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Kenmotsu 3-manifolds and gradient solitons

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The aim of this article is to characterize a Kenmotsu 3-manifold whose metric is either a gradient Yamabe soliton or gradient Einstein soliton. It is proven that in both cases this manifold is reduced to the manifold of constant sectional curvature. Finally, we verify the obtained results by an example.

Key words and phrases: Kenmotsu 3-manifold, gradient Yamabe soliton, gradient Einstein soliton.

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Introduction

Yamabe flow was introduced by R.S. Hamilton [9] in 1988 at the same time of Ricci flow. Yamabe flow is a natural geometric deformation to metrics of constant scalar curvature. A Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphism f_t generated by a vector field X on N. According to R.S. Hamilton, a Yamabe soliton is defined on a Riemannian manifold (N, g) by a vector field X satisfying

$$\frac{1}{2}\pounds_X g = (\rho - \lambda)g, \tag{1}$$

where $\pounds_X$ denotes the Lie derivative along the vector field X, ρ is the scalar curvature of g and λ is a constant. The vector field X is called the soliton field of the Yamabe soliton. A Yamabe soliton is said to be expanding, shrinking or steady if $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, respectively. If $X = Df$ for some smooth function f and D is the gradient operator of g, then the Yamabe soliton is said to be gradient Yamabe soliton and (1) takes the form

$$\nabla^2 f = (\rho - \lambda)g, \tag{2}$$

where ∇^2 is the Hessian operator of g. Gradient Yamabe solitons have been studied by several authors (see [1–3, 5, 8, 14, 16] and many others).

Definition 1 ([6]). *The metric g of the Riemannian manifold (N, g) is termed as gradient Einstein soliton in case both the function $f : N \to \mathbb{R}$ and constant $a \in \mathbb{R}$ satisfies*

$$S - \frac{1}{2}\rho g + \nabla^2 f = ag, \tag{3}$$

where S indicates the Ricci tensor. The soliton with constant function f is called trivial.

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On the other hand in [15], S. Tanno classified $(2m+1)$-dimensional connected almost contact metric manifolds N with almost contact metric structure (ϕ, ξ, η, g), whose automorphism groups have the maximum dimension $(m+1)^2$. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, N is homogeneous Sasakian manifold of constant ϕ-sectional curvature. If $c > 0$, N is global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature. If $c > 0$, N is a warped product space $\mathbb{R} \times C^\infty$. In [11], K. Kenmotsu abstracted the differential geometric properties of the third case. V.F. Kirichenko [12] obtained Kenmotsu structures from cosymplectic structures by the canonical transformations [17]. Thus, Kenmotsu manifolds represent now a well known class of almost contact metric manifolds.

In [10], author proves that, if (N, g) with dimension $m \geqslant 3$ be a compact Yamabe gradient soliton, then N is of constant scalar curvature.

In this paper, we replace compactness by considering Kenmotsu 3-manifolds and provide the following result.

Theorem 1. *A Kenmotsu 3-manifold with gradient Yamabe soliton is of constant sectional curvature* -1.

It is noticed that all compact gradient Einstein solitons are trivial [6].

Here we replace compactness by considering Kenmotsu 3-manifolds and prove the following assertion.

Theorem 2. *A Kenmotsu 3-manifold with gradient Einstein soliton is of constant sectional curvature.*

1 Preliminaries

An almost contact structure [4] on a $(2m+1)$-dimensional smooth manifold N is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-type tensor, ξ is a global vector field and η is a 1-form, such that

$$\phi^2 = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4}$$

where id denotes the identity mapping. The relation (4) implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2m$. The almost contact structure induces a natural almost complex structure J on the product manifold $N \times \mathbb{R}$ defined by $J(U, \lambda d/dt) = \big(\phi U - \lambda\xi, \eta(U)d/dt\big)$, where U is tangent to N, t is the coordinate of $\mathbb{R}$ and λ is a smooth function on $N \times \mathbb{R}$. The almost contact structure is said to be normal [13] if the almost complex structure J is integrable or equivalently $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

or equivalently, $\Phi(U, V) = g(U, \phi V)$ along with $g(U, \xi) = \eta(U)$ for all $U, V \in \chi(N)$. Then N is an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). An almost contact metric manifold is called a Kenmotsu manifold if it satisfies

$$(\nabla_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U \tag{5}$$

for all $U, V \in \chi(N)$, where ∇ is Levi-Civita connection of the Riemannian metric. A Kenmotsu manifold is normal but not Sasakian. Moreover, it is also not compact since from the formula (5) we get

$$\nabla_U \xi = U - \eta(U)\xi,$$

which gives $div\xi = 2m$. A conformal change g^* of a Riemannian metric g is called a concircular transformation [17] if geodesic circles of g are transformed into geodesic circles of g^*. Here a geodesic circle means a curve whose first curvature is constant and whose second curvature is identically zero. A cosymplectic structure is defined to be a normal almost contact metric structure (ϕ, ξ, η, g) with both the fundamental 2-form Φ and the 1-form η is closed. An almost contact metric structure is cosymplectic if and only if $\nabla\phi = 0$. In [12], V.F. Kirichenko obtained the class of Kenmotsu manifolds from cosymplectic manifolds by the canonical concircular transformations. A Kenmotsu manifold is of constant curvature -1 if and only if it is canonically concircular to $C^m \times \mathbb{R}$ (see [12]).

For a $(2m+1)$-dimensional Kenmotsu manifold, the following formulas hold:

$$K(U, V)\xi = \eta(U)V - \eta(V)U, \tag{6}$$

$$(\nabla_U \eta)V = g(U, V) - \eta(U)\eta(V),$$

$$S(\xi, \xi) = g(Q\xi, \xi) = -2m$$

for any $U, V \in \chi(N)$, where S is the Ricci tensor, Q is the Ricci operator and K is the curvature tensor. We know [7], that for a Kenmotsu 3-manifold we have

$$\begin{aligned} K(U, V)W = & \frac{\rho+4}{2}\left[g(V, W)U - g(U, W)V\right] \\ & - \frac{\rho+6}{2}\left[g(V, W)\eta(U)\xi - g(U, W)\eta(V)\xi + \eta(V)\eta(W)U - \eta(U)\eta(W)V\right], \end{aligned}$$

$$QU = \frac{1}{2}\left[(\rho+2)U - (\rho+6)\eta(U)\xi\right], \tag{7}$$

$$S(U, V) = \frac{1}{2}\left[(\rho+2)g(U, V) - (\rho+6)\eta(U)\eta(V)\right]. \tag{8}$$

An almost contact metric manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S(V, W) = ag(V, W) + b\eta(V)\eta(W)$$

for any vector field V, W on N and a, b are arbitrary functions on N. An η-Einstein manifold with b vanishing and a a constant is obviously an Einstein manifold. An η-Einstein manifold is said to be proper η-Einstein if $b \neq 0$.

Kenmotsu 3-manifolds also have been studied in [7, 18, 19] and many others.

Lemma 1 ([18])**.** *On any Kenmotsu 3-manifold (N, ϕ, ξ, η, g) we have*

$$\xi\rho = -2(\rho+6). \tag{9}$$

Lemma 2 ([7])**.** *A 3-dimensional Riemannian manifold is a manifold of constant sectional curvature -1 if and only if the scalar curvature ρ is equal to -6.*

Lemma 3 ([11, Proposition 8])**.** *Let N be an η-Einstein Kenmotsu manifold $S = ag + b\eta \otimes \eta$, and a, b be scalar functions. If either a or b is constant then the manifold becomes an Einstein manifold.*

2 Proof of the main results

Proof of Theorem 1. Let a Kenmotsu 3-manifold N admits a gradient Yamabe soliton. Then from the equation (2), we have

$$\nabla_V Df = (\rho - \lambda)V, \tag{10}$$

from which we get

$$K(U,V)Df = (U\rho)V - (V\rho)U. \tag{11}$$

Contracting the above equation, we infer

$$S(V,Df) = -2(V\rho). \tag{12}$$

Replacing U by Df in (8) and using (12) entails that

$$-2(V\rho) = \left(\frac{\rho}{2}+1\right)(Vf) - \left(\frac{\rho}{2}+3\right)(\xi f)\eta(V). \tag{13}$$

Putting $V = \xi$ in the above equation, we obtain

$$\xi\rho = \xi f. \tag{14}$$

In view of (9) and (14), we get

$$\xi f = \xi\rho = -2(\rho+6). \tag{15}$$

Taking inner product of (11) with ξ and using (6), we infer

$$(Uf)\eta(V) - (Vf)\eta(U) = (U\rho)\eta(V) - (V\rho)\eta(U). \tag{16}$$

Putting $V = \xi$ in (16) and using (14), we get

$$Uf = U\rho. \tag{17}$$

Using (17) and (15) in (13), we obtain

$$\left(\frac{\rho}{2}+3\right)\left[(V\rho) + 2(\rho+6)\eta(V)\right] = 0,$$

which implies either $\rho = -6$ or $\rho \neq -6$.

Case $\rho = -6$. Using Lemma 2 we conclude that the manifold is of constant sectional curvature -1.

Case $\rho \neq -6$. We have $V\rho = -2(\rho+6)\eta(V)$. Using (17), we get $Df = -2(\rho+6)\xi$. The equation (15) entails that

$$\xi(\xi\rho) = 4(\rho+6). \tag{18}$$

We know $\xi f = g(\xi, Df)$. Then using (10), we obtain $\xi(\xi f) = (\rho - \lambda)$. Since $\xi f = \xi\rho$, we have from the above equation

$$\xi(\xi\rho) = (\rho - \lambda). \tag{19}$$

Equation (18) and (19) together give $\rho = -\frac{1}{3}(\lambda + 24)$ = constant. Since $Vf = V\rho$ and ρ= constant, therefore we get f= constant.

Thus using f = constant in (15) implies that $\rho = -6$ which is a contradiction. This completes the proof. □

Remark. *Since the manifold N is of constant sectional curvature -1, therefore N is locally isometric to the hyperbolic space $H^3(1)$.*

Proof of Theorem 2. The equation (3) implies

$$QU - \frac{1}{2}\rho U + \nabla_U Df = \lambda U. \tag{20}$$

In view of (7) and (20), we get

$$\nabla_U Df = (\lambda - 1)U + \left(\frac{\rho}{2} + 3\right) \eta(U)\xi, \tag{21}$$

from which we get

$$K(U,V)Df = \frac{1}{2}(U\rho)\eta(V)\xi - \frac{1}{2}(V\rho)\eta(U)\xi + (\frac{\rho}{2} + 3)[\eta(V)U - \eta(U)V]. \tag{22}$$

Contracting the above equation and using (9), we infer

$$S(V, Df) = -\frac{1}{2}(V\rho). \tag{23}$$

Replacing U by Df in (8) and using (23), we obtain

$$-\frac{1}{2}(V\rho) = \left(\frac{\rho}{2} + 1\right)(Vf) - \left(\frac{\rho}{2} + 3\right)(\xi f)\eta(V). \tag{24}$$

Putting $V = \xi$ in the above equation and using (9), we get

$$\xi f = -\frac{1}{2}(\rho + 6). \tag{25}$$

Taking inner product of (22) with ξ and using (6), we infer

$$(Uf)\eta(V) - (Vf)\eta(U) = \frac{1}{2}[(U\rho)\eta(V) - (V\rho)\eta(U)]. \tag{26}$$

Putting $V = \xi$ in (26) and using (9) and (25), we get

$$-\frac{1}{2}(U\rho) = \frac{1}{2}(\rho + 6)\eta(U) - (Uf). \tag{27}$$

Using (25) and (27) in (24), we obtain

$$\left(\frac{\rho}{2} + 2\right)\left[(Vf) + \frac{1}{2}(\rho + 6)\eta(V)\right] = 0,$$

which implies either $\rho = -4$ or $\rho \neq -4$.

Case $\rho = -4$. Then the scalar curvature is constant. Using $\rho = -4$ in equation (8) gives

$$S(U,V) = -g(U,V) - \eta(U)\eta(V),$$

which implies that the manifold is an η-Einstein manifold.

Case $\rho \neq -4$. Then $Vf = -\frac{1}{2}(\rho + 6)\eta(V)$. This implies

$$Df = -\frac{1}{2}(\rho + 6)\xi. \tag{28}$$

From (9) and the above equation, we get

$$\nabla_\xi Df = (\rho + 6)\xi. \tag{29}$$

Using (29) in (21), we get $\rho = 2\lambda - 8$ = constant. Hence from (28) we can say that Df is a constant multiple of ξ, say $Df = c\xi$, c being a constant. Hence from (20), we obtain

$$QU = (2\lambda - c - 4)U + c\eta(U)\xi, \tag{30}$$

which implies that the manifold is an η-Einstein manifold.

Now, using Lemma 3, from the equation (30), we infer that N is an Einstein manifold. Since N is a Kenmotsu 3-manifold, therefore, N is of constant sectional curvature. This completes the proof. □

3 Example

We consider $N = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on N given by

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form defined by $\eta(W) = g(W, e_3)$ for all $W \in \chi(N)$ and ϕ be the $(1, 1)$-tensor defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then using the linearity of ϕ and g, we get

$$\phi^2 W = -W + \eta(W)e_3, \eta(e_3) = 1,$$

$$g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W)$$

for any $V, W \in \chi(N)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on N.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = e_2, \quad [e_1, e_3] = e_1.$$

The Riemannian connection ∇ of the metric g and using Koszul's formula, we have

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above we see that $\nabla_W \xi = W - \eta(W)\xi$ for all $W \in \chi(N)$. Hence the manifold is a Kenmotsu manifold.

Now, we have

$$K(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W. \tag{31}$$

With the help of the previous results and using (31), we obtain

$$K(e_1, e_2)e_3 = 0, \quad K(e_1, e_2)e_2 = -e_1, \quad K(e_1, e_2)e_1 = e_2,$$

$$K(e_2, e_3)e_3 = -e_2, \quad K(e_3, e_2)e_2 = -e_3,$$

$$K(e_3, e_1)e_1 = -e_3, \quad K(e_3, e_1)e_3 = e_1.$$

From the foregoing equation, we can easily calculate the non-vanishing components of the Ricci tensor S as follows

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

Hence, we get

$$\rho = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6,$$

where ρ is the scalar curvature.

Let

$$V = b_1 e_1 + b_2 e_2 + b_3 e_3, \quad U = a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where b_i, a_i, $i = 1, 2, 3$, are constants. Let $f = \log z$, $z > 0$, be a smooth function.

Now $(Vf) = b_1(e_1 f) + b_2(e_2 f) + b_3(e_3 f) = -b_3$ is a constant. This implies $\nabla^2 f = 0$. Thus g defines the gradient Yamabe soliton with $f = \log z$ and $\lambda = -6$. Therefore Theorem 1 is verified. Also, g defines the gradient Einstein soliton with $f = \log z$ and $\lambda = 1$. Therefore Theorem 2 is verified.

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Метою цієї статті є охарактеризувати 3-многовид Кенмотсу, метрика якого є або ґрадієнтним солітоном Ямабе або ґрадієнтним солітоном Айнштайна. Доведено, що в обох випадках цей многовид редукується до многовида сталої секційної кривизни. Насамкінець ми перевіряємо отримані результати за допомогою прикладу.

Ключові слова і фрази: 3-многовид Кенмотсу, ґрадієнтний солітон Ямабе, ґрадієнтний солітон Айнштайна.