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On Generalized Brioschi's Formula and its Applications

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Abstract

In this paper, we consider determinants for some families of Hessenberg matrices having various translates of the Fibonacci numbers for the nonzero entries. These determinant formulas may also be rewritten as identities involving sums of products of Fibonacci numbers and multinomial coefficients.

Keywords: Fibonacci numbers, Lucas numbers, Hessenberg matrix, Brioschi's formula, multinomial coefficient.

1 Introduction

Formulas relating determinants to Fibonacci numbers have been an object of interest for a long time, especially from the viewpoint of applications. In some cases, this sequence arises as determinants for certain families of matrices having integer entries, while in other cases this sequence is the actual entries of the matrix whose determinant is being evaluated (see, e.g., [1–6, 8] for the complete bibliography).

Consider the $n \times n$ Hessenberg matrix having the form

$$H_n(a_1, a_2, \ldots, a_n) = \begin{bmatrix} k_1 a_1 & 1 & & & & \\ k_2 a_2 & a_1 & 1 & & 0 & \\ \vdots & \vdots & \ddots & \ddots & & \\ k_{n-1} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \\ k_n a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix},$$

where $a_i \neq 0$ for at least one $i > 0$.

In [9], Zatorsky and Stefluk proved that

$$\det(H_n) = \sum_{\sigma_n = n} \frac{(-1)^{n-|s_n|}}{|s_n|} \left(\sum_{i=1}^{n} s_i k_i \right) m_n(s) a_1^{s_1} \cdots a_n^{s_n}, \qquad (1)$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s_n| = s_1 + \cdots + s_n$, $m_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ is the multinomial coefficient, and the summation is over all n-tuples $(s_1, \ldots, s_n)$ of integers $s_i \geq 0$ satisfying the Diophantine equation $\sigma_n = n$.

In the case $k_1 = \ldots = k_n = 1$ we have well-known *Brioschi's formula* [6, pp. 208–209].

Note that $s_1 + 2s_2 + \cdots + ns_n = n$ is partition of the positive integer n, where each positive integer i appears s_i times.

In the next section, we will investigate a particular case of determinants $\det(H_n)$, in which $k_i = i$. For the sake of brevity, we will use throughout the notation

$$\det(a_1, a_2, \ldots, a_n) = \det\bigl(H_n(a_1, a_2, \ldots, a_n)\bigr).$$

2 Fibonacci–Lucas multinomial identities

Let F_n denote the n-th Fibonacci number and L_n the n-th Lucas number, both satisfying the recurrence

$$w_n = w_{n-1} + w_{n-2},$$

but with the respective initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ (see [6] and the references given there).

Theorem 1. *For $n \geq 1$, the following formulas hold*:

$$\det(F_0, F_1, \ldots, F_{n-1}) = (-1)^{n-1}(L_n - 1),$$

$$\det(-F_0, -F_1, \ldots, -F_{n-1}) = 2^n + (-1)^n - L_n,$$

$$\det(F_1, F_2, \ldots, F_n) = (-1)^{n-1}\left(L_n - 1 - (-1)^n\right),$$

$$\det(F_2, F_3, \ldots, F_{n+1}) = (-1)^{n-1} L_n,$$

$$\det(F_3, F_4, \ldots, F_{n+2}) = (-1)^{n-1} L_n + 1.$$

Theorem 1 may be proved in the same way as Theorems 1 and 2 in [2].

Next, we focus on multinomial extensions of Theorem 1. Formula (1), coupled with Theorem 1 above, yields the following combinatorial identities expressing the Lucas numbers in terms of Fibonacci numbers.

Theorem 2. *For $n \geq 1$, the following formulas hold*:

$$L_n = 1 - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_0^{s_1} F_1^{s_2} \cdots F_{n-1}^{s_n},$$

$$L_n = 2^n + (-1)^n - n \sum_{\sigma_n = n} \frac{1}{|s_n|} m_n(s) F_0^{s_1} F_1^{s_2} \cdots F_{n-1}^{s_n},$$

$$L_n = 1 + (-1)^n - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_1^{s_1} F_2^{s_2} \cdots F_n^{s_n},$$

$$L_n = -n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_2^{s_1} F_3^{s_2} \cdots F_{n+1}^{s_n},$$

$$L_n = (-1)^n - n \sum_{\sigma_n = n} \frac{(-1)^{|s_n|}}{|s_n|} m_n(s) F_3^{s_1} F_4^{s_2} \cdots F_{n+2}^{s_n},$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s_n| = s_1 + \cdots + s_n$, $m_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$, and the summation is over nonnegative integers s_i satisfying equation $\sigma_n = n$.

3 Conclusion

In this paper, we evaluate several families of some Hessenberg matrices whose entries are Fibonacci numbers with sequential subscripts. In particular, we establish a connection between the Lucas and the Fibonacci sequences via Hessenberg determinants. Using generalized Brioschi's formula, we rewrite the obtained formulas as identities involving Lucas numbers, sums of products of Fibonacci numbers, and multinomial coefficients.

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