\varepsilon = 0$), and $\Phi(t, x) \in C^{(0,3)}(D)$, where $C^{(0,q)}(D)$ is a Banach space of functions $v(t, x)$ with norm

$$\|v(t, x)\|_{C^{(0,q)}(D)} = \sum_{j=0}^{q} \max_{D} \left|\frac{\partial^j v(t, x)}{\partial x^j}\right|.$$

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The shape of the domain $D$ enables us to impose the conditions of $2\pi$-periodicity in $x$ on the functions $u(t, x)$, $\Phi(t, x)$, and $f(t, x, u)$.

The solution of the problem under consideration is sought in the form of a series

$$u(t, x) = \sum_{|k| \geq 0} u_k(t) \exp(ikx). \tag{3}$$

To determine the coefficients $u_k(t)$, $k \in \mathbb{Z}$, we insert series (3) in Eq. (1) and conditions (2) and arrive at the following boundary-value problem for an infinite set of ordinary differential equations:

$$\sum_{s=0}^{n} a_s (ik)^s u_k^{(n-s)}(t) = \varepsilon f_k(t, \{u_m(t)\}) + \Phi_k(t), \qquad k \in \mathbb{Z}, \tag{4}$$

$$l_j[u_k(t)] \equiv u_k^{(j)}(0) - \mu u_k^{(j)}(T) = 0, \quad j = 0, 1, \ldots, n-1, \;\; k \in \mathbb{Z}, \tag{5}$$

where

$$f_k(t, \{u_m(t)\}) = (2\pi)^{-1} \int_0^{2\pi} f\left(t, x, \sum_{|m| \geq 0} u_m(t) \exp(imx)\right) \exp(-ikx) dx, \tag{6}$$

$$\Phi_k(t) = (2\pi)^{-1} \int_0^{2\pi} \Phi(t, x) \exp(-ikx) dx. \tag{7}$$

Let us show that problem (1), (2) is equivalent to a nonlinear integral equation.

For every $k \in \mathbb{Z}$, we consider a problem with conditions (5) for a linear equation

$$\sum_{s=0}^{n} a_s (ik)^s u_k^{(n-s)}(t) = \Phi_k(t). \tag{8}$$

According to the assumption that the operator $L$ is hyperbolic, the roots of the equation

$$\sum_{s=0}^{n} a_s \lambda^{n-s} = 0$$

denoted by $\lambda_j$, $j = 1, \ldots, n$, are real and different. Hence, the homogeneous equation corresponding to Eq. (8) has the following fundamental system of solutions:

$$u_{kj}(t) = \begin{cases} \exp(i\lambda_j kt), & k \in \mathbb{Z} \setminus \{0\}, \\ t^{j-1}, & k = 0, \end{cases} \qquad j = 1, \ldots, n.$$

Moreover, the characteristic determinant $\Delta(k)$ of problem (5), (8) is given by the formula

$$\Delta(k) = \begin{cases} (ik)^{n(n-1)/2} \prod\limits_{1 \leq p < q \leq n} (\lambda_q - \lambda_p) \sum\limits_{j=1}^{n} (1 - \mu \exp(i\lambda_j kT)), & k \in \mathbb{Z} \setminus \{0\}, \\ (1-\mu)^n \, 1!\, 2! \ldots (n-1)!, & k = 0. \end{cases} \tag{9}$$

Relation (9) implies that the determinant $\Delta(k)$ is nonzero for all $k \in \mathbb{Z}\setminus\{0\}$ if and only if at least one of the following conditions is satisfied:

(i) $|\mu| \neq 1$;

(ii) $\lambda_j kT + \varphi \neq 2\pi q,\ j = 1, \dots, n,\ q \in \mathbb{Z},\ \varphi = \arg\mu$.

Let $\Delta(k) \neq 0$ for all $k \in \mathbb{Z}$. Then the nonperturbed problem (1), (2) cannot have two different solutions (see [3], Chap. 5, Sec. 4). Moreover, for any $k \in \mathbb{Z}$, there exists a unique Green function $G_k(t, \tau)$ of problem (5), (8), and the solution of the indicated problem can be represented in the form

$$u_k^0(t) = \int_0^T G_k(t, \tau)\Phi_k(\tau)\,d\tau, \qquad k \in \mathbb{Z}. \tag{10}$$

In this case, the solution $u^0(t, x)$ of the nonperturbed problem (1), (2) is formally represented in the form of a series as

$$u^0(t, x) = \sum_{|k| \geq 0} \int_0^T G_k(t, \tau)\Phi_k(\tau)\,d\tau \exp(ikx). \tag{11}$$

In the square $K_T = \{(t, \tau) \colon 0 \leq t,\ \tau \leq T\}$ without its sides $\tau = 0$ and $\tau = T$, the functions $G_k(t, \tau)$, $k \in \mathbb{Z}$, are given by the formulas

$$G_k(t, \tau) = 2^{-1}(ik)^{1-n} \sum_{j=1}^{n} \exp(i\lambda_j k(t-\tau)) \prod_{\substack{q=1 \\ q \neq j}}^{n} (\lambda_j - \lambda_q)^{-1} \left[\operatorname{sgn}(t-\tau) + \frac{1+\mu\exp(i\lambda_j kT)}{1-\mu\exp(i\lambda_j kT)}\right], \quad k \in \mathbb{Z}\setminus\{0\}, \tag{12}$$

$$G_0(t, \tau) = (2(n-1)!)^{-1} \left\{ \operatorname{sgn}(t-\tau)(t-\tau)^{n-1} + (1-\mu)^{-n} \left(\prod_{q=1}^{n-2} q!\right)^{-1} \right.$$

$$\left. \times \sum_{j=1}^{n} \sum_{p=1}^{j} (-1)^{n-j} t^{p-1} \Delta_{jp} \frac{\tau^{n-j} + \mu(\tau - T)^{n-j}}{(n-j)!} \right\}, \tag{13}$$

where $\Delta_{jp}$, $p = 1, \dots, j$, $j = 1, \dots, n$, is the algebraic complement of the element located in the $j$th row of the $p$th column in the determinant $\det\left\| l_{j-1}[t^{p-1}] \right\|_{j,p=1}^{n}$. The definition of each function $G_k(t, \tau)$, $k \in \mathbb{Z}$, is extended to the side $\tau = 0$ $(\tau = T)$ of the square $K_T$ by continuity from the right (left).

By using the system of functions $\{G_k(t, \tau),\ k \in \mathbb{Z}\}$, we reduce problem (4), (5) to the equivalent infinite system of nonlinear integral equations

$$u_k(t) = u_k^0(t) + \varepsilon \int_0^T G_k(t, \tau) f_k(\tau, \{u_m(\tau)\})\,d\tau, \qquad k, m \in \mathbb{Z}, \tag{14}$$

where the functions $u_k^0(t)$, $k \in \mathbb{Z}$, are given by relation (10).

We denote

$$K(t,x,\tau,\xi) = (2\pi)^{-1}\sum_{|k|\geq 0} G_k(t,\tau)\exp(ik(x-\xi)). \tag{15}$$

If series (15) converges uniformly in the domain $D\times D$ and the function $u^0(t,x)$ determined by relation (11) belongs to the space $C^n(D)$, then relations (3), (6), and (14) imply that problem (1), (2) is equivalent to the nonlinear integral equation

$$u(t,x) = u^0(t,x) + \varepsilon\int_D K(t,x,\tau,\xi)f(\tau,\xi,u(\tau,\xi))\,d\tau d\xi. \tag{16}$$

**2.** The problem of convergence of series (11) and (15) is, generally speaking, connected with the problem of small denominators because the absolute values of nonzero expressions $1-\mu\exp(i\lambda_j kT)$, $j=1,\dots,n$, appearing in relation (12) for functions $G_k(t,\tau)$, $k\in\mathbb{Z}$, as denominators may be arbitrarily small for infinitely many integer numbers $k$.

Note that there are no small denominators for $|\mu|\neq 1$. This follows from the estimates

$$\begin{aligned}|1-\mu\exp(i\lambda_j kT)| &= |1-|\mu|(\cos(\lambda_j kT+\varphi)+i\sin(\lambda_j kT+\varphi))|\\ &= \sqrt{1+|\mu|^2-2|\mu|\cos(\lambda_j kT+\varphi)} \geq \big|1-|\mu|\big|, \quad j=1,\dots,n,\end{aligned} \tag{17}$$

Relations (12), (13) and estimates (17) imply that

$$\max_{0\leq t\leq T}\left|\frac{\partial^q}{\partial t^q}\int_0^T G_k(t,\tau)d\tau\right| \leq \begin{cases} 2T|k|^{1-n+q}\sum_{j=1}^{n}\Lambda_j^{(q)}\left|1-\mu\exp(i\lambda_j kT)\right|^{-1}+\delta_{nq}, & |\mu|=1,\\ T|k|^{1-n+q}\dfrac{1+|\mu|}{\big|1-|\mu|\big|}\sum_{j=1}^{n}\Lambda_j^{(q)}+\delta_{nq}, & |\mu|\neq 1,\end{cases} \quad q=0,1,\dots,n, \tag{18}$$

$$\max_{0\leq t\leq T}\left|\frac{\partial^q}{\partial t^q}\int_0^T G_0(t,\tau)d\tau\right| \leq c_0 T^{-q}, \tag{19}$$

where

$$k\in\mathbb{Z}\setminus\{0\},\qquad \Lambda_j^{(q)} = |\lambda_j|^q\prod_{\substack{m=1\\ m\neq j}}^{n}|\lambda_j-\lambda_m|^{-1},$$

$\delta_{nq}$ is the Kronecker symbol, $q=0,1,\dots,n$,

$$c_0 = T^{n-1}\left(1+|1+\mu|\sum_{j=1}^{n}\sum_{p=1}^{j}M^{j-p}|1-\mu|^{p-j-1}(2(p-1)!)^{-1}\right),$$

and

$$M = \max\{1,|\mu|\}.$$

If $|\mu| = 1$, then series (11) and (15) are, generally speaking, divergent. At the same time, in what follows, we prove that, in this case, for almost all (with respect to the Lebesgue measure in $\mathbb{R}$) numbers $\beta_j = \lambda_j T/(2\pi)$, $j = 1, \ldots, n$, small denominators insignificantly affect the convergence of these series.

**Lemma 1.** *Let* $|\mu| = 1$. *Then, for almost all (with respect to the Lebesgue measure in* $\mathbb{R}$*) numbers* $\beta = \lambda T/(2\pi)$, $\lambda \in \mathbb{R}$, *the series*

$$S = \sum_{|k|>0} |k|^{1-n} |1 - \mu \exp(i\lambda kT)|^{-1} \tag{20}$$

*converges whenever* $n \geq 3$.

***Proof.*** By using the inequality $\sin x \geq 2x/\pi$, which is true for all $x \in [0, \pi/2]$, we conclude that the following estimate holds for all real $\lambda$:

$$\begin{aligned} |1 - \mu \exp(i\lambda kT)| &= 2|\sin((\lambda kT + \varphi)/2)| \\ &= 2|\sin|\tfrac{1}{2}(\lambda|k|T + \varphi \operatorname{sgn} k) - d(k)\pi|| \\ &\geq |\tfrac{1}{2\pi}(\lambda|k|T + \varphi \operatorname{sgn} k) - d(k)| \\ &= \frac{1}{2\pi}\lambda|k|T\left|\frac{\lambda T|k| + \varphi \operatorname{sgn} k}{\lambda T|k|} - \frac{2\pi}{T\lambda}\frac{d(k)}{|k|}\right|, \end{aligned} \tag{21}$$

where $k \in \mathbb{Z}\setminus\{0\}$ and $d(k)$ is an integer number such that

$$\left|\frac{\lambda T|k| + \varphi \operatorname{sgn} k}{2\pi} - d(k)\right| \leq \frac{1}{2}. \tag{22}$$

By using inequalities (21) and (22), we obtain the following estimate for series (20):

$$S \leq \sum_{|k|>0} |k|^{1-n} |\beta|k| + \tfrac{1}{2\pi}\varphi \operatorname{sgn} k - d(k)|^{-1} = S_1 + S_2, \tag{23}$$

where

$$S_j = \sum_{k=1}^{\infty} k^{1-n} |\beta k - \tfrac{(-1)^j}{2\pi}\varphi - d_j(k)|^{-1}, \quad j = 1, 2, \tag{24}$$

and $d_j(k)$, $j = 1, 2$, is an integer number such that

$$\left|\beta k - \frac{(-1)^j \varphi}{2\pi} - d_j(k)\right|^{-1} \leq \frac{1}{2}.$$

To prove that series (24) are convergent, we use the idea of the proof of Lemma 2 in [20]. We consider the series $S_1$ and construct the series $S_1^{(p)}$ of the same form as $S_1$:

$$S_1^{(p)} = \sum_{k_q^{(p)} \in \Omega_p} \left(k_q^{(p)}\right)^{1-n} \left|\beta k_q^{(p)} + \frac{\varphi}{2\pi} - d_1\left(k_q^{(p)}\right)\right|^{-1}, \quad p \in \mathbb{N}, \tag{25}$$

where $\Omega_p \subseteq \mathbb{N}$ is the set of all $k = k_q^{(p)}$, $q = 1, 2, \ldots,$ $k_{q+1}^{(p)} > k_q^{(p)}$, satisfying the inequality

$$2^{-p-1} < \left|\beta k_q^{(p)} + \frac{\varphi}{2\pi} - d_1\left(k_q^{(p)}\right)\right| \le 2^{-p}, \quad p \in \mathbb{N}. \tag{26}$$

Clearly,

$$S_1 = \sum_{p=1}^{\infty} S_1^{(p)}$$

and, therefore, to prove the convergence of the series $S_1$, it suffices to show that

$$\sum_{p=1}^{\infty} S_1^{(p)} < \infty.$$

It follows from estimates (26) that

$$\left|\beta\left(k_{q+1}^{(p)} - k_q^{(p)}\right) - \left(d_1(k_{q+1}^{(p)}) - d_1(k_q^{(p)})\right)\right| < 2^{-p}, \quad p \in \mathbb{N}. \tag{27}$$

According to Lemma 1 in [20], for almost all $\beta$, there exists a constant $c_1 = c_1(\beta) > 0$ such that the inequality

$$\left|\beta\left(k_{q+1}^{(p)} - k_q^{(p)}\right) - \left(d_1(k_{q+1}^{(p)}) - d_1(k_q^{(p)})\right)\right| \ge c_1\left(k_{q+1}^{(p)} - k_q^{(p)}\right)^{-1-\delta}, \quad 0 < \delta < 1, \tag{28}$$

holds for all $k_q^{(p)} \in \Omega_p$.

Estimates (27) and (28) imply that

$$M_p \equiv \min_{\Omega_p}\left(k_{q+1}^{(p)} - k_q^{(p)}\right) > (2^p c_1)^{1/(1+\delta)}, \quad p \in \mathbb{N}, \tag{29}$$

for almost all $\beta$.

It is clear that, for all $k_q^{(p)} \in \Omega_p$, we have

$$k_q^{(p)} \ge (q-1)M_p + k_1^{(p)}. \tag{30}$$

It follows from Lemma 2.4 in [3] (Chap. 1) that, for almost all $\beta$ (with respect to the Lebesgue measure in $\mathbb{R}$), there exists a constant $c_2 = c_2(\beta) > 0$ such that

$$\left|\beta k_q^{(p)} + \frac{\varphi}{2\pi} - d_1\left(k_q^{(p)}\right)\right| \ge c_2\left(k_q^{(p)}\right)^{-1-\sigma}, \quad 0 < \sigma < 1, \tag{31}$$

for all $k_q^{(p)} \in \Omega_p$. Therefore, it follows from estimates (26) and (31) that

$$k_1^{(p)} \geq (2^p c_2)^{1/(1+\sigma)}, \qquad k_1^{(p)} \in \Omega_p \tag{32}$$

for almost all $\beta$.

Without loss of generality, in (32), we set $\sigma = \delta$. In view of estimates (29), (30), and (32), this enables us to write

$$k_q^{(p)} > (q-1)(2^p c_1)^{1/(1+\delta)} + (2^p c_2)^{1/(1+\delta)} > 2^{p/(1+\delta)} Cq, \tag{33}$$

where $k_q^{(p)} \in \Omega_p$ and $C = (\min\{c_1 c_2\})^{1/(1+\delta)}$.

By using relation (25) and estimates (26), (33), we conclude that

$$S_1 = \sum_{p=1}^{\infty} S_1^{(p)} < 2C^{1-n} \sum_{p=1}^{\infty} 2^{p(2+\delta-n)/(1+\delta)} \sum_{q=1}^{\infty} q^{1-n}$$

$$= 2^{(3+2\delta-n)/(1+\delta)} C^{1-n} (1 - 2^{(2+\delta-n)/(1+\delta)})^{-1} \sum_{q=1}^{\infty} q^{1-n} < \infty$$

for almost all $\beta$ (with respect to the Lebesgue measure in $\mathbb{R}$).

The convergence of the series $S_2$ is established in a similar way. Lemma 1 is proved.

Estimates (18) and Lemma 1 imply that, for all $\mu \in \mathbb{C} \setminus \{0, 1\}$ and almost all (with respect to the Lebesgue measure in $\mathbb{R}$) numbers $\beta_j = \lambda_j T/(2\pi)$, $j = 1, \ldots, n$, series (15) uniformly converges in the domain $D \times D$ for all $n \geq 3$.

Let us now show that $u^0(t, x) \in C^n(D)$. Denote

$$\gamma = c_0 \frac{T^{-n} - 1}{T^{-1} - 1}, \qquad \omega_q = \sum_{|k|>0} |k|^{-q}, \quad q = 2, 3,$$
$$D_p = \sum_{j=1}^{n} \Lambda_j^{(p)}, \qquad p = 0, 1, \ldots, n. \tag{34}$$

It follows from (7) that

$$\max_{0 \leq t \leq T} |\Phi_k(t)| \leq \tilde{\Phi} |k|^{-3}, \qquad \tilde{\Phi} = \max_D \left| \frac{\partial^3 \Phi(t, x)}{\partial x^3} \right|. \tag{35}$$

For $|\mu| = 1$, by using estimates (18), (19), and (35) and relation (11), we obtain

$$\| u^0(t, x) \|_{C^n(D)} \leq \sum_{|s| \leq n} \max_D \left| \frac{\partial^{|s|}}{\partial t^{s_1} \partial x^{s_2}} \sum_{|k| \geq 0} \int_0^T G_k(t, \tau) \Phi_k(\tau) d\tau \exp(ikx) \right|$$

$$\leq \sum_{|k| \geq 0} \sum_{|s| \leq n} |k|^{s_2} \max_{0 \leq t \leq T} \left| \frac{\partial^{s_1}}{\partial t^{s_1}} \int_0^{T} G_k(t, \tau) \Phi_k(\tau) d\tau \right|$$

$$\leq \tilde{\Phi}\left(2T\sum_{|k|>0}\sum_{|s|\leq n}\sum_{j=1}^{n}\Lambda_j^{(s_1)}|k|^{|s|-n-2}|1-\mu\exp(i\lambda_j kT)|^{-1}+\omega_3+c_0\sum_{s_1=0}^{n-1}T^{-s_1}\right)$$

$$\leq \tilde{\Phi}\left(2T\sum_{|s|\leq n}\sum_{|k|>0}\sum_{j=1}^{n}\Lambda_j^{(s_1)}|k|^{-2}|1-\mu\exp(i\lambda_j kT)|^{-1}+\omega_3+\gamma\right). \tag{36}$$

Note that, according to Lemma 1, the series

$$B_p = \sum_{|k|>0}\sum_{j=1}^{n}\Lambda_j^{(p)}|k|^{-2}|1-\mu\exp(i\lambda_j kT)|^{-1}, \qquad p=0,1,\ldots,n, \tag{37}$$

are convergent for almost all $\beta_j$, $j=1,\ldots,n$.

Hence, it follows from (36) and (37) that

$$\|u^0(t,x)\|_{C^n(D)} \leq \|\Phi(t,x)\|_{C^{(0,3)}(D)}\left(2T\sum_{p=0}^{n}(n+1-p)B_p+\omega_3+\gamma\right) \equiv \rho_1 < \infty \tag{38}$$

for $|\mu|=1$ and almost all (with respect to the Lebesgue measure) $\beta_j$, $j=1,\ldots,n$.

For $|\mu|\neq 1$, by the same reasoning, we arrive at the estimate

$$\|u^0(t,x)\|_{C^n(D)} \leq \|\Phi(t,x)\|_{C^{(0,3)}(D)}\left(T\frac{1+|\mu|}{|1-\mu|}\omega_2\sum_{p=0}^{n}(n+1-p)D_p+\omega_3+\gamma\right) \equiv \rho_2. \tag{39}$$

Thus, we have in fact proved the following theorem:

**Theorem 1.** *Let* $\Phi(t,x)\in C^{(0,3)}(D)$. *Then, for* $|\mu|=1$ *and almost all (with respect to the Lebesgue measure in* $\mathbb{R}$*) numbers* $\lambda_j T/(2\pi)$, $j=1,\ldots,n$, *and for* $|\mu|\neq 1$ *and all* $T>0$ *and* $a_s$, $s=0,1,\ldots,n$, *the nonperturbed problem (1), (2) possesses a unique solution* $u^0(t,x)\in C^n(D)$, *which can be represented in the form of series (11) and continuously depends on the function* $\Phi(t,x)$.

**3.** Consider the problem of solvability of the integral equation (16). Denote

$$\Psi_1(y) = \tilde{f}\left(2T\left(B+\sum_{j=1}^{3}y^j H_{n-3+j}\right)+\omega_3 y^3+\gamma\right),$$

$$\Psi_2(y) = \tilde{f}\left(\omega_2 T(1+|\mu|)/|1-|\mu||\left(\sum_{p=0}^{n-3}(n-2-p)D_p+\sum_{j=1}^{3}y^j W_{n-3+j}\right)+\omega_3 y^3+\gamma\right),$$

where

$$\tilde{f} = \max_{0\leq|s|\leq 4}\max_{D_1}\left|\frac{\partial^{|s|}f(t,x,u)}{\partial x^{s_1}\partial u^{s_2}}\right|, \qquad B = \sum_{p=0}^{n-3}(n-2-p)B_p,$$

$$H_q = \sum_{p=0}^{q} B_p, \quad W_q = \sum_{p=0}^{q} D_p, \quad q = n-2, n-1, n,$$

$$\varepsilon_1 = \min\left(\frac{r}{\Psi_1(1+r+\rho_1)}, \frac{1}{\Psi_1(2+r+\rho_1)}\right),$$

$$\varepsilon_2 = \min\left(\frac{r}{\Psi_2(1+r+\rho_2)}, \frac{1}{\Psi_2(2+r+\rho_2)}\right),$$

and the numbers $\gamma$, $\omega_q$, $D_p$, $B_p$, $\rho_1$, and $\rho_2$ are given by relations (34), (37)–(39).

**Theorem 2.** *Assume that $\Phi(t, x) \in C^{(0,3)}(D)$ and that the function $f(t, x, u)$ is continuous in $t$ and has bounded derivatives with respect to $x$ and $u$ up to the fourth order inclusively in the region $D_1$. Then, for $|\mu| = 1$, almost all (with respect to the Lebesgue measure in $\mathbb{R}$) numbers $\beta_j = \lambda_j T/(2\pi)$, $j = 1, \ldots, n$, and all $\varepsilon$, $|\varepsilon| < \varepsilon_1$, and for $|\mu| \neq 1$ and all $T > 0$, $a_s$, $s = 0, 1, \ldots, n$, and $\varepsilon$, $|\varepsilon| < \varepsilon_2$, Eq. (16) possesses a unique solution which belongs to the ball $\bar{S}(u^0, r) \subset C^n(D)$ and continuously depends on the function $\Phi(t, x)$.*

***Proof.*** We use the principle of contracting mappings. Consider the case $|\mu| = 1$. Equation (16) can be rewritten in the form

$$u(t, x) = A_{u^0}[u(t, x)],$$

where $A_v$ is a nonlinear integral operator

$$A_v[u(t, x)] \equiv v(t, x) + \varepsilon \int_D K(t, x, \tau, \xi) f(\tau, \xi, u(\tau, \xi))\, d\tau d\xi \tag{40}$$

defined in the ball $\bar{S}(u^0, r)$.

By $V$ we denote the collection of functions $v(t, x) \in C^n(D)$ such that

$$\| v(t, x) - u^0(t, x) \|_{C^n(D)} \le \kappa = r - |\varepsilon| \Psi_1(1 + r + \rho_1),$$

and prove that, for any function $v(t, x)$ from $V$, the operator $A_v$ maps the ball $\bar{S}(u^0, r)$ onto itself.

Note that if the function $u(t, x)$ represented in the form (3) belongs to the ball $\bar{S}(u^0, r)$, then, in view of relation (6), we obtain

$$\max_{0 \le t \le T} |f_k(t, \{u_m(t)\})| \le |k|^{-\alpha} \max_D \left| \frac{\partial^\alpha f(t, x, u(t, x))}{\partial x^\alpha} \right|, \quad \alpha = 0, 1, 2, 3. \tag{41}$$

According to the rule of differentiation of a composite function, we obtain

$$\max_D \left| \frac{\partial^\alpha f(t, x, u(t, x))}{\partial x^\alpha} \right| \le \tilde{f}(1 + \|u(t, x)\|_{C^n(D)})^\alpha$$

$$\le \tilde{f}\left(1 + \left\|u(t, x) - u^0(t, x)\right\|_{C^n(D)} + \left\|u^0(t, x)\right\|_{C^n(D)}\right)^\alpha \le \tilde{f}(1 + r + \rho_1)^\alpha, \quad \alpha = 0, 1, 2, 3. \tag{42}$$

By using relations (40) and (15) and estimates (18), (19), (41), and (42), we now get

$$\|A_v[u(t,x)] - u^0(t,x)\|_{C^n(D)}$$

$$\leq \|v(t,x)-u^0(t,x)\|_{C^n(D)} + |\varepsilon|(2\pi)^{-1}\left\|\int_D \sum_{|k|\geq 0} G_k(t,\tau)f(\tau,\xi,u(\tau,\xi))\exp(ik(x-\xi))d\tau d\xi\right\|_{C^n(D)}$$

$$\leq \kappa + |\varepsilon|(2\pi)^{-1}\sum_{|k|\geq 0}\sum_{|s|\leq n}\max_D\left|\frac{\partial^{|s|}}{\partial t^{s_1}\partial x^{s_2}}\int_0^T G_k(t,\tau)\int_0^{2\pi} f(\tau,\xi,u(\tau,\xi))\exp(ik(x-\xi))d\tau d\xi\right|$$

$$\leq \kappa + |\varepsilon|\Bigg(\sum_{|k|>0}\Bigg(\max_{D_1}|f(t,x,u(t,x))|\sum_{|s|\leq n-3}|k|^{s_2}\max_{0\leq t\leq T}\left|\frac{\partial^{s_1}}{\partial t^{s_1}}\int_0^T G_k(t,\tau)d\tau\right|$$

$$+\sum_{n-2\leq|s|\leq n}|k|^{n-s_1-3}\max_{0\leq t\leq T}\left|\frac{\partial^{s_1}}{\partial t^{s_1}}\int_0^T G_k(t,\tau)d\tau\right|\max_{D_1}\left|\frac{\partial^{|s|-n+3}f(t,x,u(t,x))}{\partial x^{|s|-n+3}}\right|\Bigg)$$

$$+\max_{D_1}|f(t\,x,u(t,x))|\sum_{s_1=0}^{n-1}\left|\frac{\partial^{s_1}}{\partial t^{s_1}}\int_0^T G_0(t,\tau)d\tau\right|\Bigg)$$

$$\leq \kappa + |\varepsilon|\tilde{f}\Bigg(2T\sum_{|s|\leq n-3}\sum_{|k|>0}\sum_{j=1}^{n}\Lambda_j^{(s_1)}|k|^{|s|-n+1}|1-\mu\exp(i\lambda_j kT)|^{-1}$$

$$+\sum_{n-2\leq|s|\leq n}(1+r+\rho_1)^{|s|-n+3}\Bigg(2T\sum_{|k|>0}\sum_{j=1}^{n}\Lambda_j^{(s_1)}|k|^{-2}|1-\mu\exp(i\lambda_j kT)|^{-1}+\delta_{|s|,n}\omega_3\Bigg)+c_0\sum_{s_1=0}^{n-1}T^{-s_1}\Bigg)$$

$$\leq \kappa + |\varepsilon|\tilde{f}\left(2T\left(B+\sum_{j=1}^{3}(1+r+\rho_1)^j H_{n-3+j}\right)+(1+r+\rho_1)^3\omega_3+\gamma\right)$$

$$= \kappa + |\varepsilon|\Psi_1(1+r+\rho_1) = r.$$

Let us now show that $A_v$ is a contraction operator for any function $v(t,x)\in V$. Assume that $u_1(t,x)$, $u_2(t,x)\in \bar{S}(u^0,r)$. We denote

$$F(t,x) \equiv f(t,x,u_1(t,x)) - f(t,x,u_2(t,x)),$$

$$\tilde{u}(t,x) \equiv \theta u_1(t,x) + (1-\theta)u_2(t,x), \qquad 0\leq\theta\leq 1.$$

In view of Lemma 1, estimates (18), (19), and (42), and the Lagrange formula of finite increments, it follows from relation (40) that

$$\|A_v[u_1(t,x)] - A_v[u_2(t,x)]\|_{C^n(D)}$$

$$\leq |\varepsilon|(2\pi)^{-1}\left\| \int_D \sum_{|k|\geq 0} G_k(t,\tau)F(\tau,\xi)\exp(ik(x-\xi))d\tau d\xi \right\|_{C^n(D)}$$

$$\leq |\varepsilon|\tilde{f}\,\|u_2(t,x)-u_1(t,x)\|_{C^n(D)}$$

$$\times\left(2TB+\gamma+\sum_{n-2\leq|s|\leq n}\ \sum_{j=0}^{|s|+3-n} C^j_{|s|+3-n}\left(1+r+\|\tilde{u}(t,x)\|_{C^n(D)}\right)^j\left(2TB_{s_1}+\delta_{|s|,n}\omega_3\right)\right)$$

$$\leq |\varepsilon|\tilde{f}\,\|u_2(t,x)-u_1(t,x)\|_{C^n(D)}\left(2T\left(B+\sum_{q=1}^{3}(2+r+\rho_1)^q H_{n-3+q}\right)+\gamma+\omega_3(2+r+\rho_1)^3\right)$$

$$= |\varepsilon|\Psi_1(2+r+\rho_1)\,\|u_2(t,x)-u_1(t,x)\|_{C^n(D)}$$

for almost all $\beta_j$, $j=1,\dots,n$.

Thus, if $|\mu|=1$ and $|\varepsilon|\Psi_1(2+r+\rho_1)\leq 1$, then $A_v$ is a contraction operator for almost all $\beta_j$, $j=1,\dots,n$.

It is obvious that the operator $A_v$ is continuous in $v$. Therefore, according to Theorems 1 and 3 in [21] (Chap. 16, Sec. 1), Eq. (16) [and, hence, problem (1), (2)] possesses a unique solution which continuously depends on the function $\Phi(t,x)$.

For $|\mu|\neq 1$, the assertion of the theorem is proved in a similar way.

Theorem 2 is proved.

***Remark 1.*** The solution of problem (1), (2) can be found as the limit of the sequence $\{u_s(t,x)\}$, where $u_0 = u^0(t,x)$, $u_{s+1}=A_{u0}[u_s(t,x)]$, $s\in\mathbb{N}$, and $A_{u0}$ is the integral operator determined by relation (40).

***Remark 2.*** The results of the present work can be generalized to the case of $p\geq 2$ spatial variables if the domain $Q$ is a $p$-dimensional torus.

## REFERENCES

1. N. A. Artem'ev, "Periodic solutions of one class of partial differential equations," *Izv. Akad. Nauk SSSR, Ser. Mat.*, No. 1, 15–50 (1937).
2. O. Vejvoda, L. Harrmann, V. Lovicar, et al., *Partial Differential Equations: Time-Periodic Solutions*, Noordhoff, Alphen an den Rijn–Sijthoff (1981).
3. B. I. Ptashnik, *Ill-Posed Boundary-Value Problems for Partial Differential Equations* [in Russian], Naukova Dumka, Kiev (1984).
4. O. Vejvoda and M. Shtedry, "Existence of classical periodic solutions of the wave equation. Relationship between the number-theoretic nature of the period and geometric properties of solutions," *Differents. Uravn.*, **20**, No. 10, 1733–1739 (1984).
5. L. G. Tret'yakova, "On the problem of 2π-periodic solutions of the equation of oscillations of a nonlinear string," *Vestn. Belorus. Univ., Ser. 1*, No. 3, 49–51 (1986).
6. E. Sinestrari and G. F. Webb, "Nonlinear hyperbolic systems with nonlocal boundary conditions," *J. Math. Anal. Appl.*, **121**, No. 2, 449–464 (1987).
7. V. V. Marinets, "On some problems for systems of nonlinear partial differential equations with nonlocal boundary conditions," *Differents. Uravn.*, **24**, No. 8, 1393–1397 (1988).
8. P. I. Plotnikov and L. N. Yungerman, "Periodic solutions of a weakly nonlinear wave equation with irrational ratio of the period to the length of the interval," *Differents. Uravn.*, **24**, No. 9, 1599–1607 (1988).

9. L. Byszewski, "Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation $u_{tx} = F(x, t, u, u_x)$," *J. Appl. Math. Stochast. Anal.*, **3**, No. 3, 163–168 (1990).
10. L. Byszewski and V. Lakshmikanthan, "Monotone iterative technique for nonlocal hyperbolic differential problems," *J. Math. Phys. Sci.*, **26**, No. 4, 345–359 (1992).
11. S. V. Zhestkov, "On the existence of ω-periodic solutions of quasilinear wave systems with $n$-spatial variables," *Vest. Akad. Navuk Belarus., Ser. Fiz.-Mat. Navuk,* No. 2, 5–11 (1994).
12. I. Ya. Kmit', "On a nonlocal problem for a quasilinear hyperbolic system of the first order with two independent variables," *Ukr. Mat. Zh.*, **45**, No. 9, 1307–1313 (1993).
13. I. Ya. Kmit', "On a problem with nonlocal (in time) conditions for hyperbolic systems," *Mat. Met. Fiz.-Mekh. Polya,* Issue 37, 21–25 (1994).
14. T. I. Kiguradze, "One boundary-value problem for hyperbolic systems," *Dokl. RAN, Mat.*, **328**, No. 2, 135–138 (1993).
15. T. I. Kiguradze, "On bounded and periodic (in a strip) solutions of quasilinear hyperbolic systems," *Differents. Uravn.*, **30**, No. 10, 1760–1773 (1994).
16. Yu. A. Mitropol'skii and L. B. Urmancheva, "On a two-point problem for systems of hyperbolic equations," *Ukr. Mat. Zh.*, **42**, No. 2, 1657–1663 (1990).
17. Yu. A. Mitropol'skii, G. P. Khoma, and M. I. Gromyak, *Asymptotic Methods for the Investigation of Quasiwave Hyperbolic Equations* [in Russian], Naukova Dumka, Kiev (1990).
18. Yu. A. Mitropol'skii and G. P. Khoma, "On periodic solutions of the second-order wave equations. V," *Ukr. Mat. Zh.*, **45**, No. 8, 1115–1121 (1993).
19. Yu. A. Mitropol'skii and N. G. Khoma, "Periodic solutions of quasilinear second-order hyperbolic equations," *Ukr. Mat. Zh.*, **47**, No. 10, 1370–1375 (1995).
20. V. I. Arnol'd, "Small denominators. I. On the mapping of a circle onto itself," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **25**, No. 1, 21–86 (1961).
21. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* [in Russian], Nauka, Moscow (1977).