ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publl. 2023, **15** (1), 95–103
doi:10.15330/cmp.15.1.95-103

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №1, С.95–103

Some convergence results for nonlinear Baskakov-Durrmeyer operators

Altin H.E.

This paper is an introduction to a sequence of nonlinear Baskakov-Durrmeyer operators (NBD_n) of the form

$$(NBD_n)(f;x) = \int_0^\infty K_n(x,t,f(t))\,dt$$

with $x \in [0,\infty)$ and $n \in \mathbb{N}$. While $K_n(x,t,u)$ provide convenient assumptions, these operators work on bounded functions, which are defined on all finite subintervals of $[0,\infty)$. This paper comprise some pointwise convergence results for these operators in certain functional spaces. As well as this study can be seen as a continuation of studies about nonlinear operators, it is the first study on nonlinear Baskakov-Durrmeyer or modified Baskakov operators, while there were more papers on linear part of the operators.

Key words and phrases: bounded variation, nonlinear operator, $(L-\psi)$ Lipschitz condition, pointwise convergence.

Bolu Abant Izzet Baysal University, 14030 Bolu, Turkey
E-mail: `erhanaltin@ibu.edu.tr`

Introduction

The focus of the approximation theory have changed for many researchers with Korovkin's main theorem to linear positive operators, which have wide potential in applications. Under the leadership of the classical Bernstein operators, which are defined on the closed interval $[0,1]$, the theory expanded to unbounded interval with Szász-Mirakyan and Baskakov operators.

The classical Baskakov operators

$$(B_n)(g;x) = \sum_{k=0}^{\infty} p_{n,k}(x)\, g\left(\frac{k}{n}\right) \tag{1}$$

with $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$, where $x \in [0,\infty)$ and $n \in \mathbb{N}$, certainly introduced to approximate functions on unbounded intervals. Just as the classical Bernstein or Szász-Mirakyan operators, classical Baskakov operators have not used to approximate integrable functions before their suitable modifications described. In this sense, R.P. Sinha et al. [13] contextured Durrmeyer-type integral modification of (1), with

$$(BD_n)(g;x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty p_{n,k}(t)\, g(t)\, dt. \tag{2}$$

УДК 517.98, 517.983.33
2020 *Mathematics Subject Classification:* 41A25, 41A35, 47G10.

Afterwards, operators (2) have been studied by many mathematicians to approximate integrable and Lebesgue integrable functions on the interval $[0, \infty)$. Some certain convergence results on functions of bounded variation for these operators can be found in [4–8]. For further reading about Baskakov operators and Durrmeyer type operators we refer the reader to [3] and [12], respectively.

In the present work, we pay attention to the nonlinear part of the operators (2), that is to say nonlinear Baskakov-Durrmeyer operators NBD_n of the form

$$(NBD_n)(g; x) = \int_0^\infty K_n(x, t, g(t))dt \tag{3}$$

with $x \in [0, \infty)$ and $n \in \mathbb{N}$, acting on Lebesgue integrable functions on $[0, \infty)$, where

$$K_n(x, t, u) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\, p_{n,k}(t) H_n(u),$$

here $H_n : \mathbb{R} \to \mathbb{R}$ is a function such that $H_n(0) = 0$ and the kernel function $K_n(x, t, u)$ satisfy convenient assumptions.

In recent years, approximation with nonlinear operators have become popular as much as its linear part. In fact, this popularity have started with the principal work of J. Musielak [11]. Thereafter, with the great contributions of C. Bardaro et al. [2], the approximation theory enlarged to the nonlinear operators. Especially some certain convergence results on functions of bounded variation for nonlinear Durrmeyer type operators can be found in [1, 9, 10].

An outline of this work is as follows. First section comprises essential notations and definitions. In Section 2, a certain result will be given, which is necessary to prove main theorems. In the last section, the main theorems of this work and their proofs are given with some corollaries. The main results of this paper deal with the pointwise convergence of nonlinear Baskakov-Durrmeyer operators NBD_n working on bounded functions, which are defined on all finite subintervals of $[0, \infty)$.

1 Preliminaries

Let Ψ be the class of all continuous and concave functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$. Let X be the set of all bounded Lebesgue measurable functions on every finite subinterval of $[0, \infty)$.

Now we establish a sequence of functions. Let $H_n : \mathbb{R} \to \mathbb{R}$ be a function with $H_n(0) = 0$ and

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Then $K_n : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$, defined by

$$K_n(x, t, u) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k}(t) H_n(u), \quad n \in \mathbb{N},$$

is a sequence of functions.

We presume the following conditions hold:

(a) $H_n : \mathbb{R} \to \mathbb{R}$ satisfies an $(L-\psi)$ *Lipschitz condition*, i.e.

$$\left|H_n(u) - H_n(v)\right| \leq \psi(|u-v|)$$

holds for every $u, v \in \mathbb{R}$ and for every $n \in \mathbb{N}$ while $\psi \in \Psi$;

(b) the function $\lambda_n(x)$, defined by

$$\lambda_n(x) := \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} K_n(x,t)\,dt,$$

have a behavior close to any fixed point $x \in (0,\infty)$, where

$$K_n(x,t) := (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(t);$$

(c) for n sufficiently large, the inequality

$$\sup_u \left|H_n(u) - u\right| \leq \frac{1}{\mu(n)}$$

holds for $u \in \mathbb{R}$ and $n \in \mathbb{N}$ with an increasing and continuous function $\mu : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} \mu(n) = \infty$.

2 Auxiliary results

Here we give a precise result, which is required while proving our theorems.

Lemma 1 ([4]). *If we define the m^{th} order moment for $BD_n(g;x)$ as*

$$T_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} (t-x)^m\, p_{n,k}(t)\,dt,$$

then

$$T_{n,1}(x) = \frac{1+2x}{n-2} \quad \text{for} \quad n>2 \quad \text{and} \quad T_{n,2}(x) = \frac{2(n-1)x(1+x)+2(1+2x)^2}{(n-2)(n-3)} \quad \text{for} \quad n>3.$$

Especially for any $\lambda > 2$ and $x > 0$ there is an integer $N(\lambda, x) > 2$ with

$$T_{n,m}(x) \leq \frac{\lambda x(1+x)}{x}$$

for all $n \geq N(\lambda, x)$. Also let for $\lambda > 2$ and $n > N(\lambda, x)$,

$$K_n(x,t) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(t),$$

then we get

$$\int_0^y K_n(x,t)\,dt \leq \frac{\lambda x(1+x)}{n(x-y)^2} \tag{4}$$

for $0 \leq y < x$ and

$$\int_z^{\infty} K_n(x,t)\,dt \leq \frac{\lambda x(1+x)}{n(z-x)^2} \tag{5}$$

for $x < z < \infty$.

3 Convergence results

We have in contemplation the following type nonlinear Baskakov-Durrmeyer operators

$$(NBD_n)(g;x) = \int_0^\infty K_n\big(x,t,g(t)\big)\,dt$$

with

$$K_n\big(x,t,g(t)\big) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(t)H_n\big(g(t)\big) = K_n(x,t)H_n\big(g(t)\big). \tag{6}$$

We presume that such an operator is defined for every $g \in Dom\ NBD_n g$, where $Dom\ NBD_n g$ is the subset of X on which $NBD_n g$ is well-defined. Some existence theorems for these type nonlinear operators can be found in [2].

Theorem 1. *Let $\psi \in \Psi$ and $g \in X$ be such that g and $\psi \circ |g|$ are functions, which have bounded variation on whole finite subinterval of $[0,\infty)$ and let $\psi(|g(t)|) < M(1+t)^\alpha$ for $t \in [0,\infty)$, $M > 0$ and $\alpha \in \mathbb{N}_0$.*

Presume $K_n(x,t,u)$ ensures conditions (a) and (b). If $\lambda > 2$ and $n > \max\{1+\alpha, N(\lambda,x)\}$, then

$$(NBD_n)(g;x) - \left[\psi\left(\left|\frac{g(x+)+g(x-)}{2}\right|\right) + \psi\left(\left|\frac{g(x+)-g(x-)}{2}\right|\right)\right]$$
$$\le \frac{x+3\lambda\,(1+x)}{nx}\sum_{k=1}^{n}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}}\psi(|g_x|) + \frac{\lambda M K_\alpha\,(1+x)^{\alpha+1}}{nx},$$

where $\bigvee_a^b \psi(|g_x|)$ is the total variation of $\psi(|g_x|)$ on $[a,b]$ and

$$g_x \equiv g_x(t) = \begin{cases} g(t)-g(x+), & x<t<\infty,\\ 0, & t=x,\\ g(t)-g(x-), & 0\le t<x. \end{cases} \tag{7}$$

Proof. For any $g \in X$, it is easily attained from (7), that

$$\begin{aligned} g(t) = {} & \frac{g(x+)+g(x-)}{2} + g_x(t)\\ & + \frac{g(x+)-g(x-)}{2}\,\mathrm{sgn}(t-x) + \delta_x(t)\left[g(x) - \frac{g(x+)+g(x-)}{2}\right], \end{aligned} \tag{8}$$

where $\delta_x(t) = \begin{cases} 1, & x=t,\\ 0, & x\ne t. \end{cases}$

Applying the operator (3) to (8), using (6) and *(a)*, we have

$$\begin{aligned} (NBD_n)(g;x) &= \int_0^\infty K_n(x,t,g(t))\,dt \le \int_0^\infty K_n(x,t)\psi(|g(t)|)\,dt\\ &\le \psi\left(\left|\frac{g(x+)+g(x-)}{2}\right|\right)\int_0^\infty K_n(x,t)\,dt + \int_0^\infty K_n(x,t)\psi(|g_x(t)|)\,dt\\ &\quad + \int_0^\infty K_n(x,t)\psi\left(\left|\frac{g(x+)-g(x-)}{2}\right||\mathrm{sgn}(t-x)|\right)dt\\ &\quad + \int_0^\infty K_n(x,t)\psi\left(\left|g(x) - \frac{g(x+)+g(x-)}{2}\right|\delta_x(t)\right)dt. \end{aligned}$$

The last integral is equal to zero from the definitions of $\delta_x(t)$. Using $|\operatorname{sgn}(t-x)| \leq 1$, we can write

$$(NBD_n)(g;x) \leq \left[\psi\left(\left|\frac{g(x+)+g(x-)}{2}\right|\right) + \psi\left(\left|\frac{g(x+)-g(x-)}{2}\right|\right)\right] \int_0^\infty K_n\,(x,t)\;dt \\ + \int_0^\infty K_n(x,t)\psi(|g_x(t)|)\,dt.$$

By virtue of $\int_0^\infty K_n(x,t)\,dt = 1$, we have

$$(NBD_n)(g;x) - \left[\psi\left(\left|\frac{g(x+)+g(x-)}{2}\right|\right) + \psi\left(\left|\frac{g(x+)-g(x-)}{2}\right|\right)\right] \leq \int_0^\infty K_n\,(x,t)\,\psi(|g_x(t)|)dt.$$

We use *(b)* to separate $[0,\infty)$ for estimation of the right hand side of the integral

$$\begin{aligned} \int_0^\infty K_n(x,t)\psi(|g_x(t)|)\,dt &= \left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} + \int_{x+\frac{x}{\sqrt{n}}}^{\infty}\right) K_n(x,t)\psi(|g_x(t)|)\,dt \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{9}$$

Using $\lambda_n(x,t) = \int_0^t K_n(x,u)\,du$ and $\int_a^b d_t\lambda_n(x,t) \leq 1$ for all $[a,b] \subseteq [0,\infty)$ we can estimate I_2 as

$$\begin{aligned} I_2 = \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} [\psi(|g_x(t)|) - \psi(|g_x(x)|)] K_n(x,t)\,dt \\ \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} \psi(|g_x|) \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} d_t\lambda_n(x,t) \\ \leq \frac{1}{n} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} \psi(|g_x|). \end{aligned} \tag{10}$$

Next we estimate I_1. By using Lebesgue-Stieltjes integration by parts with $y = x - \frac{x}{\sqrt{n}}$, we get

$$I_1 = \int_0^y \psi(|g_x(t)|)\,d_t\lambda_n(x,t) = \psi(|g_x(y)|)\lambda_n(x,y) - \int_0^y \lambda_n(x,t)\,d_t\big(\psi(|g_x(t)|)\big).$$

Using the fact that $\psi(|g_x(y)|) = \big|\psi(|g_x(y)|) - \psi(|g_x(x)|)\big| \leq \bigvee_y^x \psi(|g_x|)$ and (4) of Lemma 1, we have

$$\begin{aligned} |I_1| &\leq \bigvee_y^x \psi(|g_x|)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)\,d_t\Big(-\bigvee_t^x \psi(|g_x|)\Big) \\ &\leq \bigvee_y^x \psi(|g_x|)\frac{\lambda x(1+x)}{n(x-y)^2} + \frac{\lambda x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}\,d_t\Big(-\bigvee_t^x \psi(|g_x|)\Big). \end{aligned}$$

First using integration by parts in the last integral and then substituting $x-\frac{x}{\sqrt{t}}$ for the variable t we consequently get

$$\begin{aligned}|I_1| &\leq \frac{\lambda x(1+x)}{n}\left[\frac{1}{x^2}\bigvee_0^x \psi(|g_x|)+\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x \psi(|g_x|)\frac{2}{(x-t)^3}\,dt\right]\\ &\leq \frac{\lambda x(1+x)}{nx^2}\left[\bigvee_0^x \psi(|g_x|)+\int_1^n \bigvee_{x-\frac{x}{\sqrt{t}}}^x \psi(|fg_x|)\,dt\right]\\ &\leq \frac{\lambda x(1+x)}{nx^2}\left[\bigvee_0^x \psi(|g_x|)+\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x \psi(|g_x|)\right] \leq \frac{2\lambda\,(1+x)}{nx}\sum_{k=1}^n \bigvee_{x-\frac{x}{\sqrt{k}}}^x \psi(|g_x|).\end{aligned} \tag{11}$$

Finally we estimate I_3. Setting $z=x+\frac{x}{\sqrt{n}}$ and defining $Q_n(x,t)$ on $[0,2x]$ as

$$Q_n(x,t)=\begin{cases}1-\lambda_n\,(x,t)\,, & 0\leq t<2x,\\ 0, & t=2x,\end{cases}$$

we can write

$$\begin{aligned}I_3 &= \int_z^{2x}\psi(|g_x(t)|)\,d_t\,(\lambda_n\,(x,t))+\int_{2x}^{\infty}\psi(|g_x(t)|)\,d_t\,(\lambda_n(x,t))\\ &= -\int_z^{2x}\psi(|g_x(t)|)\,d_t\,(Q_n(x,t))-\psi(|g_x(2x)|)\int_{2x}^{\infty}K_n(x,t)\,dt\\ &\quad+\int_{2x}^{\infty}\psi(|g_x(t)|)\,d_t\,(\lambda_n(x,t))=I_{3,1}+I_{3,2}+I_{3,3}.\end{aligned}$$

For the first term, using partial integration, we get

$$I_{3,1}=-\psi(|g_x(2x)|)Q_n\,(x,2x)+\psi(|g_x(z)|)Q_n\,(x,z)+\int_z^{2x}Q_n\,(x,t)\;d_t\,(\psi(|g_x(t)|))\,.$$

Using the facts that $Q_n(x,2x)=0$ and $\psi(|g_x(z)|)\leq \bigvee_x^z \psi(|g_x|)$ and (5) of Lemma 1, we have

$$\begin{aligned}|I_{3,1}| &\leq \bigvee_x^z \psi(|g_x|)Q_n\,(x,z)+\int_z^{2x}Q_n\,(x,t)\;d_t\left(\bigvee_x^t \psi(|g_x|)\right)\\ &\leq \bigvee_x^z \psi(|g_x|)\frac{\lambda x\,(1+x)}{n\,(z-x)^2}+\frac{\lambda x\,(1+x)}{n}\int_z^{2x}\frac{1}{(t-x)^2}\,d_t\left(\bigvee_x^t \psi(|g_x|)\right).\end{aligned}$$

Once again using partial integration for the last integral and then substituting $x+\frac{x}{\sqrt{u}}$ for the variable t we consequently get

$$\begin{aligned}|I_{3,1}| &\leq \frac{\lambda x\,(1+x)}{n}\left\{\frac{1}{x^2}\bigvee_x^{2x}\psi(|g_x|)+2\int_z^{2x}\frac{1}{(t-x)^3}\bigvee_x^t \psi(|g_x|)\,dt\right\}\\ &\leq \frac{\lambda x\,(1+x)}{nx^2}\left\{\bigvee_x^{2x}\psi(|g_x|)+\int_1^n \bigvee_x^{x+\frac{x}{\sqrt{u}}}\psi(|g_x|)\,du\right\}\\ &\leq \frac{\lambda x\,(1+x)}{nx^2}\left\{\bigvee_x^{2x}\psi(|g_x|)+\sum_{k=1}^n \bigvee_x^{x+\frac{x}{\sqrt{k}}}\psi(|g_x|)\right\}\\ &\leq \frac{2\lambda\,(1+x)}{nx}\sum_{k=1}^n \bigvee_x^{x+\frac{x}{\sqrt{k}}}\psi(|g_x|).\end{aligned} \tag{12}$$

Furthermore, for $I_{3,2}$ using the similar way we obtain

$$|I_{3,2}| \leq \frac{\lambda\,(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x}^{x+\frac{x}{\sqrt{k}}} \psi(|g_x|). \tag{13}$$

Since ψ is concave and $\psi(|g(t)|) < M(1+t)^{\alpha}$, using the technique of [4], we obtain for $n > \alpha$

$$|I_{3,3}| = \left| \int_{2x}^{\infty} \psi(|g_x(t)|)\, d_t\,(\lambda_n(x,t)) \right| \leq M(n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \left((1+t)^{\alpha} + (1+x)^{\alpha} \right) p_{n,k}\,(t)\, dt.$$

Then using (4), for $n > N(\lambda, x)$, the integral $|I_{3,3}|$ is equal or smaller then

$$\begin{aligned} &\frac{2\lambda M\,(1+x)}{nx} \quad \text{if } \alpha = 0, \\ &\frac{3\lambda M\,(1+x)^2}{nx} \quad \text{if } \alpha = 1, \\ &\frac{5\lambda M\,(1+x)^3}{nx} \quad \text{if } \alpha = 2. \end{aligned}$$

Actually, whether α is even or odd, there is a constant K_{α} depending on α, with

$$|I_{3,3}| \leq \frac{\lambda M K_{\alpha}\,(1+x)^{\alpha+1}}{nx} \tag{14}$$

for all $n \geq \max\left\{(1+\alpha), N(\lambda,x)\right\}$. Editing the calculations (10), (11), (12), (13) and (14) in (9), we get the desired result. $\square$

Corollary 1. *If we choose $g \in C\,(a,b)$ in Theorem 1 for any finite subinterval (a,b) of $[0,\infty)$, then we have*

$$(NBD_n)(f;x) - \psi(|g(x)|) \leq \frac{x+3\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} \psi(|g_x|) + \frac{\lambda M K_{\alpha}(1+x)^{\alpha+1}}{nx}.$$

Theorem 2. *Let $\psi \in \Psi$ and $g \in X$ be such that g and $\psi \circ |g|$ are functions, which have bounded variation on whole finite subinterval of $[0,\infty)$ and let $\psi(|g(t)|) < M(1+t)^{\alpha}$ for $t \in [0,\infty)$, $M > 0$ and $\alpha \in \mathbb{N}_0$.*

Presume $K_n(x,t,u)$ ensures (a), (b) and (c). If $\lambda > 2$ and $n > \max\left\{1+\alpha, N\,(\lambda,x)\right\}$, then

$$\begin{aligned} \left| (NBD_n)(g;x) - \frac{g(x+) - g(x-)}{2} \right| &\leq \frac{x+3\lambda\,(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} \psi(|g_x|) \\ &\quad + \frac{\lambda M K_{\alpha}\,(1+x)^{\alpha+1}}{nx} + \psi\left(\left| \frac{g(x+) - g(x-)}{2} \right| \right) + \frac{1}{\mu\,(n)}. \end{aligned}$$

Proof. For any $g \in X$ we have

$$\begin{aligned}
&\left| (NBD_n)(f;x) - \frac{g(x+)-g(x-)}{2} \right| = \left| \int_0^\infty K_n(x,t)\, H_n(g(t))\, dt - \frac{g(x+)-g(x-)}{2} \right| \\
&\le \int_0^\infty K_n(x,t)\, \psi\left(\left| g(t) - \frac{g(x+)-g(x-)}{2} \right| \right) dt \\
&\quad + \int_0^\infty K_n(x,t) \left| H_n\left(\frac{g(x+)-g(x-)}{2} \right) - \frac{g(x+)-g(x-)}{2} \right| dt \\
&\le \int_0^\infty K_n(x,t)\, \psi(|g_x(t)|)\, dt + \int_0^\infty K_n(x,t)\, \psi\left(\left| \frac{g(x+)-g(x-)}{2} \right| |\operatorname{sgn}(t-x)| \right) dt \\
&\quad + \int_0^\infty K_n(x,t)\, \psi\left(\left| g(x) - \frac{g(x+)+g(x-)}{2} \right| \delta_x(t) \right) dt \\
&\quad + \int_0^\infty K_n(x,t) \left| H_n\left(\frac{g(x+)-g(x-)}{2} \right) - \frac{g(x+)-g(x-)}{2} \right| dt.
\end{aligned}$$

By using the required results as in the proof of the Theorem 1 and in view of *(c)*, we get the desired result. □

Corollary 2. *If we choose $g \in C(a,b)$ in Theorem 2 for any finite subinterval (a,b) of $0,\infty)$, then we have*

$$\left| (NBD_n)(g;x) - g(x) \right| \le \frac{x + 3\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} \psi(|g_x|) + \frac{\lambda M K_\alpha (1+x)^{\alpha+1}}{nx}.$$

Corollary 3. *If we choose $g \in C(a,b)$ in Theorem 2 for any finite subinterval (a,b) of $[0,\infty)$ and if we let $\psi(t) = t$ (that is, strongly Lipschitz condition), then we have*

$$\left| (NBD_n)(f;x) - f(x) \right| \le \frac{x + 3\lambda(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} |g_x| + \frac{\lambda M K_\alpha (1+x)^{\alpha+1}}{nx}.$$

References

[1] Altin H.E., Karsli H. *Some approximation properties of a nonlinear Szász-Mirakyan-Durrmeyer operators*. An. Univ. Oradea Fasc. Mat. 2021, **28** (1), 105–114.

[2] Bardaro C., Musielak J., Vinti G. Nonlinear integral operators and applications. In: Appell J. (Ed.) De Gruyter Series in Nonlinear Analysis and Applications, 9. De Gruyter, Berlin, 2003. doi:10.1515/9783110199277

[3] Bodur M., Gürel Yilmaz Ö., Aral A. *Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions*. Constr. Math. Anal. 2018, **1** (1), 1–8. doi:10.33205/cma.450708

[4] Deo N. *On the rate of convergence of modified Baskakov type operators on functions of bounded variation*. Kyungpook Math. J. 2005, **45**, 571–577.

[5] Gupta V., Arya K.V. *On the rate of pointwise convergence of modified Baskakov type operators for functions of bounded variation*. Kyungpook Math. J. 1998, **38**, 283–291.

[6] Gupta V., Kumar D. *Rate of convergence of modified Baskakov operators*. Demonstr. Math. 1997, **30** (2), 339–346.

[7] Gupta V. *On the rate of pointwise convergence of modified Baskakov operators*. Soochow J. Math. 1996, **22** (4), 543–552.

[8] Gupta V., Srivastava G.S. *An estimate of the rate of convergence of modified Baskakov operators for functions of bounded variation*. Kyungpoog Math. J. 1996, **36**, 237–247.

[9] Karsli H. *On convergence of a sequence of nonlinear Durrmeyer operators*. PanAmer. Math. J. 2014, **24** (2), 1–24.

[10] Karsli H., Tiryaki I.U., Altin H.E. *Some approximation properties of a certain nonlinear Bernstein operators*. Filomat 2014, **28** (6), 1295–1305. doi:10.2298/FIL1406295K

[11] Musielak J. On some approximation problems in modular spaces. C. R. Acad. Bulgare Sci., Sofia, 1983.

[12] Paltanea R. *Durrmeyer type operators on a simplex*. Constr. Math. Anal. 2021, **4** (2), 215–228. doi:10.33205/cma.862942

[13] Sinha R.P., Agrawal P.N., Gupta V. *On simultaneous approximation by modified Baskakov operators*. Bull. Soc. Math. Belg. Ser. B. 1991, **42** (2), 217–231.

Received 31.03.2023

Алтін Х.Е. *Деякі результати про збіжність для нелінійних операторів Баскакова-Дюрмайєра* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 95–103.

У цій статті введено послідовність нелінійних операторів Баскакова-Дюрмайєра (NBD_n) вигляду

$$(NBD_n)(f;x) = \int_0^\infty K_n(x,t,f(t))\,dt,$$

де $x \in [0,\infty)$ і $n \in \mathbb{N}$. Якщо $K_n(x,t,u)$ забезпечені зручними припущеннями, то ці оператори діють на обмежені функції, які визначені на всіх скінченних підінтервалах півосі $[0,\infty)$. У цій статті подано деякі результати про поточкову збіжність для цих операторів у певних функціональних просторах. Крім того, що це дослідження можна розглядати як продовження досліджень нелінійних операторів, це перше дослідження нелінійних операторів Баскакова-Дюрмайєра або модифікованих операторів Баскакова, при цьому лінійній частині цих операторів було присвячено багато статей.

Ключові слова і фрази: обмежена варіація, нелінійний оператор, $(L-\psi)$ умова Ліпшиця, поточкова збіжність.