SOME IDENTITIES FOR BOUBAKER POLYNOMIALS

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The Boubaker polynomials, denote by $B_n(x)$, are defined as follows:

$$B_0(x) = 1, \quad B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k},$$

where $n \geq 1$ and $\lfloor s \rfloor$ is the floor of s.

Boubaker polynomials can be represented also by recurrence relation

$$B_{n+1}(x) = xB_n(x) - B_{n-1}(x), \qquad n \geq 2,$$

with initial conditions $B_0(x) = 1$, $B_1(x) = x$, and $B_2(x) = x^2 + 2$.

The next a few members of this polynomial sequence are

$$B_3(x) = x^3 + x, \quad B_4(x) = x^4 - 2, \quad B_5(x) = x^5 - x^3 - 3x,$$
$$B_6(x) = x^6 - 2x^4 - 3x^2 + 2, \quad B_7(x) = x^7 - 3x^5 - 2x^3 + 5x,$$
$$B_8(x) = x^8 - 4x^6 + 8x^2 - 2, \quad B_9(x) = x^9 - 5x^7 + 3x^5 + 10x^3 - 7x.$$

Boubaker polynomials have many applications in different scientific fields, such as thermodynamics, cryptography, biology, nonlinear dynamics and others sciences; see the recent papers [1, 2, 7, 8] and the references given there. Solutions to several applied physics problems are based on the so-called Boubaker Polynomials Expansion Scheme (BPES), using the subsequence B_{4m} of these polynomials.

Using Trudi's formula for Toeplitz-Hessenberg determinants of a special kind, we establish the following identities for Boubaker polynomials with successive, even and odd subscripts.

Our approach is similar in spirit to [3, 4, 5, 6].

Proposition. *Let $n \geq 2$, except when noted otherwise. Then*

$$\sum_{\tau_n = n} (-1)^{T_n} p_n(t) B_1^{t_1}(x) \cdots B_n^{t_n}(x)$$
$$= (-1)^{\lfloor 3n/2 \rfloor} 2^{\frac{(-1)^n - 1}{2}} 3^{\frac{2n-3-(-1)^n}{4}} \left(x + 1 - (-1)^n (x-1)\right), \quad n \geq 1,$$

$$\sum_{\tau_n = n} (-1)^{n+T_n} p_n(t) B_2^{t_1}(x) \cdots B_{2n}^{t_n}(x) = \frac{3^n - 1}{2} x^2 + 2 \cdot 3^{n-1},$$

$$\sum_{\tau_n = n} (-1)^{T_n} p_n(t) \left(\frac{B_2(x)}{x}\right)^{t_1} \cdots \left(\frac{B_{n+1}(x)}{x}\right)^{t_n} = \frac{(-2)^n}{4x^n}(3x^2 + 4),$$

$$\sum_{\tau_n = n} (-1)^{T_n} p_n(t) \left(\frac{B_3(x)}{x}\right)^{t_1} \cdots \left(\frac{B_{2n+1}(x)}{x}\right)^{t_n} = (-1)^n 3^{n-2}(3x^2 + 4),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)\left(\frac{B_3(x)}{B_2(x)}\right)^{t_1}\cdots\left(\frac{B_{n+2}(x)}{B_2(x)}\right)^{t_n}=\frac{x^{n-2}(3x^2+4)}{(x^2+2)^n},$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)\left(\frac{B_4(x)}{B_2(x)}\right)^{t_1}\cdots\left(\frac{B_{2n+2}(x)}{B_2(x)}\right)^{t_n}=\frac{(-2)^n x^2(3x^2+4)}{4(x^2+2)^n},$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)\left(\frac{B_4(x)}{B_3(x)}\right)^{t_1}\cdots\left(\frac{B_{n+3}(x)}{B_3(x)}\right)^{t_n}=\frac{(3x^2+4)(x^2+2)^{n-2}}{(x^3+x)^n},$$

$$\sum_{\tau_n=n}p_n(t)\left(\frac{B_1(x)}{2}\right)^{t_1}\cdots\left(\frac{B_n(x)}{2}\right)^{t_n}=\frac{3^{n-2}(3x^2+4)x^{n-2}}{2^n},$$

$$\sum_{\tau_n=n}p_n(t)\left(\frac{B_2(x)}{2}\right)^{t_1}\cdots\left(\frac{B_{2n+2}(x)}{2}\right)^{t_n}=\frac{x^2(3x^2+4)(3x^2-2)^{n-2}}{2^n},$$

where $p_n(t)=\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *is the multinomial coefficient,* $T_n=t_1+\cdots+t_n$*,* $\tau_n=t_1+2t_2+\cdots+nt_n$*, and the summation is over integers* $t_j\geq 0$ *satisfying* $\tau_n=n$*.*

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