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Almost everywhere convergence of two-dimensional Walsh-Nörlund means

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The present paper the almost everywhere convergence of two-dimensional Walsh-Nörlund means is studied, when the given function belongs to the hybrid Hardy space $H_\natural$. Because the Nörlund means are a generalization of several known classical summability methods, previously known classical theorems are derived from the main theorem. In addition some new results follow in particular cases as well.

Key words and phrases: Walsh system, Nörlund mean, Hardy space, weak type inequality, almost everywhere convergence.

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1 Walsh functions

We denote the set of non-negative integers by $\mathbb{N}$. By a dyadic interval in $\mathbb{I} := [0,1)$ we mean one of the form $\left[(l-1)2^{-k}, l2^{-k}\right)$ for some $k \in \mathbb{N}$, $0 < l \leq 2^k$. For any given $k \in \mathbb{N}$ and $x \in \mathbb{I}$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. The σ-algebra generated by the dyadic intervals $\left\{I_n(x) : x \in \mathbb{I}\right\}$ will be denoted by $\mathcal{A}_n$, more precisely, we have

$$\mathcal{A}_n = \sigma\left\{\left[k2^{-n}, (k+1)2^{-n}\right) : 0 \leq k < 2^n\right\},$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$.

We also use the notation $I_n := I_n(0)$, $\overline{I}_n := \mathbb{I} \backslash I_n$, $n \in \mathbb{N}$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n = 0$ or 1. If x is a dyadic rational number, we choose the expansion, which terminates in zeros.

For any given $n \in \mathbb{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) 2^k,$$

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where $\varepsilon_k(n) = 0$ or 1 for $k \in \mathbb{N}$. This expression is called the binary expansion of n and the numbers $\varepsilon_k(n)$ are called the binary coefficients of n. Let us introduce for $1 \leq n \in \mathbb{N}$ the notation $|n| := \max\{j \in \mathbb{N}: \varepsilon_j(n) \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

Let us set the nth Walsh-Paley function at the point $x \in \mathbb{I}$ as

$$w_n(x) = (-1)^{\sum\limits_{j=0}^{\infty} \varepsilon_j(n) x_j}, \quad n \in \mathbb{N}.$$

Let us denote the logical addition on $\mathbb{I}$ by $\dotplus$. That is, for any $x, y \in \mathbb{I}$, we have

$$x \dotplus y := \sum_{n=0}^{\infty} |x_n - y_n| \, 2^{-(n+1)}.$$

The nth Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall [10, 13] that

$$D_{2^n}(x) = 2^n \, \mathbf{1}_{I_n}(x), \tag{1}$$

where $\mathbf{1}_E$ is the characteristic function of the set E.

The norm of the space $L_1(\mathbb{I}^2)$, where $\mathbb{I}^2 := [0,1) \times [0,1)$, is defined by

$$\|f\|_1 := \int_{\mathbb{I}^2} |f(x,y)| \, dxdy.$$

The space weak-$L_1(\mathbb{I}^2)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_1(\mathbb{I}^2)} := \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\}) < +\infty,$$

where μ is the Lebesgue measure.

Let $f \in L_1(\mathbb{I}^2)$. The rectangular partial sums of 2-dimensional Fourier series with respect to the Walsh system are defined by

$$S_{n,m}(f; x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \hat{f}(i,j) \, w_i(x) w_j(y),$$

where the number

$$\hat{f}(i,j) = \int_{\mathbb{I}^2} f(x,y) \, w_i(x) w_j(y) dxdy$$

is the (i,j)th Walsh-Fourier coefficient.

2 Walsh-Nörlund means

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. It is always assumed that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. We define the nth Nörlund mean of the Walsh-Fourier series by

$$t_n^{(q)}(f; x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f; x), \quad f \in L_1(\mathbb{I}), \tag{2}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$, $n \geq 1$, and

$$S_k(f;x) := \sum_{i=0}^{k-1} \left(\int_{\mathbb{I}} f w_i \right) w_i(x)$$

is the partial sums of the one-dimensional Walsh-Fourier series. In this case, the summability method generated by the sequence $\{q_k : k \geq 0\}$ is regular (see [12]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0. \tag{3}$$

The Nörlund kernels are defined by

$$F_n^{(q)}(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t).$$

The Fejér means and kernels are

$$\sigma_n(f;x) := \frac{1}{n} \sum_{k=1}^{n} S_k(f;x), \qquad K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t), \quad K_0 \equiv 0.$$

It is easily seen that the means $t_n(f)$ and $\sigma_n(f)$ can be got by convolution of f with the kernels $F_n^{(q)}$ and K_n. That is,

$$t_n^{(q)}(f;x) = \int_{\mathbb{I}} f(x \dotplus t)\, F_n^{(q)}(t)\, dt = \left(f * F_n^{(q)}\right)(x),$$
$$\sigma_n(f;x) = \int_{\mathbb{I}} f(x \dotplus t)\, K_n(t)\, dt = \left(f * K_n\right)(x).$$

It is well-known that the L_1 norms of Fejér kernels are uniformly bounded, that is, there exists a positive constant c such that

$$\|K_n\|_1 \leq c \quad \text{for all} \quad n \in \mathbb{N}. \tag{4}$$

S. Yano [18] estimated the value of c and he gave $c = 2$. Recently, in paper [15], it was shown that the exact value of c is $\frac{17}{15}$.

For sequences $\{q_k : k \in \mathbb{N}\}$ and $\{p_l : l \in \mathbb{N}\}$ of non-negative numbers the two-dimensional Nörlund means $t_{n,m}^{(q,p)}(f)$ are defined as follows

$$t_{n,m}^{(q,p)}(f;x,y) := \frac{1}{Q_n P_m} \sum_{k=1}^{n} \sum_{l=1}^{m} q_{n-k} p_{m-l} S_{k,l}(f;x,y), \quad p_0, q_0 > 0,$$

where $P_m := \sum_{k=0}^{m-1} p_k$.

The two-dimensional kernel function $F_{n,m}^{(q,p)}(x,y)$ is the product of one-dimensional kernels $F_n^{(q)}(x)$ and $F_m^{(p)}(y)$ defined by the sequences $\{q_k : k \in \mathbb{N}\}$ and $\{p_l : l \in \mathbb{N}\}$, respectively. That is,

$$t_{n,m}^{(q,p)}(f;x,y) := \left(f * \left(F_n^{(q)} \otimes F_m^{(p)}\right)\right)(x,y) = \int_{\mathbb{I}^2} f(x \dotplus s, y \dotplus t)\, F_{n,m}^{(q,p)}(s,t)\, ds\, dt,$$

where $\otimes$ denotes Kronecker's product.

The following two theorems were proved in the paper [9] and they have an important role in proving the main theorems of the presented article.

Theorem 1. *Let* $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$ *with* $n_1 > n_2 > \cdots > n_r \geq 0$. *Let us set* $n^{(0)} := n$ *and* $n^{(i)} := n^{(i-1)} - 2^{n_i}$ *for* $i = 1, \ldots, r-1$, *and* $n^{(r)} := 0$. *Then the following decomposition*

$$F_n^{(q)} = \frac{w_n}{Q_n} \sum_{j=1}^{r} Q_{n^{(j-1)}} w_{2^{n_j}} D_{2^{n_j}} - \frac{w_n}{Q_n} \sum_{j=1}^{r} w_{n^{(j-1)}} w_{2^{n_j}-1} \sum_{k=1}^{2^{n_j}-1} q_{k+n^{(j)}} D_k =: F_{n,1} + F_{n,2} \tag{5}$$

holds.

Theorem 2. *Let* $\{q_k : k \in \mathbb{N}\}$ *be a sequence of non-negative numbers. If this sequence is monotone non-increasing (in sign* $q_k \downarrow$*), then*

$$\left\| F_n^{(q)} \right\|_1 \sim \frac{1}{Q_n} \sum_{k=1}^{|n|} \left| \varepsilon_k(n) - \varepsilon_{k+1}(n) \right| Q_{2^k}. \tag{6}$$

Note that the estimation (6) is two-sided, when

$$\sup_n \frac{1}{Q_n} \sum_{k=1}^{|n|} \left| \varepsilon_k(n) - \varepsilon_{k+1}(n) \right| Q_{2^k} = \infty,$$

otherwise there is only an upper estimation.

Applying Abel's transformation we have

$$\sum_{k=1}^{2^{n_j}-1} q_{k+n^{(j)}} D_k = \sum_{k=1}^{2^{n_j}-2} \left(q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k K_k + q_{n^{(j-1)}-1} (2^{n_j} - 1) K_{2^{n_j}-1}.$$

Thus, we get

$$\begin{aligned} F_{n,2} = \frac{w_n}{Q_n} \sum_{j=1}^{r} \sum_{k=1}^{2^{n_j}-2} & w_{n^{(j-1)}} w_{2^{n_j}-1} \left(q_{k+n^{(j)}} - q_{k+n^{(j)}+1} \right) k K_k \\ & + \frac{w_n}{Q_n} \sum_{j=1}^{r} w_{n^{(j-1)}} w_{2^{n_j}-1} q_{n^{(j-1)}-1} (2^{n_j} - 1) K_{2^{n_j}-1} =: F_{n,2}^{(1)} + F_{n,2}^{(2)}. \end{aligned} \tag{7}$$

3 Operators of subsequences of Walsh-Nörlund means and H_1 space

Let $f \in L_1(\mathbb{I})$. The dyadic Hardy space $H_1(\mathbb{I})$ consists of all functions for which

$$\|f\|_{H_1} := \left\| \sup_{n \in \mathbb{N}} \left| S_{2^n}(f) \right| \right\|_1 < \infty.$$

A bounded measurable function a is an H_1 atom, if either a is constant or there exists a dyadic interval I, such that

a) $\int_I a = 0$;

b) $\|a\|_\infty \leq \mu(I)^{-1}$;

c) $\operatorname{supp} a \subset I$.

An operator T is called H_1-quasi-local, if there exists a constant $c > 0$ such that for every H_1-atom a we have

$$\int_{\mathbb{I}\setminus I} |Ta| \leq c < \infty,$$

where I is the support of the atom. We shall need the following Theorem A [13, p. 263].

An operator $T : X \to Y$ is called a σ-sublinear operator, if for any $\alpha \in \mathbb{C}$ it satisfies

$$\left|T\left(\sum_{k=1}^{\infty} f_k\right)\right| \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha||T(f)|,$$

where X is a linear space and Y is a measurable function space.

Theorem A. *Suppose that the operator T is σ-sublinear and quasi-local. If T is bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$, then*

$$\|Tf\|_1 \leq c\,\|f\|_{H_1}, \qquad f \in H_1(\mathbb{I}).$$

Let us define for the positive number K the subset $L_K(\{q_k\})$ of natural numbers by

$$L_K(\{q_k\}) := \left\{ n \in \mathbb{N} : V(n, \{q_k\}) := \frac{1}{Q_n} \sum_{k=1}^{|n|} |\varepsilon_{k+1}(n) - \varepsilon_k(n)| Q_{2^k} \leq K \right\}.$$

The following result has been proved in [8].

Theorem 3 ([8]). *Let $\{m_A : A \in \mathbb{N}\}$ be a subsequence which is not a subsequence of $L_K(\{q_k\})$ for any $K > 0$. More precisely,*

$$\sup_{A\in\mathbb{N}} \frac{1}{Q_{m_A}} \sum_{k=1}^{|m_A|} |\varepsilon_k(m_A) - \varepsilon_{k+1}(m_A)| Q_{2^k} = \infty \tag{8}$$

holds. Then the operator $t_{m_A}^{(q)}(f)$ is not uniformly bounded from $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$.

It is known [9] that if $\{q_k : k \in \mathbb{N}\}$ is a non-decreasing sequence, then the maximum operator $t_*^{(q)} := \sup_{n\in\mathbb{N}} |t_n^{(q)}|$ is bounded from the space H_1 to the space L_1. In general, the similar statement is invalid when $\{q_k : k \in \mathbb{N}\}$ is decreasing, and it is dependent on the rate of decrease. The paper [8] provides a necessary and sufficient condition for the maximum operator to be bound from the space H_1 to the space L_1. In particular, this condition reads as follows

$$\sup_{n\in\mathbb{N}} \left(\frac{1}{Q_{2^n}} \sum_{k=1}^{n} Q_{2^k} \right) < \infty. \tag{9}$$

Now, we can formulate the following problem.

Let us say the condition (9) is not fulfilled, also, there exists a subsequence $\{n_a : a \in \mathbb{N}\}$, such that

$$\sup_{a\in\mathbb{N}} \left(\frac{1}{Q_{n_a}} \sum_{k=1}^{|n_a|} \left|\varepsilon_{k-1}(n_a) - \varepsilon_k(n_a)\right| Q_{2^k} \right) < \infty. \tag{10}$$

Then is the maximal operator $\sup_{a\in\mathbb{N}} |t_{n_a}^{(q)}|$ bounded from $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$?

In general, the answer to the question is negative. In particular, the following is valid.

Theorem 4. *Let* $\{q_k : k \in \mathbb{N}\}$ *be a non-inereasing sequence. Then there exists a subsequence* $\{n_a : a \in \mathbb{N}\}$ *for which condition* (10) *is satisfied and the maximum operator* $\sup_{a\in\mathbb{N}} \left|t_{n_a}^{(q)}\right|$ *is not bounded from the space* $H_1(\mathbb{I})$ *to the space* $L_1(\mathbb{I})$.

Proof. Set $f_b := D_{2^{b+1}} - D_{2^b}$. Then it easy to see that $\sup_n \left|S_{2^n}(f_b)\right| = D_{2^b}$ and consequently,

$$\|f_b\|_{H_1} = \left\|\sup_n |S_{2^n}(f_b)|\right\|_1 = \|D_{2^b}\|_1 = 1.$$

We can write

$$\begin{aligned}
t_{2^b+2^s}^{(q)}(f_b) &= \frac{1}{Q_{2^b+2^s}} \sum_{v=1}^{2^b+2^s} q_{2^b+2^s-v} S_v(f_b) = \frac{1}{Q_{2^b+2^s}} \sum_{v=2^b}^{2^b+2^s} q_{2^b+2^s-v} S_v\left(D_{2^{b+1}} - D_{2^b}\right) \\
&= \frac{1}{Q_{2^b+2^s}} \sum_{v=2^b}^{2^b+2^s} q_{2^b+2^s-v}\left(D_v - D_{2^b}\right) = \frac{w_{2^b}}{Q_{2^b+2^s}} \sum_{v=1}^{2^s} q_{2^s-v} D_v, \quad s < b.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\left\|\sup_{0\le s<b} \left|t_{2^b+2^s}^{(q)}(f_b)\right|\right\|_1 &= \left\|\sup_{0\le s<b} \left|\frac{1}{Q_{2^b+2^s}} \sum_{v=1}^{2^s} q_{2^s-v} D_v\right|\right\|_1 \ge \sum_{t=0}^{b-1} \int_{I_t\setminus I_{t+1}} \sup_{0\le s<b} \left|\frac{1}{Q_{2^b+2^s}} \sum_{v=1}^{2^s} q_{2^s-v} D_v\right| \\
&\ge \sum_{t=0}^{b-1} \int_{I_t\setminus I_{t+1}} \left|\frac{1}{Q_{2^b+2^t}} \sum_{v=1}^{2^t} q_{2^t-v} D_v\right| = \sum_{t=0}^{b-1} \frac{1}{2^{t+1}} \frac{1}{Q_{2^b+2^t}} \sum_{v=1}^{2^t} q_{2^t-v} v \\
&\ge c \sum_{t=0}^{b-1} \frac{1}{2^{t+1}} \frac{1}{Q_{2^b}} \sum_{v=2^{t-1}}^{2^t} q_{2^t-v} v \ge \frac{c}{Q_{2^b}} \sum_{t=1}^{b} Q_{2^t}.
\end{aligned}$$

Hence,

$$\sup_{b\in\mathbb{N}} \left\|\sup_{0\le s<b} \left|t_{2^b+2^s}^{(q)}(f_b)\right|\right\|_1 = \infty.$$

Theorem 4 is proved. □

Set $t_{\#}^{(q)}(f) := \sup_{n\in\mathbb{N}} \left|t_{2^n}^{(q)}(f)\right|$. Now, we prove that the following is valid.

Theorem 5. *Let* $\{q_k : k \in \mathbb{N}\}$ *be a non-inereasing sequence. The following inequality is true*

$$\left\|t_{\#}^{(q)}(f)\right\|_1 \le c\,\|f\|_{H_1}, \quad f \in H_1(\mathbb{I}). \tag{11}$$

Proof. According to Theorem A, it suffices to prove that the sequence of operator $t^{\#}(f)$ is H_1-quasi-local and bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. The boundedness of the operator is proved in [9]. We suppose that $f \in H_1(\mathbb{I})$. Let function a be an H_1 atom. Without lost of generality we can suppose that $\text{supp}(a) \subset I_N$. Consequently, for any function g which is $\mathcal{A}_N$-measurable we have that $\int_I ag = 0$. So, we can assume that $n > N$ and it is enough to prove that the operator $t^{\#}(f)$ is H_1-quasi local. That is,

$$\sup_{n>N} \int_{\overline{I}_N} \left|a * F_{2^n}^{(q)}\right| \le c. \tag{12}$$

Let $x \in \overline{I}_N$. Then from (5) we can write

$$\begin{aligned} \left|\left(a * F_{2^n}^{(q)}\right)(x)\right| &\leq \|a\|_\infty \int_{I_N} \left|F_{2^n}^{(q)}(x \dotplus t)\right| dt \\ &\leq 2^N \int_{I_N} \left|F_{2^n,1}(x \dotplus t)\right| dt + 2^N \int_{I_N} \left|F_{2^n,2}(x \dotplus t)\right| dt. \end{aligned} \tag{13}$$

Since $F_{2^n,1} = D_{2^n}$, $t \in I_N$ and $x \notin I_N$, we have that $x \dotplus t \notin I_N$ and consequently by (1) we get $D_{2^n}(x \dotplus t) = 0$ for $n > N$, and

$$\int_{I_N} \left|F_{2^n,1}(x \dotplus t)\right| dt = 0. \tag{14}$$

Now, we estimate $F_{n,2}$ (see (7)). Since the estimation of $F_{n,2}^{(2)}$ is analogous to the estimation of $F_{n,2}^{(1)}$ it suffices to evaluate one of them. It is proved in [8] that

$$\int_{I_N} \left|F_{n,2}^{(1)}(x \dotplus t)\right| dt = J_1(n) + J_2(n) + J_3(n), \quad x \in \overline{I}_N,\ t \in I_N,$$

where

$$J_1(n) \leq \frac{c}{2^N Q_n} \sum_{j=1}^{N} q_{2^{j-1}} \sum_{m=1}^{j} \sup_{2^{m-1} \leq k < 2^m} \left(k \left|K_k\right|\right),$$

$$J_2(n) \leq \frac{c}{2^{2N}} \sum_{m=1}^{N} \sup_{2^{m-1} \leq k < 2^m} \left(k \left|K_k\right|\right),$$

$$J_3(n) \leq \frac{c}{2^{2N}} \sum_{s=0}^{N-1} \left(2^s K_{2^s}\right) + \frac{c}{2^N} \sum_{l=0}^{N-1} 2^l \mathbf{1}_{I_N(e_l)}.$$

Since (see [13])

$$\int_{\mathbb{I}} \sup_{2^{m-1} \leq k < 2^m} \left(k \left|K_k\right|\right) \leq c 2^m,$$

we get

$$\begin{aligned} \int_{\overline{I}_N} \sup_{n > 2^N} 2^N \left(\int_{I_N} \left|F_{n,2}^{(1)}(x \dotplus t)\right| dt \right) dx &\leq \frac{c}{Q_{2^N}} \sum_{j=1}^{N} q_{2^{j-1}} \sum_{m=1}^{j} \int_{\overline{I}_N} \sup_{2^{m-1} \leq k < 2^m} \left(k \left|K_k(x)\right|\right) dx \\ &+ \frac{c}{2^N} \sum_{m=1}^{N} \int_{\overline{I}_N} \sup_{2^{m-1} \leq k < 2^m} \left(k \left|K_k(x)\right|\right) dx \\ &+ \frac{c}{2^N} \sum_{s=0}^{N-1} \int_{\overline{I}_N} \left(2^s K_{2^s}(x)\right) dx \\ &+ \frac{c}{2^N} \sum_{l=0}^{N-1} 2^l \int_{\overline{I}_N} \mathbf{1}_{I_N(e_l)}(x)\, dx \\ &\leq \frac{c}{Q_{2^N}} \sum_{j=1}^{N} q_{2^{j-1}} 2^j + c. \end{aligned}$$

Since

$$\sum_{j=2}^{N} q_{2^{j-1}} 2^j \leq 4 \sum_{j=2}^{N} \sum_{l=2^{j-2}}^{2^{j-1}-1} q_l = 4 \sum_{j=1}^{2^{N-1}-1} q_j \leq 4 Q_{2^N},$$

we have

$$\int_{\overline{I}_N} \sup_{n > N} 2^N \left(\int_{I_N} \left|F_{n,2}^{(1)}(x \dotplus t)\right| dt \right) dx \leq c < \infty. \tag{15}$$

Combining (13), (14) and (15) we complete the proof of Theorem 5. □

4 Unrestricted convergence of two-dimensional Walsh-Nörlund means

Let $f \in L_1(\mathbb{I}^2)$. The hybrid maximal function is introduced by

$$f^{\natural}(x,y) := \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(t,y)dt \right|.$$

Define the space $H_{\natural}(\mathbb{I}^2)$ of Hardy type as the set of functions f such that $\|f\|_{H^{\#}} := \left\|f^{\natural}\right\|_1 < \infty$.

The positive logarithm function $\log^+$ is defined by

$$\log^+(x) := \begin{cases} \log(x), & \text{if } x > 1, \\ 0, & \text{otherwise.} \end{cases}$$

We say that the function $f \in L_1(\mathbb{I}^2)$ belongs to the logarithmic space $L\ln L(\mathbb{I}^2)$ if the integral

$$\int_{\mathbb{I}^2} \left|f\right| \log^+ \left|f\right|$$

is finite. Recall that $L\ln L(\mathbb{I}^2) \subset H_{\natural}(\mathbb{I}^2)$. Moreover, $f \in L\ln L(\mathbb{I}^2)$ if and only if $|f| \in H^1_{\natural}(\mathbb{I}^2)$.

In 1992, F. Móricz, F. Schipp and W.R. Wade proved that the Fejér means

$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} S_{i,j}(f)$$

of two-dimensional Walsh-Fourier series converge to f almost everywhere in Pringsheim sense (that is, with no restrictions on the indices other than $\min\{n,m\} \to \infty$) for all functions $f \in L\ln L(\mathbb{I}^2)$ [11]. Later, G. Gát [2] proved that the theorem of F. Móricz, F. Schipp and W.R. Wade can not be sharpened.

Hardy spaces were used by F. Weisz [16, 17] to study the almost everywhere summability of Walsh-Fourier series. In particular, it follows from theorem of F. Weisz that if $f \in H_{\natural}(\mathbb{I}^2)$, then

$$\lim_{\min\{n,m\}\to\infty} \frac{1}{A^{\alpha}_{n-1}A^{\beta}_{m-1}} \sum_{i=1}^{n}\sum_{j=1}^{m} A^{\alpha-1}_{n-i}A^{\beta-1}_{m-j} S_{ij}(f;x,y) = f(x,y) \tag{16}$$

for a.e. $(x,y) \in \mathbb{I}^2, \alpha, \beta > 0$.

The following theorem was proved by F. Móricz, F. Schipp and W.R. Wade [11] (see also [14]), which allows us to apply the one-dimensional case result for the two-dimensional case. In particular, the following has been proved.

Theorem 6 ([11]). *Let $\{V^i_n : n \in \mathbb{N}\}$, $i = 0,1$, be the sequence of $L_1(\mathbb{I})$ functions. Define one-dimensional operators $T^i f := \sup_{m\in\mathbb{N}} \left|f * V^i_m\right|$, $\widetilde{T}^i f := \sup_{m\in\mathbb{N}} \left|f * \left|V^i_m\right|\right|$ for $f \in L_1(\mathbb{I})$, $i = 0,1$, and suppose that there exist absolute constants c_0, c_1, such that*

$$\mu\Big(\big\{\widetilde{T}^0 f > \lambda\big\}\Big) \le \frac{c_0}{\lambda}\|f\|_1 \qquad \text{and} \qquad \left\|T^1 f\right\|_1 \le c_1 \|f\|_{H_1}$$

for $f \in L_1(\mathbb{I})$ and $\lambda > 0$.

If $Tf := \sup_{(n,m)\in\mathbb{N}^2} \left|f * \left(V^0_n \otimes V^1_m\right)\right|$, then

$$\mu\big(\{Tf > \lambda\}\big) \le \frac{c_0 c_1}{\lambda}\|f\|_{H_{\natural}}, \quad f \in H_{\natural}(\mathbb{I}^2),\ \lambda > 0.$$

Let us set

$$\widetilde{t}^{(q)}_{m_A}(f) := f * \left|F^{(q)}_{m_A}\right|.$$

The next theorem was proved in paper [9].

Theorem 7. *Let $\{m_A : A \in \mathbb{P}\}$ be a strictly monotone increasing sequence. Let $\{q_k : k \in \mathbb{N}\}$ be a monotone non-increasing sequence of non-negative numbers (in sign $q_k \downarrow$). If*

$$\{m_A : A \in \mathbb{N}\} \in L_K(\{q_k\}) \tag{17}$$

for some $K > 0$, then there exists a positive constant c such that

$$\sup_{\lambda>0} \lambda\mu\left(\left\{\sup_A \left|\widetilde{t}^{(q)}_{m_A}(f)\right| > \lambda\right\}\right) \le c\|f\|_1 \tag{18}$$

holds for all $f \in L_1(\mathbb{I})$ and $\lambda > 0$.

By Theorem 6, Theorem 5, Theorem 7 and (9) we have the next theorems.

Theorem 8. *Let $\{p_k : k \in \mathbb{N}\}$, $\{q_k : k \in \mathbb{N}\}$ be non-increasing sequences, such that*

$$\{n_A : A \in \mathbb{N}\} \subset L_K(\{q_k\})$$

for some $K > 0$ and

$$\sup_m \left(\frac{1}{P_{2^m}} \sum_{k=1}^{m} P_{2^k}\right) < \infty.$$

Then the maximal operator $\sup_{A,m\in\mathbb{N}} \left|f * F^{(q)}_{n_A} \otimes F^{(p)}_m\right|$ is boundend from the space $H_\natural\left(\mathbb{I}^2\right)$ to the space weak-$L_1(\mathbb{I}^2)$.

Theorem 9. *Let $\{p_k : k \in \mathbb{N}\}$, $\{q_k : k \in \mathbb{N}\}$ be non-increasing sequences, such that*

$$\{n_A : A \in \mathbb{N}\} \subset L_K(\{q_k\})$$

for some $K > 0$. Then the maximal operator $\sup_{A,m\in\mathbb{N}} \left|f * F^{(q)}_{n_A} \otimes F^{(p)}_{2^m}\right|$ is boundend from the space $H_\natural(\mathbb{I}^2)$ to the space weak-$L_1(\mathbb{I}^2)$.

Theorem 10. *Let $\{q_k : k \in \mathbb{N}\}$ be non-increasing sequence such that $\{n_A : A \in \mathbb{N}\} \subset L_K(\{q_k\})$ for some $K > 0$ and let $\{p_k : k \in \mathbb{N}\}$ be increasing (positive) sequence. Then the maximal operator $\sup_{A,m\in\mathbb{N}} \left|f * F^{(q)}_{n_A} \otimes F^{(p)}_m\right|$ is boundend from the space $H_\natural\left(\mathbb{I}^2\right)$ to the space weak-$L_1(\mathbb{I}^2)$.*

The usual density argument imply the next corollaries.

Corollary 1. *Let the conditions of Theorem 8 be satisfied. Then the two-dimensional Walsh-Nörlund means $t_{n_A,m}(f)$ converge to f almost everywhere as $\min\{n_A, m\} \to \infty$ for all functions $f \in H_\natural(\mathbb{I}^2)$.*

Corollary 2. *Let the conditions of Theorem 9 be satisfied. Then the two-dimensional Walsh-Nörlund means $t_{n_A,2^m}(f)$ converge to f almost everywhere as $\min\{n_A, 2^m\} \to \infty$ for all functions $f \in H_\natural(\mathbb{I}^2)$.*

Corollary 3. *Let the conditions of Theorem 10 be satisfied. Then the two-dimensional Walsh-Nörlund means* $t_{n_A,m}(f)$ *converge to* f *almost everywhere as* $\min\{n_A, m\} \to \infty$ *for all functions* $f \in H_\natural(\mathbb{I}^2)$.

Finally, consider the case when both sequences $\{p_k : k \in \mathbb{N}\}$ and $\{q_k : k \in \mathbb{N}\}$ are increasing and positive. In order to consider this case, we need the following lemma.

Lemma 1. *Let* $\{q_l : l \in \mathbb{N}\}$ *be a monotone non-decreasing sequence of non-negative numbers. Then for the operator* $\widetilde{t}(f) := \sup_{n\in\mathbb{N}} |f * |F_n||$ *weak type inequality* (18) *holds.*

Proof. Let the sequence $\{q_l : l \in \mathbb{N}\}$ be a monotone non-decreasing sequence of non-negative numbers. Applying Abel's transformation it is easily seen that

$$\left|F_n^{(q)}\right| \le \frac{1}{Q_n}\sum_{k=1}^{n-1}(q_{n-k}-q_{n-k-1})\,k\,|K_k| + \frac{q_0 n}{Q_n}|K_n| =: \widetilde{F}_n^{(q)}.$$

Since

$$\frac{1}{Q_n}\sum_{k=1}^{n-1}(q_{n-k}-q_{n-k-1})\,k + \frac{q_0 n}{Q_n} \le c < \infty,$$

from (4) we can prove that the operator $\widetilde{t}(f)$ is of type (L_∞, L_∞). Indeed, we can write

$$\left\|\sup_{n\in\mathbb{N}}\left|f * \left|F_n^{(q)}\right|\right|\right\|_\infty \le \left\|\sup_{n\in\mathbb{N}}\left||f| * \widetilde{F}_n^{(q)}\right|\right\|_\infty \le \|f\|_\infty \sup_{n\in\mathbb{N}}\left\|\widetilde{F}_n^{(q)}\right\|_1 \le c\,\|f\|_\infty.$$

Now, we prove that the operator $\sup_{n\in\mathbb{N}}\left|f * \widetilde{F}_n^{(q)}\right|$ is quasi-local. In particular, let $f \in L_1(\mathbb{I})$ such that $\operatorname{supp}(f) \subset I_N(u')$, $\int_{I_N(u')} f = 0$ for some dyadic interval $I_N(u')$. Then we have

$$\int_{\overline{I}_N(u')} \sup_{n\in\mathbb{N}}\left|f * \widetilde{F}_n^{(q)}\right| \le c\,\|f\|_1.$$

By the shift invariancy of the measure it can be supposed that $u' = 0$. If $n \le 2^N$, then

$$f * \widetilde{F}_n^{(q)} = 0.$$

Consequently, $n > 2^N$ can be supposed. Then we have

$$f * \widetilde{F}_n^{(q)} = \frac{1}{Q_n}\left(\sum_{k=2^N+1}^{n-1}(q_{n-k}-q_{n-k-1})\,k\Big(f * |K_k|\Big) + q_0 n\Big(f * |K_n|\Big)\right).$$

Hence,

$$\begin{aligned}\int_{\overline{I}_N}\sup_{n>2^N}\left|f * \widetilde{F}_n^{(q)}\right| &\le \sup_{n\in\mathbb{N}}\frac{1}{Q_n}\sum_{k=1}^{n-1}(q_{n-k}-q_{n-k-1})\,k\int_{\overline{I}_N}\left(\sup_{k>2^N}\int_{I_N}|f(u)|\,|K_k(x\dotplus u)|\,du\right)dx \\ &\quad + \int_{\overline{I}_N}\left(\sup_{n>2^N}\frac{q_0 n}{Q_n}\int_{I_N}|f(u)|\,|K_n(x\dotplus u)|\,du\right)dx \\ &\le c\int_{I_N}|f(u)|\left(\int_{\overline{I}_N}\sup_{k>2^N}|K_k(x\dotplus u)|\,dx\right)du.\end{aligned}$$

Since

$$\int_{\overline{I}_N} \sup_{n\geq 2^N} |K_n| < \infty, \tag{19}$$

we have

$$\int_{\overline{I}_N} \sup_{n>2^N} \left| f * \widetilde{F}_n^{(q)} \right| \leq c \, \|f\|_1 .$$

Since the sublinear operator is quasi-local and of type (L_∞, L_∞), then by standard argument (see, e.g., [13, p. 263]) it follows that the operator $\widetilde{t}\,(f)$ is of weak type (1,1). □

From Theorem 6 and Lemma 1 we get the validity of the following assertion.

Theorem 11. *Let $\{q_k : k \in \mathbb{N}\}$ and $\{p_l : l \in \mathbb{N}\}$ be monotone non-decreasing sequences of non-negative numbers. Then there exists a positive constant c such that*

$$\left| \left\{ \sup_{n,m} |t_{n,m}(f)| > \lambda \right\} \right| \leq \frac{c}{\lambda} \|f^\natural\|_1$$

holds for all $f \in H^1_\natural(\mathbb{I}^2)$.

Corollary 4. *Let the conditions of Theorem 11 be satisfied. Then the two-dimensional Walsh-Nörlund means $t_{n,m}\,(f)$ converge to f almost everywhere as $\min\{n,m\} \to \infty$ for all functions $f \in H_\natural(\mathbb{I}^2)$.*

G. Gát and G. Karagulyan [4] recently established that $L \ln L(\mathbb{I}^2)$ space is a maximum Orlicz space, in which a sequence of operators $t_{n,m}\,(f)$ can be convergent almost everywhere to f as $\min\{n,m\} \to \infty$. On the other hand, the problems of almost everywhere convergence of double Walsh-Fourier series along subsequences were studied in the papers [1,5,6].

5 Applications to various summability methods

Example 1. *Let*

$$p_j := \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j > 0, \end{cases}$$

and

$$q_j = A_j^{\alpha-1}, \quad \alpha \in (0,1)\,, \quad j \in \mathbb{N}.$$

Then

$$t_{n,m}(f;x,y) := \frac{1}{Q_n P_m} \sum_{k=1}^{n} \sum_{l=1}^{m} q_{n-k} p_{m-l} S_{k,l}(f;x,y) = \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_{k,m}(f;x,y).$$

Since the sequences $\left\{q_j : j \in \mathbb{N}\right\}$ and $\left\{p_j : j \in \mathbb{N}\right\}$ are non-increasing and $\left\{q_j : j \in \mathbb{N}\right\}$ satisfies condition (9), we get

$$\lim_{\substack{n\to\infty \\ L_M(\{p_k\})\ni m\to\infty}} \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_{k,m}(f;x,y) = f\,(x,y) \quad \text{for a.e. } x,y \in \mathbb{I},\ f \in H_\natural(\mathbb{I}^2).$$

Example 2. *Let $q_j := A_j^{\alpha-1}$, $p_j := A_j^{\beta-1}$, $\alpha, \beta \in (0,1)$. Then from Corollary 1 we obtain*

$$\lim_{\min\{n,m\}\to\infty} \frac{1}{A_{n-1}^{\alpha} A_{m-1}^{\beta}} \sum_{k=1}^{n} \sum_{l=1}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} S_{k,l}(f;x,y) = f\,(x,y) \quad \text{for a.e. } x,y \in \mathbb{I}, f \in H_\natural(\mathbb{I}^2).$$

Example 3. *Let* $q_j := j^{\alpha-1}$, $p_j := j^{\beta-1}$, $\alpha, \beta \geq 0$. *First, we consider the case when* $\alpha = \beta = 0$. *Then the Nörlund means coincide with the Nörlund logarithmic means*

$$t_{n,m}(f;x,y) := \frac{1}{Q_n P_m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S_{k,l}(f;x,y)}{(n-k)(m-l)}.$$

From Corollary 2 we have

$$\lim_{\substack{m\to\infty \\ L_K(\{q_k\})\ni n\to\infty}} \frac{1}{m\log n} \sum_{k=1}^{n-1} \sum_{l=1}^{2^m-1} \frac{S_{k,l}(f;x,y)}{(n-k)(2^m-l)} = f(x,y) \quad \text{for a. e.} \quad x,y \in \mathbb{I}, f \in H_{\natural}\left(\mathbb{I}^2\right). \tag{20}$$

We note that for the subsequence $t_{2^n,2^m}(f)$, *the Nörlund logarithmic means a.e. convergence and divergence were studied by G. Gát and the first author in the paper [3]. In particular, the following was proved.*

Theorem GG. *Let* $f \in H^{\natural}\left(\mathbb{I}^2\right)$. *Then*

$$t_{2^n,2^m}(f;x,y) \to f(x,y) \quad \text{a.e. as} \quad \min(n,m) \to \infty.$$

We also have proved that Theorem GG can not be sharpened. We note that, equality (20) in the one-dimensional case was proved by the first author in [7].

Now, we consider the case when $\alpha = 0$ and $\beta > 0$. Then from Corollary 2 we get

$$\lim_{\substack{m\to\infty \\ L_K(\{1/k\})\ni n\to\infty}} \frac{1}{m^\beta \log n} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S_{k,l}(f;x,y)}{(n-k)(m-l)^{1-\beta}} = f(x,y) \quad \text{for a.e.} \quad x,y \in \mathbb{I}, f \in H_{\natural}(\mathbb{I}^2).$$

Finally, we consider the case when $\alpha, \beta > 0$ and from Corollary 2 we get

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \frac{1}{n^\alpha m^\beta} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S_{k,l}(f;x,y)}{(n-k)^{1-\alpha}(m-l)^{1-\beta}} = f(x,y) \quad \text{for a.e.} \quad x,y \in \mathbb{I}, f \in H_{\natural}(\mathbb{I}^2).$$

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У цій статті досліджено збіжність майже скрізь двовимірних середніх Уолша-Нерлунда, коли задана функція належить гібридному простору Харді $H_\natural$. Оскільки середні Нерлунда є узагальненням кількох відомих класичних методів підсумовування, раніше відомі класичні теореми ми виводимо з основної теореми. Крім того, в окремих випадках отримано деякі нові результати.

Ключові слова і фрази: система Уолша, середня Нерлунда, простір Харді, нерівність слабкого типу, збіжність майже скрізь.