New formulae for Chebyshev polynomials of the first and second kind

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Chebyshev polynomials $\{T_n(x)\}_{n\geq 0}$ and $\{U_n(x)\}_{n\geq 0}$ of the first and second kind, respectively, are defined by the recurrence relations:

$$T_0(x) = 1,\ T_1(x) = x,\quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\quad n \geq 2.$$

$$U_0(x) = 1,\ U_1(x) = 2x,\quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),\quad n \geq 2.$$

There are many interesting properties of these polynomials [1].

Using Trudi's formula [2] for determinants and permanents of the Toeplitz – Hessenberg matrices of special kind, we obtain the new formulae for polynomials $T_n(x)$ and $U_n(x)$.

Proposition. *The following formulae are hold*:

$$\sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{\alpha} \frac{\alpha!}{s_1!\cdots s_n!} T_1^{s_1}(x) T_2^{s_2}(x)\cdots T_n^{s_n}(x) = x^{n-2}(1-x^2),\quad n \geq 2;$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{\alpha} \frac{\alpha!}{s_1!\cdots s_n!} T_2^{s_1}(x)\cdots T_{2n}^{s_n}(x) = 4x^2(1-x^2)(2x^2-1)^{n-2},\quad n \geq 2;$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{n+\alpha} \frac{\alpha!}{s_1!\cdots s_n!} U_2^{s_1}(x) U_4^{s_2}(x)\cdots U_{2n}^{s_n}(x) = 4x^2,\quad n \geq 2;$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} \frac{\alpha!}{s_1!\cdots s_n!} U_0^{s_1}(x) U_2^{s_2}(x)\cdots U_{2n-2}^{s_n}(x) = 4x^2(4x^2-1)^{n-2},\quad n \geq 2;$$

$$\sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{\alpha} \frac{\alpha!}{s_1!\cdots s_n!} U_1^{s_1}(x) U_2^{s_2}(x)\cdots U_n^{s_n}(x) = 0,\quad n \geq 3,$$

where $\alpha = s_1 + s_2 + \cdots + s_n$ *and the summation is over nonnegative integers satisfying* $s_1 + 2s_2 + \cdots + ns_n = n$.

1. J.C. Mason, D.C. Handcomb, *Chebyshev Polynomials* (Boca Raton: Chapman and Hall / CRC: 2003).
2. M. Merca, *Spec. Matrices* **1** (2013).