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Reciprocal distance Laplacian spectral properties double stars and their complements

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Several matrices are associated with graphs in order to study their properties. In such a study, researchers are interested in the spectra of the matrix under consideration, therefore, the properties are called spectral properties, with reference to the matrix. One of the interesting and hard problems in the spectral study of graphs is to order the graphs based on some spectral graph invariant, like the spectral radius, the second smallest eigenvalue, the energy, etc. Due to hardness of this problem it has been considered in the literature for small classes of graphs. Here we continue this study and add some more classes of graphs which can be ordered on the basis of spectral graph invariants. In this article, we study spectral properties of trees of diameter three, called double stars, and their complements through their reciprocal distance Laplacian eigenvalues. We give ordering of these graphs based on their reciprocal distance Laplacian spectral radius, on their second smallest reciprocal distance Laplacian eigenvalue, and on their reciprocal distance Laplacian energy.

Key words and phrases: reciprocal distance Laplacian matrix, double star, spectral ordering, energy.

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1 Introduction

All our graphs in this article are connected, simple and undirected graphs. We will use standard notations and definitions in graph theory (see, e.g., [17, 18]).

A graph is denoted by $G = G\big(V(G), E(G)\big)$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the *vertex set* and $E(G)$ is the *edge set*. The complement of G is denoted by $\overline{G}$. By K_n and $K_{1,n-1}$, we denote the complete graph and the star graph, each on n vertices, respectively.

For positive integers a and b, the *double star* $T(a, b)$ is the tree of order $n = a + b + 2$, obtained by adding an edge between the vertices of maximum degrees of the stars $K_{1,a}$ and $K_{1,b}$. The vertices of degree $a + 1$ and $b + 1$ are the *centers* of $T(a, b)$. Any tree T of diameter 3 is a double star completely defined by the degrees of its two non-pendant vertices, i.e. $T \cong T(a, b)$, where $a + 1$ and $b + 1$ are the degrees of its centers. See Figure 1 for $T(3, 5)$.

The *distance* $d(v_i, v_j)$ between two distinct vertices v_i and v_j in a connected graph G is defined as the length of the smallest path connecting them. The *diameter* of G is the largest distance among any two vertices of G. The *distance matrix* $D(G)$ of a graph G is indexed by its vertices and is defined as $\big(D(G)\big)_{i,j} = d(v_i, v_j)$. For a survey of results on the distance matrix, see [5].

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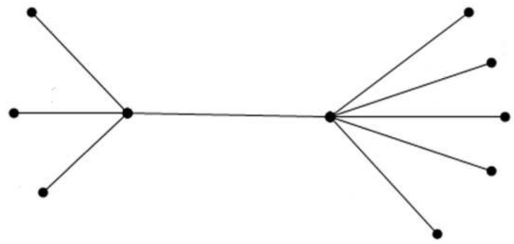

Figure 1. The double star $T(3,5)$

The *reciprocal transmission degree* of the vertex v_i is defined to be the sum of the inverse distances from v_i to all other vertices in G, i.e.

$$RTr(v_i) = \sum_{v_j \in V(G)} \frac{1}{d(v_i, v_j)}.$$

Let $RTr(G) = diag\big(RTr(v_1), RTr(v_2), \ldots, RTr(v_n)\big)$ be the diagonal matrix of reciprocal transmission degrees of G. The *reciprocal distance matrix* $RD(G)$ (also known as *Harary matrix*) is an $n \times n$ matrix whose (i,j)th entry is $\frac{1}{d(v_i,v_j)}$ if $v_i \neq v_j$ and 0 otherwise. The *Harary index* $H(G)$ of G is the sum of reciprocal distances between all unordered pairs of vertices. Clearly,

$$2H(G) = \sum_{v \in V(G)} RTr(v) = \sum_{v_i, v_j \in V(G),\ i \neq j} \frac{1}{d(v_i, v_j)} = \sum_{i=1}^{n} \sum_{j=1,\ j \neq i}^{n} \frac{1}{d(v_i, v_j)}.$$

The relation between the Harary matrix, the Harary index and the Harary energy can be seen in [13] and some recent result in this direction in [1].

Movivated by the work introducing the *signless Laplacian* matrix of a graph [14–16], and the *distance Laplacian* and *distance signless Laplacian* matrices of graphs [6–8], R. Bapat and S.K. Panda [11] defined the *reciprocal distance Laplacian* matrix as $RD^L(G) = RTr(G) - RD(G)$. Since each row sum of $RD^L(G)$ is zero, it follows that its eigenvalue is 0 and its associated eigenvector is $(1, \ldots, 1)$. The reciprocal distance Laplacian matrix is a real symmetric positive semi-definite matrix, so its eigenvalues can be indexed such that $\delta_1 \geq \cdots \geq \delta_{n-1} > \delta_n = 0$.

The set of all eigenvalues (including algebraic multiplicities) of $RD^L(G)$ is known as the reciprocal distance Laplacian spectrum (or RD^L-spectrum) of G, the largest RD^L-eigenvalue δ_1 is known as the *reciprocal distance Laplacian spectral radius* or RD^L-spectral radius of G.

The study of the spectral properties of the reciprocal distance Laplacian matrix $RD^L(G)$ of a graph G attracted the attention of several researchers. For instance, L. Medina and M. Trigo studied the problem of bounding the largest eigenvalue $\delta_1(G)$ in [27] and that of bounding the reciprocal distance Laplacian energy in [28]. Recently, the authors in [22] have extended the concept of spectral spread of a matrix to the reciprocal distance Laplacian matrix and have obtained some bounds for it. They have also obtained some estimates for the sum of the k largest reciprocal distance Laplacian eigenvalues in [22]. The reciprocal distance signless Laplacian matrix of a connected graph was introduced in [3] and its spectral properties were further studied in [2, 10, 27].

In this paper, we investigate ordering of trees of diameter 3 (double stars) and their complements on the basis of their reciprocal distance Laplacian largest eigenvalue, their second smallest reciprocal distance Laplacian eigenvalue, and their reciprocal distance Laplacian energy.

Ordering graphs on the basis of their spectra is a widely studied topic in spectral graph theory. For instance, ordering trees on the basis the distance Laplacian energy and distance Laplacian spectral radius was considered in [19, 20, 26, 29, 30]. For other orderings of graphs based on different spectra, see [24, 33] for the second largest Laplacian eigenvalue, [4, 9, 23] for the second largest signless Laplacian eigenvalue, [12,21,25,31] for the adjacency spectral radius and [32] for the adjacency energy of a signed graph.

The rest of the paper is organized as follows. In Section 2, we find the RD^L-eigenvalues of double star and its complement. We also discuss the ordering of graphs belonging to these families based on RD^L-spectral radius and the second smallest RD^L-eigenvalue. In Section 3, we find the reciprocal distance Laplacian energy of double stars and their complements. Further, we show that the graphs belonging to these families can be ordered on the basis of their reciprocal distance Laplacian energy.

2 Reciprocal distance Laplacian spectral properties of $T(a,b)$ and $\overline{T}(a,b)$

Any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as a function defined on $V(G)$, which associates every v_i to x_i, that is $X(v_i) = x_i$ for all $i = 1, 2, \ldots, n$. Also, it is easy to see that

$$X^T RD^L(G) X = \sum_{i,j,\ i\neq j} \frac{1}{d(v_i, v_j)} (x_i - x_j)^2.$$

A number δ is an eigenvalue of $RD^L(G)$ with its associated eigenvector X if and only if $X \neq 0$ and for every $v_i \in V(G)$ we have

$$\delta X(v_i) = \sum_{v_j \in V(G),\ j\neq i} \frac{1}{d(v_i, v_j)} \big(X(v_i) - X(v_j)\big), \tag{1}$$

or equivalently

$$\delta X(v_i) - RTr(v_i) = - \sum_{v_j \in V(G),\ j\neq i} \frac{1}{d(v_i, v_j)} X(v_j). \tag{2}$$

Equations (1) and (2) are known as (δ, X)-eigenequations of $RD^L(G)$.

The next result is helpful in finding some RD^L-eigenvalues of G, when G has an independent set (a set of mutually non-adjacent vertices) sharing the same neighbourhood set.

Proposition 1. *Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $S = \{v_1, v_2, \ldots, v_p\}$ be an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, \ldots, p\}$. Then the vertices of S have the same reciprocal transmission degree, say δ. Moreover, $\delta + \frac{1}{2}$ is the RD^L-eigenvalue of G with multiplicity at least $p - 1$.*

Proof. Since $S = \{v_1, v_2, \ldots, v_p\}$ is an independent set sharing the same neighbourhood, so $d(v_i, v_j) = 2$ for every $i, j \in \{1, \ldots, p\}$. We first index the independent vertices, so that the RD^L-matrix of G with given hypothesis can be written as

$$RD^L(G) = \left(\begin{array}{cccc|c} \delta & -\frac{1}{2} & \cdots & -\frac{1}{2} & B_{p\times(n-p)} \\ -\frac{1}{2} & \delta & \cdots & -\frac{1}{2} & \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & \delta & \\ \hline \multicolumn{4}{c|}{(B_{p\times(n-p)})^T} & C_{(n-p)\times(n-p)} \end{array}\right).$$

For $i = 2, 3, \ldots, p$, let $X_{i-1} = \big(-1, x_{i2}, x_{i3}, \ldots, x_{ip}, \underbrace{0, \ldots, 0}_{n-p}\big)^T = \big(X, 0_{n-p}\big)^T$ be the vector in $\mathbb{R}^n$ such that $x_{ij} = 1$ if $i = j$ and 0 otherwise, where $X = \big(-1, x_{i2}, x_{i3}, \ldots, x_{ip}\big)$ and 0_{n-p} is the zero row vector of order $n-p$. For instance, $X_1 = (-1, 1, 0, 0, \ldots, 0)^T$, $X_2 = (-1, 0, 1, 0, \ldots, 0)^T$, $X_3 = (-1, 0, 0, 1, \ldots, 0)^T$ and so on. It is clear from the definition of vectors $X_1, X_2, \ldots, X_{p-1}$ that if the kth entry in any of these vectors is non-zero, then the kth entry of rest of the vectors is zero, giving that these vectors are linearly independent vectors. We have

$$RD^L(G)X_1 = \left(\begin{array}{cccc|c} \delta & -\frac{1}{2} & \cdots & -\frac{1}{2} & \\ -\frac{1}{2} & \delta & \cdots & -\frac{1}{2} & B_{p\times(n-p)} \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & \delta & \\ \hline & & (B_{p\times(n-p)})^T & & C_{(n-p)\times(n-p)} \end{array}\right) \begin{pmatrix} -1 \\ 1 \\ \vdots \\ 0 \\ 0^T_{n-p} \end{pmatrix}$$
$$= \left(-\delta - \frac{1}{2} \quad \delta + \frac{1}{2} \quad 0 \quad \ldots 0 \quad 0 \quad \ldots \quad 0\right)^T = \left(\delta + \frac{1}{2}\right) X_1.$$

To get the last equality we have used the fact $B_{p\times(n-p)} 0^T_{n-p} = 0_p, C_{(n-p)\times(n-p)} 0^T_{n-p} = 0_{n-p}$ and $\big(B_{p\times(n-p)}\big)^T X^T = 0_{n-p}$. This last equality is true as any two vertices in the independent set S share the same neighbourhood and so if the entry in $\big(B_{p\times(n-p)}\big)^T$ corresponding to -1 is t, then the entry in $\big(B_{p\times(n-p)}\big)^T$ corresponding to 1 is also t, giving that the contribution from the product of these entries in $B_{p\times(n-p)} 0^T_{n-p}$ is zero. Proceeding similarly by considering the vectors $X_2, X_3, \ldots, X_{p-1}$, we get that these vectors are also the eigenvectors of $RD^L(G)$ corresponding to eigenvalue $\delta + \frac{1}{2}$. Since, these vectors form an independent set, the result follows. □

The following result is helpful in finding some RD^L-eigenvalues of G, when G has a clique (a set of mutually adjacent vertices) sharing same neighbourhood set outside the clique.

Proposition 2. *Let G be a connected graph on n vertices. If $S = \{v_1, v_2, \ldots, v_\eta\}$ is a clique of G such that $N(v_i) - S = N(v_j) - S$ for all $i, j \in \{1, \ldots, \eta\}$. Then the vertices of S have the same reciprocal transmission degree, say η. Moreover, $\eta + 1$ is an eigenvalue of $RD^L(G)$ with multiplicity at least $\eta - 1$.*

Proof. Similar to Proposition 1 with the same set of eigenvectors. □

Lemma 1. *Let G be a connected graph with non-zero eigenvector $X = (x_1, x_2, \ldots, x_n)^T$ corresponding to the reciprocal distance Laplacain eigenvalue δ and let $u, v \in V(G)$ be such that $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Then $x_u = x_v$.*

Proof. Clearly, $RTr(u) = RTr(v)$. By using (2), we have

$$\delta x_u - RTr(u)x_u = -\frac{1}{d(u,v)}x_v - \sum_{v_j \in V(G),\, v_j \neq v,u} \frac{1}{d(u,v_j)}x_j$$

and

$$\delta x_v - RTr(v)x_v = -\frac{1}{d(v,u)}x_u - \sum_{v_j \in V(G),\, v_j \neq v,u} \frac{1}{d(v,v_j)}x_j.$$

As $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, so $d(u, v_j) = d(v, v_j)$ for $v_j \neq u, v$. Therefore, it follows that

$$\delta(x_u - x_v) = \frac{1}{d(u,v)}(x_u - x_v),$$

which implies that $x_u = x_v$. □

The next result gives the RD^L-spectrum of $T(a,b)$.

Theorem 1. *The reciprocal distance Laplacian spectrum of $T(a,b)$, $a \leq b$, $a + b = n - 2$ consists of the eigenvalue $\frac{3a+2b+9}{6}$ with algebraic multiplicity $a - 1$, the eigenvalue $\frac{2a+3b+9}{6}$ with algebraic multiplicity $b - 1$ and the zeros of the following polynomial*

$$\frac{x}{12}\Bigg(12x^3 - (22a + 22b + 60)x^2 + \Big(12a^2 + 12b^2 + 27ab + 71a + 71b + 99\Big)x \\ - \Big(2a^3 + 2b^3 + 7a^2b + 7ab^2 + 19a^2 + 19b^2 + 40ab + 57a + 57b + 54\Big)\Bigg).$$

Proof. Let $V(T(a,b)) = \{s_1, s_2, \ldots, s_a, u, v, t_1, t_2, \ldots, t_b\}$ be the vertex set of $T(a,b)$, where s_i, $1 \leq i \leq a$, are the a pendent vertices adjacent to u and $b_i, 1 \leq i \leq b$, are the b pendent vertices adjacent to v. Clearly, $\{s_1, s_2, \ldots, s_a\}$ is an independent set of $T(a,b)$ sharing the same vertex u, such that reciprocal transmission degree of each vertex is

$$\underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{a-1} + 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{3}}_{b} = \frac{a}{2} + \frac{b}{3} + 1.$$

So, by Proposition 1, it follows that $\frac{a}{2} + \frac{b}{3} + 1 + \frac{1}{2}$ is the RD^L-eigenvalue of $T(a,b)$ with algebraic multiplicity $a - 1$. Similarly, the b pendent vertices $t_1, t_2, \ldots, t_b$ form another independent set sharing the same vertex v with each vertex of reciprocal transmission degree $\frac{a}{3} + \frac{b}{2} + 1$ and by Proposition 1, $\frac{a}{3} + \frac{b}{2} + \frac{3}{2}$ is the other RD^L-eigenvalue of G with algebraic multiplicity $b - 1$. In this way, we have found $n - 4$ reciprocal distance Laplacian eigenvalues of $T(a,b)$. Also, note that 0 is always RD^L eigenvalue of G. For the remaining three RD^L-eigenvalues of $T(a,b)$, we use (1). Let X be the eigenvector of $T(a,b)$ with $x_i = X(v_i)$ for $i = 1, 2, \ldots, n$. Then by Lemma 1, it follows that the components of X corresponding to vertices of the independent set on a vertices is same and equal to x_1 and the components of X corresponding to vertices of the independent set on b vertices is same and equal to x_4. The remaining two vertices of degree $a + 1$ and $b + 1$ are assigned x_2 and x_3, respectively. By using (1), the (δ, X)-equations of $T(a,b)$ are given by

$$\begin{aligned}
\delta x_1 &= \left(\frac{3a + 2b + 6}{6}\right)x_1 - \left(\frac{a-1}{2}\right)x_1 - x_2 - \frac{1}{2}x_3 - \frac{b}{3}x_4, \\
\delta x_2 &= -ax_1 + \left(\frac{2a + b + 3}{2}\right)x_2 - x_3 - \frac{b}{2}x_4, \\
\delta x_3 &= -\frac{a}{2}x_1 - x_2 + \left(\frac{a + 2b + 2}{2}\right)x_3 - bx_4, \\
\delta x_4 &= -\frac{a}{3}x_1 - \frac{1}{2}x_2 - x_3 + \left(\frac{2a + 3b + 6}{6}\right)x_4 - \left(\frac{b-1}{2}\right)x_4.
\end{aligned}$$

The coefficient matrix of the above system of equations is

$$\begin{pmatrix} \frac{2b+9}{6} & -1 & -\frac{1}{2} & -\frac{b}{3} \\ -a & \frac{2a+b+2}{2} & -1 & -\frac{b}{2} \\ -\frac{a}{2} & -1 & \frac{a+2b+2}{2} & -b \\ -\frac{a}{3} & -\frac{1}{2} & -1 & \frac{2a+9}{6} \end{pmatrix},$$

and its characteristic polynomial is given by $\frac{x}{12}f(x)$, where

$$\begin{aligned} f(x) = 12x^3 - (22a+22b+60)x^2 + \left(12a^2+12b^2+27ab+71a+71b+99\right)x \\ - \left(2a^3+2b^3+7a^2b+7ab^2+19a^2+19b^2+40ab+57a+57b+54\right). \end{aligned} \tag{3}$$

□

Next, we approximate the zeros of the polynomial $f(x)$.

Corollary 1. *Let $z_1 \geq z_2 \geq z_3$ be the zeros of $f(x)$. Then $z_1 \in \left(\frac{3n}{4}, n\right)$, $z_2 \in \left(\frac{n}{2}, \frac{3n}{4}\right)$ and $z_3 \in \left(\frac{n}{3}, \frac{n}{3}+1\right)$, where $a+b=n-2$ and $a \leq b$.*

Proof. For the polynomial

$$\begin{aligned} f(x) = 12x^3 - (22a+22b+60)x^2 + \left(12a^2+12b^2+27ab+71a+71b+99\right)x \\ - \left(2a^3+2b^3+7a^2b+7ab^2+19a^2+19b^2+40ab+57a+57b+54\right) \end{aligned}$$

it is easy to verify that

$$\begin{aligned} f(a+b+2) &= 2ab(2+a+b) > 0, \\ f\left(\frac{3(a+b+2)}{4}\right) &= -\frac{5}{16}(a-b)^2(2+a+b) < 0, \\ f\left(\frac{a+b+2}{2}\right) &= \frac{1}{2}(a+b+2)(ab+a+b-3) > 0, \\ f\left(\frac{a+b+2}{3}+1\right) &= \frac{1}{9}\left(2a^2+a(31b-1)+(b-1)(2b+1)\right) > 0, \\ f\left(\frac{a+b+2}{3}\right) &= -\frac{10}{9}(2+a+b)(5+a+b) < 0. \end{aligned}$$

Therefore, by the intermediate value theorem, the result follows. □

The next result shows that the reciprocal distance Laplacian spectral radius of the family $T(a,b)$ is a decreasing function of a, $1 \leq a \leq \lfloor \frac{n-2}{2} \rfloor$.

Theorem 2. *For positive integers a, b such that $a < b$ and $a+b=n-2$, we have*

$$\delta_1\big(T(a,b)\big) \leq \delta_1\big(T(a-1,b+1)\big).$$

Proof. By Theorem 1, the RD^L-spectral radius of $T(a,b)$ is the largest root of

$$\begin{aligned} f(x,a,b) = 12x^3 - (22a+22b+60)x^2 + \left(12a^2+12b^2+27ab+71a+71b+99\right)x \\ - \left(2a^3+2b^3+7a^2b+7ab^2+19a^2+19b^2+40ab+57a+57b+54\right). \end{aligned}$$

By manual calculations, we see that $f(x,a,b) - f(x,a-1,b+1) = (a-b-1)(n-3x)$. Since, $a \leq b$, so $a-b-1 \leq 0$ and by Proposition 3, we have $\frac{3n}{4} < \delta_1\big(T(a,b)\big) < n$ for all a,b, it follows that

$$\begin{aligned} f\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) &= f\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) - f\big(\delta_1\big(T(a-1,b+1)\big),a-1,b+1\big) \\ &= (a-b-1)\big(n-3\delta_1\big(T(a-1,b+1)\big)\big) > 0. \end{aligned}$$

Note that $f\big(\delta_1\big(T(a-1,b+1)\big),a-1,b+1\big) = 0$. Now, $f\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) > 0$ together with the fact $f\left(\frac{3n}{4},a,b\right) < 0$ and $f(n,a,b) > 0$, gives that the largest eigenvalue of $f(x,a,b)$ lies between $\frac{3n}{4}$ and $\delta_1\big(T(a-1,b+1)\big)$, by the intermediate value theorem. From this it is now clear that $\delta_1\big(T(a,b)\big) \leq \delta_1\big(T(a-1,b+1)\big)$. This completes the proof. $\square$

For example, consider the trees $T(2,3)$ and $T(1,4)$. Put $a = 2$ and $b = 3$ in (3). It follows that the reciprocal distance Laplacian spectral radius of $T(2,3)$ is the largest zero of the polynomial $12x^3 - 170x^2 + 772x - 1106$. By direct calculation, we see that the reciprocal distance Laplacian spectral radius of $T(2,3)$ is 6.0644826442887. Similarly, putting $a = 1$ and $b = 4$ in (3), we obtain that the reciprocal distance Laplacian spectral radius of $T(1,4)$ is the largest zero of the polynomial $12x^3 - 170x^2 + 766x - 1092$, which is 6.5. It is now clear that $\delta_1\big(T(2,3)\big) < \delta_1\big(T(1,4)\big)$.

Following assertion is the immediate consequence of Theorem 2 and gives the ordering of the trees belonging to the family $T(a,b)$ on the basis of their reciprocal distance Laplacian spectral radius.

Corollary 2. *For positive integers a,b such that $a < b$ and $a+b = n-2$, we have*

$$\delta_1\big(T(a,b)\big) \leq \delta_1\big(T(a-1,b+1)\big) \leq \cdots \leq \delta_1\big(T(1,a+b-1)\big).$$

Proceeding similar to Theorem 2 together with the fact $\frac{n}{3} < \delta_{n-1}\big(T(a,b)\big) < \frac{n}{3}+1$ for all a,b, with $f\left(\frac{n}{3}\right) < 0$ and $f\left(\frac{n}{3}+1\right) > 0$, we get the following result for second smallest reciprocal distance Laplacian eigenvalue δ_{n-1} of the family $T(a,b)$.

Theorem 3. *For positive integers a,b such that $a < b$ and $a+b = n-2$, we have*

$$\delta_{n-1}\big(T(a,b)\big) \leq \delta_{n-1}\big(T(a-1,b+1)\big) \leq \cdots \leq \delta_{n-1}\big(T(1,a+b-1)\big).$$

Note that Theorem 3 gives the ordering of the trees belonging to the family $T(a,b)$ on the basis of their second smallest reciprocal distance Laplacian eigenvalue.

The complement of tree $T(a,b)$ is denoted by $\overline{T}(a,b)$, where $a \leq b$. The following result gives the reciprocal distance Laplacian spectrum of $\overline{T}(a,b)$.

Proposition 3. *The reciprocal distance Laplacian spectrum of $\overline{T}(a,b)$, $a < b$, $a+b = n-2$, consists of the eigenvalue $\frac{2a+2b+3}{2}$ with multiplicity $a+b-2$, the simple eigenvalue 0 and the eigenvalues $y_1 \geq y_2 \geq y_3$, where $y_1 \in (n-1,n)$, $y_2 \in \left(\frac{3n}{4}, n-2\right)$ and $y_3 \in \left(\frac{n}{2}, \frac{3n}{4}\right)$.*

Proof. Labelling the vertices of $T(a,b)$ as in Theorem 1, we get that $\overline{T}(a,b)$ consists of clique on $a+b$, where a vertices share the same neighbourhood with common reciprocal transmission degree $\frac{2a+2b+1}{2}$. Likewise, b vertices of clique share the same neighbourhood with reciprocal transmission degree $\frac{2a+2b+1}{2}$. Thus by Proposition 2, it follows that $\frac{2a+2b+3}{2}$ is the

RD^L-eigenvalue with algebraic multiplicity $a+b-2$. Let X be the eigenvector of $\overline{T}(a,b)$ with $x_i = X(v_i)$ for $i = 1,2,\dots,n$. Then by Lemma 1, each coordinate of X corresponding to a vertices is equal to x_1, every coordinate of X corresponding to b vertices is x_4 and coordinates of X corresponding to vertices of degree a and b are x_3 and x_2, respectively. By using the eigenequation (1), the coefficient matrix of (δ, X)-equation is

$$\begin{pmatrix} \frac{2b+3}{2} & \frac{-1}{2} & -1 & -b \\ \frac{-a}{2} & \frac{3a+6b+2}{6} & \frac{-1}{3} & -b \\ -a & \frac{-1}{3} & \frac{6a+3b+2}{6} & \frac{-b}{2} \\ -a & -1 & \frac{-1}{2} & \frac{2a+3}{2} \end{pmatrix},$$

and its characteristic polynomial is $\dfrac{x}{12}p(x)$, where

$$\begin{aligned} p(x) = 12x^3 - x^2(30a+30b+44) + x\left(24a^2+24b^2+51ab+71a+71b+51\right) \\ - \left(6a^3+6b^3+21a^2b+21ab^2+27a^2+27b^2+60ab+39a+39b+18\right). \end{aligned}$$

Let y_1, y_2 and y_3 be the zeros of $p(x)$. Then by manual calculation, we obtain

$$\begin{aligned} & p(n) = p(a+b+2) = 2(a+b+2) = 2n > 0, \\ & p(n-1) = p(a+b+1) = a+b+1-3ab < 0, \\ & p(n-2) = p(a+b) = 6(2a+2b-ab-3) < 0, \\ & p\left(\frac{3(a+b+2)}{4}\right) = p\left(\frac{3n}{4}\right) = \frac{3}{16}(a-b)^2 n > 0, \\ & p\left(\frac{a+b+2}{2}\right) = p\left(\frac{n}{2}\right) = \frac{-n}{2}(3ab-a-b-1) < 0. \end{aligned}$$

Therefore, by intermediate value theorem, it follows that $y_1 \in \big(n-1, n\big)$, $y_2 \in \left(\frac{3n}{4}, n-2\right)$ and $y_3 \in \left(\frac{n}{2}, \frac{3n}{4}\right)$. This completes the proof. □

The next result gives that the reciprocal distance Laplacian spectral radius of the family $\overline{T}(a,b)$ is an increasing function of a, $1 \le a \le \lfloor \frac{n-2}{2} \rfloor$.

Theorem 4. *For positive integers a, b such that $a < b$ and $a+b = n-2$, we have*

$$\delta_1\big(\overline{T}(a,b)\big) \ge \delta_1\big(\overline{T}(a-1,b+1)\big).$$

Proof. By Proposition 3, the RD^L-spectral radius of $\overline{T}(a,b)$ is the largest root of

$$\begin{aligned} p(x,a,b) = 12x^3 - x^2(30a+30b+44) + x\left(24a^2+24b^2+51ab+71a+71b+51\right) \\ - \left(6a^3+6b^3+21a^2b+21ab^2+27a^2+27b^2+60ab+39a+39b+18\right). \end{aligned}$$

It is easy to see that $p(x,a,b) - p(x,a-1,b+1) = 3(a-b-1)(n-x)$.

Since $a < b$, by Proposition 3, we have $n-1 < \delta_1\big(T(a,b)\big) < n$ for all a, b. It follows that

$$\begin{aligned} p\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) &= p\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) - p\big(\delta_1\big(T(a-1,b+1)\big),a-1,b+1\big) \\ &= 3(a-b-1)\big(n-\delta_1\big(T(a-1,b+1)\big)\big) < 0. \end{aligned}$$

Note that $p\big(\delta_1\big(T(a-1,b+1)\big),a-1,b+1\big) = 0$. Now, $p\big(\delta_1\big(T(a-1,b+1)\big),a,b\big) < 0$ together with $p(n-1,a,b) < 0$ and $p(n,a,b) > 0$ implies that $\delta_1\big(\overline{T}(a,b)\big) \ge \delta_1\big(\overline{T}(a-1,b+1)\big)$. This completes the proof. □

Following assertion is the immediate consequence of Theorem 4 and gives the ordering of the graphs belonging to the family $\overline{T}(a,b)$ on the basis of their reciprocal distance Laplacian spectral radius.

Corollary 3. *For positive integers a, b such that $a < b$ and $a + b = n - 2$, we have*

$$\delta_1\big(\overline{T}(a,b)\big) \geq \delta_1\big(\overline{T}(a-1,b+1)\big) \geq \cdots \geq \delta_1\big(\overline{T}(1,a+b-1)\big).$$

Proceeding similar to Theorem 4 together with the fact $\frac{n}{2} < \delta_{n-1}\big(\overline{T}(a,b)\big) < \frac{3n}{4}$ for all a, b, with $f(\frac{n}{2}) < 0$ and $f(\frac{3n}{4}) > 0$, we get the following result for second smallest reciprocal distance Laplacian eigenvalue of $\overline{T}(a,b)$.

Theorem 5. *For positive integers a, b such that $a < b$ and $a + b = n - 2$, we have*

$$\delta_{n-1}\big(\overline{T}(a,b)\big) \geq \delta_{n-1}\big(\overline{T}(a-1,b+1)\big).$$

The next result is the immediate consequence of Theorem 5 and gives the ordering of the graphs belonging to the family $\overline{T}(a,b)$ on the basis of their second smallest reciprocal distance Laplacian eigenvalue.

Corollary 4. *For positive integers a, b such that $a < b$ and $a + b = n - 2$, we have*

$$\delta_{n-1}\big(\overline{T}(a,b)\big) \geq \delta_{n-1}\big(\overline{T}(a-1,b+1)\big) \geq \cdots \geq \delta_{n-1}\big(\overline{T}(1,a+b-1)\big).$$

3 Reciprocal Distance Laplacian energy ordering of $T(a,b)$ and $\overline{T}(a,b)$

In this section, we discuss the reciprocal distance Laplacian energy of the families $T(a,b)$ and $\overline{T}(a,b)$. We prove that the graphs belonging to these families can be ordered on the basis of their reciprocal distance Laplacian energy.

The reciprocal distance Laplacian energy is defined by

$$E_{RD^L}(G) = \sum_{i=1}^{n} \left| \delta_i - \frac{2H(G)}{n} \right|,$$

which is the same as a sum of singular values of the real symmetric matrix $RD^L(G) - \frac{2H(G)}{n} I_n$, where I_n is the identity matrix. Note that $\sum\limits_{i=1}^{n} \left(\delta_i - \frac{2H(G)}{n} \right) = 0$. We observe that $E_{RD^L}(G)$ is actually the trace norm of the matrix $RD^L(G) - \frac{2H(G)}{n} I_n$ and is related with the Ky Fan k-norm of $RD^L(G)$ (see Lemma 2 below). It is important problem in matrix theory and in spectral graph theory to characterize the linear operators having maximum and minimum norms. For some recent works on $E_{RD^L}(G)$, we refer to [28]. Therefore, the following problem can be of interest.

Problem 1. *Among all connected graphs G of order n determine the graphs which attain the extremal values for the trace norm $\left\| RD^L(G) - \frac{2H(G)}{n} I_n \right\|_*$.*

In general, this problem is very hard and as such any partial characterization among a class of graphs with some special symmetry will be highly appreciable.

Let σ be the largest positive integer such that $\delta_\sigma \geq \frac{2H(G)}{n}$. That is, σ is the positive integer with $\delta_\sigma \geq \frac{2H(G)}{n}$ and $\delta_{\sigma+1} < \frac{2H(G)}{n}$. It is clear that σ gives the number of reciprocal distance Laplacian eigenvalues of G, which lie in $\left[0, \frac{2H(G)}{n}\right]$, and the eigenvalues, which lie in $\left[\frac{2H(G)}{n}, n\right]$. It is an interesting and hard problem in linear algebra to find the distribution of the eigenvalues of a given matrix. The problem of distribution of eigenvalues of a given matrix has been considered for many graph matrices and various interesting results are obtained. Like other graph matrices, the following problem can be of interest for the matrix $RD^L(G)$.

Problem 2. *Among all connected graphs G of order n with a given parameter α (like the number of edges, the independence number, the matching number, the chromatic number, the vertex covering number, the average Harary index, etc.) determine the number of reciprocal distance Laplacian eigenvalues in the interval* $[0, \alpha]$.

The next result shows that we express the reciprocal distance Laplacian energy $E_{RD^L}(G)$ in terms of Ky Fan k-norm of the matrix $RD^L(G)$.

Lemma 2. *The reciprocal distance Laplacian energy of G satisfies the following relation*

$$E_{RD^L}(G) = 2\left(\sum_{i=1}^{\sigma} \delta_i - \frac{2\sigma H(G)}{n}\right) = 2 \max_{1\leq k\leq n} \left(\sum_{i=1}^{k} \delta_i - \frac{2kH(G)}{n}\right),$$

where $\sum\limits_{i=1}^{k} \delta_i$ is the sum of first k largest RD^L-eigenvalues (Ky Fan k-norm) of G and σ is the number of RD^L-eigenvalues in $\left[0, \frac{2H(G)}{n}\right]$.

Proof. Let σ be the largest positive integer such that $\delta_\sigma \geq \frac{2H(G)}{n}$. Then by the definition of reciprocal distance Laplacian energy $E_{RD^L}(G)$ and the fact $2H(G) = \sum\limits_{i=1}^{n} \delta_i$, we have

$$E_{RD^L}(G) = \sum_{i=1}^{n} \left|\delta_i - \frac{2H(G)}{n}\right| = \sum_{i=1}^{\sigma} \left(\delta_i - \frac{2H(G)}{n}\right) + \sum_{i=\sigma+1}^{n} \left(\frac{2H(G)}{n} - \delta_i\right) \sum_{i=1}^{\sigma} \delta_i - \frac{4\sigma H(G)}{n}$$
$$+ 2H(G) - \sum_{i=\sigma+1}^{n} \delta_i = 2\left(\sum_{i=1}^{\sigma} \delta_i - \frac{2\sigma H(G)}{n}\right).$$

Next, we are required to prove that $2\left(\sum\limits_{i=1}^{\sigma} \delta_i - \frac{2\sigma H(G)}{n}\right) = 2 \max\limits_{1\leq k\leq n} \left(\sum\limits_{i=1}^{k} \delta_i - \frac{2kH(G)}{n}\right)$. Since $\delta_i < \dfrac{2H(G)}{n}$ for $i \geq \sigma + 1$, we have

$$\begin{aligned} \sum_{i=1}^{k} \delta_i - \frac{2kH(G)}{n} &= \sum_{i=1}^{\sigma} \delta_i + \sum_{i=\sigma+1}^{k} \delta_i - \frac{2kH(G)}{n} \\ &< \sum_{i=1}^{\sigma} \delta_i + (k-\sigma)\frac{2H(G)}{n} - k\frac{2H(G)}{n} = \sum_{i=1}^{\sigma} \delta_i - \frac{2\sigma H(G)}{n} \end{aligned}$$

for $k > \sigma$. Similarly, for $k \leq \sigma$, it can be easily verified that $\sum\limits_{i=1}^{k} \delta_i - k\frac{2H(G)}{n} \leq \sum\limits_{i=1}^{\sigma} \delta_i - \frac{2\sigma H(G)}{n}$, that finishes the proof. $\square$

The next result calculates σ of $T(a, b)$ for $a < b$, while the case $a = b$ is discussed in Theorem 6.

Proposition 4. *For the tree $T(a,b)$ with $a<b$ and $a+b=n-2$, the value of σ is $b+1$.*

Proof. Let $\delta_1 \geq \delta_2 \geq \cdots \delta_{n-1} > \delta_n = 0$ be the reciprocal distance Laplacian eigenvalues of $T(a,b)$ and let $\sigma = \sigma(T(a,b))$ be the number of reciprocal distance Laplacian eigenvalues of $T(a,b)$, which are greater than or equal to $\frac{2H(T(a,b))}{n}$. Since the sum of diagonal entries of the matrix $RD^L(G)$ equals twice the Harary index, thus

$$\begin{aligned} 2H(T(a,b)) &= a\left(\frac{a}{2}+\frac{b}{3}+1\right)+b\left(\frac{a}{3}+\frac{b}{2}+1\right)+a+1+\frac{b}{2}+\frac{a}{2}+b+1 \\ &= \frac{3a^2+3b^2+4ab+15a+15b+12}{6}, \end{aligned}$$

and so average Harary index is $\frac{2H(T(a,b))}{n} = \frac{3a^2+3b^2+4ab+15a+15b+12}{6(a+b+2)}$. By Lemma 1 and Corollary 1, the reciprocal distance Laplacian spectrum of $T(a,b)$ consists of the eigenvalue $\frac{2a+3b+9}{6}$ with multiplicity $b-1$, the eigenvalue $\frac{2b+3a+9}{6}$ with multiplicity $a-1$, the simple eigenvalue 0 and the eigenvalues $z_1 \geq z_2 \geq z_3$ with $z_1 \in \left(\frac{3n}{4}, n\right)$, $z_2 \in \left(\frac{n}{2}, \frac{3n}{4}\right)$ and $z_3 \in \left(\frac{n}{3}, \frac{n}{3}+1\right)$. It is clear that $\delta_1 = z_1$, $\delta_{n-1} = z_3$ for $a \geq 4$. The RD^L-spectral radius of $T(a,b)$ is always greater than $\frac{2H(T(a,b))}{n}$. For the eigenvalue z_3, we see that $\frac{n}{3}+1 < \frac{2H(T(a,b))}{n}$. For the eigenvalue z_2, we see that $\frac{a+b+2}{2} \geq \frac{3a^2+3b^2+4ab+15a+15b+12}{6(a+b+2)} = \frac{2H(T(a,b))}{n}$ gives that $2ab \geq 3a+3b$, which further gives that $b \geq \frac{3a}{2a-3}$. Since $\frac{3a}{2a-3} < a$ for all $a \geq 4$, it follows that $z_2 \geq \frac{2H(T(a,b))}{n}$. For the eigenvalue $\frac{2a+3b+9}{6}$, we see that $\frac{2a+3b+9}{6} \geq \frac{3a^2+3b^2+4ab+15a+15b+12}{6(a+b+2)} = \frac{2H(T(a,b))}{n}$ always holds. Lastly, for the eigenvalue $\frac{2b+3a+9}{6}$, we see that $\frac{2b+3a+9}{6} < \frac{3a^2+3b^2+4ab+15a+15b+12}{6(a+b+2)} = \frac{2H(T(a,b))}{n}$ gives that $b^2-(a-2)b-6>0$. It is easy to see that this last inequality always holds for $b>a$. Thus with this discussion, it follows that $\sigma = 1+1+b-1 = b+1$. This completes the proof. □

The next result gives the reciprocal distance Laplacian energy of $T(a,b)$.

Theorem 6. *For the graph $T(a,b)$ with $n-2=a+b$, the reciprocal distance Laplacian energy is*

$$E_{RD^L}(T(a,b)) = \frac{2(b+1)(ab+3)}{3n} + \left(\frac{5b+6a-ab+12}{3}\right) - 2\delta_{n-1} \quad \textit{if} \quad a<b,$$

and

$$E_{RD^L}(T(a,b)) = 4b + \frac{1}{6}\sqrt{25b^2+66b+9} + \frac{28}{3(b+1)} \quad \textit{if} \quad a=b\geq 4.$$

Proof. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} > \delta_n = 0$, be the RD^L-eigenvalues of $T(a,b)$. Then by Theorem 1, Proposition 4 and Lemma 2, the reciprocal distance Laplacian energy of $T(a,b)$ is

$$\begin{aligned} E_{RD^L}(T(a,b)) &= 2\left(\sum_{i=1}^{b+1}\delta_i - \frac{2(b+1)H(T(a,b))}{n}\right) \\ &= 2\left(z_1+z_2+(b-1)\frac{2a+3b+9}{6} - \frac{(b+1)(3a^2+3b^2+4ab+15a+15b+12)}{6(a+b+2)}\right) \\ &= 2\left(z_1+z_2+(b-1)\frac{2a+3b+9}{6} - (b+1)\left(\frac{a+b+3}{2}-\frac{ab+3}{3n}\right)\right). \end{aligned}$$

From (3), we see that $z_1 + z_2 + z_3 = \frac{11a+11b+30}{6}$ and it is clear that z_3 is the smallest RD^L-eigenvalue of $T(a,b)$. Thus, the reciprocal distance Laplacian energy of $T(a,b)$, $a < b$, is given by

$$E_{RD^L}(T(a,b)) = \frac{2(b+1)(ab+3)}{3n} - \left(\frac{ab-5b-6a-12}{3}\right) - 2\delta_{n-1}.$$

If $a = b$, then $n = 2b+2$ and from Theorem 1 it follows that the RD^L-spectrum of $T(b,b)$ is

$$\left\{\frac{1}{12}\left(13b+21 \pm \sqrt{25b^2+66b+9}\right), \frac{3(b+1)}{2}, \left(\frac{5b+9}{6}\right)^{[2b-2]}, 0\right\}$$

and average of the RD^L-eigenvalues is $\frac{2H(T(b,b))}{n} = \frac{5b^2+15b+6}{6(b+1)} = \frac{5b+15}{6} - \frac{14}{6(b+1)}$. Also, the RD^L-spectral radius $\frac{1}{12}\left(13b+21+\sqrt{25b^2+66b+9}\right)$ is always greater than $\frac{2H(T(b,b))}{n}$. Further, it is clear that $\frac{3(b+1)}{2} \geq \frac{5b^2+15b+6}{6(b+1)}$ and $\frac{5b+9}{6} < \frac{5b^2+15b+6}{6(b+1)}$ for $b \geq 4$. Thus, we have $\sigma = 2$ and so the reciprocal distance Laplacian energy of $T(a,b)$ is

$$\begin{aligned} E_{RD^L}\big(T(b,b)\big) &= 2\left(\frac{1}{12}\left(13b+21+\sqrt{25b^2+66b+9}\right) + \frac{3(b+1)}{2} - \frac{2(5b^2+15b+6)}{6b+6}\right) \\ &= 4b + \frac{1}{6}\sqrt{25b^2+66b+9} + \frac{28}{3(b+1)}. \end{aligned}$$

This completes the proof. □

Since $\frac{n}{3} < \delta_{n-1} < \frac{n}{3}+1$, the following result is the immediate consequence of Theorem 6 and states that $E_{RD^L}\big(T(a,b)\big)$ lies in an interval of length 2.

Corollary 5. *The reciprocal distance Laplacian energy of $T(a,b)$ with $a < b$ and $n-2 = a+b$ satisfies the following relations*

$$E_{RD^L}\big(T(a,b)\big) > \frac{2(b+1)(ab+3)}{3n} + \left(\frac{3b+4a-ab+2}{3}\right)$$

and

$$E_{RD^L}(T(a,b)) < \frac{2(b+1)(ab+3)}{3n} + \left(\frac{3b+4a-ab+8}{3}\right).$$

For the family of trees $T(a-t, b+t)$, where $t = 0, 1, \ldots, a-3$, the following result shows that the reciprocal distance Laplacian energy of $T(a-t, b+t)$ is strictly decreasing function of t for some values of n and strictly increasing function of t for some other values of n.

Theorem 7. *For $t = 0,1,2,\ldots,a-3$, $a+b = n-2$ and $b > a \geq 3$, the reciprocal distance Laplacian energy $E_{RD^L}\big(T(a-t,b+t)\big)$ of the family $T(a-t,b+t)$ is a decreasing function of t, provided that $2n \geq 6a-3l-3k+3+\sqrt{\psi(a,l,k)}$ and an increasing function of t, provided that $2n < 6a-3l-3k-3+\sqrt{\psi(a,l,k)-72a+36k+36l}$, where*

$$\psi(a,l,k) = 12a^2 - 12a(k+l+1) + k^2 + l^2 + 10kl + 6k + 6l + 17$$

and $0 \leq l < k \leq a-3$ are positive integers.

Proof. If $b > a \geq 3$, then by Theorem 6, the reciprocal distance Laplacian energy of $T(a,b)$ is given by

$$E_{RD^L}\big(T(a,b)\big) = \frac{2(b+1)(ab+3)}{3n} - \left(\frac{ab-5b-6a-12}{3}\right) - 2\delta_{n-1}\big(T(a,b)\big), \qquad (4)$$

where $\delta_{n-1}\big(T(a,b)\big)$ is the second smallest reciprocal distance Laplacian eigenvalue of $T(a,b)$. Since $b = n-a-2$, therefore using this in (4), we can rewrite it as

$$E_{RD^L}\big(T(a,b)\big) = \left(\frac{a+5}{3}\right)n - a^2 - a + \frac{8}{3} + \frac{1}{3n}\left(2a^3+6a^2-2a-6\right) - 2\delta_{n-1}(T(a,b)). \qquad (5)$$

Consider the trees $T_1 = T(a-l, b+l)$ and $T_2 = T(a-k, b+k)$, where $0 \leq l < k$. From (5), we get

$$\begin{aligned} E_{RD^L}(T_1) - E_{RD^L}(T_2) = 2\big(\delta_{n-1}(T_2) - \delta_{n-1}(T_1)\big) + (k-l)\Big(\frac{n}{3} + k + l - 2a - 1 \\ + \frac{1}{3n}\big(6a^2 - 6ak - 6al + 12a + 2k^2 + 2kl - 6k + 2l^2 - 6l - 2\big)\Big). \end{aligned} \qquad (6)$$

Since $a \geq 3$, it follows from Theorem 3 that $\delta_{n-1}(T_2) > \delta_{n-1}(T_1)$. Also,

$$(k-l)\Big(\frac{n}{3} + k + l - 2a - 1 + \frac{1}{3n}\big(6a^2 - 6ak - 6al + 12a + 2k^2 + 2kl - 6k + 2l^2 - 6l - 2\big)\Big) \geq 0$$

gives that $2n \geq 3(2a-l-k+1) + \sqrt{12a^2 - 12a(k+l+1) + k^2 + l^2 + 10kl + 6k + 6l + 17}$. This shows that $E_{RD^L}(T_1) > E_{RD^L}(T_2)$, provided that $2n \geq 6a - 3l - 3k + 3 + \sqrt{\psi(a,l,k)}$. Again using the fact that $\frac{n}{3} < \delta_{n-1}(T_1), \delta_{n-1}(T_2) < \frac{n}{3} + 1$, we get $\delta_{n-1}(T_2) - \delta_{n-1}(T_1) < 1$. From this together with (6), we get $E_{RD^L}(T_1) < E_{RD^L}(T_2)$, provided that

$$\begin{aligned} (k-l)\Big(\frac{n}{3} + \frac{2}{k-l} + k + l - 2a - 1 \\ + \frac{1}{3n}\big(6a^2 - 6ak - 6al + 12a + 2k^2 + 2kl - 6k + 2l^2 - 6l - 2\big)\Big) < 0. \end{aligned}$$

The above inequality gives

$$n^2 - \left(6a + 3 - 3k - 3l - \frac{6}{k-l}\right)n + \left(6a^2 - 6ak - 6al + 12a + 2k^2 + 2kl - 6k + 2l^2 - 6l - 2\right) < 0.$$

Since $k - l \geq 1$, therefore this inequality will hold provided that the inequality

$$n^2 - (6a - 3k - 3l - 3)n + \left(6a^2 - 6ak - 6al + 12a + 2k^2 + 2kl - 6k + 2l^2 - 6l - 2\right) < 0$$

holds. From this inequality we get

$$2n < 6a - 3l - 3k - 3 + \sqrt{12a^2 - a(12k + 12l + 84) + k^2 + l^2 + 10kl + 42k + 42l + 17}.$$

Thus it follows that $E_{RD^L}(T_1) < E_{RD^L}(T_2)$, provided that

$$2n < 6a - 3l - 3k - 3 + \sqrt{\psi(a,l,k) - 72a + 36k + 36l}.$$

This completes the proof in this case. □

In particular, taking $l = 1$ and $k = 2$, it follows from Theorem 7 that

$$E_{RD^L}\big(T(a-1, b+1)\big) > E_{RD^L}\big(T(a-2, b+2)\big),$$

provided that $2n \geq 6a - 6 + \sqrt{12a^2 - 48a + 60}$ and

$$E_{RD^L}\big(T(a-1, b+1)\big) < E_{RD^L}\big(T(a-2, b+2)\big),$$

provided that $2n < 6a - 12 + \sqrt{12a^2 - 120a + 168}$. That is,

$$E_{RD^L}\big(T(a-1, b+1)\big) > E_{RD^L}\big(T(a-2, b+2)\big)$$

holds, provided that $n \geq 3a - 3 + \sqrt{3a^2 - 12a + 15}$ and

$$E_{RD^L}\big(T(a-1, b+1)\big) < E_{RD^L}\big(T(a-2, b+2)\big),$$

provided that $n < 3a - 6 + \sqrt{3a^2 - 30a + 42}$.

The following observation is immediate consequence of Theorem 7 and gives that we can order the trees belonging to the family $T(a, b)$ on the basis of their reciprocal distance Laplacian energies.

Corollary 6. *Among all trees in $T(a, b)$ with $b > a \geq 3$ and $a + b = n - 2$ the following holds.*

(i) If $2n \geq 3(2a - l - k + 1) + \sqrt{\psi(a, l, k)}$, where $\psi(a, l, k)$ is defined in Theorem 7, then the tree $T(3, n-3)$ has the minimum reciprocal distance Laplacian energy, while as the tree $T\left(\lfloor\frac{n-2}{2}\rfloor, \lceil\frac{n-2}{2}\rceil\right)$ if n is odd and the tree $T\left(\lfloor\frac{n-2}{2}\rfloor - 1, \lceil\frac{n-2}{2}\rceil + 1\right)$ if n is even has the maximum reciprocal distance Laplacian energy.

(ii) If $n > 2a + 2$ and $2n < 6a - 3l - 3k - 3 + \sqrt{\psi(a, l, k) - 72a + 36k + 36l}$, then the tree $T\left(\lfloor\frac{n-3}{2}\rfloor, \lceil\frac{n-3}{2}\rceil\right)$ if n is odd and the tree $T\left(\lfloor\frac{n-3}{2}\rfloor - 1, \lceil\frac{n-3}{2}\rceil + 1\right)$ if n is even has the minimum reciprocal distance Laplacian energy, while as the tree $T(3, n-3)$ has the maximum reciprocal distance Laplacian energy.

The next result gives the reciprocal distance Laplacian energy of the complement of $T(a, b)$.

Theorem 8. *For the graph $\overline{T}(a, b)$ with $n - 2 = a + b$, $3 \leq a < b$, the reciprocal distance Laplacian energy is $E_{RD^L}\big(\overline{T}(a, b)\big) = 2\delta_1 + n - \frac{28}{3} + \frac{4}{n}$ and*

$$E_{RD^L}\big(\overline{T}(b, b)\big) = \frac{1}{6}\left(33b - 23 + \sqrt{9b^2 + 66b + 25}\right) - \frac{2(2b-1)}{3b+3} \quad \textit{if } a = b.$$

Proof. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} > \delta_n = 0$ be the reciprocal distance Laplacian eigenvalues of $\overline{T}(a, b)$. Then by Proposition 3, we have $\delta_1 = y_1$, $\delta_i = a + b + \frac{3}{2}$ for $i = 2, 3, \ldots, a + b - 1$, $\delta_{a+b} = y_2$, $\delta_{a+b+1} = y_3$ and $\delta_{a+b+2} = 0$, where $y_1 \in \big(n-1, n\big), y_2 \in \left(\frac{3n}{4}, n-2\right)$ and $y_3 \in \left(\frac{n}{2}, \frac{3n}{4}\right)$. Let $\sigma = \sigma\big(\overline{T}(a, b)\big)$ be the number of reciprocal distance Laplacian eigenvalues of $\overline{T}(a, b)$, which are greater than or equal to $\frac{2H(\overline{T}(a,b))}{n}$. The average of RD^L-eigenvalues of $\overline{T}(a, b)$ is $\frac{2H(\overline{T}(a,b))}{n} = \frac{3a^2+3b^2+6ab+6a+6b+2}{3(a+b+2)}$. The RD^L-spectral radius δ_1 of any graph G is always greater or equal to $\frac{2H(G)}{n}$. For the eigenvalue $\frac{2a+2b+3}{2}$, we see that $\frac{2a+2b+3}{2} \geq \frac{3a^2+3b^2+6ab+6a+6b+2}{3(a+b+2)} = \frac{2H(\overline{T}(a,b))}{n}$, provided that $9a + 9b + 14 \geq 0$, which is always true. For the eigenvalues y_2, we see that $n-2=a+b<\frac{3a^2+3b^2+6ab+6a+6b+2}{3(a+b+2)}=\frac{2H(\overline{T}(a,b))}{n}$, giving that $-2<0$, which always true. From this discussion it follows that $\sigma=1+a+b-2=a+b-1$. Therefore, the reciprocal distance Laplacian energy of $\overline{T}(a,b)$ is given by

$$\begin{aligned}
E_{RD^L}\big(\overline{T}(a,b)\big) &= 2\left(\sum_{i=1}^{a+b-1}\delta_i-\frac{2(a+b-1)H\big(\overline{T}(a,b)\big)}{n}\right)\\
&=2\left(\delta_1+(a+b-2)\left(a+b+\frac{3}{2}\right)-(a+b-1)\frac{3a^2+3b^2+6ab+6a+6b+2}{3(a+b+2)}\right)\\
&=2\left(\delta_1+(a+b-2)\left(a+b+\frac{3}{2}\right)-(a+b-1)\left(a+b+\frac{2}{3(a+b+2)}\right)\right)\\
&=2\delta_1+n-\frac{28}{3}+\frac{4}{n}.
\end{aligned}$$

If $a=b$, then by Proposition 3 the RD^L-spectrum of $\overline{T}(b,b)$ is given by

$$\left\{\frac{1}{12}\Big(21b+13\pm\sqrt{9b^2+66b+25}\Big),\left(\frac{4b+3}{2}\right)^{[2b-2]},\frac{3(b+1)}{2},0\right\}$$

and $\frac{2H(\overline{T}(b,b))}{n}=\frac{12b^2+12b+2}{6(b+1)}$. It is easy to verify that $\frac{4b+3}{2}\geq\frac{2H(\overline{T}(b,b))}{n}$ and $\frac{3(b+1)}{2}<\frac{2H(\overline{T}(b,b))}{n}$ for $b\geq 3$. Thus, $\sigma=2b-1$ and reciprocal distance Laplacian energy is given by

$$E_{RD^L}\big(\overline{T}(b,b)\big)=\frac{1}{6}\Big(33b-23+\sqrt{9b^2+66b+25}\Big)-\frac{2(2b-1)}{3b+3}.$$

This completes the proof. □

For the family $\overline{T}(a,b)$, we have the following result which gives ordering of graphs belonging to this family based on their reciprocal distance Laplacian energies.

Theorem 9. *Among all graphs in $\overline{T}(a,b)$ with $b>a\geq 3$ and $a+b=n-2$, the graph $\overline{T}(3,n-3)$ has the minimum reciprocal distance Laplacian energy, while the graph $\overline{T}\left(\lfloor\frac{n-2}{2}\rfloor,\lceil\frac{n-2}{2}\rceil\right)$ if n is odd and the graph $\overline{T}\left(\lfloor\frac{n-2}{2}\rfloor-1,\lceil\frac{n-2}{2}\rceil+1\right)$ if n is even has the maximum reciprocal distance Laplacian energy.*

Proof. For $t=0,1,2,\ldots,a-3$, $a+b=n-2$ and $3\leq a<b$, consider the family $\overline{T}(a-t,b+t)$. To prove the result, we need to show that the reciprocal distance Laplacian energy of $\overline{T}(a-t,b+t)$ is an decreasing function of t. Let $H_1=\overline{T}(a-l,b+l)$ and $H_2=\overline{T}(a-k,b+k)$, where $0\leq l<k\leq a-3$ are any to members of the family $\overline{T}(a-t,b+t)$. Clearly, the reciprocal distance Laplacian energy of $\overline{T}(a-t,b+t)$ is an decreasing function of t, if we show $E_{RD^L}(H_1)>E_{RD^L}(H_2)$. From Theorem 8, it is clear that

$$\begin{aligned}
E_{RD^L}(H_1)-E_{RD^L}(H_2)&=\left(2\delta_1(H_1)+n-\frac{28}{3}+\frac{4}{n}\right)-\left(2\delta_1(H_2)+n-\frac{28}{3}+\frac{4}{n}\right)\\
&=2\big(\delta_1(H_1)-\delta_2(H_2)\big).
\end{aligned}$$

Since $a\geq 3$, it follows from Theorem 4 that $\delta_1(H_1)>\delta_1(H_2)$. With this the result follows. □

The following result is immediate consequence from Theorems 8 and 9.

Theorem 10. *Let $G_1 = \overline{T}(a_1, b_1)$ and $G_2 = \overline{T}(a_2, b_2)$ with $a_1 + b_1 = a_2 + b_2 = n - 2$, $b_1 > a_1 \geq 3$ and $b_2 > a_2 \geq 3$. Then the following assertions are equivalent:*

(i) $\delta_1(G_1) > \delta_1(G_2)$,

(ii) $E_{RD^L}(G_1) > E_{RD^L}(G_2)$,

(iii) $a_1 > a_2$.

From this result it is clear that for the family of graphs $\overline{T}(a, b)$, the reciprocal distance spectral radius and the reciprocal distance Laplacian energy increases with increase in a.

4 Conclusion

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with complex entries. The Ky-Fan k-norm of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ is defined as $\|M\|_k = \sum_{i=1}^{k} \sigma_i(M)$, where $\sigma_1(M) \geq \cdots \geq \sigma_n(M)$ are the singular values of M. For $k = 1$, the Ky-Fan k-norm gives the spectral norm of M. For $k = n$, the Ky-Fan k-norm gives the trace norm of M. It is well known that for a symmetric matrix M, we have $\sigma_i(M) = |\lambda_i(M)|$, $\sigma_i(M)$ is the ith singular value and $\lambda_i(M)$ is the ith eigenvalue of M. Using the fact that the matrix $RD^L(G)$ is symmetric and positive semi-definite, it follows that the spectral radius of the matrix $RD^L(G)$ is same as its spectral norm. Also, in the light of this definition, it follows that the reciprocal distance Laplacian energy $E_{RD^L}(G)$ of a connected graph G is the trace norm of the matrix $RD^L(G) - \frac{2H(G)}{n} I_n$, where I_n is the identity matrix of order n. It is an interesting problem in matrix theory to determine among a given class of matrices the matrix (or the matrices) which attain the maximum value and the minimum value for the spectral norm/trace norm. Another problem is to order the matrices among a given class of matrices based on their spectral norm/trace norm. These problems are considered for many graph matrices and a number of papers can be found in the literature in this direction.

In this article our aim was to study these problems for the reciprocal distance Laplacian matrix of a connected graph. Formally, we considered these problems for the reciprocal distance Laplacian matrix of trees of diameter three, called double stars, and their complements. We give ordering of these graphs based on their reciprocal distance Laplacian spectral radius, on their second smallest reciprocal distance Laplacian eigenvalue, and on their reciprocal distance Laplacian energy.

It will be an interesting direction for the researchers to add some new class of graphs which can be ordered based on their spectral norm and/or trace norm of the reciprocal distance Laplacian matrix of a connected graph.

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Ґані Х.А., Ратхер Б.А., Аучіче М. *Подвійні зіркові графи зі спектральними властивостями Лапласа взаємної відстані та їх доповнення* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 576–593.

Декілька матриць можуть бути асоційовані з графами для вивчення їхніх властивостей. У такому дослідженні дослідників цікавлять спектри матриць, що розглядаються, тому відповідні властивості називаються спектральними властивостями відносно матриць. Однією з цікавих і складних проблем у спектральному дослідженні графів є задача впорядкування графів на основі деяких інваріантів спектрального графа, таких як спектральний радіус, друге найменше власне значення, енергія тощо. Через складність цієї проблеми вона розглядалася в літературі для малої кількості класів графів. Тут ми продовжуємо ці дослідження та додаємо ще кілька класів графів, які можна впорядкувати на основі інваріантів спектральних графів. У цій статті ми вивчаємо спектральні властивості дерев діаметра три, які називаються подвійними зірковими графами, та їх доповнення через власні лапласівські значення взаємної відстані. Ми впорядковуємо ці графи на основі спектрального радіуса Лапласа взаємної відстані, другого найменшого власного значення Лапласа взаємної відстані та лапласівської енергії взаємної відстані.

Ключові слова і фрази: матриця Лапласа взаємної відстані, подвійний зірковий граф, спектральне впорядкування, енергія.