ISSN (Print): 2077-9879
ISSN (Online): 2617-2658

Eurasian Mathematical Journal

2018, Volume 9, Number 4

Founded in 2010 by
the L.N. Gumilyov Eurasian National University
in cooperation with
the M.V. Lomonosov Moscow State University
the Peoples' Friendship University of Russia (RUDN University)
the University of Padua

Starting with 2018 co-funded
by the L.N. Gumilyov Eurasian National University
and
the Peoples' Friendship University of Russia (RUDN University)

Supported by the ISAAC
(International Society for Analysis, its Applications and Computation)
and
by the Kazakhstan Mathematical Society

Published by

the L.N. Gumilyov Eurasian National University
Astana, Kazakhstan

Aims and Scope

The Eurasian Mathematical Journal (EMJ) publishes carefully selected original research papers in all areas of mathematics written by mathematicians, principally from Europe and Asia. However papers by mathematicians from other continents are also welcome.

From time to time the EMJ publishes survey papers.

The EMJ publishes 4 issues in a year.

The language of the paper must be English only.

The contents of EMJ are indexed in Scopus, Web of Science (ESCI), Mathematical Reviews, MathSciNet, Zentralblatt Math (ZMATH), Referativnyi Zhurnal – Matematika, Math-Net.Ru.

The EMJ is included in the list of journals recommended by the Committee for Control of Education and Science (Ministry of Education and Science of the Republic of Kazakhstan) and in the list of journals recommended by the Higher Attestation Commission (Ministry of Education and Science of the Russian Federation).

Information for the Authors

Submission. Manuscripts should be written in LaTeX and should be submitted electronically in DVI, PostScript or PDF format to the EMJ Editorial Office via e-mail (eurasianmj@yandex.kz).

When the paper is accepted, the authors will be asked to send the tex-file of the paper to the Editorial Office.

The author who submitted an article for publication will be considered as a corresponding author. Authors may nominate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not guaranteed.

Copyright. When the paper is accepted, the copyright is automatically transferred to the EMJ. Manuscripts are accepted for review on the understanding that the same work has not been already published (except in the form of an abstract), that it is not under consideration for publication elsewhere, and that it has been approved by all authors.

Title page. The title page should start with the title of the paper and authors' names (no degrees). It should contain the Keywords (no more than 10), the Subject Classification (AMS Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes), and the Abstract (no more than 150 words with minimal use of mathematical symbols).

Figures. Figures should be prepared in a digital form which is suitable for direct reproduction.

References. Bibliographical references should be listed alphabetically at the end of the article. The authors should consult the Mathematical Reviews for the standard abbreviations of journals' names.

Authors' data. The authors' affiliations, addresses and e-mail addresses should be placed after the References.

Proofs. The authors will receive proofs only once. The late return of proofs may result in the paper being published in a later issue.

Offprints. The authors will receive offprints in electronic form.

Publication Ethics and Publication Malpractice

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the EMJ implies that the work described has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The EMJ follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct ($http://publicationethics.org/files/u2/New_Code.pdf$). To verify originality, your article may be checked by the originality detection service CrossCheck http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the EMJ.

The Editorial Board of the EMJ will monitor and safeguard publishing ethics.

The procedure of reviewing a manuscript, established by the Editorial Board of the Eurasian Mathematical Journal

1. **Reviewing procedure**

1.1. All research papers received by the Eurasian Mathematical Journal (EMJ) are subject to mandatory reviewing.

1.2. The Managing Editor of the journal determines whether a paper fits to the scope of the EMJ and satisfies the rules of writing papers for the EMJ, and directs it for a preliminary review to one of the Editors-in-chief who checks the scientific content of the manuscript and assigns a specialist for reviewing the manuscript.

1.3. Reviewers of manuscripts are selected from highly qualified scientists and specialists of the L.N. Gumilyov Eurasian National University (doctors of sciences, professors), other universities of the Republic of Kazakhstan and foreign countries. An author of a paper cannot be its reviewer.

1.4. Duration of reviewing in each case is determined by the Managing Editor aiming at creating conditions for the most rapid publication of the paper.

1.5. Reviewing is confidential. Information about a reviewer is anonymous to the authors and is available only for the Editorial Board and the Control Committee in the Field of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan (CCFES). The author has the right to read the text of the review.

1.6. If required, the review is sent to the author by e-mail.

1.7. A positive review is not a sufficient basis for publication of the paper.

1.8. If a reviewer overall approves the paper, but has observations, the review is confidentially sent to the author. A revised version of the paper in which the comments of the reviewer are taken into account is sent to the same reviewer for additional reviewing.

1.9. In the case of a negative review the text of the review is confidentially sent to the author.

1.10. If the author sends a well reasoned response to the comments of the reviewer, the paper should be considered by a commission, consisting of three members of the Editorial Board.

1.11. The final decision on publication of the paper is made by the Editorial Board and is recorded in the minutes of the meeting of the Editorial Board.

1.12. After the paper is accepted for publication by the Editorial Board the Managing Editor informs the author about this and about the date of publication.

1.13. Originals reviews are stored in the Editorial Office for three years from the date of publication and are provided on request of the CCFES.

1.14. No fee for reviewing papers will be charged.

2. **Requirements for the content of a review**

2.1. In the title of a review there should be indicated the author(s) and the title of a paper.

2.2. A review should include a qualified analysis of the material of a paper, objective assessment and reasoned recommendations.

2.3. A review should cover the following topics:

- compliance of the paper with the scope of the EMJ;
- compliance of the title of the paper to its content;
- compliance of the paper to the rules of writing papers for the EMJ (abstract, key words and phrases, bibliography etc.);
- a general description and assessment of the content of the paper (subject, focus, actuality of the topic, importance and actuality of the obtained results, possible applications);
- content of the paper (the originality of the material, survey of previously published studies on the topic of the paper, erroneous statements (if any), controversial issues (if any), and so on);

- exposition of the paper (clarity, conciseness, completeness of proofs, completeness of bibliographic references, typographical quality of the text);

- possibility of reducing the volume of the paper, without harming the content and understanding of the presented scientific results;

- description of positive aspects of the paper, as well as of drawbacks, recommendations for corrections and complements to the text.

2.4. The final part of the review should contain an overall opinion of a reviewer on the paper and a clear recommendation on whether the paper can be published in the Eurasian Mathematical Journal, should be sent back to the author for revision or cannot be published.

Web-page

The web-page of EMJ is www.emj.enu.kz. One can enter the web-page by typing Eurasian Mathematical Journal in any search engine (Google, Yandex, etc.). The archive of the web-page contains all papers published in EMJ (free access).

Subscription

For Institutions

- US$ 200 (or equivalent) for one volume (4 issues)
- US$ 60 (or equivalent) for one issue

For Individuals

- US$ 160 (or equivalent) for one volume (4 issues)
- US$ 50 (or equivalent) for one issue.

The price includes handling and postage.

The Subscription Form for subscribers can be obtained by e-mail:

eurasianmj@yandex.kz

The Eurasian Mathematical Journal (EMJ)
The Astana Editorial Office
The L.N. Gumilyov Eurasian National University
Building no. 3
Room 306a
Tel.: +7-7172-709500 extension 33312
13 Kazhymukan St
010008 Astana, Kazakhstan

The Moscow Editorial Office
The Peoples' Friendship University of Russia
(RUDN University)
Room 515
Tel.: +7-495-9550968
3 Ordzonikidze St
117198 Moscow, Russia

SULTANAEV YAUDAT TALGATOVICH

(to the 70th birthday)

On 19th July 2018 was 70th birthday of Yaudat Talgatovich Sultanaev, doctor of science (1990), professor (1991), honorary scientist of the Russian Federation, laureate of State award of the Republic of Bashkortostan in the field of science and technology, professor of the Bashkir State Pedagogical University, member of the Editorial Board of the Eurasian Mathematical Journal.

Ya.T. Sultanaev was born in the sity of Orsk. In 1971 he graduated from the Bashkir State University and then completed his postgraduate studies in the Moscow State University. Ya.T. Sultanaev's scientific supervisors were distinguished mathematicians A.G. Kostyuchenko and B.M. Levitan.

Ya.T. Sultanaev is a famous specialist in the spectral theory of differential operators and the qualitative theory of ordinary differential equations.

He obtained bilateral Tauberian theorems of Keldysh type, completely solved the problem on spectral assymptotics for semi-bounded ordinary differential operators, suggested a new method of investigation of assymptotic behaviour of solutions to singular differential equations which allowed him to essentially weaken the conditions on coefficients.

Jointly with V.A. Sadovnichii and A.M. Akhtyamov, he investigated inverse spectral problems with non-separated boundary conditions.

He published more than 70 papers in leading mathematical journals.

Among pupils of Ya.T. Sultanaev there are more than 20 candidates of science and one doctor of science.

The Editorial Board of the Eurasian Mathematical Journal congratulates Yaudat Talgatovich on the occasion of his 70th birthday and wishes him good health and new achievements in mathematics and mathematical education.

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 9, Number 4 (2018), 61 – 67

ON DETERMINANTS AND PERMANENTS OF SOME TOEPLITZ-HESSENBERG MATRICES WHOSE ENTRIES ARE JACOBSTHAL NUMBERS

T. Goy

Communicated by V.I. Burenkov

Key words: Jacobsthal sequence, Toeplitz-Hessenberg matrix, determinant, permanent, multinomial coefficient.

AMS Mathematics Subject Classification: 15B36, 11B37, 11B05.

Abstract. In this paper, we study some families of Toeplitz-Hessenberg determinants and permanents the entries of which are Jacobsthal numbers. These studies have led us to discover new identities for Jacobsthal numbers.

DOI: https://doi.org/10.32523/2077-9879-2018-9-4-61-67

1 Introduction

The *Jacobsthal sequence* $(J_n)_{n\geq 0}$ is the sequence of integers satisfying the recurrence relation $J_{n+2} = J_{n+1} + 2J_n$, $J_0 = 0$, $J_1 = 1$. The first Jacobsthal numbers are [21]:

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, \ldots .$$

The Jacobsthal number at a specific point in the sequence may be calculated directly using the closed-form equation

$$J_n = \frac{2^n - (-1)^n}{3}, \quad n \geq 0,$$

and it also can be expressed in the following form

$$J_n = \frac{1 - (-1)^n}{2} + \left\lfloor \frac{2^n}{3} \right\rfloor, \quad n \geq 0.$$

Jacobsthal sequence has a rich history and many remarkable properties, such as counting microcontroller skip instructions [7] and counting the number of ways to tile a $3\times(n-1)$ rectangle with 2×2 and 1×1 tiles [12].

As examples of recent work involving the Jacobsthal numbers and its various generalizations, see [1]–[6], [8]–[11], [16]–[19], [22]–[26]. For instance, Köken and Bozkurt in [18] defined a n-square Jacobsthal matrix and using this matrix derived some properties of Jacobsthal numbers. In [16], Jhala et al. obtained Binet formula for k-Jacobsthal numbers and established some properties for these numbers. Uygun [22] defined (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas matrix sequences. Also Uygun and Eldogan [23] defined k-Jacobsthal and k-Jacobsthal-Lucas matrix sequence and investigated some properties of these sequences. Akbulak and Öteleş [1] defined two n-square upper Hessenberg matrices one of which corresponds to the adjacency matrix a directed pseudo graph and investigated relations between determinants and permanents of these Hessenberg matrices and sum formulas of the Jacobsthal sequences. Cılasun [5] introduced recurrence relation for multiple-counting Jacobsthal sequences and showed their application with Fermat's little theorem. Particular cases of Jacobsthal and Jacobsthal-Lucas numbers were investigated earlier by Horadam [13]–[15].

2 Toeplitz-Hessenberg matrices and related formulas

A *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0, a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}, \tag{2.1}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

We expand the determinant $\det(M_n)$ and permanent $\operatorname{per}(M_n)$ according to the first row repeatedly. Then we obtain the following recurrent formulas for determinants and permanents of the Toeplitz-Hessenberg matrix (2.1):

$$\det(M_n) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det(M_{n-k}) \tag{2.2}$$

and

$$\operatorname{per}(M_n) = \sum_{k=1}^{n} a_0^{k-1} a_k \operatorname{per}(M_{n-k}), \tag{2.3}$$

where, by definition, $\det(M_0) = 1$ and $\operatorname{per}(M_0) = 1$.

The following results are known as Trudi's formulas (see, for example, [20]).

Theorem 2.1. *Let n be a positive integer. Then*

$$\det(M_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-a_0)^{n-(t_1+t_2+\cdots+t_n)} p_n(t) a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n} \tag{2.4}$$

and

$$\operatorname{per}(M_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} a_0^{n-(t_1+t_2+\cdots+t_n)} p_n(t) a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}, \tag{2.5}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$, and $p_n(t) = \frac{(t_1+t_2+\cdots+t_n)!}{t_1!t_2!\cdots t_n!}$ is the multinomial coefficient.

3 Determinants and permanents of the Toeplitz-Hessenberg matrices whose entries are Jacobsthal numbers

Denote by

$$D(a_1, a_2, \ldots, a_n) = \det(M_n(1, a_1, a_2, \ldots, a_n)),$$
$$P(a_1, a_2, \ldots, a_n) = \operatorname{per}(M_n(1, a_1, a_2, \ldots, a_n)).$$

Proposition 3.1. *For all $n \geq 1$, the following formulas hold:*

$$D(J_1, J_2, \ldots, J_n) = \big(1 - (-1)^n\big) \cdot 2^{\frac{n-3}{2}}, \tag{3.1}$$

$$P(J_1, J_2, \ldots, J_n) = \frac{\sqrt{3}}{6}\Big((1+\sqrt{3})^n - (1-\sqrt{3})^n\Big). \tag{3.2}$$

Proof. We prove only formula (3.1), because the proof of (3.2) is similar. For simplicity of notation, we write D_n instead of $D(J_1, J_2, \ldots, J_n)$. Thus, we must prove that

$$D_n = \det\begin{pmatrix} J_1 & 1 & \cdots & 0 & 0 \\ J_2 & J_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ J_{n-1} & J_{n-2} & \cdots & J_1 & 1 \\ J_n & J_{n-1} & \cdots & J_2 & J_1 \end{pmatrix} = \big(1 - (-1)^n\big) \cdot 2^{\frac{n-3}{2}}.$$

We use induction on n. For $n = 1$ formula (3.1) holds. Suppose that assertion holds for all $k \leq n-1$ and proof its validity for n. From (2.2) we have

$$\begin{aligned} D_n &= \sum_{i=1}^{n}(-1)^{i+1} J_i D_{n-i} \\ &= J_1 D_{n-1} + \sum_{i=2}^{n}(-1)^{i+1}\left(J_{i-1} + 2J_{i-2}\right) D_{n-i} \\ &= D_{n-1} + \sum_{i=2}^{n}(-1)^{i+1} J_{i-1} D_{n-i} + 2\sum_{i=2}^{n}(-1)^{i+1} J_{i-2} D_{n-i} \\ &= D_{n-1} + \sum_{i=1}^{n-1}(-1)^{i+2} J_i D_{n-i-1} + 2\sum_{i=0}^{n-2}(-1)^{i+3} J_i D_{n-i-2} \\ &= D_{n-1} - D_{n-1} + 2D_{n-2} \\ &= 2\big(1 - (-1)^{n-2}\big) \cdot 2^{\frac{n-5}{2}} \\ &= \big(1 - (-1)^n\big) \cdot 2^{\frac{n-3}{2}}. \end{aligned}$$

Therefore, (3.1) holds for all positive integers. □

Proposition 3.2. *For all $n \geq 1$, the following formulas hold:*

$$D(J_0, J_1, \ldots, J_{n-1}) = (-1)^{n-1} F_{n-1}, \tag{3.3}$$

where F_n is the n-th Fibonacci number;

$$D(J_1, J_3, \ldots, J_{2n-1}) = \frac{(-1)^{n-1}}{2}\Big((2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}\Big), \tag{3.4}$$

$$D(J_2, J_4, \ldots, J_{2n}) = n(-2)^{n-1}, \tag{3.5}$$

$$D(J_0, J_2, \ldots, J_{2n-2}) = \frac{\sqrt{5}}{5}\left(\left(\frac{5+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{5-\sqrt{5}}{2}\right)^{n-1}\right), \tag{3.6}$$

$$D(J_3, J_5, \ldots, J_{2n+1}) = (-2)^{n-1} + 2\delta_{n1}, \tag{3.7}$$

where δ_{n1} is the Kronecker symbol.

Proof. We prove only (3.3), the other one can be prove in the same way. To simplify notation, we write S_n instead of $D(J_0, J_1, \dots, J_{n-1})$. Thus, we must prove that

$$S_n = \det\begin{pmatrix} J_0 & 1 & \cdots & 0 & 0 \\ J_1 & J_0 & \cdots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ J_{n-2} & J_{n-3} & \cdots & J_0 & 1 \\ J_{n-1} & J_{n-2} & \cdots & J_1 & J_0 \end{pmatrix} = (-1)^{n-1}F_{n-1}, \qquad n \geq 1.$$

To prove it, we use induction on n. For $n = 1$ the identity is trivial. Assuming (3.3) to hold for $n-1$, we proved it for n. Using (2.2), we get

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1}(-1)^k J_k S_{n-k-1} \\ &= J_0 S_{n-1} - J_1 S_{n-2} + \sum_{k=2}^{n-1}(-1)^k J_k S_{n-k-1} \\ &= -S_{n-2} + \sum_{k=2}^{n-1}(-1)^k \left(J_{k-1} + 2J_{k-2}\right) S_{n-k-1} \\ &= -S_{n-2} + \sum_{k=1}^{n-1}(-1)^k J_{k-1} S_{n-k-1} + 2\sum_{k=2}^{n-1}(-1)^k J_{k-2} S_{n-k-1} \\ &= -S_{n-2} + \sum_{k=0}^{n-2}(-1)^{k+1} J_k S_{n-k-2} + 2\sum_{k=0}^{n-3}(-1)^{k+2} J_k S_{n-k-3} \\ &= -S_{n-2} - S_{n-1} + 2S_{n-2} \\ &= S_{n-2} - S_{n-1} \\ &= (-1)^{n-3}F_{n-3} - (-1)^{n-2}F_{n-2} \\ &= (-1)^{n-1}F_{n-1}. \end{aligned}$$

Therefore, (3.3) holds for all positive integers. □

The similar formulas hold true for permanents of Toeplitz-Hessenberg matrix whose entries are Jacobsthal numbers (sequential elements, elements with even and odd subscripts).

Proposition 3.3. *For all $n \geq 1$, the following formulas hold:*

$$P(J_0, J_1, \dots, J_{n-1}) = \frac{\sqrt{13}}{13}\left(\left(\frac{1+\sqrt{13}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{13}}{2}\right)^{n-1}\right), \tag{3.8}$$

$$P(J_1, J_3, \dots, J_{2n-1}) = \frac{1}{6}\left((3+\sqrt{3})^n + (3-\sqrt{3})^n\right), \tag{3.9}$$

$$P(J_2, J_4, \dots, J_{2n}) = \frac{\sqrt{5}}{10}\left((3+\sqrt{5})^n - (3-\sqrt{5})^n\right), \tag{3.10}$$

$$P(J_0, J_2, \dots, J_{2n-2}) = \frac{\sqrt{13}}{13}\left(\left(\frac{5+\sqrt{13}}{2}\right)^{n-1} - \left(\frac{5-\sqrt{13}}{2}\right)^{n-1}\right), \tag{3.11}$$

$$P(J_3, J_5, \dots, J_{2n+1}) = \frac{1}{16}\left((4+\sqrt{8})^{n+1} - (4-\sqrt{8})^{n+1}\right). \tag{3.12}$$

Proof. The proof uses mathematical induction on n and is similar to the proof of (3.1) or (3.3). □

4 Main formulas

Using (2.4) for determinants in (3.1), (3.3), (3.4), (3.5), (3.6), (3.7), and using (2.5) for permanents in (3.2), (3.8), (3.9), (3.10), (3.11), (3.12), we obtain the following Jacobsthal identities.

Proposition 4.1. *For all $n \geq 1$, the following formulas hold:*

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_1^{t_1} J_2^{t_2} \cdots J_n^{t_n} = \left((-1)^n - 1\right) \cdot 2^{\frac{n-3}{2}},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_1^{t_1} J_2^{t_2} \cdots J_n^{t_n} = \frac{\sqrt{3}}{6}\left((1+\sqrt{3})^n - (1-\sqrt{3})^n\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_0^{t_1} J_1^{t_2} \cdots J_{n-1}^{t_n} = -F_{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_1^{t_1} J_3^{t_2} \cdots J_{2n-1}^{t_n} = -\frac{1}{2}\left((2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_2^{t_1} J_4^{t_2} \cdots J_{2n}^{t_n} = -n2^{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_0^{t_1} J_2^{t_2} \cdots J_{n-2}^{t_n} = \frac{\sqrt{5}}{5}\left(\left(\frac{-5+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{-5-\sqrt{5}}{2}\right)^{n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^T p_n(t) J_3^{t_1} J_5^{t_2} \cdots J_{2n+1}^{t_n} = -2^{n-1} + (-1)^n 2\delta_{n1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_0^{t_1} J_1^{t_2} \cdots J_{n-1}^{t_n} = \frac{\sqrt{13}}{13}\left(\left(\frac{1+\sqrt{13}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{13}}{2}\right)^{n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_1^{t_1} J_3^{t_2} \cdots J_{2n-1}^{t_n} = \frac{1}{6}\left((3+\sqrt{3})^n + (3-\sqrt{3})^n\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_2^{t_1} J_4^{t_2} \cdots J_{2n}^{t_n} = \frac{\sqrt{5}}{10}\left((3+\sqrt{5})^n - (3-\sqrt{5})^n\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_0^{t_1} J_2^{t_2} \cdots J_{2n-2}^{t_n} = \frac{\sqrt{13}}{13}\left(\left(\frac{5+\sqrt{13}}{2}\right)^{n} - \left(\frac{5-\sqrt{13}}{2}\right)^{n}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) J_3^{t_1} J_5^{t_2} \cdots J_{2n+1}^{t_n} = \frac{1}{16}\left((4+\sqrt{8})^{n+1} - (4-\sqrt{8})^{n+1}\right),$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$, $T = t_1 + t_2 + \cdots + t_n$, $p_n(t) = \dfrac{(t_1+t_2+\cdots+t_n)!}{t_1!t_2!\cdots t_n!}$, F_n is the n-th Fibonacci number, and δ_{n1} is the Kronecker symbol.

Acknowledgments

I am grateful to my advisor Prof. Roman Zatorsky for his valuable comments and suggestions. I also would like to thank the referee for his/her careful reading and many helpful suggestions.

References

[1] M. Akbulak, A. Öteleş, *On the sums of Pell and Jacobsthal numbers by matrix method*. Bull. Iranian Math. Soc. 40 (2014), no. 4, 1017–1025.

[2] İ. Aktaş, H. Köse, *Hessenberg matrices and the Pell-Lucas and Jacobsthal numbers*. Int. J. Pure Appl. Math. 101 (2015), no. 3, 425–432.

[3] G. Cerda-Morales, *Matrix representation of the q-Jacobsthal numbers*, Proyecciones. 31 (2012), no. 4, 345–354.

[4] Z. Čerin, *Sums of squares and products of Jacobsthal numbers*. J. Integer Seq. 10 (2007), no. 2, Article 07.2.5.

[5] M.H. Cılasun, *Generalized multiple counting Jacobsthal sequences of Fermat pseudoprimes*. J. Integer Seq. 19 (2016), no. 2, Article 16.2.3.

[6] C.K. Cook, M.R. Bacon, *Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations*. Ann. Math. Informat. 41 (2013), 27–39.

[7] A. Daşdemir, *On the Jacobsthal numbers by matrix method*. SDU Journal of Science. 7 (2012), no. 1, 69–76.

[8] Ö. Deveci, E. Karaduman, G. Sağlam, *The Jacobsthal sequences in finite groups*. Bull. Iranian Math. Soc. 42 (2016), no. 1, 79–89.

[9] G.B. Djordjević, *Some generalizations of the Jacobsthal numbers*. Filomat. 24 (2010), no. 2, 143–151.

[10] G.B. Djordjević, H.M. Srivastava, *Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers*. Math. Comput. Model. 42 (2005), no. 9–10, 1049–1056.

[11] D.D. Frey, J.A. Sellers, *Jacobsthal numbers and alternating sign matrices*. J. Integer Seq. 3 (2000), no. 2, Article 00.2.3.

[12] S. Heubach, *Tiling an m-by-n area with squares of size up to k-by-k* ($m \leq 5$). Congr. Numer. 140 (1999), 43–64.

[13] A.F. Horadam, *Jacobsthal and Pell curves*. Fibonacci Quart. 34 (1996), no. 1, 79–83.

[14] A.F. Horadam, *Jacobsthal representation numbers*. Fibonacci Quart. 34 (1996), no. 1, 40–54.

[15] A.F. Horadam, *Jacobsthal representation polynomials*. Fibonacci Quart. 35 (1997), no. 2, 137–148.

[16] D. Jhala, K. Sisodiya, G.P.S. Rathore, *On some identities for k-Jacobsthal numbers*. Int. J. Math. Anal. 7 (2013), no. 12, 551–556.

[17] D. Jhala, G.P.S. Rathore, K. Sisodiya, *Some properties of k-Jacobsthal numbers with arithmetic indexes*. Turkish J. Anal. Number Theory. 2 (2014), no. 4, 119–124.

[18] F. Köken, D. Bozkurt, *On the Jacobsthal numbers by matrix methods*. Int. J. Contemp. Math. Sci. 34 (2008), no. 3, 605–614.

[19] H.-C. Li, *A proof about the Jacobsthal and Jacobsthal-Lucas sequences*. Far East J. Math. Sci. 2 (2012), no. 2, 605–614.

[20] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*. Spec. Matrices. 1 (2013), 10–16.

[21] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org.

[22] Ş. Uygun, *The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences*. Appl. Math. Sci. 9 (2015), no. 70, 3467–3476.

[23] Ş. Uygun, H. Eldogan, *k-Jacobsthal and k-Jacobsthal Lucas matrix sequences*. Int. Math. Forum. 11 (2016), no. 3, 145–154.

[24] F. Yılmaz, D. Bozkurt, *The generalized order-k Jacobsthal numbers*. Int. J. Contemp. Math. Sci. 34 (2013), no. 4, 1685–1694.

[25] F. Yılmaz, D. Bozkurt, *The adjacency matrix of one type of directed graph and the Jacobsthal numbers and their determinantal representation*. J. Appl. Math. 2012 (2012), Article ID 423163.

[26] R. Zatorsky, T. Goy, *Parapermanents of triangular matrices and some general theorems on number sequences*. J. Integer Seq. 19 (2016), no. 2, Article 16.2.2.

Taras Goy
Faculty of Mathematics and Computer Sciences
Vasyl Stefanyk Precarpathian National University
57 Shevchenko St.,
76018 Ivano-Frankivsk, Ukraine
E-mail: tarasgoy@yahoo.com

Received: 29.11.2016
Revised version: 21.07.2018