

Binomial sum relations involving Fibonacci and Lucas numbers

Kunle Adegoke

Department of Physics and Engineering Physics,
Obafemi Awolowo University, Ile-Ife, Nigeria
adegoke00@gmail.com

Robert Frontczak

Independent Researcher
Reutlingen, Germany
robert.frontczak@web.de

Taras Goy

Faculty of Mathematics and Computer Science
Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine
taras.goy@pnu.edu.ua

Abstract

In this paper, we introduce relations between binomial sums involving (generalized) Fibonacci and Lucas numbers, and different kinds of binomial coefficients. We also present some relations between sums with two and three binomial coefficients. In the course of exploration we rediscover a few relations presented as problem proposals.

2020 Mathematics Subject Classification: 11B37, 11B39.

Keywords: Binomial coefficient; central binomial coefficient; Fibonacci number; Lucas number; Horadam sequence; recurrence relation.

1 Introduction and motivation

The literature on Fibonacci numbers is immensely rich. There exist dozens of articles and problem proposals dealing with binomial sums involving these sequences as (weighted) summands. We attempt to give a short survey, not claiming completeness. The following binomial sums have been studied (X_n stands for a (weighted) Fibonacci or Lucas number, alternating or non-alternating, or a product of them):

- Standard form and variants of it [2, 6, 12, 14, 19, 24, 26, 27]

$$\sum_{k=0}^n \binom{n}{k} X_k;$$

- Forms coming from the Waring formula and studied by Gould [17], for instance,

$$\sum_{k=0}^{n-1} \frac{n}{n-k} \binom{n-k}{k} X_k;$$

- Forms introduced by Filipponi [15]

$$\sum_{k=0}^n \binom{2n-k-1}{k} X_k;$$

- Forms introduced by Jennings [22]

$$\sum_{k=0}^n \binom{n+k}{2k} X_k;$$

- Forms introduced by Kilic and Ionascu [23]

$$\sum_{k=0}^n \binom{2n}{n+k} X_k;$$

- Forms studied recently by Bai, Chu and Guo [9]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n}{n-2k} X_k \quad \text{and} \quad \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n+2}{n-2k} X_k;$$

- Forms studied by the authors in the recent paper [4]

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} X_k;$$

- Forms studied by the authors in the recent paper [3]

$$\sum_{k=0}^n \frac{n}{n+k} \binom{n+k}{n-k} X_k \quad \text{and} \quad \sum_{k=0}^n \frac{k}{n+k} \binom{n+k}{n-k} X_k.$$

We note that

$$\sum_{k=0}^n \binom{n+k}{2k} X_k = \sum_{k=0}^n \frac{n}{n+k} \binom{n+k}{n-k} X_k + \sum_{k=0}^n \frac{k}{n+k} \binom{n+k}{n-k} X_k.$$

Let $(W_j(a, b; p, q))_{j \geq 0}$ be the Horadam sequence [21] defined for all non-negative integers j by the recurrence

$$W_0 = a, \quad W_1 = b; \quad W_j = pW_{j-1} - qW_{j-2}, \quad j \geq 2, \quad (1)$$

where a, b, p and q are arbitrary complex numbers, with $p \neq 0$ and $q \neq 0$. Extension of the definition of (W_j) to negative subscripts is provided by writing the recurrence relation as

$$W_{-j} = \frac{1}{q}(pW_{-j+1} - W_{-j+2}).$$

Two important cases of (W_j) are the Lucas sequences of the first kind, $(U_j(p, q)) = (W_j(0, 1; p, q))$, and of the second kind, $(V_j(p, q)) = (W_j(2, p; p, q))$, so that

$$U_0 = 0, \quad U_1 = 1, \quad U_j = pU_{j-1} - qU_{j-2}, \quad j \geq 2;$$

and

$$V_0 = 2, \quad V_1 = p, \quad V_j = pV_{j-1} - qV_{j-2}, \quad j \geq 2.$$

The most well-known Lucas sequences are the Fibonacci sequence $(F_j) = (U_j(1, -1))$ and the sequence of Lucas numbers $(L_j) = (V_j(1, -1))$.

The Binet formulas for sequences (U_j) , (V_j) and (W_j) in the non-degenerate case, $\Delta = p^2 - 4q > 0$, are

$$U_j = \frac{\tau^j - \sigma^j}{\sqrt{\Delta}}, \quad V_j = \tau^j + \sigma^j, \quad W_j = A\tau^j + B\sigma^j, \quad (2)$$

with $A = \frac{b - a\sigma}{\sqrt{\Delta}}$ and $B = \frac{a\tau - b}{\sqrt{\Delta}}$, where

$$\tau = \tau(p, q) = \frac{p + \sqrt{\Delta}}{2}, \quad \sigma = \sigma(p, q) = \frac{p - \sqrt{\Delta}}{2}$$

are the distinct zeros of the characteristic polynomial $x^2 - px + q$ of the Horadam sequence (1).

The Binet formulas for the Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\sqrt{5}}, \quad L_j = \alpha^j + \beta^j, \quad (3)$$

where $\alpha = \tau(1, -1) = (1 + \sqrt{5})/2$ is the golden ratio and $\beta = \sigma(1, -1) = -1/\alpha$.

The sequences $(F_n)_{n \geq 0}$ and $(L_n)_{n \geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [32] as entries A000045 and A000032, respectively. For more information on them we recommend the books by Koshy [25] and Vajda [34], among others.

In this paper, we introduce relations between binomial sums involving (generalized) Fibonacci and Lucas numbers, and different kinds of binomial coefficients. We also present some relations between sums with two and three binomial coefficients. In the course of exploration we rediscover a few relations presented as problem proposals.

We will make use of the following known results.

Lemma 1. *If a, b, c and d are rational numbers and λ is an irrational number, then*

$$a + b\lambda = c + d\lambda \iff a = c, b = d.$$

Lemma 2. *For any integer s ,*

$$q^s + \tau^{2s} = \tau^s V_s, \quad q^s - \tau^{2s} = -\Delta \tau^s U_s, \quad (4)$$

$$q^s + \sigma^{2s} = \sigma^s V_s, \quad q^s - \sigma^{2s} = \Delta \sigma^s U_s. \quad (5)$$

In particular,

$$(-1)^s + \alpha^{2s} = \alpha^s L_s, \quad (-1)^s - \alpha^{2s} = -\sqrt{5}\alpha^s F_s, \quad (6)$$

$$(-1)^s + \beta^{2s} = \beta^s L_s, \quad (-1)^s - \beta^{2s} = \sqrt{5}\beta^s F_s. \quad (7)$$

Lemma 3. *Let r and d be any integers. Then*

$$V_{r+s} - \tau^r V_s = -\Delta \sigma^s U_r, \quad (8)$$

$$V_{r+s} - \sigma^r V_s = \Delta \tau^s U_r, \quad (9)$$

$$U_{r+s} - \tau^r U_s = \sigma^s U_r, \quad (10)$$

$$U_{r+s} - \sigma^r U_s = \tau^s U_r. \quad (11)$$

In particular, [19],

$$L_{r+s} - L_r \alpha^s = -\sqrt{5}\beta^r F_s, \quad L_{r+s} - L_r \beta^s = \sqrt{5}\alpha^r F_s, \quad (12)$$

$$F_{r+s} - F_r \alpha^s = \beta^r F_s, \quad F_{r+s} - F_r \beta^s = \alpha^r F_s. \quad (13)$$

Lemma 4. *For any integer j ,*

$$A\tau^j - B\sigma^j = \frac{W_{j+1} - qW_{j-1}}{\Delta}, \quad (14)$$

$$A\sigma^j + B\tau^j = q^j W_{-j}. \quad (15)$$

Proof. See [5, Lemma 1] for a proof of (14). Identity (15) is a consequence of the Binet formula. \square

2 Relations from a classical polynomial identity

The first binomial sum relations follow from the next classical polynomial identity which we state in the next lemma.

Lemma 5 ([29, Identity 6.21]). *If x is a complex variable and m and n are non-negative integers, then*

$$\sum_{k=0}^n \binom{m-n+k}{k} (1+x)^{n-k} x^k = \sum_{k=0}^n \binom{m+1}{k} x^k, \quad x \neq -1. \quad (16)$$

According to Gould [29], identity (16) is due to Laplace. In addition, we note that the binomial theorem is a special case of (16) which occurs at $m = n - 1$.

Using

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

and replacing m by $m - 1$ we have the equivalent and useful form of Lemma 5:

$$\sum_{k=0}^n (-1)^k \binom{n-m}{k} (1+x)^{n-k} x^k = \sum_{k=0}^n \binom{m}{k} x^k, \quad x \neq -1.$$

Theorem 1. *If r, s and t are any integers and m and n are non-negative integers, then*

$$\sum_{k=0}^n \binom{m-n+k}{k} U_{r+s}^k U_s^{n-k} W_{t+r(n-k)} = \sum_{k=0}^n (-q^s)^{n-k} \binom{m+1}{k} U_{r+s}^k U_r^{n-k} W_{t-s(n-k)}.$$

Proof. Set $x = -U_{r+s}/(U_r \sigma^s)$ in (16), use (11) and multiply through by τ^t , obtaining

$$\sum_{k=0}^n \binom{m-n+k}{k} U_{r+s}^k U_s^{n-k} \tau^{r(n-k)+t} = (-1)^t \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} U_{r+s}^k U_r^{n-k} \sigma^{s(n-k)-t}.$$

Similarly, setting $x = -U_{r+s}/(U_r \tau^s)$ in (16), using (10) and multiplying through by σ^t , yields

$$\sum_{k=0}^n \binom{m-n+k}{k} U_{r+s}^k U_s^{n-k} \sigma^{r(n-k)+t} = (-1)^t \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} U_{r+s}^k U_r^{n-k} \tau^{s(n-k)-t}.$$

The results follow by combining these identities according to the Binet formulas (2) and Lemma 4. \square

In particular,

$$\sum_{k=0}^n \binom{m-n+k}{k} U_{r+s}^k U_s^{n-k} V_{r(n-k)+t} = q^t \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} U_{r+s}^k U_r^{n-k} V_{s(n-k)-t},$$

and

$$\sum_{k=0}^n \binom{m-n+k}{k} U_{r+s}^k U_s^{n-k} U_{r(n-k)+t} = q^t \sum_{k=0}^n (-1)^{n-k+1} \binom{m+1}{k} U_{r+s}^k U_r^{n-k} U_{s(n-k)-t};$$

with the special cases

$$\sum_{k=0}^n \binom{m-n+k}{k} F_{r+s}^k F_s^{n-k} L_{r(n-k)+t} = \sum_{k=0}^n (-1)^{n-k-t} \binom{m+1}{k} F_{r+s}^k F_r^{n-k} L_{s(n-k)-t}, \quad (17)$$

$$\sum_{k=0}^n \binom{m-n+k}{k} F_{r+s}^k F_s^{n-k} F_{r(n-k)+t} = - \sum_{k=0}^n (-1)^{n-k-t} \binom{m+1}{k} F_{r+s}^k F_r^{n-k} F_{s(n-k)-t}. \quad (18)$$

Corollary 2. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^n (-q^s)^{n-k} \binom{n+1}{k} U_{r+s}^k U_r^{n-k} W_{t-s(n-k)} = \sum_{k=0}^n U_{r+s}^k U_s^{n-k} W_{t+r(n-k)}.$$

Proof. Set $m = n$ in Theorem 1. □

In particular,

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} U_{r+s}^k U_r^{n-k} V_{s(n-k)-t} &= \frac{1}{q^t} \sum_{k=0}^n U_{r+s}^k U_s^{n-k} V_{r(n-k)+t}, \\ \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} U_{r+s}^k U_r^{n-k} U_{s(n-k)-t} &= -\frac{1}{q^t} \sum_{k=0}^n U_{r+s}^k U_s^{n-k} U_{r(n-k)+t}; \end{aligned}$$

with

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} F_{r+s}^k F_r^{n-k} L_{s(n-k)-t} &= (-1)^t \sum_{k=0}^n F_{r+s}^k F_s^{n-k} L_{r(n-k)+t}, \\ \sum_{k=0}^n (-1)^{n-k} \binom{n+1}{k} F_{r+s}^k F_r^{n-k} F_{s(n-k)-t} &= (-1)^{t-1} \sum_{k=0}^n F_{r+s}^k F_s^{n-k} F_{r(n-k)+t}. \end{aligned}$$

Corollary 3. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^n (-q^s)^{n-k} \binom{n}{k} U_{r+s}^k U_r^{n-k} W_{t-s(n-k)} = U_s^n W_{t+rn}.$$

Proof. Set $m = n - 1$ in Theorem 1. □

In particular,

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} U_{r+s}^k U_r^{n-k} V_{s(n-k)-t} &= \frac{U_s^n V_{rn+t}}{q^t}, \\ \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} U_{r+s}^k U_r^{n-k} U_{s(n-k)-t} &= -\frac{U_s^n U_{rn+t}}{q^t}; \end{aligned}$$

with the special cases

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F_{r+s}^k F_r^{n-k} L_{s(n-k)-t} = (-1)^t F_s^n L_{rn+t}, \quad (19)$$

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} F_{r+s}^k F_r^{n-k} F_{s(n-k)-t} = (-1)^{t+1} F_s^n F_{rn+t}. \quad (20)$$

We mention that identities (19) and (20) exhibit strong similarities to those derived by Hoggatt, Phillips and Leonard in [20].

Corollary 4. *If m and n are non-negative integers and r is any integer, then*

$$\sum_{k=0}^n (-q^r)^k \binom{m-n+k}{k} W_{r(n-2k)} = \sum_{k=0}^n (-q^r)^k \binom{m+1}{k} V_r^{n-k} W_{-rk}.$$

Proof. Make the substitutions $r \mapsto 2r$, $s \mapsto -r$ and $t \mapsto -rn$ in Theorem 1 and simplify. \square

In particular,

$$\begin{aligned} \sum_{k=0}^n (-q^r)^k \binom{m-n+k}{k} V_{r(n-2k)} &= \sum_{k=0}^n (-1)^k \binom{m+1}{k} V_r^{n-k} V_{rk}, \\ \sum_{k=0}^n (-q^r)^k \binom{m-n+k}{k} U_{r(n-2k)} &= \sum_{k=0}^n (-1)^{k+1} \binom{m+1}{k} V_r^{n-k} U_{rk}; \end{aligned}$$

with the special cases

$$\begin{aligned} \sum_{k=0}^n (-1)^{k(r+1)} \binom{m-n+k}{k} L_{r(n-2k)} &= \sum_{k=0}^n (-1)^k \binom{m+1}{k} L_r^{n-k} L_{rk}, \\ \sum_{k=0}^n (-1)^{k(r+1)} \binom{m-n+k}{k} F_{r(n-2k)} &= \sum_{k=0}^n (-1)^{k+1} \binom{m+1}{k} L_r^{n-k} F_{rk}. \end{aligned}$$

By making appropriate substitutions in Theorem 1 many new sum relations can be established. For example, setting $r = 1$, $s = -2$, and $t = 1$ (or $r = -1$, $s = 2$ and $t = -1$) in (17) gives

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} L_{n-k+1} = \sum_{k=0}^n (-1)^k \binom{m+1}{k} L_{2(n-k)+1}$$

which at $m = 2n$ gives

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{n-k+1} = \sum_{k=0}^n (-1)^k \binom{2n+1}{k} L_{2(n-k)+1}. \quad (21)$$

The corresponding Fibonacci sums from (18) are of exactly the same structure

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} F_{n-k+1} = \sum_{k=0}^n (-1)^k \binom{m+1}{k} F_{2(n-k)+1}$$

with the special case

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{n-k+1} = \sum_{k=0}^n (-1)^k \binom{2n+1}{k} F_{2(n-k)+1}. \quad (22)$$

Another example is the relation

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} L_{n-3k} = \sum_{k=0}^n (-1)^k \binom{m+1}{k} 2^{n-k} L_{2k}$$

which at $m = 2n$ gives

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{n-3k} = \sum_{k=0}^n (-1)^k \binom{2n+1}{k} 2^{n-k} L_{2k}, \quad (23)$$

and its Fibonacci counterparts:

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{m-n+k}{k} F_{n-3k} &= \sum_{k=1}^n (-1)^{k-1} \binom{m+1}{k} 2^{n-k} F_{2k}, \\ \sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{n-3k} &= \sum_{k=1}^n (-1)^{k-1} \binom{2n+1}{k} 2^{n-k} F_{2k}. \end{aligned}$$

Theorem 5. *If m and n are non-negative integers and r, s are any integers, then*

$$\sum_{k=0}^n (-q^s)^k \binom{m-n+k}{k} U_r^k U_s^{n-k} W_{r(n-k)-sk+t} = \sum_{k=0}^n (-q^s)^k \binom{m+1}{k} U_r^k U_{r+s}^{n-k} W_{t-sk}.$$

Proof. Set $x = -U_r \sigma^s / U_{r+s}$ in (16), use (11) and multiply through by τ^t , obtaining

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} q^{sk} U_r^k U_s^{n-k} \tau^{rn-(r+s)k+t} = q^t \sum_{k=0}^n (-1)^k \binom{m+1}{k} U_r^k U_{r+s}^{n-k} \sigma^{sk-t}.$$

Similarly, setting $x = -U_r \tau^s / U_{r+s}$ in (16), using (13) and multiplying through by σ^t , yields

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} q^{sk} U_r^k U_s^{n-k} \sigma^{rn-(r+s)k+t} = q^t \sum_{k=0}^n (-1)^k \binom{m+1}{k} U_r^k U_{r+s}^{n-k} \tau^{sk-t}.$$

Now, the result follows immediately upon combining according to the Binet formulas (2). \square

In particular,

$$\sum_{k=0}^n (-q^s)^k \binom{m-n+k}{k} U_r^k U_s^{n-k} V_{rn-(r+s)k+t} = q^t \sum_{k=0}^n (-1)^k \binom{m+1}{k} U_r^k U_{r+s}^{n-k} V_{sk-t}, \quad (24)$$

$$\sum_{k=0}^n (-q^s)^k \binom{m-n+k}{k} U_r^k U_s^{n-k} U_{rn-(r+s)k+t} = -q^t \sum_{k=0}^n (-1)^k \binom{m+1}{k} U_r^k U_{r+s}^{n-k} U_{sk-t}; \quad (25)$$

with the special cases

$$\begin{aligned}\sum_{k=0}^n (-1)^{k(s-1)} \binom{m-n+k}{k} F_r^k F_s^{n-k} L_{rn-(r+s)k+t} &= \sum_{k=0}^n (-1)^{k(s-1)} \binom{m+1}{k} F_r^k F_{r+s}^{n-k} L_{t-sk}, \\ \sum_{k=0}^n (-1)^{k(s-1)} \binom{m-n+k}{k} F_r^k F_s^{n-k} F_{rn-(r+s)k+t} &= \sum_{k=0}^n (-1)^{k(s-1)} \binom{m+1}{k} F_r^k F_{r+s}^{n-k} F_{t-sk}.\end{aligned}$$

Corollary 6. *If n is a non-negative integer and r and s are any integers, then*

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{n+1}{k} U_r^k U_{r+s}^{n-k} V_{sk-t} &= \frac{1}{q^t} \sum_{k=0}^n (-q^s)^k U_r^k U_s^{n-k} V_{rn-(r+s)k+t}, \\ \sum_{k=0}^n (-1)^k \binom{n+1}{k} U_r^k U_{r+s}^{n-k} U_{sk-t} &= -\frac{1}{q^t} \sum_{k=0}^n (-q^s)^k U_r^k U_s^{n-k} U_{rn-(r+s)k+t}.\end{aligned}$$

Proof. Set $m = n$ in (24) and (25). □

In particular,

$$\begin{aligned}\sum_{k=0}^n (-1)^{k(s-1)} \binom{n+1}{k} F_r^k F_{r+s}^{n-k} L_{t-sk} &= \sum_{k=0}^n (-1)^{k(s-1)} F_r^k F_s^{n-k} L_{rn-(r+s)k+t}, \\ \sum_{k=0}^n (-1)^{k(s-1)} \binom{n+1}{k} F_r^k F_{r+s}^{n-k} F_{t-sk} &= \sum_{k=0}^n (-1)^{k(s-1)} F_r^k F_s^{n-k} F_{rn-(r+s)k+t}.\end{aligned}$$

We mention that setting $m = n - 1$ in Theorem 5 gives again Corollary 3.

Corollary 7. *If m and n are non-negative integers and r is any integer, then*

$$\begin{aligned}\sum_{k=0}^n \binom{m-n+k}{k} V_r^k V_{r(n-k)} &= \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} V_r^k V_{r(n-k)}, \\ \sum_{k=0}^n \binom{m-n+k}{k} V_r^k U_{r(n-k)} &= \sum_{k=0}^n (-1)^{n-k+1} \binom{m+1}{k} V_r^k U_{r(n-k)}.\end{aligned}$$

Proof. Make the substitutions $r \mapsto 2r$, $s \mapsto -r$, $t \mapsto -rn$ in (24), (25) and simplify. □

In particular,

$$\begin{aligned}\sum_{k=0}^n \binom{m-n+k}{k} L_r^k L_{r(n-k)} &= \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} L_r^k L_{r(n-k)}, \\ \sum_{k=0}^n \binom{m-n+k}{k} L_r^k F_{r(n-k)} &= \sum_{k=0}^n (-1)^{n-k+1} \binom{m+1}{k} L_r^k F_{r(n-k)}.\end{aligned}$$

Theorem 8. *If m and n are non-negative integers and s, t are integers, then*

$$\begin{aligned} \sum_{k=0}^{2n} \binom{m-2n+k}{k} 2^{2n-k} V_s^k W_{s(2n-k)+t} &= W_t \sum_{k=0}^n \binom{m+1}{2k} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k} \\ &\quad + (W_{t+1} - qW_{t-1}) \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k)} U_s^{2(n-k)+1} V_s^{2k-1}, \\ \sum_{k=0}^{2n-1} \binom{m-2n+k+1}{k} 2^{2n-k-1} V_s^k W_{s(2n-k-1)+t} &= W_t \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k-1} \\ &\quad + (W_{t+1} - qW_{t-1}) \sum_{k=0}^{n-1} \binom{m+1}{2k} \Delta^{2(n-k-1)} U_s^{2(n-k)-1} V_s^{2k}. \end{aligned}$$

Proof. Set $x = V_s/(\Delta U_s)$ in (16) and multiply through by τ^t to obtain

$$\begin{aligned} \sum_{k=0}^n \binom{m-n+k}{k} 2^{n-k} \tau^{s(n-k)+t} V_s^k \\ = \tau^t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m+1}{2k} \Delta^{n-2k} U_s^{n-2k} V_s^{2k} + \tau^t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m+1}{2k-1} \Delta^{n-2k+1} U_s^{n-2k+1} V_s^{2k-1}. \end{aligned} \quad (26)$$

Similarly, set $x = -V_s/(\Delta U_s)$ in (16) and multiply through by σ^t to obtain

$$\begin{aligned} (-1)^n \sum_{k=0}^n \binom{m-n+k}{k} 2^{n-k} \sigma^{s(n-k)+t} V_s^k \\ = \sigma^t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m+1}{2k} \Delta^{n-2k} U_s^{n-2k} V_s^{2k} - \sigma^t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m+1}{2k-1} \Delta^{n-2k+1} U_s^{n-2k+1} V_s^{2k-1}. \end{aligned} \quad (27)$$

Combine (26) and (27) according to the Binet formula while making use also of (14). Consider the cases $n \mapsto 2n$ and $n \mapsto 2n-1$, in turn. \square

In particular,

$$\begin{aligned} \sum_{k=0}^{2n} \binom{m-2n+k}{k} 2^{2n-k} V_s^k V_{s(2n-k)+t} \\ = V_t \sum_{k=0}^n \binom{m+1}{2k} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k} + U_t \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k+1)} U_s^{2(n-k)+1} V_s^{2k-1}, \end{aligned} \quad (28)$$

$$\begin{aligned} \sum_{k=0}^{2n} \binom{m-2n+k}{k} 2^{2n-k} V_s^k U_{s(2n-k)+t} \\ = U_t \sum_{k=0}^n \binom{m+1}{2k} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k} + V_t \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k)} U_s^{2(n-k)+1} V_s^{2k-1}, \end{aligned} \quad (29)$$

$$\begin{aligned}
& \sum_{k=0}^{2n-1} \binom{m-2n+k+1}{k} 2^{2n-k-1} V_s^k V_{s(2n-k-1)+t} \\
&= U_t \sum_{k=0}^{n-1} \binom{m+1}{2k} \Delta^{2(n-k)} U_s^{2(n-k)-1} V_s^{2k} + V_t \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k-1}
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
& \sum_{k=0}^{2n-1} \binom{m-2n+k+1}{k} 2^{2n-k-1} V_s^k U_{s(2n-k-1)+t} \\
&= V_t \sum_{k=0}^{n-1} \binom{m+1}{2k} \Delta^{2(n-k-1)} U_s^{2(n-k)-1} V_s^{2k} + U_t \sum_{k=1}^n \binom{m+1}{2k-1} \Delta^{2(n-k)} U_s^{2(n-k)} V_s^{2k-1}.
\end{aligned} \tag{31}$$

with the special cases

$$\begin{aligned}
& \sum_{k=0}^{2n} \binom{m-2n+k}{k} 2^{2n-k} L_s^k L_{s(2n-k)+t} \\
&= L_t \sum_{k=0}^n \binom{m+1}{2k} 5^{n-k} L_s^{2k} F_s^{2(n-k)} + F_t \sum_{k=1}^n \binom{m+1}{2k-1} 5^{n-k+1} L_s^{2k-1} F_s^{2(n-k)+1}, \\
& \sum_{k=0}^{2n} \binom{m-2n+k}{k} 2^{2n-k} L_s^k F_{s(2n-k)+t} \\
&= F_t \sum_{k=0}^n \binom{m+1}{2k} 5^{n-k} L_s^{2k} F_s^{2(n-k)} + L_t \sum_{k=1}^n \binom{m+1}{2k-1} 5^{n-k} L_s^{2k-1} F_s^{2(n-k)+1}, \\
& \sum_{k=0}^{2n-1} \binom{m-2n+k+1}{k} 2^{2n-k-1} L_s^k L_{s(2n-k-1)+t} \\
&= F_t \sum_{k=0}^{n-1} \binom{m+1}{2k} 5^{n-k} L_s^{2k} F_s^{2(n-k)-1} + L_t \sum_{k=1}^n \binom{m+1}{2k-1} 5^{n-k} L_s^{2k-1} F_s^{2(n-k)}, \\
& \sum_{k=0}^{2n-1} \binom{m-2n+k+1}{k} 2^{2n-k-1} L_s^k F_{s(2n-k-1)+t} \\
&= L_t \sum_{k=0}^{n-1} \binom{m+1}{2k} 5^{n-k-1} L_s^{2k} F_s^{2(n-k)-1} + F_t \sum_{k=1}^n \binom{m+1}{2k-1} 5^{n-k} L_s^{2k-1} F_s^{2(n-k)}.
\end{aligned}$$

Note that in (28)–(31), we used (see, for example, [1, Identities (1.16), (1.17)])

$$U_{t+1} - qU_{t-1} = V_t, \quad V_{t+1} - qV_{t-1} = \Delta^2 U_t. \tag{32}$$

Lemma 6. *If x is a complex variable and m, n are non-negative integers, then*

$$\sum_{k=0}^n \binom{m-n+k}{k} x^k = \sum_{k=0}^n \binom{m+1}{k} x^k (1-x)^{n-k} \tag{33}$$

Proof. Use the transformation $\frac{x}{1+x} \mapsto x$ in (16). □

Theorem 9. *If m and n are non-negative integers and s, t are integers, then*

$$\begin{aligned} L_t \sum_{k=0}^n \binom{m-2n+2k}{2k} \frac{F_s^{2(n-k)} L_s^{2k}}{5^k} - F_t \sum_{k=1}^n \binom{m-2n+2k-1}{2k-1} \frac{F_s^{2(n-k)+1} L_s^{2k-1}}{5^{k-1}} \\ = \left(\frac{4}{5}\right)^n \sum_{k=0}^{2n} (-1)^k \binom{m+1}{k} \frac{L_s^k L_{s(2n-k)+t}}{2^k}, \end{aligned} \quad (34)$$

$$\begin{aligned} F_t \sum_{k=0}^n \binom{m-2n+2k}{2k} \frac{F_s^{2(n-k)} L_s^{2k}}{5^k} - L_t \sum_{k=1}^n \binom{m-2n+2k-1}{2k-1} \frac{F_s^{2(n-k)+1} L_s^{2k-1}}{5^k} \\ = \left(\frac{4}{5}\right)^n \sum_{k=0}^{2n} (-1)^k \binom{m+1}{k} \frac{L_s^k F_{s(2n-k)+t}}{2^k}, \end{aligned} \quad (35)$$

$$\begin{aligned} L_t \sum_{k=1}^n \binom{m-2n+2k}{2k-1} \frac{F_s^{2(n-k)} L_s^{2k-1}}{5^k} - F_t \sum_{k=0}^{n-1} \binom{m-2n+2k+1}{2k} \frac{F_s^{2(n-k)-1} L_s^{2k}}{5^k} \\ = \left(\frac{4}{5}\right)^n \sum_{k=0}^{2n-1} (-1)^{k-1} \binom{m+1}{k} \frac{L_s^k L_{s(2n-k-1)+t}}{2^{k+1}}, \end{aligned} \quad (36)$$

$$\begin{aligned} F_t \sum_{k=1}^n \binom{m-2n+2k}{2k-1} \frac{F_s^{2(n-k)} L_s^{2k-1}}{5^k} - L_t \sum_{k=0}^{n-1} \binom{m-2n+2k+1}{2k} \frac{F_s^{2(n-k)-1} L_s^{2k}}{5^{k+1}} \\ = \left(\frac{4}{5}\right)^n \sum_{k=0}^{2n-1} (-1)^{k-1} \binom{m+1}{k} \frac{L_s^k F_{s(2n-k-1)+t}}{2^{k+1}}. \end{aligned} \quad (37)$$

Proof. Set $x = L_s/(\sqrt{5}F_s)$ in (33) to obtain

$$\sum_{k=0}^n \binom{m-n+k}{k} (\sqrt{5})^{n-k} F_s^{n-k} L_s^k = \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} 2^{n-k} L_s^k \beta^{s(n-k)},$$

so that

$$\begin{aligned} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \frac{F_s^{n-2k} L_s^{2k}}{(\sqrt{5})^{2k}} + \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \frac{F_s^{n-2k+1} L_s^{2k-1}}{(\sqrt{5})^{2k-1}} \\ = \left(\frac{2}{\sqrt{5}}\right)^n \sum_{k=0}^n (-1)^{n-k} \binom{m+1}{k} \frac{L_s^k \beta^{s(n-k)}}{2^k}. \end{aligned} \quad (38)$$

Writing $2n$ for n in (38) (after multiplying through by β^t) and comparing the coefficients of $\sqrt{5}$ produces (34) and (35). Writing $2n-1$ for n gives (36) and (37). □

Corollary 10. *If m and n are non-negative integers and s is an integer, then*

$$\begin{aligned} \sum_{k=0}^n \binom{m-2n+2k}{2k} 5^{n-k} F_s^{2(n-k)} L_s^{2k} &= \sum_{k=0}^{2n} (-1)^k \binom{m+1}{k} 2^{2n-k-1} L_s^k L_{s(2n-k)}, \\ \sum_{k=1}^n \binom{m-2n+2k-1}{2k-1} 5^{n-k} F_s^{2(n-k)+1} L_s^{2k-1} &= \sum_{k=0}^{2n} (-1)^{k+1} \binom{m+1}{k} 2^{2n-k-1} L_s^k F_{s(2n-k)}, \\ \sum_{k=1}^n \binom{m-2n+2k}{2k-1} 5^{n-k} F_s^{2(n-k)} L_s^{2k-1} &= \sum_{k=0}^{2n-1} (-1)^{k+1} \binom{m+1}{k} 2^{2n-k-2} L_s^k L_{s(2n-k-1)}, \\ \sum_{k=0}^{n-1} \binom{m-2n+2k+1}{2k} 5^{n-k-1} F_s^{2(n-k)-1} L_s^{2k} &= \sum_{k=0}^{2n-1} (-1)^k \binom{m+1}{k} 2^{2n-k-2} L_s^k F_{s(2n-k-1)}. \end{aligned}$$

Corollary 11. *If n is a non-negative integer and s is any integer, then*

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 2^{2n-1-k} L_s^k L_{s(2n-k)} &= 5^n F_s^{2n}, \\ \sum_{k=0}^{2n} (-1)^{k+1} \binom{2n+1}{k} 2^{2n-k-1} L_s^k F_{s(2n-k)} &= \sum_{k=1}^n 5^{n-k} F_s^{2(n-k)+1} L_s^{2k-1}, \\ \sum_{k=0}^{2n-1} (-1)^{k+1} \binom{2n}{k} 2^{2n-k-2} L_s^k L_{s(2n-k-1)} &= \sum_{k=1}^n 5^{n-k} F_s^{2(n-k)} L_s^{2k-1}, \\ \sum_{k=0}^{2n-1} (-1)^k \binom{2n}{k} 2^{2n-k-2} L_s^k F_{s(2n-k-1)} &= \sum_{k=0}^{n-1} 5^{n-k-1} F_s^{2(n-k)-1} L_s^{2k}. \end{aligned}$$

3 Relations from a recent identity by Alzer

In 2015 Alzer [8], building on the work of Aharonov and Elias [7], studied the polynomial

$$P_n(x) = (1-x)^{n+1} \sum_{k=0}^n \binom{n+k}{k} x^k, \quad x \in \mathbb{C}. \quad (39)$$

Among other things he showed that

$$P_n(x) = 1 - x + (1-2x) \sum_{k=0}^{n-1} \binom{2k+1}{k} x^{k+1} (1-x)^{k+1}. \quad (40)$$

Such a polynomial identity immediately offers many appealing Fibonacci and Lucas sum relations as can be seen from the next series of theorems.

Theorem 12. For each non-negative integer n we have the relations

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{n+1-k} = 1 - 2 \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k}, \quad (41)$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{n+1-k} = 1. \quad (42)$$

Proof. Set $x = \alpha$ and $x = \beta$ in (39) and (40), respectively, and combine according to the Binet formulas (3). \square

Comparing (22) with (41), and (21) with (42), we find

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{k} F_{2(n-k)+1} = 1 - 2 \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k}$$

and

$$\sum_{k=0}^n (-1)^k \binom{2n+1}{k} L_{2(n-k)+1} = 1.$$

Theorem 13. For each non-negative integer n we have the relations

$$\sum_{k=0}^n \binom{n+k}{k} F_{n+2k+1} = (-1)^n - \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2k+1}{k} F_{3(k+2)},$$

$$\sum_{k=0}^n \binom{n+k}{k} L_{n+2k+1} = (-1)^n - \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2k+1}{k} L_{3(k+2)}.$$

Proof. Set $x = \alpha^2$ and $x = \beta^2$ in (39) and (40), respectively, and combine according to the Binet formulas (3). \square

The next theorem generalizes Theorem 12.

Theorem 14. For non-negative integers n and m we have the relations

$$\sum_{k=0}^n (-1)^{mk} \binom{n+k}{k} \frac{F_{m(n+1-k)}}{L_m^k} = F_m L_m^n \left(1 + 2 \sum_{k=0}^{n-1} \frac{(-1)^{m(k+1)}}{L_m^{2(k+1)}} \binom{2k+1}{k} \right),$$

$$\sum_{k=0}^n (-1)^{mk} \binom{n+k}{k} \frac{L_{m(n+1-k)}}{L_m^k} = L_m^{n+1}. \quad (43)$$

Proof. Set $x = \alpha^m/L_m$ and $x = \beta^m/L_m$ in (39) and (40), respectively, and combine according to the Binet formulas. \square

When $m = 1$ then Theorem 14 reduces to Theorem 12. As additional examples we state the next relations:

$$\sum_{k=0}^n \binom{n+k}{k} 2^{-k} = 2^n,$$

which also appears in Alzer's paper [8] as Eq. (1.4), and

$$\begin{aligned} \sum_{k=0}^n \binom{n+k}{k} \frac{F_{2(n+1-k)}}{3^k} &= 3^n \left(1 + 2 \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{1}{9^{k+1}} \right), \\ \sum_{k=0}^n \binom{n+k}{k} \frac{L_{2(n+1-k)}}{3^k} &= 3^{n+1}. \end{aligned}$$

Theorem 15. *For non-negative integer n and any integers m and t , we have the relations*

$$\begin{aligned} q^{mn} \sum_{k=0}^n \binom{n+k}{k} \frac{W_{mk+t}}{V_m^k} \\ = V_m^n W_{mn+t} - V_m^n U_m (W_{m(n+1)+t+1} - qW_{m(n+1)+t-1}) \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{q^{mk}}{V_m^{2(k+1)}}. \end{aligned}$$

Proof. Set $x = \tau^m/V_m$ and $x = \sigma^m/V_m$ in (39) and (40), respectively, and combine according to the Binet formulas, while making use of Lemma 4. \square

In particular,

$$\begin{aligned} q^{mn} \sum_{k=0}^n \binom{n+k}{k} \frac{U_{mk+t}}{V_m^k} \\ = V_m^n U_{mn+t} - V_m^n U_m (U_{m(n+1)+t+1} - qU_{m(n+1)+t-1}) \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{q^{mk}}{V_m^{2(k+1)}}, \\ q^{mn} \sum_{k=0}^n \binom{n+k}{k} \frac{V_{mk+t}}{V_m^k} \\ = V_m^n V_{mn+t} - V_m^n U_m (V_{m(n+1)+t+1} - qV_{m(n+1)+t-1}) \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{q^{mk}}{V_m^{2(k+1)}}; \end{aligned}$$

with the special cases

$$\begin{aligned} \sum_{k=0}^n \binom{n+k}{k} \frac{F_{mk+t}}{L_m^k} &= (-1)^{mn} L_m^n F_{mk+t} - L_m^n F_m L_{m(n+1)+t} \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{(-1)^{m(n-k)}}{L_m^{2(k+1)}}, \\ \sum_{k=0}^n \binom{n+k}{k} \frac{L_{mk+t}}{L_m^k} &= (-1)^{mn} L_m^n L_{mk+t} - 5L_m^n F_m F_{m(n+1)+t} \sum_{k=0}^{n-1} \binom{2k+1}{k} \frac{(-1)^{m(n-k)}}{L_m^{2(k+1)}}. \end{aligned}$$

Theorem 16. For each non-negative integer n we have the relations

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} \frac{F_{2(n+1)-k}}{2^k} = 2^n - \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k} 2^{n-2k-1} F_{k+2},$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} \frac{L_{2(n+1)-k}}{2^k} = 3 \cdot 2^n - \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k} 2^{n-2k-1} L_{k+2}.$$

Proof. Set $x = \alpha/2$ and $x = \beta/2$ in (39) and (40), respectively, and combine according to the Binet formulas. \square

Theorem 17. For each non-negative integer n we have the relations

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{2(n+1)+k} = 1 - \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k} F_{3(k+2)},$$

$$\sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{2(n+1)+k} = 3 - \sum_{k=0}^{n-1} (-1)^k \binom{2k+1}{k} L_{3(k+2)}.$$

Proof. Set $x = 1/\alpha$ and $x = 1/\beta$ in (39) and (40), respectively, and combine according to the Binet formulas. \square

Remark. Combining Theorem 13 with Theorem 17 gives the relations

$$\sum_{k=0}^n \binom{n+k}{k} F_{n+1+2k} = \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{k} F_{2(n+1)+k},$$

$$\sum_{k=0}^n \binom{n+k}{k} L_{n+1+2k} = 2(-1)^n + \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{k} L_{2(n+1)+k}.$$

Theorem 18. For each non-negative integer n we have the relations

$$3^n \sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{2(n+1+2k)} = 1 - \sum_{k=0}^{n-1} (-3)^k \binom{2k+1}{k} (3F_{6k+8} + F_{6k+10}),$$

$$3^n \sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{2(n+1+2k)} = 3 - \sum_{k=0}^{n-1} (-3)^k \binom{2k+1}{k} (3L_{6k+8} + L_{6k+10}).$$

Proof. Set $x = -\alpha^4$ and $x = -\beta^4$ in (39) and (40), respectively, and combine according to the Binet formulas. \square

The last Theorem in this set involves mixed identities.

Theorem 19. For each non-negative integer n we have the relations

$$2^n \sum_{k=0}^n (-1)^k \binom{n+k}{k} F_{2(n+1)+3k} = 1 - \sum_{k=0}^{n-1} (-2)^k \binom{2k+1}{k} L_{5k+8},$$

$$2^n \sum_{k=0}^n (-1)^k \binom{n+k}{k} L_{2(n+1)+3k} = 3 - 5 \sum_{k=0}^{n-1} (-2)^k \binom{2k+1}{k} F_{5k+8}.$$

Proof. Set $x = -\alpha^3$ and $x = -\beta^3$ in (39) and (40), respectively, and combine according to the Binet formulas. \square

As a final remark in this section we note that some of the identities presented in this section follow also from the following lemma.

Lemma 7 ([29, Identities 6.22, 6.23]). *If x is a complex variable and m, n are non-negative integers, then*

$$\sum_{k=0}^n \binom{n+k}{k} ((1-x)^{n+1} x^k + x^{n+1} (1-x)^k) = 1, \quad x \neq 0, \quad (44)$$

$$\sum_{k=0}^n \binom{n+k}{k} ((1-x)^{n+1} + x^{n+1-k} (1-x)^k) = x^{n+1}, \quad x \neq 0, x \neq 1. \quad (45)$$

For instance, identity (42) is an immediate consequence of (44) at $x = \alpha$. Also, (43) follows easily from (44).

4 Relations involving two central binomial coefficients

Lemma 8. *Let x be a complex variable. Then*

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} x^{2k} = \sum_{k=0}^n \binom{n}{k}^2 (1+x)^{2k} (1-x)^{2(n-k)}. \quad (46)$$

Proof. From Riordan's book [30] it is known that for the polynomial

$$A_n(t) = \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} t^k$$

we have the relation

$$A_n((2t-1)^2) = 4^n \sum_{k=0}^n \binom{n}{k}^2 t^{2k} (1-t)^{2(n-k)}.$$

Set $x = 2t - 1$ and simplify. \square

Theorem 20. For each integer r and each non-negative integer n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{2k+r} &= \sum_{k=0}^n \binom{n}{k}^2 F_{6k-2n+r}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{2k+r} &= \sum_{k=0}^n \binom{n}{k}^2 L_{6k-2n+r}.\end{aligned}$$

Proof. Set $x = \alpha$ and $x = \beta$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas (3). \square

Theorem 21. For each integer r and each non-negative integer n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{4k+r} &= F_{2n+r} \sum_{k=0}^n \binom{n}{k}^2 5^k, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{4k+r} &= L_{2n+r} \sum_{k=0}^n \binom{n}{k}^2 5^k.\end{aligned}$$

Proof. Set $x = \alpha^2$ and $x = \beta^2$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

We note the following particular results:

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{2(2k-n)} &= 0, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{2(2k-n)} &= 2 \sum_{k=0}^n \binom{n}{k}^2 5^k.\end{aligned}$$

Theorem 22. For each integer r and each non-negative integer n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 4^{n-k} F_{2k+r} &= \sum_{k=0}^n \binom{n}{k}^2 5^k F_{6k-4n+r}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 4^{n-k} L_{2k+r} &= \sum_{k=0}^n \binom{n}{k}^2 5^k L_{6k-4n+r}.\end{aligned}$$

Proof. Set $x = \alpha/2$ and $x = \beta/2$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 23. For each integer r and each non-negative integer n we have the relations

$$\begin{aligned}F_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^k 4^{n-k} &= \sum_{k=0}^n \binom{n}{k}^2 F_{6(2k-n)+r}, \\ L_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^k 4^{n-k} &= \sum_{k=0}^n \binom{n}{k}^2 L_{6(2k-n)+r}.\end{aligned}$$

Proof. Set $x = \sqrt{5}/2$ and $x = -\sqrt{5}/2$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 24. *For each integer r and each non-negative integer n we have the relations*

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 F_{2(n+k)+r}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 L_{2(n+k)+r}.\end{aligned}$$

Proof. Set $x = \alpha^3$ and $x = \beta^3$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 25. *For each integer r and each non-negative integer n we have the relations*

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{8k+r} &= F_{4n+r} \sum_{k=0}^n \binom{n}{k}^2 9^k 5^{n-k}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{8k+r} &= L_{4n+r} \sum_{k=0}^n \binom{n}{k}^2 9^k 5^{n-k}.\end{aligned}$$

Proof. Set $x = \alpha^4$ and $x = \beta^4$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

In particular,

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{4(2k-n)} = 0$$

and

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{4(2k-n)} = 2 \sum_{k=0}^n \binom{n}{k}^2 9^k 5^{n-k}.$$

We proceed with some identities involving an additional parameter.

Theorem 26. *If n is a non-negative integer and r, s are any integers, then*

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} q^{2s(n-k)} W_{4ks+r} = W_{2ns+r} \sum_{k=0}^n \binom{n}{k}^2 \Delta^{2k} V_s^{2(n-k)} U_s^{2k}.$$

Proof. Set $x = \sigma^s/\tau^s$ and $x = \tau^s/\sigma^s$, in turn, in Lemma 8, multiply through by τ^r and σ^r , respectively, and combine according to the Binet formulas. \square

In particular,

$$\begin{aligned} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} q^{2s(n-k)} U_{4ks+r} &= U_{2ns+r} \sum_{k=0}^n \binom{n}{k}^2 \Delta^{2k} V_s^{2(n-k)} U_s^{2k}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} q^{2s(n-k)} V_{4ks+r} &= V_{2ns+r} \sum_{k=0}^n \binom{n}{k}^2 \Delta^{2k} V_s^{2(n-k)} U_s^{2k}, \end{aligned}$$

with the special cases

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} F_{4sk+r} = F_{2ns+r} \sum_{k=0}^n \binom{n}{k}^2 (5F_s^2)^{n-k} L_s^{2k}, \quad (47)$$

$$\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} L_{4sk+r} = L_{2ns+r} \sum_{k=0}^n \binom{n}{k}^2 (5F_s^2)^{n-k} L_s^{2k}. \quad (48)$$

Remark. Note that Theorems 21 and 25 are particular cases of (47) and (48) at $s = 1$ and $s = 2$, respectively.

Theorem 27. For integers r and $s \geq 1$, and each non-negative integer n we have the relations

$$\begin{aligned} F_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} (5F_s^2)^k L_s^{2(n-k)} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 F_{2s(2k-n)+r}, \\ L_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} (5F_s^2)^k L_s^{2(n-k)} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 L_{2s(2k-n)+r}. \end{aligned}$$

Proof. Set $x = \sqrt{5}F_s/L_s$ and $x = -\sqrt{5}F_s/L_s$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 28. For integers r and $s \geq 1$, and each non-negative integer n we have the relations

$$\begin{aligned} F_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} (5F_s^2)^{n-k} L_s^{2k} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 F_{2s(2k-n)+r}, \\ L_r \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} (5F_s^2)^{n-k} L_s^{2k} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 L_{2s(2k-n)+r}. \end{aligned}$$

Proof. Set $x = L_s/(\sqrt{5}F_s)$ and $x = -L_s/(\sqrt{5}F_s)$ in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 29. For each integer r and each non-negative integer n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^{n-k} F_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 4^k F_{2k+r}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^{n-k} L_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 4^k L_{2k+r}.\end{aligned}$$

Proof. Set $x = \alpha^3/\sqrt{5}$ and $x = \beta^3/\sqrt{5}$, in turn, in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas, using also the fact that $\sqrt{5} - \alpha^3 = -2$ and $\sqrt{5} + \alpha^3 = 4\alpha$. \square

Theorem 30. For each integer r , and each non-negative integer n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 9^{n-k} F_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 5^k F_{4k-2n+r}, \\ \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 9^{n-k} L_{6k+r} &= 4^n \sum_{k=0}^n \binom{n}{k}^2 5^k L_{4k-2n+r}.\end{aligned}$$

Proof. Set $x = \alpha^3/3$ and $x = \beta^3/3$, in turn, in Lemma 8, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas, using also the fact that $3 - \alpha^3 = 2\beta$ and $3 + \alpha^3 = 2\sqrt{5}\alpha$. \square

Theorem 31. If n is a non-negative integer and r, s are any integers, then

$$\sum_{k=0}^n \binom{n}{k}^2 q^{2s(n-k)} W_{4sk+r} = \frac{W_{2sn+r}}{4^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \Delta^{2k} V_s^{2(n-k)} U_s^{2k}.$$

Proof. Set $x = \Delta U_s/V_s$ in Lemma 8 and multiply through by σ^r . Repeat for $x = -\Delta U_s/V_s$ and multiply through by τ^r . Now combine the resulting equations using the Binet formula. \square

In particular,

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 q^{2s(n-k)} U_{4sk+r} &= \frac{U_{2sn+r}}{4^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \Delta^{2k} V_s^{2(n-k)} U_s^{2k}, \\ \sum_{k=0}^n \binom{n}{k}^2 q^{2s(n-k)} V_{4sk+r} &= \frac{V_{2sn+r}}{4^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} \Delta^{2k} V_s^{2(n-k)} U_s^{2k},\end{aligned}$$

with the special cases

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 F_{4sk+r} &= \frac{F_{2sn+r}}{4^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^k L_s^{2(n-k)} F_s^{2k}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_{4sk+r} &= \frac{L_{2sn+r}}{4^n} \sum_{k=0}^n \binom{2k}{k} \binom{2(n-k)}{n-k} 5^k L_s^{2(n-k)} F_s^{2k}.\end{aligned}$$

5 Another class of identities with squared binomial coefficients

Lemma 9 ([28]). *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^n \binom{n}{k}^2 x^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (x-1)^{n-k}. \quad (49)$$

Theorem 32. *Let r, s and m be arbitrary integers with $r \neq 0$. Then for each non-negative integer n we have the relations*

$$\begin{aligned} \sum_{k=0}^n (-1)^{k(s+1)} \binom{n}{k}^2 \left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+m} \\ = \sum_{k=0}^n (-1)^{(s+1)(n-k)} \binom{n}{k} \binom{n+k}{k} \left(\frac{F_{r+s}}{F_r}\right)^{n-k} F_{s(n-k)+m}, \end{aligned} \quad (50)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^{k(s+1)} \binom{n}{k}^2 \left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+m} \\ = \sum_{k=0}^n (-1)^{(s+1)(n-k)} \binom{n}{k} \binom{n+k}{k} \left(\frac{F_{r+s}}{F_r}\right)^{n-k} L_{s(n-k)+m}. \end{aligned} \quad (51)$$

Proof. Set $x = -F_s\beta^r/(F_r\alpha^s)$ and $x = -F_s\alpha^r/(F_r\beta^s)$, respectively, in Lemma 9, and use Lemma 3. Multiply through by α^m and β^m , respectively, and combine according to the Binet formulas. \square

Corollary 33. *For each integer m and each non-negative integer n we have*

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k}^2 F_{k+m} &= (-1)^{m+1} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{n-k-m}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_{k+m} &= (-1)^m \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{n-k-m}. \end{aligned}$$

Proof. Set $r = 2$ and $s = -1$ in Theorem 32. \square

The case $m = 0$ in Corollary 33 was proposed by Carlitz as a problem in the Fibonacci Quarterly [10] (with a typo).

Corollary 34. *For each integer m and each non-negative integer n we have*

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k}^2 F_{2k+m} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} F_{n-k+m}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_{2k+m} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} L_{n-k+m}. \end{aligned}$$

Proof. Set $r = s = 1$ in Theorem 32. □

The case $m = 0$ in Corollary 34 was proposed by Carlitz as another problem in the Fibonacci Quarterly [11].

Corollary 35. *For each integer m and each non-negative integer n we have*

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 F_{3k+m} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} 2^{n-k} F_{n-k+m}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_{3k+m} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} 2^{n-k} L_{n-k+m}.\end{aligned}$$

Proof. Set $r = 2$ and $s = 1$ in Theorem 32. □

Corollary 36. *For each integer m and each non-negative integer n we have*

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{n}{k}^2 F_{k+m} &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{2(n-k)+m}, \\ \sum_{k=0}^n (-1)^k \binom{n}{k}^2 L_{k+m} &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{2(n-k)+m}.\end{aligned}$$

Proof. Set $r = -1$ and $s = 2$ in Theorem 32. □

Corollary 37. *For each integer r and each non-negative integer n we have*

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{n}{k}^2 F_{3k+m} &= \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{2(n-k)+m}, \\ \sum_{k=0}^n (-1)^k \binom{n}{k}^2 L_{3k+m} &= \sum_{k=0}^n (-2)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{2(n-k)+m}.\end{aligned}$$

Proof. Set $r = 1$ and $s = 2$ in Theorem 32. □

Corollary 38. *For each integer r and each non-negative integer n we have*

$$\begin{aligned}\sum_{k=0}^n (-1)^{(r+1)k} \binom{n}{k}^2 F_{2rk+m} &= \sum_{k=0}^n (-1)^{(r+1)(n-k)} \binom{n}{k} \binom{n+k}{k} L_r^{n-k} F_{r(n-k)+m}, \\ \sum_{k=0}^n (-1)^{(r+1)k} \binom{n}{k}^2 L_{2rk+m} &= \sum_{k=0}^n (-1)^{(r+1)(n-k)} \binom{n}{k} \binom{n+k}{k} L_r^{n-k} L_{r(n-k)+m}.\end{aligned}$$

Proof. Set $s = r$ in Theorem 32. □

Corollary 39. For each integer r and each non-negative integer n we have

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{n}{k}^2 L_r^k F_{3rk+m} &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (L_{2r} + (-1)^r)^{n-k} F_{2r(n-k)+m}, \\ \sum_{k=0}^n (-1)^k \binom{n}{k}^2 L_r^k L_{3rk+m} &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (L_{2r} + (-1)^r)^{n-k} L_{2r(n-k)+m}.\end{aligned}$$

Proof. Set $s = 2r$ in Theorem 32 and make use of $F_{3r}/F_r = L_{2r} + (-1)^r$. \square

Theorem 40. For each non-negative integers r , m and n we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 L_m^{n-k} F_{mk+r} &= (-1)^{r-1} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_m^k F_{m(n-k)-r}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_m^{n-k} L_{mk+r} &= (-1)^r \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_m^k L_{m(n-k)-r}.\end{aligned}$$

Proof. Set $x = \alpha^m/L_m$ and $x = \beta^m/L_m$ in Lemma 9, multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

When $m = 1$ then Theorem 40 reduces to Corollary 33. When $m = 2$ then

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 3^{n-k} F_{2k+r} &= (-1)^{n-r-1} \sum_{k=0}^n (-3)^k \binom{n}{k} \binom{n+k}{k} F_{2(n-k)-r}, \\ \sum_{k=0}^n \binom{n}{k}^2 3^{n-k} L_{2k+r} &= (-1)^{n-r} \sum_{k=0}^n (-3)^k \binom{n}{k} \binom{n+k}{k} L_{2(n-k)-r}.\end{aligned}$$

Theorem 41. For each non-negative integer n , any odd integer m and any integer r we have the relations

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k}^2 F_{2mk+r} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} L_m^{n-k} F_{m(n-k)+r}, \\ \sum_{k=0}^n \binom{n}{k}^2 L_{2mk+r} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} L_m^{n-k} L_{m(n-k)+r}.\end{aligned}$$

Proof. Set $x = \alpha^{2m}$ and $x = \beta^{2m}$, m odd, in Lemma 9, and use the facts that $\alpha^{2m} - 1 = \alpha^m L_m$ and $\beta^{2m} - 1 = \beta^m L_m$. Multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

When $m = 1$ then Theorem 41 reduces to Corollary 34.

We conclude this section with a sort of inverse relations compared to those from Theorem 32.

Theorem 42. For each non-negative integer n and any integers m, r and s , we have the relations

$$\begin{aligned} \sum_{k=0}^n (-1)^{sk} \binom{n}{k}^2 \left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+m} &= \sum_{k=0}^n (-1)^{s(n-k)} \binom{n}{k} \binom{n+k}{k} \left(\frac{F_s}{F_r}\right)^{n-k} F_{(r+s)(n-k)+m}, \\ \sum_{k=0}^n (-1)^{sk} \binom{n}{k}^2 \left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+m} &= \sum_{k=0}^n (-1)^{s(n-k)} \binom{n}{k} \binom{n+k}{k} \left(\frac{F_s}{F_r}\right)^{n-k} L_{(r+s)(n-k)+m}. \end{aligned}$$

Proof. Set $x = F_{r+s}/(\alpha^s F_r)$ and $x = F_{r+s}/(\beta^s F_r)$, respectively, in Lemma 9, and use Lemma 3. Multiply through by α^m and β^m , respectively, and combine according to the Binet formulas. \square

6 More identities with two binomial coefficients

Lemma 10 ([18, Identity (3.17)]). If n is a non-negative integer, m is any real number and x is any complex variable, then

$$\sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} (x-1)^k = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} x^k. \quad (52)$$

In particular,

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (x-1)^k = \sum_{k=0}^n \binom{n}{k}^2 x^k \quad (53)$$

$$\sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} (x-1)^k = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} x^k. \quad (54)$$

Proof. The first particular case is obvious. The second follows upon setting $m = n - 1/2$ in (52) and using

$$\binom{n-1/2}{k} = \frac{\binom{2n}{n} \binom{n}{k}}{2^{2k} \binom{2(n-k)}{n-k}}$$

with $0 \leq k \leq n$ (see [18, Identity (Z.45)]). \square

Remark. Comparing (49) with (53) we immediately get an “identity” of the form

$$\sum_{k=0}^n \binom{n}{k} \binom{2n-k}{n} (x-1)^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (x-1)^{n-k}.$$

Such an identity does not contain any new information as the identities can be trivially transformed into each other by reindexing $\sum_{k=0}^n a_k = \sum_{k=0}^n a_{n-k}$. This shows that the binomial sum relations from the previous section are actually special instances of those derived now.

Theorem 43. Let r, s and p be arbitrary integers with $r \neq 0$, and let m is any real number. Then for all non-negative integer n we have the relations

$$\begin{aligned} \sum_{k=0}^n (-1)^{(s+1)k} \binom{n}{k} \binom{m+n-k}{n} \left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+p} \\ = \sum_{k=0}^n (-1)^{(s+1)k} \binom{n}{k} \binom{m}{k} \left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+p}, \end{aligned} \quad (55)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^{(s+1)k} \binom{n}{k} \binom{m+n-k}{n} \left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+p} \\ = \sum_{k=0}^n (-1)^{(s+1)k} \binom{n}{k} \binom{m}{k} \left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+p}. \end{aligned} \quad (56)$$

In particular,

$$\begin{aligned} \sum_{k=0}^n (-1)^{(s+1)k} \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+p} \\ = \binom{2n}{n} \sum_{k=0}^n (-1)^{(s+1)k} \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} \left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+p}, \\ \sum_{k=0}^n (-1)^{(s+1)k} \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+p} \\ = \binom{2n}{n} \sum_{k=0}^n (-1)^{(s+1)k} \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} \left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+p}. \end{aligned}$$

Proof. Set $x = -F_s \beta^r / (F_r \alpha^s)$ and $x = -F_s \alpha^r / (F_r \beta^s)$, respectively, in (52), and use Lemma 3. Multiply through by α^p and β^p , respectively, and combine according to the Binet formulas. \square

Corollary 44. If n is a non-negative integer, m is any real number and p is any integer, then

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} F_{p-k} &= \sum_{k=0}^n \binom{n}{k} \binom{m}{k} F_{p+k}, \\ \sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} L_{p-k} &= \sum_{k=0}^n \binom{n}{k} \binom{m}{k} L_{p+k}. \end{aligned}$$

In particular,

$$\sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} F_{p-k} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} F_{p+k},$$

$$\sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_{p-k} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} L_{p+k}.$$

Corollary 45. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} F_{k+p} &= \sum_{k=0}^n \binom{n}{k} \binom{m}{k} F_{2k+p}, \\ \sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} L_{k+p} &= \sum_{k=0}^n \binom{n}{k} \binom{m}{k} L_{2k+p}. \end{aligned}$$

In particular,

$$\begin{aligned} \sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} F_{k+p} &= \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} F_{2k+p}, \\ \sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_{k+p} &= \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} L_{2k+p}. \end{aligned}$$

Corollary 46. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\sum_{k=0}^n (\pm 2)^k \binom{n}{k} \binom{m+n-k}{n} F_{\left(\frac{3\mp 1}{2}\right)k+p} = \sum_{k=0}^n (\pm 1)^k \binom{n}{k} \binom{m}{k} F_{3k+p}, \quad (57)$$

$$\sum_{k=0}^n (\pm 2)^k \binom{n}{k} \binom{m+n-k}{n} L_{\left(\frac{3\mp 1}{2}\right)k+p} = \sum_{k=0}^n (\pm 1)^k \binom{n}{k} \binom{m}{k} L_{3k+p}. \quad (58)$$

In particular,

$$\begin{aligned} \sum_{k=0}^n (\pm 2)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} F_{\left(\frac{3\mp 1}{2}\right)k+p} &= \binom{2n}{n} \sum_{k=0}^n (\pm 1)^k \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} F_{3k+p}, \\ \sum_{k=0}^n (\pm 2)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_{\left(\frac{3\mp 1}{2}\right)k+p} &= \binom{2n}{n} \sum_{k=0}^n (\pm 1)^k \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} L_{3k+p}. \end{aligned}$$

Corollary 47. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\begin{aligned} \sum_{k=0}^n (-3)^k \binom{n}{k} \binom{m+n-k}{n} F_{2k+p} &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m}{k} F_{4k+p}, \\ \sum_{k=0}^n (-3)^k \binom{n}{k} \binom{m+n-k}{n} L_{2k+p} &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m}{k} L_{4k+p}. \end{aligned}$$

In particular,

$$\sum_{k=0}^n (-3)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} F_{2k+p} = \binom{2n}{n} \sum_{k=0}^n (-1)^k \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} F_{4k+p},$$

$$\sum_{k=0}^n (-3)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_{2k+p} = \binom{2n}{n} \sum_{k=0}^n (-1)^k \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} L_{4k+p}.$$

Theorem 48. If n is a non-negative integer, m is any real number and s, r are any integers, then

$$\sum_{k=0}^n (-1)^{k(s-1)} \binom{n}{k} \binom{m+n-k}{n} \frac{F_{r-sk}}{L_s^k} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \frac{F_{sk+r}}{L_s^k},$$

$$\sum_{k=0}^n (-1)^{k(s-1)} \binom{n}{k} \binom{m+n-k}{n} \frac{L_{r-sk}}{L_s^k} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \frac{L_{sk+r}}{L_s^k}.$$

In particular,

$$\sum_{k=0}^n (-1)^{k(s-1)} \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \frac{F_{r-sk}}{L_s^k} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} \frac{F_{sk+r}}{L_s^k},$$

$$\sum_{k=0}^n (-1)^{k(s-1)} \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \frac{L_{r-sk}}{L_s^k} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} \frac{L_{sk+r}}{L_s^k}.$$

Proof. Set $x = \alpha^s/L_s$ and $x = \beta^s/L_s$ in (52), multiply through by α^r and β^r , respectively, and combine according to the Binet formulas. \square

Theorem 49. If n is a non-negative integer, m is any real number, r is any integer and s is an odd integer, then

$$\sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} L_s^k F_{sk+r} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} F_{2sk+r},$$

$$\sum_{k=0}^n \binom{n}{k} \binom{m+n-k}{n} L_s^k L_{sk+r} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} L_{2sk+r}.$$

In particular,

$$\sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_s^k F_{sk+r} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} F_{2sk+r},$$

$$\sum_{k=0}^n \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} L_s^k L_{sk+r} = \binom{2n}{n} \sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}} 2^{2(n-k)} L_{2sk+r}.$$

Proof. Set $x = \alpha^{2s}$ in (52) and use the fact that $\alpha^{2s} - 1 = \alpha^s L_s$ if s is an odd integer. \square

Theorem 50. *Let n be a non-negative integer and let m be a real number. Let r, s, p and t be any integers. Then*

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+n-k}{n} \frac{L_{r+s}^k}{L_r^k} L_{s(p-k)+t} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{m}{2k} \frac{5^k F_s^{2k}}{L_r^{2k}} L_{sp-2k(r+s)+t} \\ &+ (-1)^r \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} \binom{m}{2k-1} \frac{5^k F_s^{2k-1}}{L_r^{2k-1}} F_{sp-(2k-1)(r+s)+t}, \end{aligned} \quad (59)$$

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+n-k}{n} \frac{L_{r+s}^k}{L_r^k} F_{s(p-k)+t} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{m}{2k} \frac{5^k F_s^{2k}}{L_r^{2k}} F_{sp-2k(r+s)+t} \\ &+ (-1)^r \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} \binom{m}{2k-1} \frac{5^{k-1} F_s^{2k-1}}{L_r^{2k-1}} L_{sp-(2k-1)(r+s)+t}. \end{aligned} \quad (60)$$

Proof. Set $x = \sqrt{5}\beta^r F_s / (L_r \alpha^s)$ in (52) and use (12) to obtain

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+n-k}{n} L_r^{p-k} L_{r+s}^k \alpha^{s(p-k)+t} \\ = \sum_{k=0}^n (-1)^{kr} \binom{n}{k} \binom{m}{k} (\sqrt{5})^k L_r^{p-k} F_s^k \alpha^{sp-(r+s)k+t}, \end{aligned}$$

from which the results follow. \square

Corollary 51. *Let n be a non-negative integer; let m be a real number and let s and p be any integers. Then*

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+n-k}{n} \frac{2^{k-1} L_{sk}}{L_s^k} &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{m}{2k} \frac{5^k F_s^{2k}}{L_s^{2k}}, \\ \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+n-k}{n} \frac{2^{k-1} F_{sk}}{L_s^k} &= - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} \binom{m}{2k-1} \frac{5^{k-1} F_s^{2k-1}}{L_s^{2k-1}}. \end{aligned}$$

Proof. Set $t = -sp$ and $r = -s$ in (59) and (60). \square

Corollary 52. *Let n be a non-negative integer and let s and p be any integers. Then*

$$\begin{aligned} \sum_{k=0}^n (-2)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \frac{L_{sk}}{L_s^k} &= \binom{2n}{n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n}{2k}^2}{\binom{2(n-2k)}{n-2k}} \frac{5^k 2^{2(n-2k)+1} F_s^{2k}}{L_s^{2k}}, \\ \sum_{k=0}^n (-2)^k \frac{\binom{n}{k} \binom{2(2n-k)}{2n-k} \binom{2n-k}{n}}{\binom{2(n-k)}{n-k}} \frac{F_{sk}}{L_s^k} &= - \binom{2n}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\binom{n}{2k-1}^2}{\binom{2(n-2k+1)}{n-2k+1}} \frac{2^{2(n-2k+1)+1} 5^{k-1} F_s^{2k-1}}{L_s^{2k-1}}. \end{aligned}$$

7 Still other classes of identities with two binomial coefficients

Lemma 11 ([18, Identity (3.18)]). *If n is a non-negative integer, m is any real number and x, y are any complex variables, then*

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} (x-y)^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} x^{n-k} y^k. \quad (61)$$

Theorem 53. *If n is a non-negative integer, m is any real number and r, s and t are any integers, then*

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} q^{rk} U_r^{n-k} U_s^k W_{t-(r+s)k+sn} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} q^{rk} U_{r+s}^{n-k} U_s^k W_{t-rk}.$$

Proof. Set $x = U_{r+s}$ and $y = \tau^r U_s$ in (61) and multiply by σ^t to obtain

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} \sigma^{s(n-k)+t} \tau^{rk} U_r^{n-k} U_s^k = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \sigma^t \tau^{rk} U_{r+s}^{n-k} U_s^k. \quad (62)$$

Similarly obtain

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} \sigma^{rk} \tau^{s(n-k)+t} U_r^{n-k} U_s^k = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} \sigma^{rk} \tau^t U_{r+s}^{n-k} U_s^k. \quad (63)$$

Combine (62) and (63) using the Binet formula. □

In particular,

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} q^{rk} U_r^{n-k} U_s^k V_{t-(r+s)k+sn} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} q^{rk} U_{r+s}^{n-k} U_s^k V_{t-rk}$$

and

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{k} q^{rk} U_r^{n-k} U_s^k U_{t-(r+s)k+sn} = \sum_{k=0}^n \binom{n}{k} \binom{m}{k} q^{rk} U_{r+s}^{n-k} U_s^k U_{t-rk};$$

with the special cases

$$\begin{aligned} \sum_{k=0}^n (-1)^{rk} \binom{n}{k} \binom{m+k}{k} F_r^{n-k} F_s^k L_{t-(r+s)k+sn} &= \sum_{k=0}^n (-1)^{rk} \binom{n}{k} \binom{m}{k} F_{r+s}^{n-k} F_s^k L_{t-rk}, \\ \sum_{k=0}^n (-1)^{rk} \binom{n}{k} \binom{m+k}{k} F_r^{n-k} F_s^k F_{t-(r+s)k+sn} &= \sum_{k=0}^n (-1)^{rk} \binom{n}{k} \binom{m}{k} F_{r+s}^{n-k} F_s^k F_{t-rk}. \end{aligned}$$

Lemma 12 ([18, Identity (3.84)]). *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2n-2k} x^k = \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} (x-1)^k. \quad (64)$$

Theorem 54. *If n is a non-negative integer and r, s, t and m are any integers, then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} U_s^{m-k} U_{r+s}^k W_{t+r(m-k)} \\ &= \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} q^{sk} U_s^{m-k} U_r^k W_{t+r(m-k)-sk}. \end{aligned}$$

Proof. Set $x = U_{r+s}/(\tau^r U_s)$ in (64); use (10) and multiply through the resulting equation by τ^t to obtain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} U_s^{m-k} U_{r+s}^k \tau^{t+r(m-k)} \\ &= \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} U_s^{m-k} U_r^k \tau^{t+r(m-k)} \sigma^{sk}. \end{aligned} \quad (65)$$

Similarly obtain

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} U_s^{m-k} U_{r+s}^k \sigma^{t+r(m-k)} \\ &= \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} U_s^{m-k} U_r^k \sigma^{t+r(m-k)} \tau^{sk}. \end{aligned} \quad (66)$$

Combine (65) and (66) according to the Binet formula. \square

In particular,

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} U_s^{m-k} U_{r+s}^k U_{t+r(m-k)} \\ &= \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} q^{sk} U_s^{m-k} U_r^k U_{t+r(m-k)-sk} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} U_s^{m-k} U_{r+s}^k V_{t+r(m-k)} \\ &= \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} q^{sk} U_s^{m-k} U_r^k V_{t+r(m-k)-sk}; \end{aligned}$$

with the special cases

$$\begin{aligned}
& \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} F_s^{m-k} F_{r+s}^k F_{t+r(m-k)} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} q^{sk} F_s^{m-k} F_r^k F_{t+r(m-k)-sk}, \\
& \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} 2^{2(n-k)} F_s^{m-k} F_{r+s}^k L_{t+r(m-k)} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{2(n-k)}{n-k} \binom{2k}{k} q^{sk} F_s^{m-k} F_r^k L_{t+r(m-k)-sk}.
\end{aligned}$$

Lemma 13 ([16, 31]). *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1+x)^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k. \quad (67)$$

Theorem 55. *If n is a non-negative integer and s, t and m are any integers, then*

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} W_{2s(m-k)+t} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} q^{sk} W_{2s(m-k)+t}, \quad (68)$$

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{V_s^k W_{sk+t}}{q^{sk}} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{W_{2sk+t}}{q^{sk}}. \quad (69)$$

Proof. To prove (68), set $x = q^s/\tau^{2s}$ and $x = q^s/\sigma^s$, in turn, in (67) and use (4). Combine the resulting equations according to the Binet formula. For (69), use $x = \tau^{2s}/q^s$ and $x = \sigma^s/q^s$, in turn, in (67). \square

In particular,

$$\begin{aligned}
& \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} U_{2s(m-k)+t} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} q^{sk} U_{2s(m-k)+t}, \\
& \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{V_s^k U_{sk+t}}{q^{sk}} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{U_{2sk+t}}{q^{sk}},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} V_{2s(m-k)+t} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} q^{sk} V_{2s(m-k)+t}, \\
& \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \frac{V_s^k V_{sk+t}}{q^{sk}} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{V_{2sk+t}}{q^{sk}};
\end{aligned}$$

and the special cases

$$\begin{aligned}\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} F_{2s(m-k)+t} &= \sum_{k=0}^n (-1)^{sk} \binom{n}{k} \binom{n+k}{k} F_{2s(m-k)+t}, \\ \sum_{k=0}^n (-1)^{n+(s-1)k} \binom{n}{k} \binom{n+k}{k} L_s^k F_{sk+t} &= \sum_{k=0}^n (-1)^{sk} \binom{n}{k} \binom{n+k}{k} F_{2sk+t}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} L_{2s(m-k)+t} &= \sum_{k=0}^n (-1)^{sk} \binom{n}{k} \binom{n+k}{k} L_{2s(m-k)+t}, \\ \sum_{k=0}^n (-1)^{n+(s-1)k} \binom{n}{k} \binom{n+k}{k} L_s^k L_{sk+t} &= \sum_{k=0}^n (-1)^{sk} \binom{n}{k} \binom{n+k}{k} L_{2sk+t}.\end{aligned}$$

Next we present an obvious extension of (33) and some associated Fibonacci–Lucas sums.

Lemma 14. *Let x and y be complex variables. Let m and n be non-negative integers and let r be any integer. Then*

$$\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} x^{n-k-r} y^k = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} (x-y)^{n-k-r} y^k. \quad (70)$$

Theorem 56. *If m and n are non-negative integers and s, r and t are any integers, then*

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} q^{s(n-k-r)} W_{2sk+t} \\ = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} V_s^{n-k-r} W_{s(n+k-r)+t}.\end{aligned}$$

Proof. Choose $x = q^s$ and $y = \tau^{2s}$ in (70), use (4) and multiply through by τ^t to obtain

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} q^{s(n-k-r)} \tau^{2sk+t} \\ = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} \tau^{s(n+k-r)+t} V_s^{n-k-r}.\end{aligned} \quad (71)$$

Similarly obtain

$$\begin{aligned}\sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} q^{s(n-k-r)} \sigma^{2sk+t} \\ = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} \sigma^{s(n+k-r)+t} V_s^{n-k-r},\end{aligned} \quad (72)$$

The result follows from (71), (72) and the Binet formula. \square

In particular,

$$\begin{aligned}
& \sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} q^{s(n-k-r)} U_{2sk+t} \\
&= \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} V_s^{n-k-r} U_{s(n+k-r)+t}, \\
& \sum_{k=0}^n (-1)^k \binom{m-n+k}{k} \binom{n-k}{r} q^{s(n-k-r)} V_{2sk+t} \\
&= \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} V_s^{n-k-r} V_{s(n+k-r)+t};
\end{aligned}$$

with the special cases

$$\begin{aligned}
& \sum_{k=0}^n (-1)^{k(s+1)} \binom{m-n+k}{k} \binom{n-k}{r} L_{2sk+t} \\
&= \frac{(-1)^{s(n-r)}}{L_s^r} \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} L_s^{n-k} L_{s(n+k-r)+t}, \\
& \sum_{k=0}^n (-1)^{k(s+1)} \binom{m-n+k}{k} \binom{n-k}{r} F_{2sk+t} \\
&= \frac{(-1)^{s(n-r)}}{L_s^r} \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} L_s^{n-k} F_{s(n+k-r)+t}.
\end{aligned}$$

Theorem 57. *Let m, n, r, s and t be integers with $n \geq 0$. If n and r have the same parity then*

$$\begin{aligned}
& W_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r} U_s^{n-2k-r} V_s^{2k} \\
& - (W_{t+1} - qW_{t-1}) \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r} U_s^{n-2k-r+1} V_s^{2k-1} \\
& = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k W_{s(n-k-r)+t},
\end{aligned}$$

while if n and r have different parities, then

$$\begin{aligned}
& (W_{t+1} - qW_{t-1}) \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r-1} U_s^{n-2k-r} V_s^{2k} \\
& - W_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k+1-r} U_s^{n-2k-r+1} V_s^{2k-1} \\
& = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k W_{s(n-k-r)+t}.
\end{aligned}$$

Proof. Set $y = V_s$ and $x = \Delta U_s$ in (70), use Lemma 4 and the summation identity

$$\sum_{j=0}^n f_j = \sum_{j=0}^{\lfloor n/2 \rfloor} f_{2j} + \sum_{j=1}^{\lfloor n/2 \rfloor} f_{2j-1}.$$

□

In particular, if n and r have the same parity then

$$\begin{aligned}
& U_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} U_s^{n-2k-r} V_s^{2k} \Delta^{n-2k-r} \\
& - V_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r} U_s^{n-2k-r+1} V_s^{2k-1} \\
& = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k U_{s(n-k-r)+t}
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
& V_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r} U_s^{n-2k-r} V_s^{2k} \\
& - U_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+2} U_s^{n-2k-r+1} V_s^{2k-1} \\
& = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k V_{s(n-k-r)+t};
\end{aligned}$$

while if n and r have different parities, then

$$\begin{aligned}
& V_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r-1} U_s^{n-2k-r} V_s^{2k} \\
& \quad - U_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r+1} V_s^{2k-1} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k U_{s(n-k-r)+t},
\end{aligned}$$

and

$$\begin{aligned}
& U_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r} V_s^{2k} \\
& \quad - V_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r+1} V_s^{2k-1} \quad (74) \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k U_{s(n-k-r)+t};
\end{aligned}$$

with the special cases:

$$\begin{aligned}
& L_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& \quad - F_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r)/2-k+1} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} L_s^k L_{s(n-k-r)+t},
\end{aligned}$$

$$\begin{aligned}
& F_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& \quad - L_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} L_s^k F_{s(n-k-r)+t};
\end{aligned}$$

while if n and r have different parities then

$$\begin{aligned}
& F_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& \quad - L_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} L_s^k L_{s(n-k-r)+t}, \\
& L_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& \quad - F_t \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r-1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \quad = \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} L_s^k F_{s(n-k-r)+t}.
\end{aligned}$$

Note that in simplifying (73)–(74), we used (32).

Corollary 58. *If m, n, r and s are integers, then*

$$\begin{aligned}
& \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r-1} L_s^k L_{s(n-k-r)} \\
& = \begin{cases} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k}, & \text{if } n-r \text{ is even;} \\ - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1}, & \text{otherwise;} \end{cases} \\
& \sum_{k=0}^n (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r-1} L_s^k F_{s(n-k-r)} \\
& = \begin{cases} - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r)/2-k} F_s^{n-2k-r+1} L_s^{2k-1}, & \text{if } n-r \text{ is even;} \\ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k}, & \text{otherwise.} \end{cases}
\end{aligned}$$

8 Identities with three binomial coefficients

Concerning identities with three binomial coefficients some classical Fibonacci (Lucas) examples exist. For instance, Carlitz [13] presented the identities

$$\sum_{k=0}^n \binom{n}{k}^3 F_k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} F_{2n-3k}$$

and

$$\sum_{k=0}^n \binom{n}{k}^3 L_k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} L_{2n-3k}.$$

In addition, Zeitlin [35, 37] derived

$$\begin{aligned} \sum_{k=0}^{2n} \binom{2n}{k}^3 F_{2k} &= F_{2n} \sum_{k=0}^n \frac{(2n+k)!}{(k!)^3 (2n-2k)!} 5^{n-k}, \\ \sum_{k=0}^{2n} \binom{2n}{k}^3 L_{2k} &= L_{2n} \sum_{k=0}^n \frac{(2n+k)!}{(k!)^3 (2n-2k)!} 5^{n-k}. \end{aligned}$$

In his solution to Carlitz' proposal from above Zeitlin [36] proved *mutatis mutandis* the identity

$$\sum_{k=0}^n \binom{n}{k}^3 (-q)^{n-k} p^k W_k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} p^k (-q)^k W_{2n-3k}.$$

His results are based on the polynomial identity

$$\sum_{k=0}^n \binom{n}{k}^3 x^k = \sum_{2k \leq n} \frac{(n+k)!}{(k!)^3 (n-2k)!} x^k (x+1)^{n-2k}.$$

In this section we provide more examples of this kind using “Zeitlin’s identity” in its equivalent form given in the next lemma.

Lemma 15 ([18, Identity (6.7)], [33]). *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^n \binom{n}{k}^3 x^k = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} x^k (1+x)^{n-2k}. \quad (75)$$

Theorem 59. *If n is a non-negative integer and r and t are any integers, then*

$$\sum_{k=0}^n \binom{n}{k}^3 q^{rk} W_{r(n-2k)+t} = W_t \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} q^{rk} V_r^{n-2k}.$$

Proof. Set $x = \tau^r/\sigma^r$ and $x = \sigma^r/\tau^r$, in turn, in (75). Combine according to the Binet formula. \square

Corollary 60. *If n is a non-negative integer and r is any integer, then*

$$\sum_{k=0}^n \binom{n}{k}^3 q^{rk} U_{r(n-2k)} = 0.$$

In particular,

$$\sum_{k=0}^n \binom{n}{k}^3 (-1)^{rk} F_{r(n-2k)} = 0.$$

Theorem 61. *If n is a non-negative integer and r, s and t are any integers, then*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k}^3 U_{r+s}^{n-k} U_s^k W_{t+rk} \\ &= \sum_{k=0}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} q^{s(n-2k)} U_{r+s}^k U_s^k U_r^{n-2k} W_{t+rk-s(n-2k)}. \end{aligned}$$

Proof. Set $x = -\tau^r U_s/U_{r+s}$ and $x = -\sigma^r U_s/U_{r+s}$ in (75), in turn, bearing in mind (10) and (11). Combine the resulting equations using the Binet formula and Lemma 4. \square

Corollary 62. *If n is a non-negative integer and r and t are any integers, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 V_r^{n-k} W_{t+rk} = \sum_{k=0}^n (-1)^k \binom{n+k}{2k} \binom{2k}{k} \binom{n-k}{k} q^{r(n-2k)} V_r^k W_{t+r(3k-n)}.$$

9 Concluding comments

Further identities with two binomial coefficients can be derived from Lemma 16 which is a generalization of (16).

Lemma 16. *Let x and y be complex variables. Let m and n be non-negative integers and let r be any integer. Then*

$$\sum_{k=0}^n \binom{m-n+k}{k} \binom{n-k}{r} (x+y)^{n-k-r} y^k = \sum_{k=0}^n \binom{m+1}{k} \binom{n-k}{r} x^{n-k-r} y^k.$$

References

- [1] K. Adegoke, A master identity for Horadam numbers. Preprint arXiv: 1903.11057, 2019, 14 p.

- [2] K. Adegoke, Weighted sums of some second-order sequences. *Fibonacci Quart.* **56**, (2018), 252–262.
- [3] K. Adegoke, R. Frontczak, T. Goy, Binomial Fibonacci sums from Chebyshev polynomials. Preprint arXiv: 2308.04567, 2023, 25 p.
- [4] K. Adegoke, R. Frontczak, T. Goy, New binomial Fibonacci sums. Preprint arXiv: 2210.12159v1, 2022, 20 p. In press in *Palestine J. Math.* **7** (2023).
- [5] K. Adegoke, R. Frontczak, T. Goy, Special formulas involving polygonal numbers and Horadam numbers. *Carpathian Math. Publ.* **13** (2021), 207–216.
- [6] K. Adegoke, A. Olatinwo, S. Ghosh, Cubic binomial Fibonacci sums. *Electron. J. Math.* **2** (2021), 44–51.
- [7] D. Aharonov, U. Elias, A binomial identity via differential equations. *Amer. Math. Monthly* **120** (2013), 462–466.
- [8] H. Alzer, On a combinatorial sum. *Indag. Math.* **26** (2015), 519–525.
- [9] M. Bai, W. Chu, D. Guo, Reciprocal formulae among Pell and Lucas polynomials. *Mathematics* **10** (2022), 2691.
- [10] L. Carlitz, Problem H-97. *Fibonacci Quart.* **4** (1966), 332.
- [11] L. Carlitz, Problem H-106. *Fibonacci Quart.* **5** (1967), 70.
- [12] L. Carlitz, Some classes of Fibonacci sums. *Fibonacci Quart.* **16** (1978), 411–425.
- [13] L. Carlitz, Problem H-180. *Fibonacci Quart.* **9** (1971), 62.
- [14] L. Carlitz, H. H. Ferns, Some Fibonacci and Lucas identities. *Fibonacci Quart.* **1** (1970), 61–73.
- [15] P. Filipponi, Some binomial Fibonacci identities. *Fibonacci Quart.* **33** (1995), 251–257.
- [16] H. W. Gould, A curious identity which is not so curious. *Math. Gaz.* **88** (2004), 87.
- [17] H. W. Gould, A Fibonacci formula of Lucas and its subsequent manifestations and rediscoveries. *Fibonacci Quart.* **15** (1977), 25–29.
- [18] H. W. Gould, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, USA, 1972.
- [19] V. E. Hoggatt, Jr., M. Bicknell, Some new Fibonacci identities. *Fibonacci Quart.* **2** (1964), 121–133.
- [20] V. E. Hoggatt, Jr., J. W. Phillips, H. T. Leonard, Jr., Twenty-four master identities. *Fibonacci Quart.* **9** (1971), 1–17.

- [21] A. F. Horadam, Basic properties of a certain generalized sequence of numbers. *Fibonacci Quart.* **3** (1965), 161–176.
- [22] D. Jennings, Some polynomial identities for the Fibonacci and Lucas numbers. *Fibonacci Quart.* **31** (1993), 134–137.
- [23] E. Kilic, E. J. Ionascu, Certain binomial sums with recursive coefficients. *Fibonacci Quart.* **48** (2010), 161–167.
- [24] E. Kilic, N. Ömür, Y. T. Ulutas, Binomial sums whose coefficients are products of terms of binary sequences. *Util. Math.* **84** (2011), 45–52.
- [25] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, 2001.
- [26] J. W. Layman, Certain general binomial-Fibonacci sums. *Fibonacci Quart.* **15** (1977), 362–366.
- [27] C. T. Long, Some binomial Fibonacci identities, In: *Applications of Fibonacci Numbers*; Bergum G.E., Philippou A.N., Horadam A.F. (Eds.), Springer, 1990, pp. 241–254.
- [28] L. Mozer, Problem E799. *Amer. Math. Monthly* **55** (1948), 30.
- [29] J. Quaintance (Ed.), *Combinatorial Identities: Table I: Intermediate Techniques for Summing Finite Series*, From the seven unpublished manuscripts of H.W. Gould, 2010.
- [30] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.
- [31] S. Simon, A curious identity. *Math. Gaz.* **85** (2001), 296–298.
- [32] N. J. A. Sloane (Ed.), *The On-Line Encyclopedia of Integer Sequences*. Available online at <https://oeis.org>.
- [33] Z. Sun, On sums involving products of three binomial coefficients. *Acta Arith.* **156** (2012), 123–141.
- [34] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Press, 2008.
- [35] D. Zeitlin, Problem H-191. *Fibonacci Quart.* **10** (1972), 185–186.
- [36] D. Zeitlin, Solution to Problem H-180. *Fibonacci Quart.* **10** (1972), 284–287.
- [37] D. Zeitlin, Solution to Problem H-191. *Fibonacci Quart.* **10** (1972), 631–633.