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Reduction of invertible matrices by two-sided transformations from Zelisko groups to a simpler form

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Zelisko group originated in the study of invertible matrices that reduce a matrix to the Smith form. In the calculations related to finding the Smith form of matrix product, their greatest common divisor and least common multiple, the problem of reducing an invertible matrices by two-sided transformations from Zelisko groups to a simpler form arises. In the article, exactly such a form was obtained. We also establish the relationship between the stable range of a ring and the representation of invertible matrices as a product of three factors, two of which belong to Zelisko groups.

Key words and phrases: Zelisko group, Smith form, reduction of matrices, transforming matrix.

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1 Introduction

By default, we denote by $M_n(R)$, $\mathrm{GL}_n(R)$ the ring and the complete linear group of $n \times n$ matrices over a ring R, respectively.

Let R be a commutative ring without zero divisors (commutative domain) over which each matrix D admits diagonal reduction, i.e. there exist invertible matrices P_D and Q_D of appropriate sizes, such that

$$P_D D Q_D = \mathrm{diag}\ (\delta_1, \dots, \delta_n) =: \Delta, \quad \text{where} \ \ \delta_i | \delta_{i+1}, \quad i = 1, \dots, n-1.$$

Due to I. Kaplanskii [5], R is called an *elementary divisor domain*. The matrix Δ is called the *Smith form*, and P_D, Q_D are left and right *transforming matrices* for the matrix D. A matrix P_D is ambiguously defined: each matrix from the coset $\mathbf{G}_\Delta P_D$ is again the left transforming matrix for the matrix D, where $\mathbf{G}_\Delta$ is the multiplicative group (Zelisko group) defined as follows

$$\mathbf{G}_\Delta = \{K \in \mathrm{GL}_n(R) : \exists\ K_1 \in \mathrm{GL}_n(R) \ \text{ such that } \ K\Delta = \Delta K_1\},$$

see [2,3,12]. If $\det\ \Delta \neq 0$, the group $\mathbf{G}_\Delta$ consist of all invertible matrices of the form

$$\left\| \begin{matrix} h_{11} & h_{12} & \dots & h_{1.n-1} & h_{1n} \\ \frac{\delta_2}{\delta_1} h_{21} & h_{22} & \dots & h_{2.n-1} & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\delta_n}{\delta_1} h_{n1} & \frac{\delta_n}{\delta_2} h_{n2} & \dots & \frac{\delta_n}{\delta_{n-1}} h_{n.n-1} & h_{nn} \end{matrix} \right\|. \tag{1}$$

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The situation is similar with the right transforming matrices for the matrix D: each matrix from coset $Q_D\mathbf{G}_\Delta^T$ is again the right transforming matrix for the matrix D, where $\mathbf{G}_\Delta^T$ is the group obtained from $\mathbf{G}_\Delta$ by transposing of its elements.

It should be noted that the Zelisko group $\mathbf{G}_\Delta$ has played an important role in solving the problem of separating a regular factor from a polynomial matrix over a field, which was one of the urgent problems of the middle of the last century [6].

The determinant of the matrix has the multiplicative property: determinant of the product of two matrices is equal to the product of the factors determinants. Smith form, generally speaking, does not have this property. Indeed,

$$A := \operatorname{diag}(1,\ 2), \quad B := \operatorname{diag}(2,\ 1)$$

have the Smith form $\operatorname{diag}(1,\ 2)$, however, their product $AB = \operatorname{diag}(2,\ 2)$, which is simultaneously the Smith form of the product of these matrices, does not coincide with the product of their Smith forms, namely the matrix $\operatorname{diag}(1,\ 4)$. However, under certain conditions, Smith form has the multiplicative property. The following result sheds light on the nature of the multiplicative of the Smith form.

Theorem 1 ([10]). *Let R be an elementary divisor domain and*

$$A = P_A^{-1}\mathrm{E}Q_A^{-1}, \quad B = P_B^{-1}\Phi Q_B^{-1} \in M_n(R),$$

where E, Φ are Smith forms of this matrices. The Smith form of the matrix AB is equal to $\mathrm{E}\Phi$ if and only if $Q_A^{-1}P_B^{-1} = LH$, where $L \in \mathbf{G}_\mathrm{E}^T$, $H \in \mathbf{G}_\Phi$.

As follows from the above example, not every invertible matrix can be represented as a product of matrices from $\mathbf{G}_\mathrm{E}^T$ and $\mathbf{G}_\Phi$. In this regard, the question arises: what is the simpler form of invertible matrix with respect to left transformations from $\mathbf{G}_\mathrm{E}^T$ and right from $\mathbf{G}_\Phi$. If R is an elementary divisor domain of stable range 1.5 (see definition below) the Theorem 5 gives the answer to this question.

The study of the Smith form of the greatest common divisor (g.c.d.) and the least common multiple (l.c.m.) of matrices requires establishing the condition under which the invertible matrix is the product of elements from $\mathbf{G}_\Phi$ and $\mathbf{G}_\mathrm{E}$. Theorem 4 shows a simpler form of such matrix with respect to the left transformations from $\mathbf{G}_\Phi$ and to the right from $\mathbf{G}_\mathrm{E}$.

2 Auxiliary statements

Throughout this article, unless specifically stated, R will denote a commutative elementary divisor domain. By $S_{(i)}$ we denote $i \times i$ submatrix of a matrix $S = \|s_{ij}\| \in M_n(R)$ of the form

$$S_{(i)} := \begin{Vmatrix} s_{n-i+1.1} & s_{n-i+1.2} & \dots & s_{n-i+1.i} \\ s_{n-i+2.1} & s_{n-i+2.2} & \dots & s_{n-i+2.i} \\ \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{ni} \end{Vmatrix}, \quad i = 1, \dots, n-1.$$

By d-matrix we will denote a diagonal matrix, in which each previous diagonal element divides its next one; $(a,\ b)$ denotes g.c.d. of a and b, $|A|$ denotes determinant of A.

Lemma 1. *Let $\Phi := \operatorname{diag}\ (\varphi_1, \ldots, \varphi_n)$ be nonsingular d-matrix over R and $S \in \mathrm{GL}_n(R)$. If $H \in \mathbf{G}_\Phi$, then*

$$\left(\frac{\varphi_{i+1}}{\varphi_i}, \left|S_{(i)}\right|\right) = \left(\frac{\varphi_{i+1}}{\varphi_i}, \left|(SH)_{(i)}\right|\right), \quad i = 1, \ldots, n-1.$$

Proof. The first i columns of H have the form (see (1))

$$H_i := \left\|\begin{array}{ccccc} h_{11} & h_{12} & \ldots & h_{1.i-1} & h_{1i} \\ \frac{\varphi_2}{\varphi_1}h_{21} & h_{22} & \ldots & h_{2.i-1} & h_{2i} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\varphi_i}{\varphi_1}h_{i1} & \frac{\varphi_i}{\varphi_2}h_{i2} & \ldots & \frac{\varphi_i}{\varphi_{i-1}}h_{i.i-1} & h_{ii} \\ \hline \frac{\varphi_{i+1}}{\varphi_1}h_{i+1.1} & \frac{\varphi_{i+1}}{\varphi_2}h_{i+1.2} & \ldots & \frac{\varphi_{i+1}}{\varphi_{i-1}}h_{i+1.i-1} & \frac{\varphi_{i+1}}{\varphi_i}h_{i+1.i} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\varphi_n}{\varphi_1}h_{n1} & \frac{\varphi_n}{\varphi_2}h_{n2} & \ldots & \frac{\varphi_n}{\varphi_{i-1}}h_{n.i-1} & \frac{\varphi_n}{\varphi_i}h_{ni} \end{array}\right\| =: \left\|\begin{array}{c} K_i \\ K'_i \end{array}\right\|.$$

All elements of K'_i are divided onto $\frac{\varphi_{i+1}}{\varphi_i}$. It follows that all $i \times i$ submatrices of H_i, with exception of K_i, contain at least one row, all elements of which are divided onto $\frac{\varphi_{i+1}}{\varphi_i}$. Therefore, all minors determinant of order i of H_i, with exception of $|K_i|$, are divided onto $\frac{\varphi_{i+1}}{\varphi_i}$. The matrix H_i is invertible. Consequently, g.c.d. of all minors determinant of order i of this matrix are equal to 1. Hence

$$\left(|K_i|, \frac{\varphi_{i+1}}{\varphi_i}\right) = 1. \tag{2}$$

The last i rows of S have the form $\left\|\begin{array}{cc} S_{(i)} & S'_{(i)} \end{array}\right\|$. Then

$$(SH)_{(i)} = \left\|\begin{array}{cc} S_{(i)} & S'_{(i)} \end{array}\right\| \cdot \left\|\begin{array}{c} K_i \\ K'_i \end{array}\right\|.$$

Using the Binet-Cauchy formula, we get

$$\left|(SH)_{(i)}\right| = \left|S_{(i)}\right| |K_i| + d_i,$$

where d_i is sum of all minors determinant of order i of $\left\|\begin{array}{cc} S_{(i)} & S'_{(i)} \end{array}\right\|$, except minor $|S_{(i)}|$, by the corresponding minors of $\left\|\begin{array}{c} K_i \\ K'_i \end{array}\right\|$. Taking into account that $d_i = \frac{\varphi_{i+1}}{\varphi_i}d'_i$, $d'_i \in R$, we have

$$\left|SH_{(i)}\right| = \frac{\varphi_{i+1}}{\varphi_i}d'_i + \left|S_{(i)}\right| |K_i|.$$

Considering (2), we have

$$\left(\frac{\varphi_{i+1}}{\varphi_i}, \left|(SH)_{(i)}\right|\right) = \left(\frac{\varphi_{i+1}}{\varphi_i}, \frac{\varphi_{i+1}}{\varphi_i}d'_i + \left|S_{(i)}\right| |K_i|\right) = \left(\frac{\varphi_{i+1}}{\varphi_i}, \left|S_{(i)}\right| |K_i|\right) = \left(\frac{\varphi_{i+1}}{\varphi_i}, \left|S_{(i)}\right|\right).$$

□

Theorem 2. *Let $S \in \mathrm{GL}_n(R)$ and $\Phi := \operatorname{diag}\ (\varphi_1, \dots, \varphi_n)$ be a nonsingular d-matrix. The group $\mathbf{G}_\Phi$ contains a matrix H such that*

$$SH = \begin{Vmatrix} t_{11} & t_{12} & \dots & t_{1.n-1} & 1 \\ t_{21} & t_{22} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix} \tag{3}$$

if and only if

$$\left(\frac{\varphi_{i+1}}{\varphi_i},\ \left| S_{(i)} \right| \right) = 1, \quad i = 1, \dots, n-1. \tag{4}$$

Proof. Necessity. Let $H \in \mathbf{G}_\Phi$, moreover $SH =: T$ is a matrix of the form (3). Based on the Lemma 1,

$$\left(\frac{\varphi_{i+1}}{\varphi_i},\ \left| S_{(i)} \right| \right) = \left(\frac{\varphi_{i+1}}{\varphi_i},\ \left| T_{(i)} \right| \right), \quad i = 1, \dots, n-1.$$

Since $\left| T_{(i)} \right| = \pm 1$, $i = 1, \dots, n-1$, equations (4) are fulfilled.

Sufficiency. Let $S := \left\| s_{ij} \right\| \in \mathrm{GL}_2(R)$. Since

$$\left(\frac{\varphi_2}{\varphi_1},\ s_{21} \right) = 1 \ \text{ and } \ (s_{22},\ s_{21}) = 1,$$

we have

$$\left(\frac{\varphi_2}{\varphi_1} s_{22},\ s_{21} \right) = 1.$$

The ring R is commutative finitely generated principal ideal domain (Bézout domain). It follows that there are u, v, such that

$$s_{22} \frac{\varphi_2}{\varphi_1} v + s_{21} u = 1.$$

Therefore,

$$K := \begin{Vmatrix} u & -s_{22} \\ \frac{\varphi_2}{\varphi_1} v & s_{21} \end{Vmatrix}$$

is an element of $\mathbf{G}_\Phi$. So

$$SK = \begin{Vmatrix} * & d \\ 1 & 0 \end{Vmatrix}.$$

The martrix SK is invertible. Consequently, d is an invertible element of R. Therefore, $K\operatorname{diag}\left(1,\ d^{-1}\right)$ will be the desired one. Thus, the theorem statement is correct for second order matrices.

Assume the correctness of this statement for all matrices of order less than n and consider a matrix $S := \left\| s_{ij} \right\| \in \mathrm{GL}_n(R)$. It follows from (4) that

$$\left(\frac{\varphi_{i+1}}{\varphi_i},\ (s_{n1}, s_{n2}, \dots, s_{ni}) \right) = 1, \quad i = 1, \dots, n-1. \tag{5}$$

Consider the row
$$\left\| \begin{matrix} s_{n1} & \frac{\varphi_2}{\varphi_1}s_{n2} & \frac{\varphi_3}{\varphi_1}s_{n3} & \dots & \frac{\varphi_n}{\varphi_1}s_{nn} \end{matrix} \right\|.$$
Step by step, using (5), we get
$$\begin{aligned}\left(s_{n1}, \quad \frac{\varphi_2}{\varphi_1}s_{n2}, \quad \frac{\varphi_3}{\varphi_1}s_{n3}, \quad \dots, \quad \frac{\varphi_n}{\varphi_1}s_{nn}\right) &= \left(s_{n1}, \quad \frac{\varphi_2}{\varphi_1}\left(s_{n2}, \quad \frac{\varphi_3}{\varphi_2}s_{n3}, \quad \dots, \quad \frac{\varphi_n}{\varphi_2}s_{nn}\right)\right) \\ &= \left(s_{n1}, \ s_{n2}, \ \frac{\varphi_3}{\varphi_2}s_{n3}, \ \dots, \ \frac{\varphi_n}{\varphi_2}s_{nn}\right) \\ &= \left((s_{n1}, \ s_{n2}), \ \frac{\varphi_3}{\varphi_2}\left(s_{n3}, \ \frac{\varphi_4}{\varphi_3}s_{n4}, \ \dots, \ \frac{\varphi_n}{\varphi_3}s_{nn}\right)\right) \\ &= \left(s_{n1}, \quad s_{n2}, \quad s_{n3}, \quad \frac{\varphi_4}{\varphi_3}s_{n4} \quad \dots, \quad \frac{\varphi_n}{\varphi_3}s_{nn}\right) = \cdots \\ &= (s_{n1}, s_{n2}, \dots, s_{nn}) = 1.\end{aligned}$$
There are $u_{n1}, \ u_{n2}, \ \dots, \ u_{nn} \in R$, such that
$$s_{n1}u_{11} + \frac{\varphi_2}{\varphi_1}u_{21}s_{n2} + \frac{\varphi_3}{\varphi_1}u_{31}s_{n3} + \ldots + \frac{\varphi_n}{\varphi_1}u_{n1}s_{nn} = 1.$$
Consequently, the column
$$\left\| \begin{matrix} u_{11} & \frac{\varphi_2}{\varphi_1}u_{21} & \frac{\varphi_3}{\varphi_1}u_{31} & \dots & \frac{\varphi_n}{\varphi_1}u_{n1} \end{matrix} \right\|^T$$
is unimodular. On the basis of [5, Theorem 3.7], it can be complemented to an invertible matrix of the form
$$H_1 := \left\| \begin{matrix} u_{11} & u_{12} & u_{13} & \dots & u_{1.n-1} & u_{1n} \\ \frac{\varphi_2}{\varphi_1}u_{21} & u_{22} & u_{23} & \dots & u_{2.n-1} & u_{2n} \\ \frac{\varphi_3}{\varphi_1}u_{31} & 0 & u_{33} & \dots & u_{3.n-1} & u_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\varphi_{n-1}}{\varphi_1}u_{n-1.1} & 0 & \dots & 0 & u_{n-1.n-1} & u_{n-1.n} \\ \frac{\varphi_n}{\varphi_1}u_{n1} & 0 & \dots & 0 & 0 & u_{nn} \end{matrix} \right\|,$$
where $H_1 \in \mathbf{G}_\Phi$. Then
$$SH_1 = \left\| \begin{matrix} c_{11} & c_{12} & \dots & c_{1n} \\ \dots & \dots & \dots & \dots \\ c_{n-1.1} & c_{n-1.2} & \dots & c_{n-1.n} \\ 1 & c_{n2} & \dots & c_{nn} \end{matrix} \right\| =: S_1$$
and
$$S_1 \underbrace{\left\| \begin{matrix} 1 & -c_{n2} & \dots & -c_{nn} \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{matrix} \right\|}_{H_2} = \left\| \begin{array}{c|ccc} c_{11} & c'_{12} & \dots & c'_{1n} \\ \dots & \dots & \dots & \dots \\ c_{n-1.1} & c'_{n-1.2} & \dots & c'_{n-1.n} \\ \hline 1 & 0 & \dots & 0 \end{array} \right\| =: \left\| \begin{matrix} C_{11} & C_{12} \\ 1 & \mathbf{0} \end{matrix} \right\|.$$
Since $H_1H_2 \in \mathbf{G}_\Phi$, according to Lemma 1, we get
$$\left(\frac{\varphi_{i+1}}{\varphi_i}, \left| \begin{matrix} C_{11} & C_{12} \\ 1 & \mathbf{0} \end{matrix} \right|_{(i)}\right) = 1, \quad i = 1, \dots, n-1.$$

It follows that

$$\left(\frac{\varphi_{i+1}}{\varphi_i},\ |C_{12}|_{(i)}\right) = 1, \quad i = 2, \ldots, n-1. \tag{6}$$

Consider the matrix $\Phi_1 := \operatorname{diag}(\varphi_2,\ \varphi_3,\ \ldots, \varphi_n)$. A matrix C_{12} is invertible and satisfies (6). According to the induction assumption, the group $\mathbf{G}_{\Phi_1}$ contains a matrix N, such that $C_{12}N$ has the form (3). Since $\|1 \bigoplus N\| \in \mathbf{G}_\Phi$, the matrix $H_1H_2\|1 \bigoplus N\|$ will be the matrix that reduces S to the desired form. □

3 Main results

A commutative ring R is said to be of *stable range* 1.5 (see [9, 11]) if for each $a, b \in R$ and $0 \neq c \in R$ satisfying $(a, b, c) = 1$, there exists $r \in R$ with

$$(a + br, c) = 1.$$

The notion of a ring of stable range 1.5 is modification of Bass stable range concept [1]. Commutative principal ideal domains, adequate rings [4], factorial rings has stable range 1.5. According to [8, Theorem 2.1], commutative Bézout domains of stable range 1.5 is an elementary divisor domains.

Denote by $U_n^{lw}(R)$ the group of lower unitriangular $n \times n$ matrices over R, and by $Ad_n^{up}(R)$ the set of matrices of the form (3) over R.

Theorem 3. *Let R be a commutative Bézout domain and $\Phi := \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$ be a nonsingular d-matrix. The following conditions are equivalent:*

1) *R has stable range 1.5;*

2) $\mathrm{GL}_2(R) = U_2^{lw}(R)Ad_2^{up}(R)\mathbf{G}_\Phi$ *for all matrices Φ;*

3) $\mathrm{GL}_n(R) = U_n^{lw}(R)Ad_n^{up}(R)\mathbf{G}_\Phi$ *for all matrices Φ and for all $n \geq 2$.*

Proof. 1) ⇒ 2). Let $A := \|a_{ij}\| \in \mathrm{GL}_2(R)$. Then

$$(a_{11},\ a_{21}) = 1 \quad\Longrightarrow\quad \left(a_{11},\ a_{21},\ \frac{\varphi_2}{\varphi_1}\right) = 1.$$

There exists $r \in R$ such that

$$\left(a_{11}r + a_{21},\ \frac{\varphi_2}{\varphi_1}\right) = 1.$$

So

$$\left\|\begin{matrix} 1 & 0 \\ r & 1 \end{matrix}\right\| A = \left\|\begin{matrix} a_{11} & a_{12} \\ a_{11}r + a_{21} & a_{12}r + a_{22} \end{matrix}\right\| =: A_1.$$

The matrix A_1 satisfies the conditions of Theorem 2. Therefore, there is $H \in \mathbf{G}_\Phi$ such that

$$A_1H = \left\|\begin{matrix} * & 1 \\ 1 & 0 \end{matrix}\right\|.$$

Thus,

$$A = \left\| \begin{matrix} 1 & 0 \\ -r & 1 \end{matrix} \right\| \left\| \begin{matrix} * & 1 \\ 1 & 0 \end{matrix} \right\| H^{-1},$$

i.e. $\mathrm{GL}_2(R) = U_2^{lw}(R) Ad_2^{up}(R)\mathbf{G}_\Phi$.

2) $\Rightarrow$ 1). Let $(a,b,c) = 1$, where $abc \neq 0$, and

$$a = (a,b)a_1, \quad b = (a,b)b_1, \quad (a_1, b_1) = 1.$$

There are $u, v \in R$, such that

$$a_1 u + b_1 v = 1.$$

So

$$A := \left\| \begin{matrix} a_1 & -v \\ b_1 & u \end{matrix} \right\|$$

is invertible matrix. Consider the nonsingular d-matrix

$$\Phi := \left\| \begin{matrix} 1 & 0 \\ 0 & c \end{matrix} \right\|.$$

According to the theorem assumption, A is a product of three matrices, namely $A = UVH$, where $U \in U_2^{lw}(R)$, $V \in Ad_2^{up}(R)$, $H \in \mathbf{G}_\Phi$. Noting that

$$U^{-1} := \left\| \begin{matrix} 1 & 0 \\ r & 1 \end{matrix} \right\|, \quad V := \left\| \begin{matrix} q & 1 \\ 1 & 0 \end{matrix} \right\|, \quad H^{-1} := \left\| \begin{matrix} h_{11} & h_{12} \\ ch_{21} & h_{22} \end{matrix} \right\|,$$

we get

$$V = U^{-1}AH^{-1} = \left\| \begin{matrix} * & * \\ h_{11}(ra_1 + b_1) + ch_{21}(u - rv) & * \end{matrix} \right\| = \left\| \begin{matrix} q & 1 \\ 1 & 0 \end{matrix} \right\|.$$

Consequently,

$$h_{11}(ra_1 + b_1) + ch_{21}(u - rv) = 1.$$

It follows that $(ra_1 + b_1, c) = 1$. Since, $((a,b), c) = 1$, we have

$$\big((a,b)\,(ra_1 + b_1)\,, c\big) = (ra + b, c) = 1.$$

The case $a = 0$ or $b = 0$ is obvious. Therefore, R has stable range 1.5.

The implication 3) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 3). For second order matrices, as was just proved, our statement is correct. Let us assume that it is correct for matrices of order less than n. Since 2) $\Leftrightarrow$ 1) is proved, R is a Bézout domain of stable range 1.5. Let $A := \|a_{ij}\| \in \mathrm{GL}_n(R)$. Then

$$(a_{11},\, a_{21},\, \ldots,\, a_{n1}) = 1 \implies \left(a_{11},\, a_{21},\, \ldots,\, a_{n1},\, \frac{\varphi_n}{\varphi_1}\right) = 1.$$

By [8, Property 1.19], there are $r_1, \ldots, r_{n-1} \in R$ such that

$$\left(a_{11}r_1 + a_{21}r_2 + \ldots + a_{n-1.1}r_{n-1} + a_{n1},\, \frac{\varphi_n}{\varphi_1}\right) = 1.$$

Consider the matrix

$$U_n := \left\| \begin{matrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ r_1 & \dots & r_{n-1} & 1 \end{matrix} \right\|.$$

All elements of the last row of invertible matrix $AU_n =: \|b_{ij}\|$ satisfy the condition

$$\big(b_{n1},\ (b_{n2},\ \dots,\ b_{nn})\big) = 1.$$

Since $\left(b_{n1},\ \frac{\varphi_n}{\varphi_1}\right) = 1$, then

$$\left(b_{n1},\ \frac{\varphi_n}{\varphi_1}(b_{n2},\ \dots,\ b_{nn})\right) = 1.$$

It follows that

$$\left(b_{n1},\ \frac{\varphi_2}{\varphi_1}b_{n2},\ \dots,\ \frac{\varphi_n}{\varphi_1}b_{nn}\right) = 1.$$

Similarly, to prove the sufficiency of Theorem 2, there exists $H_n \in \mathbf{G}_\Phi$, such that

$$U_nAH_n = \left\| \begin{array}{c|ccc} c_{11} & c_{12} & \dots & c_{1n} \\ \dots & \dots & \dots & \dots \\ c_{n-1.1} & c_{n-1.2} & \dots & c_{n-1.n} \\ \hline 1 & 0 & \dots & 0 \end{array} \right\| =: \left\| \begin{matrix} C_{11} & C_{12} \\ 1 & \mathbf{0} \end{matrix} \right\|.$$

Consider d-matrix $\Phi_1 := \operatorname{diag}(\varphi_2,\ \dots, \varphi_n)$. Since $C_{12} \in \mathrm{GL}_{n-1}(R)$, according to the assumption, $C_{12} = U_{n-1}V_{n-1}H_{n-1}$, where $U_{n-1} \in U^{lw}_{n-1}(R)$, $V_{n-1} \in Ad^{up}_{n-1}(R)$, $H_{n-1} \in \mathbf{G}_{\Phi_1}$. Thus,

$$\begin{aligned} U_nAH_n = \left\| \begin{matrix} C_{11} & C_{12} \\ 1 & \mathbf{0} \end{matrix} \right\| &= \left\| \begin{matrix} C_{11} & U_{n-1}V_{n-1}H_{n-1} \\ 1 & \mathbf{0} \end{matrix} \right\| \\ &= \underbrace{\left\| \begin{matrix} U_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{matrix} \right\|}_{M} \underbrace{\left\| \begin{matrix} U_{n-1}^{-1}C_{11} & V_{n-1} \\ 1 & \mathbf{0} \end{matrix} \right\|}_{S} \underbrace{\left\| \begin{matrix} 1 & \mathbf{0} \\ \mathbf{0} & H_{n-1} \end{matrix} \right\|}_{N} = MSN. \end{aligned}$$

So, A can be written in the form

$$A = \left(U_n^{-1}M\right) S \left(NH_n^{-1}\right).$$

Noting that $U_n^{-1}M \in U_n^{lw}(R)$, $S \in Ad_n^{up}(R)$, $NH_n^{-1} \in \mathbf{G}_\Phi$, we get

$$\mathrm{GL}_n(R) = U_n^{lw}(R)Ad_n^{up}(R)\mathbf{G}_\Phi.$$

The theorem is proved. □

Denote by $K(\alpha)$ the set of representatives of $R/R\alpha$, where $\alpha \in R$.

Theorem 4. *Let R be a commutative Bézout domain of stable range 1.5 and*

$$\mathrm{E} := \operatorname{diag}\,(\varepsilon_1,\ \varepsilon_2,\ \ldots,\ \varepsilon_n)\,,\quad \Phi := \operatorname{diag}\,(\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_n)$$

are nonsingular d-matrices over R and $S \in \mathrm{GL}_n(R)$. There are matrices $L \in \mathbf{G}_{\mathrm{E}}^{T}$, $H \in \mathbf{G}_{\Phi}$, such that

$$LSH = \left\| \begin{matrix} t_{11} & t_{12} & \ldots & t_{1.n-1} & 1 \\ t_{21} & t_{22} & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ t_{n-1.1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{matrix} \right\|, \tag{7}$$

where $t_{ij} \in K(\mu_{ij})$, $\mu_{ij} := \left(\frac{\varepsilon_{n+1-j}}{\varepsilon_i},\ \frac{\varphi_{n+1-i}}{\varphi_j}\right)$, $i,j = 1,\ldots,n-1$, $i+j \le n$.

Proof. At first we consider the case of second order matrices. Since $U_2^{lw}(R) \subset \mathbf{G}_{\mathrm{E}}^{T}$, according to the item 2) of Theorem 3, there exist matrices $L_1 \in \mathbf{G}_{\mathrm{E}}^{T}$, $H_1 \in \mathbf{G}_{\Phi}$, such that

$$L_1SH_1 = \left\| \begin{matrix} p_{11} & 1 \\ 1 & 0 \end{matrix} \right\|.$$

Let $p_{11} \equiv t_{11} \pmod{\mu_{11}}$, where $t_{11} \in K(\mu_{11})$. That is, $p_{11} - t_{11} = \mu_{11}r_{11}$, $r_{11} \in R$. Since $\mu_{11} = \left(\frac{\varepsilon_2}{\varepsilon_1}, \frac{\varphi_2}{\varphi_1}\right)$, there exist $u, v \in R$, such that

$$\mu_{11} = \frac{\varepsilon_2}{\varepsilon_1}u + \frac{\varphi_2}{\varphi_1}v.$$

Hence,

$$p_{11} - \frac{\varepsilon_2}{\varepsilon_1}ur_{11} - \frac{\varphi_2}{\varphi_1}vr_{11} = t_{11}.$$

So,

$$\left\| \begin{matrix} 1 & -\frac{\varepsilon_2}{\varepsilon_1}ur_{11} \\ 0 & 1 \end{matrix} \right\| \left\| \begin{matrix} p_{11} & 1 \\ 1 & 0 \end{matrix} \right\| \left\| \begin{matrix} 1 & 0 \\ -\frac{\varphi_2}{\varphi_1}vr_{11} & 1 \end{matrix} \right\| = \left\| \begin{matrix} t_{11} & 1 \\ 1 & 0 \end{matrix} \right\|.$$

Assume that our statement is correct for matrices of order less than n. Since $U_n^{lw}(R) \subset \mathbf{G}_{\mathrm{E}}^{T}$, according to the item 3) of Theorem 3, there are $L_1 \in \mathbf{G}_{\mathrm{E}}^{T}$, $H_1 \in \mathbf{G}_{\Phi}$, such that

$$L_1SH_1 = \left\| \begin{array}{cccc|c} p_{11} & p_{12} & \ldots & p_{1.n-1} & 1 \\ \hline p_{21} & p_{22} & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_{n-1.1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{array} \right\| = \left\| \begin{matrix} S_{11} & 1 \\ S_{21} & \mathbf{0} \end{matrix} \right\| =: S_1.$$

Consider d-matrices

$$\mathrm{E}_1 := \operatorname{diag}\left(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\right), \quad \Phi_n := \operatorname{diag}\left(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\right).$$

Since $S_{21} \in \mathrm{GL}_{n-1}$, according to the induction assumption, there are $L_2 \in \mathbf{G}_{\mathrm{E}_1}^{T}$, $H_2 \in \mathbf{G}_{\Phi_n}$, such that $L_2S_{21}H_2$ has the form (7). Then

$$\|1 \oplus L_2\|\, S_1\, \|H_2 \oplus 1\| = \left\| \begin{matrix} g_{11} & g_{12} & \cdots & g_{1.n-2} & g_{1.n-1} & 1 \\ t_{21} & t_{22} & \ldots & t_{2.n-2} & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ t_{n-1.1} & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{matrix} \right\| =: S_2,$$

where $t_{ij} \in K\left(\mu_{ij}\right)$. Let

$$g_{1.n-1} \equiv t_{1.n-1} (\text{mod } \mu_{1.n-1}), \ \text{ where } \ t_{1.n-1} \in K(\mu_{1.n-1}), \ \mu_{1.n-1} = \left(\frac{\varepsilon_2}{\varepsilon_1}, \ \frac{\varphi_n}{\varphi_{n-1}}\right).$$

By analogy of the case of the second order matrices, there are $l_{21}, \ h_{n.n-1} \in R$, such that

$$g_{1.n-1} - t_{1.n-1} = \frac{\varepsilon_2}{\varepsilon_1} l_{21} + \frac{\varphi_n}{\varphi_{n-1}} h_{n.n-1}.$$

Then

$$L_3 := \left\| \begin{matrix} 1 & -\frac{\varepsilon_2}{\varepsilon_1} l_{21} \\ 0 & 1 \end{matrix} \right\| \oplus I_{n-2} \in \mathbf{G}_{\mathrm{E}}^T, \quad H_3 := I_{n-2} \oplus \left\| \begin{matrix} 1 & 0 \\ -\frac{\varphi_n}{\varphi_{n-1}} h_{n.n-1} & 1 \end{matrix} \right\| \in \mathbf{G}_\Phi$$

and

$$L_3 S_2 H_3 = \left\| \begin{matrix} g'_{11} & g'_{12} & \dots & g'_{1.n-2} & t_{1.n-1} & 1 \\ t_{21} & t_{22} & \dots & t_{2.n-2} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n-1.1} & 1 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{matrix} \right\| =: S_3.$$

Similarly, there are $l_{31}, \ h_{n.n-2} \in R$, such that

$$g'_{1.n-2} - t_{1.n-2} = \frac{\varepsilon_3}{\varepsilon_1} l_{31} + \frac{\varphi_n}{\varphi_{n-2}} h_{n.n-2}, \quad t_{1.n-2} \in K(\mu_{1.n-2}), \quad \mu_{1.n-2} = \left(\frac{\varepsilon_3}{\varepsilon_1}, \ \frac{\varphi_n}{\varphi_{n-2}}\right).$$

Then

$$L_4 := \left\| \begin{matrix} 1 & 0 & -\frac{\varepsilon_3}{\varepsilon_1} l_{31} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\| \oplus I_{n-3} \in \mathbf{G}_{\mathrm{E}}^T, \quad H_4 := I_{n-3} \oplus \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\varphi_n}{\varphi_{n-2}} h_{n.n-2} & 0 & 1 \end{matrix} \right\| \in \mathbf{G}_\Phi$$

and

$$L_4 S_3 H_4 = \left\| \begin{matrix} g''_{11} & g''_{12} & \dots & g''_{1.n-3} & t_{1.n-2} & t_{1.n-1} & 1 \\ t_{21} & t_{22} & \dots & t_{2.n-3} & t_{2.n-2} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n-1.1} & 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \end{matrix} \right\|.$$

Continuing the described process, we reduce S to the form (7). The theorem is proved. □

Theorem 5. *Let R be a commutative Bézout domain of stable range 1.5 and*

$$\mathrm{E} := \operatorname{diag}\left(\varepsilon_1, \ \varepsilon_2, \ \dots, \ \varepsilon_n\right), \quad \Phi := \operatorname{diag}\left(\varphi_1, \ \varphi_2, \ \dots, \ \varphi_n\right)$$

are nonsingular d-matrices over R and $S \in \mathrm{GL}_n(R)$.

There are $H \in \mathbf{G}_\Phi$, $L \in \mathbf{G}_{\mathrm{E}}$, such that

$$HSL = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ k_{21} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ k_{n-1.1} & k_{n-1.2} & \dots & 1 & 0 \\ k_{n1} & k_{n2} & \dots & k_{n.n-1} & 1 \end{matrix} \right\|, \tag{8}$$

where $t_{ij} \in K(\nu_{ij})$, $\nu_{ij} = \left(\frac{\varphi_i}{\varphi_j}; \ \frac{\varepsilon_i}{\varepsilon_j}\right)$, $i = 2, \dots, n$, $j = 1, \dots, n-1$, $i > j$.

Proof. By [8, Theorem 2.14], there are matrices $H_1 \in \mathbf{G}_\Phi$, $L_1 \in \mathbf{G}_{\mathrm{E}}$, such that

$$H_1 S L_1 = \left\| \begin{array}{ccccc} 1 & 0 & \dots & 0 & 0 \\ q_{21} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q_{n-1.1} & q_{n-1.2} & \dots & 1 & 0 \\ q_{n1} & q_{n2} & \dots & q_{n.n-1} & 1 \end{array} \right\| =: S_1.$$

Reduction of the matrix S_1 to the form (8) by transformations from the groups $\mathbf{G}_\Phi$, $\mathbf{G}_{\mathrm{E}}$ is not much differs from the methods used in the final part of the proof of the Theorem 4, and therefore we will not quote it. The theorem is proved. □

In connection with this result, we note the work of V. Petrychkovych [7] concerning two-sided transformations of matrices over adequate rings.

Remark 1. *The matrix of the form* (7) *is not the canonical form of S with respect to left transformations from $\mathbf{G}_{\mathrm{E}}^T$ and right from $\mathbf{G}_\Phi$. Similarly, the matrix of the form* (8) *is not the canonical form of S with respect to left transformations from $\mathbf{G}_\Phi$ and right from $\mathbf{G}_{\mathrm{E}}$.*

Example 1. *Let $R = \mathbb{Z}$ and*

$$\Phi = \mathrm{E} := \mathrm{diag}\,(1, 8), \quad S_1 := \left\| \begin{array}{cc} 6 & 1 \\ 1 & 0 \end{array} \right\|, \quad S_2 := \left\| \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right\|.$$

Since $\mu_{21} = (8,\ 8) = 8$, $K(8) = \{0, 1, \dots, 7\}$, S_1, S_2 are matrices of the form (7). *However, $\mathbf{G}_{\mathrm{E}}^T$ contains the matrix $H := \left\| \begin{array}{cc} -1 & 8 \\ 0 & 1 \end{array} \right\|$, and $\mathbf{G}_\Phi$ the matrix $L := \left\| \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right\|$, such that $HS_1L = S_2$.*

Example 2. *Let $R = \mathbb{Z}$ and*

$$\Phi := \mathrm{diag}\,(1, 12), \quad \mathrm{E} := \mathrm{diag}\,(1, 18), \quad S_1 := \left\| \begin{array}{cc} 1 & 0 \\ 4 & 1 \end{array} \right\|, \quad S_2 := \left\| \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right\|.$$

Since $\nu_{21} = (12,\ 18) = 6$, $K(6) = \{0, 1, \dots, 5\}$, S_1, S_2 are matrices of the form (8). *However, $\mathbf{G}_\Phi$ contains the matrix $H := \left\| \begin{array}{cc} 13 & -2 \\ 12 \cdot 6 & -11 \end{array} \right\|$, and $\mathbf{G}_{\mathrm{E}}$ the matrix $L := \left\| \begin{array}{cc} -7 & 2 \\ -18 & 5 \end{array} \right\|$, such that $HS_1L = S_2$.*

Remark 2. *It is easy to check that in the ring of lower triangular matrices over Bézout domain the matrix of the form* (8) *is the canonical form with respect to left transformations from $\mathbf{G}_\Phi$ and right from $\mathbf{G}_{\mathrm{E}}$.*

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Романів А.М., Щедрик В.П. *Зведення оборотних матриць двосторонніми перетвореннями з груп Зеліска до простішого вигляду* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 500–511.

Група Зеліска виникла при дослідженні оборотних матриць, що зводять задану матрицю до її форми Сміта. При обчисленнях, пов'язаних зі знаходженням форми Сміта добутку матриць, їх найбільшого спільного дільника та найменшого спільного кратного постає задача зведення оборотних матриць двосторонніми перетвореннями з груп Зеліска до простішого вигляду. В роботі отримано власне такий вигляд. Також встановлено взаємозв'язок між стабільним рангом кільця та зображенням оборотних матриць у вигляді добутку трьох співмножників, два з яких належать групам Зеліска.

Ключові слова і фрази: група Зеліска, форма Сміта, редукція матриць, перетворювальна матриця.