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V. M. Gavrylkiv

ON THE AUTOMORPHISM GROUP OF THE SUPEREXTENSION OF A SEMIGROUP

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A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any set $B \supset A$ belongs to $\mathcal{A}$. An upfamily $\mathcal{L}$ of subsets of X is said to be *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. A linked upfamily $\mathcal{M}$ of subsets of X is *maximal linked* if $\mathcal{M}$ coincides with each linked upfamily $\mathcal{L}$ on X that contains $\mathcal{M}$. The *superextension* $\lambda(X)$ consists of all maximal linked upfamilies on X. Any associative binary operation $*\colon X \times X \to X$ can be extended to an associative binary operation $*\colon \lambda(X) \times \lambda(X) \to \lambda(X)$. In the paper we study automorphisms of superextensions of semigroups and describe the automorphism groups of superextensions of null semigroups, almost null semigroups, right zero semigroups and left zero semigroups. Also we find the automorphism groups of superextensions of all semigroups S of order $|S| \leq 3$.

Introduction. In this paper we investigate the automorphism group of the superextension $\lambda(S)$ of a semigroup S. The thorough study of various extensions of semigroups was started in [15] and continued in [1]–[10], [16]–[21]. The largest among these extensions is the semigroup $\upsilon(S)$ of all upfamilies on S. A family $\mathcal{A}$ of non-empty subsets of a set X is called an *upfamily* if for each set $A \in \mathcal{A}$ any subset $B \supset A$ belongs to $\mathcal{A}$. Each family $\mathcal{B}$ of non-empty subsets of X generates the upfamily $\langle \mathcal{B} \rangle = \{A \subset X \colon \exists B \in \mathcal{B}\ (B \subset A)\}$. An upfamily $\mathcal{F}$ a *filter* if it is closed under taking finite intersections. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. The family $\beta(X)$ of all ultrafilters on a set X is called the Stone-Čech compactification of X, see [22], [25]. An ultrafilter $\langle \{x\} \rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Identifying each point $x \in X$ with the principal ultrafilter $\langle \{x\} \rangle$, we can consider $X \subset \beta(X) \subset \upsilon(X)$. It was shown in [15] that any associative binary operation $*\colon S \times S \to S$ can be extended to an associative binary operation $*\colon \upsilon(S) \times \upsilon(S) \to \upsilon(S)$, defined by the formula

$$\mathcal{L} * \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a \colon L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$

for upfamilies $\mathcal{L}, \mathcal{M} \in \upsilon(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$.

The semigroup $\upsilon(S)$ contains many other important extensions of S. In particular, it contains the semigroup $\lambda(S)$ of maximal linked upfamilies on S. The space $\lambda(S)$ is well-known in General and Categorial Topology as the *superextension* of S, see [24]– [26]. An

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upfamily $\mathcal{L}$ of subsets of S is *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. The family of all linked upfamilies on S is denoted by $N_2(S)$. It is a subsemigroup of $\upsilon(S)$. The superextension $\lambda(S)$ consists of all maximal elements of $N_2(S)$, see [14], [15].

For a finite set X the cardinality of the set $\lambda(X)$ grows very quickly as $|X|$ tends to infinity. The calculation of the cardinality of $\lambda(X)$ seems to be a difficult combinatorial problem related to the still unsolved Dedekind's problem of calculation of the number $M(n)$ of monotone upfamilies on an n-element set, see [12].

We were able to calculate the cardinalities of $\lambda(X)$ only for sets X of cardinality $|X| \leq 6$, see [10]. The results of (computer) calculations are presented in Table 1.

$\lvert X\rvert$	1	2	3	4	5	6
$\lvert\lambda(X)\rvert$	1	2	4	12	81	2646

Table 1: The cardinality of $\lambda(X)$ for a set X of cardinality $|X| \leq 6$.

Each map $f\colon X \to Y$ induces the map (see [14])

$$\lambda f\colon \lambda(X) \to \lambda(Y), \quad \lambda f\colon \mathcal{M} \mapsto \big\langle f(M) \subset Y\colon M \in \mathcal{M}\big\rangle.$$

If $\varphi\colon S \to S'$ is a homomorphism of semigroups, then $\lambda\varphi\colon \lambda(S) \to \lambda(S')$ is a homomorphism as well, see [17].

A non-empty subset I of a semigroup S is called an *ideal* if $IS \cup SI \subset I$. An ideal I of a semigroup S is said to be *proper* if $I \neq S$. A proper ideal M of S is *maximal* if M coincides with each proper ideal I of S that contains M. It is easy to see that for every $n \in \mathbb{N}$ the subset $S^{\cdot n} = \{x_1 \cdot \ldots \cdot x_n\colon x_1, \ldots, x_n \in S\}$ is an ideal in S.

An element z of a semigroup S is called a *zero* (resp. a *left zero*, a *right zero*) in S if $az = za = z$ (resp. $za = z$, $az = z$) for any $a \in S$. An element e of a semigroup S is called an *idempotent* if $ee = e$. By $E(S)$ we denote the set of all idempotents of a semigroup S.

Recall that an *isomorphism* between S and S' is bijective function $\psi\colon S \to S'$ such that $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in S$. If there exists an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted $S \cong S'$. An isomorphism $\psi\colon S \to S$ is called an *automorphism* of a semigroup S. By $\mathrm{Aut}\,(S)$ we denote the automorphism group of a semigroup S.

An *antiisomorphism* between S and S' is bijective function $\psi\colon S \to S'$ such that $\psi(xy) = \psi(y)\psi(x)$ for all $x, y \in S$. If there exist an antiisomorphism between S and S', then S and S' are said to be *antiisomorphic*, denoted $S \cong_a S'$. If $(S, *)$ is a semigroup, then $(S, \star)$, where $x \star y = y * x$, is a semigroup as well. The semigroups $(S, *)$ and $(S, \star)$ are called *opposite*. It is easy to see that opposite semigroups are antiisomorphic.

A subset A of a semigroup S is called *characteristic* if $\psi(A) = A$ for any automorphism ψ of S. It is easy to see that the set $E(S)$ is characteristic and so are the ideals $S^{\cdot n}$ for all $n \in \mathbb{N}$.

Following the algebraic tradition, we take for a model of a cyclic group of order n the multiplicative group $C_n = \{z \in \mathbb{C}\colon z^n = 1\}$ of n-th roots of 1.

For a set X by S_X we denote the group of all bijections of X.

A semigroup $\langle a\rangle = \{a^n\}_{n\in\mathbb{N}}$ generated by a single element a is called *monogenic* or *cyclic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite monogenic semigroup $S = \langle a\rangle$ also has the simple structure, see [11], [23]. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{m+r-1}\}$ and $m + r - 1 = |S|$;
- for any $i, j \in \omega$ the equality $a^{r+i} = a^{r+j}$ holds if and only if $i \equiv j \mod m$;
- $\{a^r, a^{r+1}, \dots, a^{m+r-1}\}$ is a cyclic and maximal subgroup of S.

By $\mathrm{M}_{r,m}$ we shall denote a finite monogenic semigroup of index r and period m. It is well-known that $\mathrm{Aut}\,(\mathrm{M}_{r,m}) \cong C_1$ for $r \geq 2$ and

$$\mathrm{Aut}\,(\mathrm{M}_{1,m}) \cong \mathrm{Aut}\,(C_m) \cong C_{\varphi(m)},$$

where $\varphi(m)$ is the value of Euler's function for $m \in \mathbb{N}$.

1. Extending automorphisms from a semigroup to its superextension. In this section we observe that each automorphism of a semigroup S can be extended to an automorphism of its superextension $\lambda(S)$ and the automorphism group $\mathrm{Aut}\,(\lambda(S))$ of the superextension of a semigroup S contains a subgroup, isomorphic to the automorphism group $\mathrm{Aut}\,(S)$ of S. The following statements are corollaries of the functoriality of the superextension in the category of semigroups.

Proposition 1. *If $\psi\colon S \to T$ is an isomorphism, then $\lambda\psi\colon \lambda(S) \to \lambda(T)$ is an isomorphism as well.*

Corollary 1. *If $\psi\colon S \to S$ is an automorphism of a semigroup S, then $\lambda\psi\colon \lambda(S) \to \lambda(S)$ is an automorphism of the superextension $\lambda(S)$.*

Corollary 2. *The automorphism group $\mathrm{Aut}\,(\lambda(S))$ of the superextension of a semigroup S contains as a subgroup an isomorphic copy of the automorphism group $\mathrm{Aut}\,(S)$ of S.*

Corollary 2 motivates a question: *is the automorphism group $\mathrm{Aut}\,(S)$ of a semigroup S normal in the automorphism group $\mathrm{Aut}\,(\lambda(S))$ of its superextension $\lambda(S)$?* In the next sections we shall provide many counterexamples to this question.

Remark 1. In contrast to the preservation homomorphisms by the functor of superextension, antiisomorphisms are not preserved by this functor. Indeed, consider the symmetric group $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ with operation $\circ$ of composition of permutations. Let $(S_3)^{op}$ be its opposite group with operation $*$ defined by $x * y = y \circ x$ for $x, y \in S_3$. Observe that the identity map $\psi\colon S_3 \to (S_3)^{op}$, $\psi(x) = x$, is an antiisomorphism, inducing the identity map $\lambda\psi\colon \lambda(S_3) \to \lambda((S_3)^{op})$. Consider the maximal linked upfamilies

$$\mathcal{L} = \langle\{(12), (23)\}, \{(12), (123)\}, \{(23), (123)\}\rangle,$$
$$\mathcal{M} = \langle\{(12), (23)\}, \{(12), (13)\}, \{(13), (23)\}\rangle,$$

and observe that

$$\{(1), (123)\} = (12) \circ \{(12), (23)\} \cup (23) \circ \{(13), (23)\} \in \mathcal{L} \circ \mathcal{M}$$

but $\{(1), (123)\} \notin \mathcal{M} * \mathcal{L}$. Therefore,

$$\lambda\psi(\mathcal{L} \circ \mathcal{M}) = \mathcal{L} \circ \mathcal{M} \neq \mathcal{M} * \mathcal{L} = \lambda\psi(\mathcal{M}) * \lambda\psi(\mathcal{L}).$$

2. The automorphism groups of the semigroups $\lambda(LO_X)$, $\lambda(RO_X)$, $\lambda(O_X)$, $\lambda(AO_X)$, $\lambda((O_X)^{+1})$ and $\lambda((O_X)^{+0})$.

2.1. The semigroups $\lambda(LO_X)$ and $\lambda(RO_X)$. A semigroup S is said to be a *left (right) zero semigroup* if $ab = a$ ($ab = b$) for any $a, b \in S$. By LO_X and RO_X we denote the left zero semigroup and the right zero semigroup on a set X, respectively. If X is finite of cardinality $|X| = n$, then instead of LO_X and RO_X we write LO_n and RO_n, respectively.

Proposition 2. *If X is a left zero semigroup or a right zero semigroup, then* $\mathrm{Aut}\,(\lambda(X))$ *is isomorphic to the symmetric group* $S_{\lambda(X)}$.

Proof. In [21, Theorem 3] it was shown that the superextension of a right (left) zero semigroup is a right (left) zero semigroup as well. Each permutation on a right (left) zero semigroup is an automorphism. Indeed,

$$\psi(xy) = \psi(y) = \psi(x)\psi(y), \;\; \psi(xy) = \psi(x) = \psi(x)\psi(y)$$

for any elements x and y of the right zero semigroup and the left zero semigroup, respectively. Therefore, $\mathrm{Aut}\,(\lambda(X)) \cong S_{\lambda(X)}$. □

Using Proposition 2 and the values of $|\lambda(X)|$ from Table 1, we list in Table 2 present the automorphism groups of the semigroups $\lambda(LO_n)$ and $\lambda(RO_n)$ for $n \leq 6$.

n	1	2	3	4	5	6
$\mathrm{Aut}\,(LO_n)$, $\mathrm{Aut}\,(RO_n)$	C_1	C_2	S_3	S_4	S_5	S_6
$\mathrm{Aut}\,(\lambda(LO_n))$, $\mathrm{Aut}\,(\lambda(RO_n))$	C_1	C_2	S_4	S_{12}	S_{81}	S_{2646}

Table 2: The automorphism groups of superextensions of semigroups LO_n and RO_n for $n \leq 6$.

2.2. The semigroups $\lambda(O_X)$. A semigroup S is called a *null semigroup* if there exists an element $z \in S$ such that $xy = z$ for any $x, y \in S$. In this case the element z is the zero of S. All null semigroups on the same set are isomorphic. By O_X we denote a null semigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of O_X we write O_n.

Proposition 3. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$. The automorphism group of the semigroup $\lambda(O_X)$ is isomorphic to the symmetric group* $S_{\lambda(O_X)\setminus\{z\}}$.

Proof. In [21, Theorem 1] it was proved that the superextension of a null semigroup is a null semigroup with the same zero z. Taking into account that z is the zero of the semigroup $\lambda(O_X)$, we conclude that $\psi(z) = z$ for any $\psi \in \mathrm{Aut}\,(\lambda(O_X))$. Each permutation on the set $\lambda(O_X) \setminus \{z\}$ determines an automorphism of $\lambda(O_X)$. Indeed, $\psi(xy) = z = \psi(x)\psi(y)$ for any elements $x, y \in \lambda(O_X)$. Therefore, $\mathrm{Aut}\,(\lambda(O_X)) \cong S_{\lambda(O_X)\setminus\{z\}}$. □

Using Proposition 3 and values of $|\lambda(X)|$ from Table 1 we list in Table 3 the automorphism groups of the semigroups $\lambda(O_n)$ for $n \leq 6$.

2.3. The semigroups $\lambda(AO_X)$. A semigroup S is said to be an *almost null semigroup* if there exist distinct elements $a, z \in S$ such that $aa = a$ and $xy = z$ for any $(x, y) \in S \times S \setminus \{(a, a)\}$. In this case the element z is the zero of S and a is the unique idempotent in $S \setminus \{z\}$. All almost null semigroups on the same set are isomorphic. By AO_X we denote an almost null

n	1	2	3	4	5	6
$\mathrm{Aut}\,(O_n)$	C_1	C_1	C_2	S_3	S_4	S_5
$\mathrm{Aut}\,(\lambda(O_n))$	C_1	C_1	S_3	S_{11}	S_{80}	S_{2645}

Table 3: The automorphism groups of superextensions of semigroups O_n for $n \leq 6$.

semigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of AO_X we write AO_n.

It easy to check that the automorphism groups of the semigroup $\lambda(AO_2)$ is trivial. In the following proposition we describe the automorphism groups of the semigroups $\lambda(AO_X)$ on a set X of cardinality $|X| \geq 3$.

Proposition 4. *Let z be the zero of the almost null semigroup AO_X on a set X of cardinality $|X| \geq 3$. The group $\mathrm{Aut}\,(\lambda(AO_X))$ is isomorphic to the symmetric group $S_{\lambda(AO_X)\setminus\{a,z\}}$, where a is the idempotent in $AO_X \setminus \{z\}$.*

Proof. In [21, Theorem 2] it was shown that the superextension of an almost null semigroup is an almost null semigroup as well. Taking into account that z is the zero of the semigroup $\lambda(OA_X)$ and a is the unique idempotent in $\lambda(OA_X) \setminus \{z\}$, we conclude that $\psi(z) = z$ and $\psi(a) = a$ for any $\psi \in \mathrm{Aut}\,(\lambda(AO_X))$. Each permutation on the set $\lambda(AO_X) \setminus \{a, z\}$ defines an automorphism. Indeed, $\psi(aa) = \psi(a) = a = aa = \psi(a)\psi(a)$ and $\psi(xy) = z = \psi(x)\psi(y)$ for any $(x, y) \in \lambda(OA_X) \times \lambda(OA_X) \setminus \{(a, a)\}$. Therefore, $\mathrm{Aut}\,(\lambda(OA_X)) \cong S_{\lambda(AO_X)\setminus\{a,z\}}$. □

Using Proposition 4 and the values of $|\lambda(X)|$ from Table 1 we list in Table 4 the automorphism groups of the semigroups $\lambda(AO_n)$ for $n \in \{2, 3, 4, 5, 6\}$.

n	2	3	4	5	6
$\mathrm{Aut}\,(AO_n)$	C_1	C_1	C_2	S_3	S_4
$\mathrm{Aut}\,(\lambda(AO_n))$	C_1	C_2	S_{10}	S_{79}	S_{2644}

Table 4: The automorphism groups of superextensions of semigroups AO_n for $n \leq 6$.

2.4. The semigroups $\lambda((O_X)^{+1})$ and $\lambda((O_X)^{+0})$. Let S be a semigroup and $e \notin S$. The binary operation defined on S can be extended to $S \cup \{e\}$ putting $es = se = s$ for all $s \in S \cup \{e\}$. By S^{+1} we denote the monoid $S \cup \{e\}$ obtained from S by adjoining the extra identity e (regardless of whether S is or is not a monoid). If ψ is an automorphism of the semigroup S^{+1}, then $\psi(e) = e$. Consequently, each automorphism of S can be uniquely extended to an automorphism of the semigroup S^{+1}, and hence $\mathrm{Aut}\,(S^{+1}) \cong \mathrm{Aut}\,(S)$.

Proposition 5. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$. Then $\lambda((O_X)^{+1}) \cong (O_{\lambda((O_X)^{+1})\setminus\{e\}})^{+1}$ and $\mathrm{Aut}\,(\lambda((O_X)^{+1})) \cong S_{\lambda((O_X)^{+1})\setminus\{z,e\}}$, where e is the extra identity adjointed to O_X.*

Proof. It is easy to see that the principal ultrafilter generated by e is the identity of $\lambda((O_X)^{+1})$. Let $\mathcal{L}, \mathcal{M} \in \lambda((O_X)^{+1}) \setminus \{e\}$. Then there exist $L \in \mathcal{L}$ and $M \in \mathcal{M}$ such that $e \notin L$ and $e \notin M$. Consequently, $LM = \{z\}$ and the linkedness of $\mathcal{L} * \mathcal{M}$ implies that $\mathcal{L} * \mathcal{M} = \langle\{z\}\rangle$. This proves the isomorphism $\lambda((O_X)^{+1}) \cong (O_{\lambda((O_X)^{+1})\setminus\{e\}})^{+1}$.

Therefore,

$$\mathrm{Aut}\,(\lambda((O_X)^{+1})) \cong \mathrm{Aut}\,(O_{\lambda((O_X)^{+1})\setminus\{e\}})^{+1}) \cong \mathrm{Aut}\,(O_{\lambda((O_X)^{+1})\setminus\{e\}})) \cong S_{\lambda((O_X)^{+1})\setminus\{e,z\}}$$

according to Proposition 3. □

Let S be a semigroup and $0 \notin S$. The binary operation defined on S can be extended to $S \cup \{0\}$ putting $0s = s0 = 0$ for all $s \in S \cup \{0\}$. By S^{+0} we denote the semigroup $S \cup \{0\}$ obtained from S by adjoining the extra zero 0 (regardless of whether S has or has not the zero). If ψ is an automorphism of the semigroup S^{+0}, then $\psi(0) = 0$. Consequently, each automorphism of S can be uniquely extended to an automorphism of the semigroup S^{+0}. Therefore, $\mathrm{Aut}\,(S^{+0}) \cong \mathrm{Aut}\,(S)$.

Proposition 6. *Let z be the zero of the null semigroup O_X on a set X of cardinality $|X| \geq 2$. Then $\lambda((O_X)^{+0}) \cong (O_{\lambda((O_X)^{+0})\setminus\{0\}})^{+0}$ and $\mathrm{Aut}\,(\lambda((O_X)^{+0})) \cong S_{\lambda((O_X)^{+0})\setminus\{z,0\}}$, where 0 is the extra zero adjointed to O_X.*

Proof. It is easy to see that the principal ultrafilter generated by 0 is the zero of $\lambda((O_X)^{+0})$. Let $\mathcal{L}, \mathcal{M} \in \lambda((O_X)^{+0}) \setminus \{0\}$. Then there exist $L \in \mathcal{L}$ and $M \in \mathcal{M}$ such that $0 \notin L$ and $0 \notin M$. Consequently, $LM = \{z\}$ and the linkedness of $\mathcal{L} * \mathcal{M}$ implies that $\mathcal{L} * \mathcal{M} = \langle\{z\}\rangle$. This proves the isomorphism $\lambda((O_X)^{+0}) \cong (O_{\lambda((O_X)^{+0})\setminus\{0\}})^{+0}$.

Therefore,

$$\mathrm{Aut}\,(\lambda((O_X)^{+0})) \cong \mathrm{Aut}\,(O_{\lambda((O_X)^{+0})\setminus\{0\}})^{+0}) \cong \mathrm{Aut}\,(O_{\lambda((O_X)^{+0})\setminus\{0\}})) \cong S_{\lambda((O_X)^{+0})\setminus\{0,z\}}$$

according to Proposition 3. □

Using Propositions 5-6 and the values of $|\lambda(X)|$ from Table 1 we list in Table 5 the automorphism groups of the semigroups $\lambda((O_n)^{+1}))$ and $\lambda((O_n)^{+0}))$ for $n \leq 5$.

n	1	2	3	4	5
$\mathrm{Aut}\,((O_n)^{+1})$, $\mathrm{Aut}\,((O_n)^{+0})$	C_1	C_1	C_2	S_3	S_4
$\mathrm{Aut}\,(\lambda((O_n)^{+1}))$, $\mathrm{Aut}\,(\lambda((O_n)^{+0}))$	C_1	C_2	S_{10}	S_{79}	S_{2644}

Table 5: The automorphism groups of superextensions of semigroups $(O_n)^{+1}$ and $(O_n)^{+0}$ for $n \leq 5$.

3. The automorphism groups of superextensions of semigroups S of order $|S| \leq 3$. It is well-known that there are exactly five pairwise non-isomorphic semigroups having two elements: C_2, L_2, O_2, LO_2, RO_2, where L_2 is the semilattice $\{0,1\}$ indowed with the operation of minimum. Superextensions $\lambda(S)$ of semigroups S of order $|S| \leq 2$ consist only of principal ultrafilters, and hence $\lambda(S) \cong S$. Therefore,

$$\mathrm{Aut}\,(\lambda(C_1)) \cong \mathrm{Aut}\,(C_1) \cong C_1,\ \mathrm{Aut}\,(\lambda(C_2)) \cong \mathrm{Aut}\,(C_2) \cong C_1,$$
$$\mathrm{Aut}\,(\lambda(L_2)) \cong \mathrm{Aut}\,(L_2) \cong C_1,\ \mathrm{Aut}\,(\lambda(O_2)) \cong \mathrm{Aut}\,(O_2) \cong C_1,$$
$$\mathrm{Aut}\,(\lambda(LO_2)) \cong \mathrm{Aut}\,(LO_2) \cong S_2 \cong C_2,\ \mathrm{Aut}\,(\lambda(RO_2)) \cong \mathrm{Aut}\,(RO_2) \cong S_2 \cong C_2.$$

In the remaining part of the paper we concentrate on describing the structure of the automorphism groups of superextensions $\lambda(S)$ of three-element semigroups S. Among 19683 different binary operations on a three-element set $S = \{a, b, c\}$ there are exactly 113 operations which are associative, see [13]. In other words, there exist exactly 113 three-element semigroups, and many of these are isomorphic so that there are essentially only 24 pairwise non-isomorphic semigroups of order 3.

For a three-element semigroup S the semigroup $\lambda(S)$ contains the three principal ultrafilters and the maximal linked upfamily $\triangle = \{A \subset S : |A| \geq 2\}$. Therefore, we can write $\lambda(S) = S \cup \{\triangle\}$.

3.1. The automorphism groups of superextensions of commutative semigroups of order 3. In this subsection we describe the structure of the automorphism groups of superextensions of commutative three-element semigroups. Among 24 pairwise non-isomorphic semigroups of order 3 there are 12 commutative semigroups. Superextensions of all commutative semigroups of order 3 are commutative as well, see [21].

Up to isomorphism, the cyclic group C_3 is a unique group of order 3. There are three pairwise non-isomorphic monogenic semigroups of order 3: $\mathrm{M}_{1,3}$, $\mathrm{M}_{2,2}$ and $\mathrm{M}_{3,1}$. Also the monogenic semigroup $\mathrm{M}_{3,2}$ contains the 3-element characteristic ideal $\mathrm{M}_{3,2}^{\cdot 2}$. The monogenic semigroup $\mathrm{M}_{1,3}$ of index 1 is isomorphic to the cyclic group C_3. The automorphism groups of superextensions of groups and finite monogenic semigroups and their ideals were described in [8] and [9]:

$$\mathrm{Aut}\,(\lambda(C_3)) \cong \mathrm{Aut}\,(\lambda(\mathrm{M}_{1,3})) \cong C_2,\ \mathrm{Aut}\,(\lambda(\mathrm{M}_{2,2})) \cong C_2,$$
$$\mathrm{Aut}\,(\lambda(\mathrm{M}_{3,2}^{\cdot 2}) \cong C_2,\ \mathrm{Aut}\,(\lambda(\mathrm{M}_{3,1})) \cong C_1.$$

The structure of the automorphism groups of superextensions of semigroups O_3, AO_3, $(O_2)^{+1}$ and $(O_2)^{+0}$ was described in Section 2. According to Propositions 3–6 we have the following isomorphisms.

$$\mathrm{Aut}\,(\lambda(O_3)) \cong S_3,\ \mathrm{Aut}\,(\lambda(AO_3)) \cong S_2 \cong C_2,\ \mathrm{Aut}\,(\lambda((O_2)^{+1})) \cong C_2,\ \mathrm{Aut}\,(\lambda((O_2)^{+0}) \cong C_2.$$

Consider the three-element semigroups $(C_2)^{+0}$ and $(C_2)^{+1}$. In [21] it was shown that

$$\lambda((C_2)^{+0}) \cong ((C_2)^{+0})^{+0} \text{ and } \lambda((C_2)^{+1}) \cong ((C_2)^{+1})^{+1}.$$

Therefore,

$$\mathrm{Aut}\,(\lambda((C_2)^{+0})) \cong \mathrm{Aut}\,(((C_2)^{+0})^{+0}) \cong \mathrm{Aut}\,((C_2)^{+0}) \cong \mathrm{Aut}\,(C_2) \cong C_1,$$
$$\mathrm{Aut}\,(\lambda((C_2)^{+1})) \cong \mathrm{Aut}\,(((C_2)^{+1})^{+1}) \cong \mathrm{Aut}\,((C_2)^{+1}) \cong \mathrm{Aut}\,(C_2) \cong C_1.$$

Let us recall that a *semilattice* is a commutative semigroup of idempotents. By L_n we denote the linear semilattice $\{0, 1, \ldots, n\}$ of order n, endowed with the operation of minimum. There are two non-isomorphic semilattices of order 3: the linear semilattice L_3 and non-linear semilattice $V = \{0, a, b\}$, where 0 is the zero of V and $ab = ba = 0$.

The superextensions of semilattices were studied in [4]. In particular, it was shown that $\lambda(L_3) \cong L_4$, and hence
$$\mathrm{Aut}\,(\lambda(L_3)) \cong \mathrm{Aut}\,(L_4) \cong C_1.$$

Consider the non-linear semilattice $V = \{0, a, b\}$. Taking into account that

$$\triangle * \triangle \ni a\{0, b\} \cup b\{0, a\} = \{0\},$$

we conclude that $\triangle^2 = 0$. Consequently, $\triangle$ is a unique non-idempotent element in $\lambda(V)$, and hence $\psi(\triangle) = \triangle$ for any $\psi \in \mathrm{Aut}\,(\lambda(V))$. Therefore, each automorphism of V can be uniquely extended to an automorphism of $\lambda(V)$ and

$$\mathrm{Aut}\,(\lambda(V)) \cong \mathrm{Aut}\,(V) \cong C_2.$$

We summarize the obtained results on the automorphism groups $\mathrm{Aut}\,(\lambda(S))$ of superextensions of commutative three-element semigroups S in Table 6.

S	C_3	$\mathrm{M}_{2,2}$	$\mathrm{M}_{3,1}$	$\mathrm{M}^{:2}_{3,2}$	O_3	AO_3	O_2^{+1}	O_2^{+0}	C_2^{+1}	C_2^{+0}	L_3	V
$\mathrm{Aut}\,(S)$	C_2	C_1	C_1	C_1	C_2	C_1	C_1	C_1	C_1	C_1	C_1	C_2
$\mathrm{Aut}\,(\lambda(S))$	C_2	C_2	C_1	C_2	S_3	C_2	C_2	C_2	C_1	C_1	C_1	C_2

Table 6: The automorphism groups of the superextensions of commutative semigroups S of order 3.

Remark 2. Analyzing Table 6 one can see that in general case for semigroups S and T the isomorphism $\mathrm{Aut}\,(S) \cong \mathrm{Aut}\,(T)$ does not imply $\mathrm{Aut}\,(\lambda(S)) \cong \mathrm{Aut}\,(\lambda(T))$ and the isomorphism $\mathrm{Aut}\,(\lambda(S)) \cong \mathrm{Aut}\,(\lambda(T))$ does not imply $\mathrm{Aut}\,(S) \cong \mathrm{Aut}\,(T)$.

3.2. The automorphism groups of superextensions of non-commutative semigroups of order 3. There are 12 pairwise non-isomorphic non-commutative three-element semigroups. Non-commutative semigroups are divided into the pairs of opposite semigroups that are antiisomorphic. The automorphism groups of opposite semigroups are isomorphic.

The structure of the automorphism group of superextensions of a left zero semigroup and a right zero semigroup was described in Proposition 2:

$$\mathrm{Aut}\,(\lambda(LO_3)) \cong S_4 \text{ and } \mathrm{Aut}\,(\lambda(RO_3)) \cong S_4.$$

Consider the semigroups $(LO_2)^{+0}$, $(RO_2)^{+0}$, $(LO_2)^{+1}$ and $(RO_2)^{+1}$. In [21] it was proved that

$$\lambda((LO_2)^{+0}) \cong (LO_3)^{+0}, \quad \lambda((RO_2)^{+0}) \cong (RO_3)^{+0},$$
$$\lambda((LO_2)^{+1}) \cong ((LO_2)^{+1})^{+1}, \quad \lambda((RO_2)^{+1}) \cong ((RO_2)^{+1})^{+1}.$$

Therefore,

$$\mathrm{Aut}\,(\lambda((LO_2)^{+0})) \cong \mathrm{Aut}\,((LO_3)^{+0}) \cong \mathrm{Aut}\,(LO_3) \cong S_3,$$
$$\mathrm{Aut}\,(\lambda((RO_2)^{+0})) \cong \mathrm{Aut}\,((RO_3)^{+0}) \cong \mathrm{Aut}\,(RO_3) \cong S_3,$$
$$\mathrm{Aut}\,(\lambda((LO_2)^{+1})) \cong \mathrm{Aut}\,(((LO_2)^{+1})^{+1}) \cong \mathrm{Aut}\,((LO_2)^{+1}) \cong \mathrm{Aut}\,(LO_2) \cong S_2 \cong C_2,$$
$$\mathrm{Aut}\,(\lambda((RO_2)^{+1})) \cong \mathrm{Aut}\,(((RO_2)^{+1})^{+1}) \cong \mathrm{Aut}\,((RO_2)^{+1}) \cong \mathrm{Aut}\,(RO_2) \cong S_2 \cong C_2.$$

We have to consider the remaining three pairs of superextensions of opposite three-element semigroups. In [21] it was shown that superextensions of opposite three-element semigroups are opposite semigroups as well. Since the opposite semigroups have isomorphic automorphism groups, we shall describe the automorphism group of superextensions defined by the left Cayley tables in each pair of the following opposite superextensions.

Let us show that the semigroups $L_2^{\leftarrow}$, $(L_2^{\leftarrow})^{op}$ and their superextensions $\lambda(L_2^{\leftarrow})$, $\lambda((L_2^{\leftarrow})^{op})$ defined by the following Cayley tables have the trivial automorphism groups.

$\circ$	a	b	c	$\triangle$
a	a	a	a	a
b	a	b	a	a
c	c	c	c	c
$\triangle$	a	$\triangle$	a	a

$\circ$	a	b	c	$\triangle$
a	a	a	c	a
b	a	b	c	$\triangle$
c	a	a	c	a
$\triangle$	a	a	c	a

Let $\psi\colon \lambda(L_2^{\leftarrow}) \to \lambda(L_2^{\leftarrow})$ be an automorphism. Taking into account that b is the unique right identity of $\lambda(L_2^{\leftarrow})$ and $\triangle$ is the unique element of $\lambda(L_2^{\leftarrow}) \setminus E(\lambda(L_2^{\leftarrow}))$, we conclude that $\psi(b) = b$ and $\psi(\triangle) = \triangle$. Then

$$\psi(a) = \psi(ba) = \psi(b)\psi(a) = b\psi(a) \in b\{a, c\} = \{a\}.$$

Consequently, $\psi(c) = c$ and hence ψ is the identity automorphism. Therefore, $\mathrm{Aut}\,(\lambda(L_2^{\leftarrow})) \cong C_1$ and hence $\mathrm{Aut}\,(L_2^{\leftarrow}) \cong C_1$.

Consider the next pair of opposite three-element semigroups $\mathrm{M}_{2,1}^{\leftarrow}$ and $(\mathrm{M}_{2,1}^{\leftarrow})^{op}$ and their opposite superextensions defined by the following Cayley tables.

$\circ$	a	b	c	$\triangle$
a	b	b	b	b
b	b	b	b	b
c	a	b	c	$\triangle$
$\triangle$	b	b	b	b

$\circ$	a	b	c	$\triangle$
a	b	b	a	b
b	b	b	b	b
c	b	b	c	b
$\triangle$	b	b	$\triangle$	b

Let $\psi\colon \lambda(\mathrm{M}_{2,1}^{\leftarrow}) \to \lambda(\mathrm{M}_{2,1}^{\leftarrow})$ be an automorphism. Taking into account that b is the zero of $\lambda(\mathrm{M}_{2,1}^{\leftarrow})$ and c is the unique idempotent of $\lambda(\mathrm{M}_{2,1}^{\leftarrow}) \setminus \{b\}$, we conclude that $\psi(b) = b$ and $\psi(c) = c$. It is easy to check that the map $\varphi\colon \lambda(\mathrm{M}_{2,1}^{\leftarrow}) \to \lambda(\mathrm{M}_{2,1}^{\leftarrow})$ defined by

$$\varphi\colon a \mapsto \triangle,\ \ \varphi\colon b \mapsto b,\ \ \varphi\colon c \mapsto c,\ \ \varphi\colon \triangle \mapsto a,$$

is an automorphism. Therefore, $\mathrm{Aut}\,(\lambda(\mathrm{M}_{2,1}^{\leftarrow})) \cong C_2$. It is easy to see that the identity automorphism is the unique automorphism of the semigroup $\mathrm{M}_{2,1}^{\leftarrow}$ and hence $\mathrm{Aut}\,(\mathrm{M}_{2,1}^{\leftarrow}) \cong C_1$.

The last two non-commutative opposite superextensions of three-element semigroups $LO_2^{\leftarrow}$ and $(LO_2^{\leftarrow})^{op}$ are given by the following Cayley tables.

$\circ$	a	b	c	$\triangle$
a	a	a	a	a
b	b	b	b	b
c	a	a	a	a
$\triangle$	a	a	a	a

$\circ$	a	b	c	$\triangle$
a	a	b	a	a
b	a	b	a	a
c	a	b	a	a
$\triangle$	a	b	a	a

Let $\psi\colon \lambda(LO_2^{\leftarrow}) \to \lambda(LO_2^{\leftarrow})$ be an automorphism of the semigroup $\lambda(LO_2^{\leftarrow})$ given by the left Cayley table. Since $\{a, b\}$ is a subsemigroup of left zeros of $\lambda(LO_2^{\leftarrow})$, then $\psi(\{a, b\}) =$

$\{a, b\}$, and hence $\psi(\{c, \triangle\}) = \{c, \triangle\}$. One can check that each permutation of $\{a, b\}$ and each permutation of $\{c, \triangle\}$ determine an automorphism of $\lambda(LO_2^{\leftarrow})$. Therefore, $\mathrm{Aut}\,(\lambda(LO_2^{\leftarrow})) \cong C_2 \times C_2$. By the same arguments we can show that $\mathrm{Aut}\,(LO_2^{\leftarrow}) \cong C_2$.

The obtained results on the automorphism groups of superextensions of non-commutative three-element semigroups are summed up in the Table 7.

S	LO_3, RO_3	$(LO_2)^{+0}$, $(RO_2)^{+0}$	$(LO_2)^{+1}$, $(RO_2)^{+1}$	$L_2^{\leftarrow}$, $(L_2^{\leftarrow})^{op}$	$\mathrm{M}_{2,1}^{\leftarrow}$, $(\mathrm{M}_{2,1}^{\leftarrow})^{op}$	$LO_2^{\leftarrow}$, $(LO_2^{\leftarrow})^{op}$
$\mathrm{Aut}\,(S)$	S_3	C_2	C_2	C_1	C_1	C_2
$\mathrm{Aut}\,(\lambda(S))$	S_4	S_3	C_2	C_1	C_2	$C_2 \times C_2$

Table 7: The automorphism groups of superextensions of 3-element non-commutative semigroups.

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Faculty of Mathematics and Computer Science
Vasyl Stefanyk Precarpathian National University
vgavrylkiv@gmail.com

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