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STRUCTURE OF THE FUNDAMENTAL SOLUTION OF CAUCHY PROBLEM FOR KOLMOGOROV SYSTEMS OF SECOND-ORDER

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Abstract. We study a structure of the fundamental solution of the Cauchy problem for a class of ultra parabolic equations with a finite number of groups of variables with degenerated parabolicity.

Keywords: Kolmogorov systems, fundamental solutions, degenerate parabolic equations.

1. Introduction

In this paper we investigate the fundamental solution to the Cauchy problem (FSCP) for a class of systems of Kolmogorov equations [1, 2] which are a natural generalization of diffusion equation with inertia.

The equations that generalize Kolmogorov equations have been studied in many papers, especially a detailed description of the theories of the diffusion equations with inertia is presented in [3-5]. The great interest to study the behavior of solutions of Cauchy problem and boundary problems for Kolmogorov equations caused their wide application in Financial Mathematics for calculating the price of Asian options and volatility characteristics [6, 7].

We consider the system of equations with arbitrary number of groups of variables for which the parabolicity is degenerated and research the structure of FSCP. I particular we obtained exact dependence and types of shifts on lines of levels for FSCP of systems and model equations.

2. Notations and Formulation of the Problem

Let n, n_0 be fixed natural numbers and $n_0 > 1, x \in R^{n_0}, (x,s) = \sum_{j=1}^{n_0} x_j s_j, \quad x^* = (x_2, ..., x_{n_0})$. Consider the following system of equations of the form

$$\partial_t u_\nu(t,x) - \sum_{j=1}^{n_0-1} x_j \, \partial_{x_{j+1}} \ u_\nu(t,x) = \sum_{k=0}^{2} \sum_{r=1}^{n} a_k^{\nu r}(t,x) \partial_{x_1^k}^k \, u_r(t,x), \ \nu = \overline{1,n}. \ x \in \Pi_{(0,T]}, \qquad (2.1)$$

where $\Pi_{(0,T]} = \{(t,x), \ \ t \in (0,T], \ \ T > 0, \ \ x \in R^{n_0}\}$.
Assume that the coefficients $a_k^{nur}(t,x)$ of the system are complex-valued functions such that

$$\partial_t \, \omega_\nu(t,x) = \sum_{k=0}^{2} \sum_{r=1}^{n} a_k^{\nu r}(t,x) \partial_{x_1^k}^k \, \omega_r(t,x), \ \nu = \overline{1,n} \qquad (2.2)$$

system (2.2) is uniformly parabolic in Petrovsky means in $\Pi_{[0,T]}$ and $(x_2, ..., x_{n_0})$ are considered as parameters. For convenience, we write the system (2.1) in matrix form:

$$\partial_t u(t,x) - \sum_{j=1}^{n_0-1} x_j \partial_{x_{j+1}} \ u(t,x) = \sum_{k=0}^{2} a_k(t,x) \, \partial_{x_1^k}^k \, u(t,x).$$

Find the solution of system (2.1) which satisfies the initial condition

$$u(t,x)|_{t=\tau} = u_0(x), \ x \in R^{n_0}, \ 0 \le \tau < t \le T, \qquad (2.3)$$

where τ is a given number and $u_0 = col(u_{01}, (x), ..., u_{0n}, (x))$ is a given matrix column.

3. The Solution of Cauchy Problem for Systems with Constant Coefficients

Let us consider Cauchy problem for system (2.1) in which coefficients a_2^{nur} are constants $a_1^{\nu r} \equiv 0, \ a_0^{\nu r} = 0, \ \nu = \overline{1,n}, \ r = \overline{1,n}$.

$$\partial_t u_\nu - \sum_{j=1}^{n_0-1} x_j \, \partial_{x_{j+1}} \, u_\nu(t,x) = \sum_{r=1}^{n} \ a_2^{\nu r} \partial_{x_1^2}^2 (t,x) \ u_r(t,x), \ \ \nu = \overline{1,n.} \qquad (3.1)$$

$$u_r(t,x)|_{t=\tau} = u_{0\,r}(x), \ x \in R^{n_0}, \ r = \overline{1,n}, \ 0 \le \tau < t \le T, \qquad (3.2)$$

where $u_{0r}(x)$ are sufficiently smooth compactly supported functions.
Let λ be roots $\lambda_1, ..., \lambda_n$ of equation $det\{(a_2^{\nu r} \ (is)^2)_{\nu,\, r=1}^n - \lambda I\} = 0$, where I is the identity matrix of order n, i is the imaginary unit and $Re\lambda(s) \le -\delta_0 \ s_1^2, \ s_1 \in R^1$ with some constant $\delta_0 > 0$.
Using the Fourier transform we can reduce the Cauchy problem (3.1), (3.2) to the Cauchy problem for systems of differential equations in partial derivatives of the first order. For this components $u_1, ..., u_n$ solutions of Cauchy problem (3.1), (3.2) will be sought in the form of an

inverse Fourier transform on s of unknown functions $v_1, ..., v_n$, namely

$$u(t,x) := F^{-1}[v_r(t,s)](t,x) := (2\pi)^{-n_0/2} \int\limits_{R^{n_0}} exp\{i(x,s)\} v_r(t,s) ds,$$
$$0 \le \tau < t \le T, x \in R^{n_0}, r = \overline{1,n}.$$

Taking into account the equality

$$\partial_t\, F^{-1}[v_r] = F^{-1}[\partial_t v_r]; x_j \partial_{x_{j+1}} F^{-1}[v_r] = F^{-1}[-s_{j+1}\partial_{s_j} v_r],$$
$$\partial^2_{x_1^2}\, F^{-1}[v_r] = F^{-1}[-s_1^2\, v_r], \partial_{x_1} F^{-1}[v_r] = F^{-1}[is_1 v_r]$$

we obtain for $v_1, ..., v_n$ the following Cauchy problem

$$\partial_t v_r(t,s) + \sum_{j=1}^{n_0-1} s_{j+1}\partial_{s_j}\, v_r\, (t,s) = -\sum_{k=1}^{n} a_2^{rk}\, s_1^2\, v_k\, (t,s). \tag{3.3}$$

$$v_r\, (t,s)\mid_{t=r} = v_{0r}(s),\ s \in R^{n_0},\ r = \overline{1,n},\ 0 \le \tau < t \le T. \tag{3.4}$$

Since functions $u_{0r}(x)$ are quite smooth and compactly supported their Fourier transforms are analytic functions for which the inequality true

$$|v_{0r}(s)| \le c(1+|s|)^{-m},\ s \in R^{n_0},\ m \ge n_0 + 1, \tag{3.5}$$

where $v_{or}(s) := F[u_{0r}(x)]$.

In problems (3.3), (3.4) s^* - parameter. The system (3.3) consists of differential equations in partial derivatives of the first order and these equations have the same basic parts. According to [8, p. 146-148] this system is equivalent to a homogeneous linear differential equation with first-order partial derivatives for functions ω with $n + n_0$ independent variables $t, s_1, ..., s_{n_0-1}, v_1, ..., v_n$,

$$\partial_t \omega + \sum_{j=1}^{n_0-1} s_{j+1}\partial_{s_j}\omega + \sum_{r,\,l=1}^{n} a_2^{rl} s_1^2 v_r \partial_{v_l}\omega = 0,$$

which is equivalent to the system of ordinary differential equations:

$$dt = \frac{ds_1}{s_2} = \frac{ds_2}{s_3} = ... = \frac{ds_{n_0-1}}{s_{n_0}} = \frac{dv_1}{\sum\limits_{n=1}^{n} -a_2^{1r}\, s_1^2\, v_r} = ... = \frac{dv_n}{\sum\limits_{r=1}^{n} -a_2^{nr}\, s_1^2\, v_r}$$

.

Let us select $n_0 + n - 1$ independent integrals, in this system from $dt = \frac{ds_{n_0-1}}{s_{n_0}}$ we can find

$$s_{n_0-1} = ts_{n_0} + c_1 \tag{3.6}$$

and from $dt = \frac{ds_{n_0-2}}{s_{n_0-1}}$, taking into account (3.6), we have

$$s_{n_0-2} = t^2 s_{n_0}/2 + tc_1 + c_2 \tag{3.7}$$

and from $dt = \frac{ds_{n_0-k}}{s_{n_0-(k-1)}}$ for $k = \overline{3, n_0 - 1}$ we obtain

$$s_{n_0-k} = \frac{t^k}{k!} s_{n_0} + \frac{t^{k-1}}{(k-1)!} c_1 + \frac{t^{k-2}}{(k-2)!} c_2 + ... + c_k. \tag{3.8}$$

Using (3.6) - (3.8), we write

$$s = (s_1, s_2, ..., s_{n_0-(k-1)}, ..., s_{n_0}) = (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \frac{t^{n_0-2}}{(n_0-2)!}c_1 + ... + c_{n_0-1}, ..., \frac{t^{k-1}}{(k-1)!}s_{n_0} + \frac{t^{k-2}}{(k-2)!}c_1 + ... + c_{k-1}, ts_{n_0} + c_1, s_{n_0}). \quad (3.9)$$

Substituting (3.9) into the system of equations

$$dv_r = -\sum_{l=1}^{n} a_2^{rl} s_1^2 v_l dt, r = \overline{1,n} \quad (3.10)$$

we obtain the system of equations (3.10) for the characteristics of (3.6)-(3.8):

$$dv_r(t, P(t, s_{n_0}, c)) = -\sum_{l=1}^{n} a_2^{rl} \, (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!}c_{k-1})^2 v_l dt, \quad (3.11)$$

where

$$P(t, s_{n_0}, c) := (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!} \, c_{k-1}, ..., ts_{n_0} + c_1, s_{n_0}),$$

with the initial condition

$$v_r(t, P(t, s_{n_0}, c)|_{t=\tau} = v_{0r} \, (P(\tau, s_{n_0}, c)), \; r = \overline{1,n}. \quad (3.12)$$

Problem (3.11), (3.12) has a unique solution for $0 \leq \tau < t \leq T < +\infty$. Solution of Cauchy problem (3.16), (3.17) can be written as

$$v(t, P(t, s_{n_0}, c)) = Q(t, \tau, P(\tau, s_{n_0}, c))v_0(P(\tau, s_{n_0}, c)), \quad (3.13)$$

where $Q(t, \tau, P(\tau, s_{n_0}, c))$ is a normal matrix solutions of (3.11), $Q(t, \tau, P(\tau, s_{n_0}, c))|_{t=\tau} = I$.

Since the matrix

$$A(t) = (-a_2^{rl}(\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!}c_{k-1})^2)_{r,\, l=1}^{n}$$

commutes with $\int_\tau^t A(\tau)d\tau$, then

$$Q(t, \tau, P(\tau, s_{n_0}, c)) = exp\{-\int_\tau^t A(\beta)d\beta\} = exp\{-A_1 \int_\tau^t (\frac{\beta^{n_0-1}s_{n_0}}{(n_0-1)!} + \sum_{k=2}^{n_0} \frac{\beta^{n_{0-k}}}{(n_{0-k})!} \, c_{k-1})^2 d\beta\} \; ,$$

where $A_1 = (a_2^{r\ell})_{r,\, l=1}^{n}$.

We use the method of mathematical induction to find $c_k, \; k = \overline{1, n_0-1}$, with (3.6) - (3.7), for c_k is true $c_k, \; k = \overline{1, n_0-1}$, for c_k formula:

$$c_k = \sum_{j=0}^{k} (-t)^j s_{n_0-k+j}/j!, \; k = \overline{1, n_0}.$$

Valid, from (3.6) - (3.8) we have:

$$c_1 = s_{n_0-1} - ts_{no},$$

$$c_2 = s_{n_0} - tc_1 - \frac{t^2}{2!}\, s_{n_0} = s_{n_0-2} - ts_{n_0-1} + \frac{t^2}{2!} s_{n_0},$$

similar

$$c_3 = s_{n_0-3} - \frac{t^3}{3!} s_{n_0} - \frac{t^2}{2!} c_1 - tc_2 = s_{n_0-3} - ts_{n_0-2} + \frac{t^2}{2!} s_{n_0-1} - \frac{t^3}{3!} s_{n_0}, ...$$

Let $c_{k-1} = s_{n_0-k-1} - ts_{n_o-k+2} + ... + \frac{(-t)^j}{j!} s_{n_0-k+j+1} + ... + \frac{(-t)^{k-1}}{(k-1)!} s_{n_0}$, then for c_k we obtain

$$\begin{aligned} c_k &= s_{n_0-k} - \frac{t^k}{k!} s_{n_0} - \frac{t^{k-1}}{(k-1)!} c_1 - ... - tc_{k-1} = s_{n_0-k} - \frac{t^k}{(k)!} s_{n_0} - \frac{t^{k-1}}{(k-1)!}(s_{n_0-1} - ts_{n_0}) - t^{k-2} \\ &\times (s_{n_0-2} - ts_{n_0-1} + \frac{t^2}{2!} s_{n_0}) - ... - t(s_{n_0-k+1} - ts_{n_0-k+2} + ... + \frac{(-t)^j}{j!} s_{n_0-k+j+1} + ... \\ &+ \frac{(-t)^{k-1}}{(k-1)!} s_{n_0}) = s_{n_0-k} + s_{n_0} \frac{(-t)^k}{k!} + s_{n_0-1} \frac{(-t)^{k-1}}{(k-1)!} + ... + (-t) s_{n_0-k+1}, \end{aligned}$$

so we have

$$c_k = \sum_{j=0}^{k} (-t)^j s_{n_0-k+j}/j!,\ k = \overline{1,n}. \tag{3.14}$$

Substituting (3.14) into (3.13) we obtain

$$\begin{aligned} v(t,s) &= \exp\{-A_1 \int_\tau^t (\beta^{n_0-1} s_{n_0}/(n_0-1)! + \sum_{j=0}^{k} \beta^{n_0-k} (\sum_{j=0}^{k-1} s_{n_0-j} (-t)^{k-j-1}/(k-1-j)! \\ &\times ((n_0-1)!))^2 d\beta\}\ v_0(s_1 + (\tau-t)s_2 + (\tau-t)^2 s_3/2! + ... + (\tau-t)^{n_0-1} s_{n_0}/(n_0-1)!, s_2 \\ &+ (\tau-t)s_3 + ... + (\tau-t)^{n_0-2} s_{n_0}/(n_0-2)!, ..., s_{n_0-1} + (\tau-t)s_{n_0}, s_{n_0}). \end{aligned}$$

After the reduction of similar terms in the exponent exp, we will have

$$\begin{aligned} v(t,s) &= \exp\{-A_1 \int_\tau^t (s_1 + (\beta-t)s_2 + ... + (\beta-t)^{n_0-1} s_{n_0}/(n_0-1)!)^2 d\beta\} v_0(s_1 + \tau - t)s_2 + ... \\ &+ (\tau-t)^{n_0-1}! s_{n_0}/(n_0-1)!, s_2 + (\tau-t)s_3 + ... + (\tau-t)^{n_0-2} s_{n_0}/(n_0-2)!, ..., s_{n_0-1} \\ &+ (\tau-t)s_{n_0}, s_{n_0}). \end{aligned}$$

Find $u(t,x)$:

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^{n_0/2}} \int_{R^{n_0}} \exp\{i(x,s) - A_1 \int_\tau^t (s_1 + (\beta-t)s_2 + ... + (\beta-t)^{n_0-1} s_{n_0}/(n_0-1)!)^2 d\beta \\ &\times v_0(s_1 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-1}\, s_{n_0}/(n_0-1)!, ...s_{n_0-1} + (\tau-t)s_{n_0}, s_{n_0}) ds. \end{aligned} \tag{3.15}$$

Changing the variables in (3.15) by

$$s_1 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-1} s_{n_0}/(n_0-1)! = \alpha_1,$$
$$s_2 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-2}\, s_{n_0}/(n_0-2)! = \alpha_2,$$

$$s_{n_0-1} + (\tau - t)s_{n_0} = \alpha_{n_0-1};$$

$$s_{n_0} = \alpha_{n_0},$$

or

$$s_1 = \alpha_1 - (\tau - t)\alpha_2 + ... + (-1)^{n-1} \frac{(\tau - t)^{n_0-1}}{(n_0-1)!}\alpha_{n_0},$$

$$s_2 = \alpha_2 - (\tau - t)\alpha_3 + ... + (-1)^{n-2} (\tau - t)^{n_0-2}\alpha_{n_0}/(n_0-2)!,$$

$$s_k = \alpha_k - (\tau - t)\alpha_{k+1} + ... + (-1)^{n_0-k} (\tau - t)^{n_0-k}/(n_0-k)!,$$

$$s_{n_0-1} = \alpha_{n_0-1} - (\tau - t)\alpha_{n_0},$$

$$s_{n_0} = \alpha_{n_0},$$

we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^{n_0/2}} \int\limits_{R^{n_0}} \exp\{i\alpha_1 x_1 + i\alpha_2(x_2 - (\tau - t)x_1) + i\alpha_3\ (x_3 - (\tau - t)x_2 + (\tau - t)^2 x_1/2!) \\ &+ \quad ... + i\alpha_k \sum_{j=0}^{k-1} (-1)^j\ x_{k-j}(\tau - t)^j/j! + ... + i\alpha_{n_0} \sum_{j=0}^{n_0-1} (-1)^j\ x_{n_0-j}(\tau - t)^j/j! \\ &- \quad A_1 \int\limits_{\tau}^{t} (\alpha_1 + (\beta - \tau)\alpha_2 + (\beta - \tau)^2\alpha_3/2! + ... + (\beta - \tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta\} v_0(\alpha) d\alpha. \end{aligned}$$

$v_0(\alpha) = Fu_0(x)$, since

$$u(t,x) = \int_{R^{n_0}} G(t-\tau, x-\xi; x) u_0(\xi) d\xi, \tag{3.16}$$

where $G(t-\tau, x-\xi; x)$ is the fundamental solution of Cauchy problem and has the form:

$$\begin{aligned} G(t-\tau, x-\xi; x) &= (2\pi)^{-\frac{n_0}{2}} \int_{R^{n_0}} \exp\{i\alpha_1(x_1 - \xi_1) + i\alpha_2(x_2 - \xi_2 - (\tau - t)x_1) \\ &+ \quad i\alpha_3(x_3 - \xi_3 - (\tau - t)x_2 + (\tau - t)^2 x_1/2!) + ... \\ &+ \quad i\alpha_k(\sum_{j=0}^{k-1}(-1)^j\ x_{k-j}(\tau - t)^j/j! - \xi_k) + ... \\ &+ \quad i\alpha_{n_0}(\sum_{j=0}^{n_0-1} (-1)^j\ x_{n_0-j}(\tau - t)^j/j! - \xi_{n_0}) - \int_{\tau}^{t} (\alpha_1 + (\beta - \tau)\alpha_2 + ... \\ &+ \quad (\beta - \tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta\} d\alpha. \end{aligned} \tag{3.17}$$

4. The Study of Behavior of the Fundamental Solution of Cauchy Problem

In order to investigate the behavior of $G(t-\tau, x-\xi; x)$ we compute integral

$$I = \int_{\tau}^{t} (\alpha_1 + (\beta-\tau)\alpha_2 + ... + (\beta-\tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta. \qquad (4.1)$$

Having replacement $(\beta-\tau)(\tau-t)^{-1} = \theta$, we obtain

$$I = \int_{0}^{1} (\alpha_1 + \theta(t-\tau)\alpha_2 + ... + \theta^{n_0-1}(t-\tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\theta(t-\tau).$$

Denoting

$$\alpha_1(t-\tau)^{\frac{1}{2}} = s_1, \quad \alpha_2(t-\tau)^{\frac{3}{2}} = s_2, \quad ..., \quad \alpha_k(t-\tau)^{\frac{2k-1}{2}}/(k-1)! = s_k, \quad ...,$$

$$\alpha_n(t-\tau)^{\frac{2n_0-1}{2}}/(n_0-1)! = s_{n_0},$$

we have

$$\begin{aligned} I &= \int_0^1 (s_1 + \theta s_2 + ... + \theta^{n_0-1}s_{n_0})^2 d\theta = s_1^2 + s_2^2/3 + s_3^2/5 + ... + s_{n_0}^2/(2n_0-1) \\ &+ 2\sum_{j=2}^{n_0} s_1 s_j/j + 2\sum_{j=3}^{n_0} s_2 s_j/(j+1) + ... + 2\sum_{j=k+1}^{n_0} s_k s_j/(k+j-1) + ... \\ &+ 2s_{n_0-1}s_{n_0}/(2n_0-2). \end{aligned} \qquad (4.2)$$

In (4.2) to select the perfect square $s_1, s_2, ..., s_{n_0}$, will have:

$$\begin{aligned} I &= (\sum_{j=1}^{n_0} s_j/j)^2 + 3(\sum_{j=2}^{n_0} \frac{(j-1)s_j}{j(j+1)})^2 + \frac{1}{180}(\sum_{k=3}^{n_0} \frac{s_k 30(k-1)(k-2)}{k(k+1)(k+2)})^2 \\ &+ (\sum_{k=4}^{n_0} \frac{s_k(k-1)(k-2)(k-3)}{k(k+1)(k+2)(k+3)})^2 + ... + (2j-1)(\sum_{k=j}^{n_0} \frac{s_k(k-1)...(k-(j-1))}{k(k+1)...(k+j-1)})^2 + ... \\ &+ (2n_0-3)(\sum_{k=-1}^{n_0} \frac{s_k(k-1)...(k-(n_0-2))}{k(k+1)...(k+n_0-1)})^2 + (2n_0-1)s_0^2 \frac{(n_0-1)^2(n_0-2)^2...2^2}{n_0^2(n_0+1)^2...(2n_0-1)^2} \end{aligned} \qquad (4.3)$$

Using (4.3) $G(t-\tau, x-\xi; x)$ written as:

$$\begin{aligned}
G(t-\tau, x-\xi; x) &= (2\pi)^{-n_0} \int_{R^{n_0}} \exp\{is_1(x_1-\xi_1)(t-\tau)^{-1/2} + is_2(x_2-\xi_2-(\tau-t)x_1) \\
&\times (t-\tau)^{-3/2} + 2!is_3(x_3-\xi_3-(\tau-t)x_2+(t-\tau)^2x_1/2!)(t-\tau)^{-5/2} + ... \\
&+ (k-1)!is_k(\sum_{j=0}^{k-1}(-1)^j x_{k-j}(\tau-t)^j/j! - \xi_k)(t-\tau)^{-(2k-1)/2} \\
&+ is_{n_0}(\sum_{j=0}^{n_0-1}(-1)^j x_{n_0-1}(\tau-t)^j/j! - \xi_{n_0})(t-\tau)^{-(2n_0-1)/2}(n_0-1)! \\
&- A_1[(\sum_{j=1}^{n_0} s_j/j)^2 + 3(\sum_{j=2}^{n_0}\frac{(j-1)s_j}{j(j+1)})^2 + 5(\sum_{j=3}^{n_0}\frac{s_j(j-1)(j-2)}{j(j+1)(j+2)})^2 + ... \\
&+ (2k-1)(\sum_{j=k}^{n_0}\frac{s_j(j-1)...(j-k+1)}{j(j+1)...(j+k-1)})^2 + ... + (2n_0-3) \\
&\times (\sum_{j=n_0-1}^{n_0}\frac{s_j(j-1)...(j-(n_0+2))}{j(j+1)...(j+n_0-2)})^2 + (2n_0-1)(\frac{s_{n_0}(n_0-1)!}{n_0...(2n_0-1)})^2\}]ds \\
&\times (t-\tau)^{-n_0^2/2} 2!...(n_0-1)!
\end{aligned} \tag{4.4}$$

Consider the system

$$\begin{aligned}
\sum_{j=1}^{n_0}\frac{s_j}{j} &= \alpha_1, \\
\sum_{j=2}^{n_0}\frac{(j-1)s_j}{j(j+1)} &= \alpha_2, \\
&\dots \\
\sum_{j=k}^{n_0}\frac{s_j(j-1)...(j-k+1)}{j(j+1)...(j+k-1)} &= \alpha_k, \\
&\dots \\
\frac{s_{n_0-1}(n_0-2)!}{(n_0-1)...(2n_0-3)} + \frac{s_{n_0}(n_0-1)!}{n_0...(2n_0-2)} &= \alpha_{n_0-1} \\
\frac{s_{n_0}(n_0-1)!}{n_0...(2n_0-1)} &= \alpha_{n_0}.
\end{aligned} \tag{4.5}$$

If we solve (4.5) we obtain

$$s_1 = \alpha_1 - 3\alpha_2 + 5\alpha_3 - 7\alpha_4 + ... + (-1)^{n_0-1}(2n_0 - 1)\alpha_{n_0},$$
$$\frac{s_2}{2\cdot 3} = \alpha_2 - 5\alpha_3 + \frac{4\cdot 7}{2!}\alpha_4 - \frac{4\cdot 5\cdot 9}{3!}\alpha_5 + \frac{4\cdot 5\cdot 6\cdot 11}{4!}\alpha_6 + ... + \frac{(-1)^{n_0-2}}{(n_0-2)!}4\cdot 5...n_0(2n_0-1)\alpha_{n_0},$$
$$\dots$$
$$\frac{s_k(k-1)!}{k...(2k-1)} = \alpha_k - (2k+1)\alpha_{k+1} + \frac{2k(2k+3)}{2!}\alpha_{k+2} - \frac{2k(2k+1)(2k+5)}{3!}\alpha_{k+3}$$
$$+ \frac{2k(2k+1)(2k+2)(2k+7)}{4!}\alpha_{k+4} + ... + \frac{(-1)^{j-k}2k(2k+1)...(j+k-2)(2j-1)}{(j-k)!}\alpha_j + ...$$
$$+ \frac{(-1)^{n_0-k}2k(2k+1)...(n_0+k-2)(2n_0-1)}{(n_0-k)!}\alpha_{n_0},$$
$$\dots$$
$$\frac{s_{n_0-1}(n_0-2)!}{(n_0-1)...(2n_0-3)} = \alpha_{n_0-1} - \alpha_{n_0}(2n_0-1),$$
$$\frac{s_{n_0}(n_0-1)!}{n_0...(2n_0-1)} = \alpha_{n_0}.$$

With this system we find $s_k(k-1)!, k=\overline{1,n_0}$ and substitute in (4.4).

$$\begin{aligned}
G(t-\tau, x-\xi; x) &= (2\pi)^{-n_0}\int\limits_{R^{n_0}} \exp\{-A_1\sum_{k=1}^{n_0}(2k-1)\alpha_k^2 + i[\alpha_1 - 3\alpha_2 + 5\alpha_3 + ... \\
&+ (-1)^{n_0-1}(2n_0-1)\alpha_{n_0}](x_1-\xi_1)(t-\tau)^{-1/2} + 2\cdot 3(t-\tau)^{-3/2}(x_2-\xi_2 \\
&- (\tau-t)x_1)i[\alpha_2 - 5\alpha_3 + \frac{4\cdot 7}{2!}\alpha_4 - \frac{4\cdot 5\cdot 9}{3!}\alpha_5 + ... + \frac{(-1)^{n_0-2}}{(n_0-2)!}4\cdot 5... \\
&\times\ n_0(2n_0-1)\alpha_{n_0}] + k...(2k-1)(t-\tau)^{-(2k-1)/2}i[\alpha_k - (2k+1)\alpha_{k+1} \\
&+ \frac{2k(2k+3)}{2!}\alpha_{k+2} - \frac{2k(2k+1)(2k+5)}{3!}\alpha_{k+3}\frac{2k(2k+1)(2k+2)(2k+7)}{4!}\alpha_{k+4} \\
&+ ... + \frac{(-1)^{j-k}2k(2k+1)...(j+k-2)(2j-1)}{(j-k)!}\alpha_j + ... \\
&+ \frac{(-1)^{n_0-k}2k(2k+1)...(n_0+k-2)(2n_0-1)}{(n_0-k)!}\alpha_{n_0}] \\
&\times\ (\sum_{j=0}^{k_0-1}(-1)^j x_{k_0-j}(\tau-t)^j/j! - \xi_k) + ... \\
&+ n_0(n_0+1)...(2n_0-1)(t-\tau)^{-(2n_0-1)/2}i\alpha_{n_0} \\
&\times\ (\sum_{j=0}^{n_0-1}(-1)^j x_{n_0-j}(\tau-t)^j/j! - \xi_{n_0})\}d\alpha(t-\tau)^{-n_0^2/2} \\
&\times\ \prod_{k=1}^{n_0}k(k+1)...(2k-1). \qquad (4.6)
\end{aligned}$$

In (4.6) with respect to group the similar terms α_j, we have:

$$
\begin{aligned}
G(t-\tau, x &- \xi; x) = (2\pi)^{-n_0} \int_{R^{n_0}} \exp\{-A_1 \sum_{k=1}^{n_0} (2k-1)\alpha_k^2 + i\alpha_1 (t-\tau)^{-1/2}(x_1 - \xi_1) \\
&+ i\alpha_2 (t-\tau)^{-3/2} 6[x_2 - \xi_2 + (x_1+\xi_1)(t-\tau)/2] + i\alpha_3 (t-\tau)^{-5/2} 60[x_3 - \xi_3 \\
&+ (t-\tau)(x_2+\xi_2)/2 + (t-\tau)^2 (x_1 - \xi_1)/12] + 1540 i\alpha_4 (t-\tau)^{-7/2}[x_4 - \xi_4 \\
&+ (x_3+\xi_3)(t-\tau)/2 + (x_2-\xi_2)(t-\tau)^2/10 + (t-\tau)^3(x_1+\xi_1)/120] + ... \\
&+ n_0(n_0+1)...(2n_0-1) i\alpha_{n_0}[\sum_{j=0}^{n_0-1} x_{n_0-j}(t-\tau)^j/j - \xi_{n_0} - (t-\tau) \\
&\times (\sum_{j=0}^{n_0-2} x_{n_0-j-1}(t-\tau)^j/j - \xi_{n_0-1}) + (t-\tau)^2 \\
&\times (\sum_{j=0}^{n_0-3} x_{n_0-j-2}(t-\tau)^j/j - \xi_{n_0-2})(n-2)/4(2n_0-3) + ... \\
&+ (-1)^{(n_0-k)} \frac{(t-\tau)^{(n_0-k)}}{(n_0-k)!} \frac{2k...(2k+1)...(2k+(n_0-k)-2)(2k+2(n_0-k)-1)}{n_0...(2n_0-1)} \\
&\times (\sum_{j=0}^{k-1} x_{k-j}(t-\tau)^j/j! - \xi_k) + ... + (-1)^{(n_0-2)} \frac{(t-\tau)^{(n_0-2)}(x_2 - \xi_2 + (t-\tau)x_1)}{2(n+1)...(2n-3)} \\
&+ (-1)^{(n_0-1)} \frac{(t-\tau)^{(n_0-1)}(x_1-\xi_1)}{n_0...(2n_0-2)}]\} d\alpha (t-\tau)^{-n_0^2/2} \prod_{k=1}^{n_0} k(k+1)...(2k-1). \quad (4.7)
\end{aligned}
$$

Remark 1. Each coefficient of the $i\alpha_k$ can be reduced to the form:

$$
\begin{aligned}
& i\alpha_k (t-\tau)^{-(2k-1)/2} k...(2k-1)[x_k - \xi_k + (t-\tau)(x_{k-1} + \xi_{k-1}/2) + ... + (x_{k-j} - (-1)^j \xi_{k-j}) \\
& \times (t-\tau)^j \frac{2j(2j+1)...(k+j-2)}{j!k...(2k-2)} + ... + (x_1 - (-1)^{k-1}\xi_1)(t-\tau)^{k-1} \frac{1}{j!k(k+1)...(2k-2)}] \\
& = i\alpha_k k...(2k-1)[x_k - \xi_k + (t-\tau)(x_{k-1} - \xi_{k-1})/2 + ... + (x_{k-j} - (-1)^j \xi_{k-j}) \\
& \times (t-\tau) \frac{(j+1)...(k+j-2)}{(j-1)!(k-1)k...(2k-3)} + \frac{(x_1 - (-1)^{k-1}\xi_1)(t-\tau)^{k-1}}{2(k-1)k...(2k-3)}].
\end{aligned}
$$

From (4.7) it follows: $G(t-\tau, x-\xi; x)$ is the Fourier transform of

$$
I_1(\sigma) = \exp\{-A_1 \sum_{k=1}^{n_0} (2k-1)\alpha_k^2\},\ \sigma \in R^{n_0},
$$

according to selected points, with parabolic [9], we obtain estimates for $I_1(\alpha + i\beta), \alpha \in R^{n_0}, \beta \in R^{n_0}$,

$$|I_1(\alpha+i\beta)| \le C\exp\{-c_0\sum_{k=0}^{n_0}\alpha_k^2 + c\sum_{k=0}^{n_0}\beta_k^2\},$$

where positive constants C, c_0, c depends on n_0, n, constant parabolic δ_0, $\max\limits_{1\le r,\, s\le n}|a_2^{rv}|$.

Fourier transform of I_1 is an entire function for which the derivatives satisfy estimation at $t>\tau, x\in R^{n_0}, \xi\in R^{n_0}$:

$$\begin{aligned}
\|\partial_{x_j}^m G(t &- \tau, x-\xi; x)| \le C_m(t-\tau)^{-\frac{n_0^2}{2}-\frac{(2j-1)m}{2}}\exp\{-c_0^*[|x_1-\xi_1|^2 4^{-1}(t-\tau)^{-1} + 3|x_2-\xi_2 \\
&+ (t-\tau)(x_1+\xi_1)/2|^2(t-\tau)^{-3} + 180|x_3-\xi_3+(x_2+\xi_2)(t-\tau)/2+(t-\tau)^2(x_1 \\
&- \xi_1)/12|^2(t-\tau)^{-5} + 25200|x_4-\xi_4+(x_3+\xi_3)(t-\tau)/2+(t-\tau)^2(x_2-\xi_2)/10 \\
&+ (x_1+\xi_1)(t-\tau)^3/120|^2(t-\tau)^{-7}+...+(k-1)^2k^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)} \\
&\times |\sum_{j=0}^{k-1}x_{k-j}(t-\tau)^j/j! - \xi_k - (t-\tau)(\sum_{j=0}^{k-2}x_{k-1-j}(t-\tau)^j/j! - \xi_{k-1})/2+... \\
&+ (-1)^{k-l}\frac{(t-\tau)^{k-l}}{(k-l)!}\frac{2l(2l+1)...(2l+(k-l)-2)(2l+2(k-l)-1)}{k...(2k-1)} \\
&\times (\sum_{j=0}^{l-1}x_{l-j}(t-\tau)^j/j! - \xi_l) + ... + \frac{(-1)^{k-2}(t-\tau)^{k-2}(x_2-\xi_2+(t-\tau)x_1)}{2(k+1)...(2k-3)} \\
&+ \frac{(-1)^{k-1}(t-\tau)^{k-1}(x_1-\xi_1)}{k...(2k-2)}|^2 + ... + (n_0-1)^2n_0^2(n_0+1)^2...(2n_0-3)^2 \\
&\times (2n_0-1)(t-\tau)^{-\frac{2n_0-1}{2}}|\sum_{j=0}^{n_0-1}x_{n_0-j}(t-\tau)^j/j! - \xi_{n_0} \\
&- (t-\tau)(\sum_{j=0}^{n_0-2}x_{n_0-1-j}(t-\tau)^j/j! - \xi_{n_0-1})/2 + (t-\tau)^2 \\
&\times (\sum_{j=0}^{n_0-3}x_{n_0-2-j}(t-\tau)^j/j! - \xi_{n_0-2})\frac{(n_0-2)}{4(2n_0-3)} + ... + (-1)^{n_0-k} \\
&\times (\sum_{j=0}^{k-1}x_{k-j}(t-\tau)^j/j! - \xi_k)\frac{(t-\tau)^{n_0-k}}{(n_0-k)!}\frac{2k...(n_0+k-2)}{n_0...(2n_0-2)} \\
&\times (\sum_{j=0}^{k-1}x_{k-j}(t-\tau)^j/j! - \xi_k) + ... + (-1)^{n_0-2}\frac{(t-\tau)^{n_0-2}(x_2-\xi_2+(t-\tau)x_1)}{2(n_0+1)...(2n_0-3)} \\
&+ \frac{(-1)^{n_0-1}(t-\tau)^{n_0-1}(x_1-\xi_1)}{n_0...(2n_0-3)}]^2
\end{aligned} \tag{4.8}$$

where positive constants C_m, c_0^* dependent $n_0, j, m, \delta, \sup\limits_{r,\,\nu}|a_2^{rv}|, T, j=\overline{1,n_0}$. After remark 1, formula (4.8) can be written as follows

$$
\begin{aligned}
|\partial_{x_j}^m G(t &- \tau, x-\xi; x)| \le C_m (t-\tau)^{-\frac{n_0^2+(2j-1)m}{2}} \exp\{-c_0^*[|x_1-\xi_1|^2 4^{-1}(t-\tau)^{-1} + 3|x_2-\xi_2 \\
&+ (x_1+\xi_1)2^{-1}(t-\tau)|^2(t-\tau)^{-3} + 180|x_3-\xi_3+(x_2+\xi_2)(t-\tau)/2+(t-\tau)^2 \\
&\times (x_1-\xi_1)/12|^2(t-\tau)^{-5} + 2520|x_4-\xi_4+(x_3+\xi_3)(t-\tau)/2 \\
&+ (t-\tau)^2(x_2-\xi_2)/10+(x_1+\xi_1)(t-\tau)^3/120|^2(t-\tau)^{-7}+... \\
&+ (k-1)^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_k-\xi_k+(t-\tau)(x_{k-1}+\xi_{k-1})/2+... \\
&+ (x_{k-j}-(-1)^j\xi_{k-j})(t-\tau)^j(j+1)...(k+j-2)/(j-1)!(k-1)k...(2k-3)+... \\
&+ (x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1}/(2(k-1)k...(2k-3))|^2+... \\
&+ (n_0-1)^2...(2n_0-3)^2(2n_0-1)(t-\tau)^{-(2n_0-1)}|x_{n_0}-\xi_{n_0} \\
&+ (x_{n_0-1}+\xi_{n_0-1})(t-\tau)/2+...+(t-\tau)^{n_0-1}(x_1-(-1)^{n_0-1}\xi_1)/(2(n_0-1) \\
&\times (2n_0-3))|^2]\}, t-\tau>0, x\in R^{n_0}, \xi\in R^n, m\in N\cup\{0\}.
\end{aligned}
$$

Remark 2. Estimates (4.8) are exact, because for system considered one equation $n=1$

$$
\partial_t u(t,x) - \sum_{j=1}^{n_0-1} x_j \partial_{x_{j+1}} u(t,x) = \partial^2_{x_1^2} u(t,x)
$$

we obtain $G(t-\tau, x-\xi; x)$ at $t>\tau$, in the form

$$
\begin{aligned}
G(t &- \tau, x-\xi; x) = 2^{n_0}\pi^{-\frac{n_0}{2}} \prod_{k=1}^{n_0} k(k+1)...(2k-1)^{-\frac{1}{2}}(t-\tau)^{-\frac{2n_0-1}{2}} \exp\{-|x_1-\xi_1|^2 4^{-1} \\
&\times (t-\tau)^{-1} - 3|x_2-\xi_2+(x_1+\xi_1)2^{-1}(t-\tau)|^2(t-\tau)^{-3} - 180|x_3-\xi_3 \\
&+ (x_2+\xi_2)(t-\tau)2^{-1}+(t-\tau)^2(x_1-\xi_1)12^{-1}|^2(t-\tau)^{-5} - 2520|x_4-\xi_4 \\
&+ (x_3+\xi_3)(t-\tau)2^{-1}+(t-\tau)^2(x_2-\xi_2)10^{-1}+(x_1+\xi_1)(t-\tau)^3 120^{-1}|^2 \\
&\times (t-\tau)^{-7} - ... - k^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_k-\xi_k \\
&+ (t-\tau)(x_{k-1}+\xi_{k-1})2^{-1}+...+(x_{k-j}-(-1)^j\xi_{k-j})(t-\tau)^j(j+1)... \\
&\times (k+j-2)/(j-1)!(k-1)k...(2k-3)+...+(x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1} \\
&\times (2(k-1)k...(2k-3))^{-1}|^2 - ... - n_0^2...(2n_0-2)^2(2n_0-1)(t-\tau)^{-(2n_0-1)} \\
&\times |x_{n_0}-\xi_{n_0}-(x_{n_0-1}+\xi_{n_0-1})(t-\tau)2^{-1}+... \\
&+ (t-\tau)^{n_0-1}(x_1-(-1)^{n_0-1}\xi_1)(2(n_0-1)...(2n_0-3))^{-1}|^2\}
\end{aligned} \tag{4.9}
$$

In particular, from (4.9) with $x=x_1,\ x_2=y,\ (n_0=2)$ have FSCP for the diffusion equation with inertia if $n_0=\overline{2,5}$ obtain the results of [10-12]. Repeating the arguments of this work for the equation

$$
\partial_t u(t,x) - \sum_{\nu=1}^{p}\sum_{j=1}^{n_\nu-1} x_{\nu j}\partial_{x_{\nu(j+1)}} u(t,x) = \sum_{\nu=1}^{p} \partial^2_{x^2_{\nu 1}} u(t,x) + \sum_{\nu=p+1}^{m} \partial^2_{x^2_{\nu 1}} u(t,x), t>\tau,
$$

$$
x = (x_{11},...,x_{1n_1}; x_{21},...,x_{2n_2}; x_{p1},...,x_{pn_p}; x_{(p+1)1},...,x_{m1}),
$$

$$
n_1 \ge n_2 \ge ... \ge n_p > 1, p>1, m\ge p,
$$

we obtain an analogue of the formula (4.9):

$$
\begin{aligned}
G(t & - \tau, x-\xi; x) = \prod_{\nu=1}^{p}(2\sqrt{\pi})^{n_\nu+m-p}(t-\tau)^{-\frac{2n_\nu-1+m-p}{2}}\prod_{k=1}^{n_\nu}k(k+1)...(2k-1)^{-\frac{1}{2}} \\
\times & \exp\{-\sum_{j=1}^{p}[|x_{\nu1}-\xi_{\nu1}|^2 4^{-1}(t-\tau)^{-1}+3|x_{\nu2}-\xi_{\nu2}+(x_{\nu1}+\xi_{\nu1})2^{-1}(t-\tau)|^2(t-\tau)^{-3} \\
+ & 180|x_{\nu3}-\xi_{\nu3}+(x_{\nu2}+\xi_{\nu2})(t-\tau)2^{-1}+(t-\tau)^2(x_{\nu1}-\xi_{\nu1})12^{-1}|^2(t-\tau)^{-5} \\
+ & 2520|x_{\nu4}-\xi_{\nu4}+(x_{\nu3}+\xi_{\nu3})(t-\tau)2^{-1}+(t-\tau)^2(x_{\nu2}-\xi_{\nu2})10^{-1}+(x_{\nu1}+\xi_{\nu1})(t-\tau)^3 \\
\times & 120^{-1}|^2(t-\tau)^{-7}+...+k^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_{\nu k}-\xi_{\nu k}+(t-\tau)(x_{\nu(k-1)} \\
+ & \xi_{\nu(k-1)})2^{-1}+...+(x_{\nu(k-j)}-(-1)^j\xi_{\nu(k-j)})(t-\tau)^j(j+1)...(k+j-2)/(j-1)!(k-1)k... \\
\times & (2k-3)+...+(x_{\nu1}-(-1)^{k-1}\xi_{\nu1})(t-\tau)^{k-1}(2(k-1)k...(2k-3))^{-1}|^2+...+n_\nu^2... \\
\times & (2n_\nu-2)^2(2n_\nu-1)(t-\tau)^{-(2n_\nu-1)}|x_{\nu n_\nu}-\xi_{\nu n_\nu}-(x_{\nu(n_\nu-1)}+\xi_{\nu(n_\nu-1)})(t-\tau)2^{-1}+... \\
+ & (t-\tau)^{n_\nu-1}(x_{\nu1}-(-1)^{n_\nu-1}\xi_{\nu1})(2(n_\nu-1)...(2n_\nu-3))^{-1}|^2]-\sum_{\nu=p-1}^{n}|x_{\nu1}-\xi_{\nu1}|^2 4^{-1} \\
\times & (t-\tau)^{-1}\}, x\in R^n,\ \xi\in R^n, n=\sum_{\nu=1}^{p}n_\nu+m-p.
\end{aligned}
$$

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Буртняк І.В., Малицька Г.П. Структура фундаментального розв'язку задачі Коші для систем Колмогорова другого порядку. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 9-22.

Досліджується структура фундаментального розв'язку задачі Коші для одного класу систем ультрапараболічних рівнянь, що мають скінчену кількість груп змінних за якими вироджується параболічність.

Ключові слова: системи Колмогорова, фундаментальний розв'язок, вироджені параболічні рівняння.