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RECURRENCE RELATIONS FOR THE SQUARES OF THE HORADAM NUMBERS AND SOME ASSOCIATED CONSEQUENCES

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ABSTRACT. We derive recurrence relations for the squares of the Horadam numbers w_n^2, where the Horadam sequence w_n is such that the numbers w_n, for $n \in \mathbb{Z}$, are defined recursively by $w_0 = a$, $w_1 = b$, $w_n = pw_{n-1} - qw_{n-2}$ $(n \geq 2)$, where a, b, p and q are arbitrary complex numbers with $p \neq 0$ and $q \neq 0$. Some related results emanating from the recurrence relations such as reciprocal sums, partial sums, and sums with double binomial coefficients are also presented.

1. Introduction

The Horadam sequence $w_n = w_n(a, b; p, q)$ is defined, for all integers, by the recurrence relation [8]

$$w_0 = a, \; w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

with

$$w_{-n} = \frac{1}{q}(pw_{-n+1} - w_{-n+2}),$$

where a, b, p and q are arbitrary complex numbers with non-zero p and q.

The sequence w_n generalizes many important number and polynomial sequences, for instance, the Fibonacci sequence $F_n = w_n(0, 1; 1, -1)$, the Lucas sequence $L_n = w_n(2, 1; 1, -1)$, the Pell sequence $P_n = w_n(0; 1; 2; -1)$, the Chebyshev polynomials of the first and second kind given by

$$T_n(x) = w_n(1, x; 2x, 1) \quad \text{and} \quad U_n(x) = w_n(1, 2x; 2x, 1), \quad \text{respectively.}$$

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The sequences F_n and L_n are special cases of the *gibonacci sequence* (generalized Fibonacci sequence) $G_n(a,b) = w_n(a,b;1,-1)$. The sequence G_n was studied by A. Horadam [7] in 1961 under the notation H_n. Sequences

$$u_n(p,q) = w_n(0,1;p,q) \quad \text{and} \quad v_n(p,q) = w_n(2,p;p,q)$$

are called the Lucas sequences of the first kind and of the second kind, respectively.

The Binet formulas for sequences u_n, v_n and w_n in the non-degenerated case, $p^2 - 4q > 0$, are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n, \qquad w_n = \frac{b - a\beta}{\alpha - \beta}\alpha^n + \frac{a\alpha - b}{\alpha - \beta}\beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are the distinct zeros of the characteristic polynomial $x^2 - px + q$.

More recent results on Horadam numbers can be found in [1,3–6,17], among others. Also, we refer the reader to the survey papers [13,14]. Properties of Lucas sequences can be found in [16]. The books [11, 12, 21] are excellent reference materials on Fibonacci and Lucas numbers.

Our purpose in this paper is to derive recurrence relations for the squares of the Horadam numbers. Based on these relations, we will present some related results, such as reciprocal summation identities, partial sum formulas and summation identities involving double binomial coefficients and the squares of the Horadam numbers.

2. Recurrence relations for the squares of Horadam numbers

The following identities were derived by Horadam [8]:

$$w_{m+r} = u_{r+1}w_m - qu_r w_{m-1}, \tag{1}$$

$$v_r w_m = w_{m+r} + q^r w_{m-r} \tag{2}$$

and

$$w_{n-r}w_{m+n+r} = w_n w_{m+n} + q^{n-r} e u_r u_{m+r},$$

where $e = pab - qa^2 - b^2$.

In our first theorem we state the recurrence relations for squares of Horadam numbers, which will be the key identities in the subsequent parts of the paper.

Theorem 2.1. *Let m and r be integers. Then*

$$pw_{m+r}^2 - u_r u_{r+1} w_{m+1}^2 = -pqu_{r+1}u_{r-1}w_m^2 + q^3 u_r u_{r-1} w_{m-1}^2 \tag{3}$$

and

$$w_{m+r}^2 - q^{3r} w_{m-2r}^2 = \left(v_r^2 - q^r\right)\left(w_m^2 - q^r w_{m-r}^2\right). \tag{4}$$

P r o o f. Squaring identity (1) and the recurrence relation $w_{m+1} = pw_m - qw_{m-1}$ and eliminating the cross-term $w_m w_{m-1}$ between the results gives identity (3).

To derive (4), replace m by $m - r$ in the identity (2) and rearrange to obtain

$$q^r w_{m-2r} = v_r w_{m-r} - w_m \,. \tag{5}$$

Now identity (2) can be rearranged as

$$w_{m+r} = v_r w_m - q^r w_{m-r} \,. \tag{6}$$

Squaring both (5) and (6) and eliminating the cross-term $w_m w_{m-r}$ between the resulting expressions yields identity (4). $\square$

Choosing $p = 1$ and $q = -1$ in relation (3) gives

$$G_{m+r}^2 = F_r F_{r+1} G_{m+1}^2 + F_{r+1} F_{r-1} G_m^2 - F_r F_{r-1} G_{m-1}^2$$

of which the familiar identity

$$G_{m+2}^2 = 2G_{m+1}^2 + 2G_m^2 - G_{m-1}^2$$

is a particular case (see [9]).

The gibonacci version of identity (4) is

$$G_{m+r}^2 - (-1)^r G_{m-2r}^2 = \big(L_r^2 - (-1)^r\big)\big(G_m^2 - (-1)^r G_{m-r}^2\big)$$

or, using $F_{3r} = \big(L_r - (-1)^r\big)F_r$ (see [11, p. 112, Formula 108]), equivalently,

$$\frac{G_{m+r}^2 - (-1)^r G_{m-2r}^2}{G_m^2 - (-1)^r G_{m-r}^2} = \frac{F_{3r}}{F_r}.$$

Since $u_n(2x, 1) = U_{n-1}(x)$ and $v_n(2x, 1) = 2T_n(x)$, the Chebyshev versions of (3) and (4), respectively, are

$$\begin{aligned} 2xT_{m+r}^2(x) = {} & U_{r-1}(x)U_r(x)T_{m+1}^2(x) \\ & - 2xU_{r-2}(x)U_r(x)T_m^2(x) + U_{r-2}(x)U_{r-1}(x)T_{m-1}^2(x), \end{aligned}$$

$$\begin{aligned} 2xU_{m+r}^2(x) = {} & U_{r-1}(x)U_r(x)U_{m+1}^2(x) \\ & - 2xU_{r-2}(x)U_r(x)U_m^2(x) + U_{r-2}(x)U_{r-1}(x)U_{m-1}^2(x) \end{aligned}$$

and

$$T_{m+r}^2(x) - T_{m-2r}^2(x) = \big(4T_r^2(x) - 1\big)\big(T_m^2(x) - T_{m-r}^2(x)\big),$$

$$U_{m+r}^2(x) - U_{m-2r}^2(x) = \big(4T_r^2(x) - 1\big)\big(U_m^2(x) - U_{m-r}^2(x)\big).$$

For the last identities we obtain

$$\frac{T_{m+r}^2(x) - T_{m-2r}^2(x)}{T_m^2(x) - T_{m-r}^2(x)} = \frac{U_{m+r}^2(x) - U_{m-2r}^2(x)}{U_m^2(x) - U_{m-r}^2(x)}.$$

Also, these relations yield as a special case

$$T_{m+1}^2(x) - T_{m-2}^2(x) = \big(4x^2 - 1\big)\big(T_m^2(x) - T_{m-1}^2(x)\big)$$

and

$$U_{m+1}^2(x) - U_{m-2}^2(x) = \big(4x^2 - 1\big)\big(U_m^2(x) - U_{m-1}^2(x)\big).$$

3. Some reciprocal series

In this section, we will use the following companion sequences:

$$h_n = w_n(a, b; p, -1), \quad s_n = u_n(p, -1), \quad r_n = w_n(a, b; p, 1), \quad t_n = u_n(p, 1).$$

From (3) the following result is an immediate consequence.

Theorem 3.1. *For $n \geq 1$, we have*

$$\sum_{m=2}^{n} \frac{h_{2m}^2 - s_{m-1} h_m^2 s_{m+1}}{s_{m-1} h_{m-1}^2 s_m^2 h_m^2 s_{m+1} h_{m+1}^2} = \frac{1}{p}\left(\frac{1}{pb^2(pb+a)^2} - \frac{1}{s_n h_n^2 s_{n+1} h_{n+1}^2}\right), \tag{7}$$

$$\sum_{m=2}^{\infty} \frac{h_{2m}^2 - s_{m-1} h_m^2 s_{m+1}}{s_{m-1} h_{m-1}^2 s_m^2 h_m^2 s_{m+1} h_{m+1}^2} = \frac{1}{p^2 b^2 (pb+a)^2}$$

and

$$\sum_{m=2}^{n} \frac{(-1)^m \big(r_{2m}^2 + t_{m-1} r_m^2 t_{m+1}\big)}{t_{m-1} r_{m-1}^2 t_m^2 r_m^2 t_{m+1} r_{m+1}^2} = \frac{1}{p}\left(\frac{1}{pb^2(pb-a)^2} + \frac{(-1)^n}{t_n r_n^2 t_{n+1} r_{n+1}^2}\right), \tag{8}$$

$$\sum_{m=2}^{\infty} \frac{(-1)^m \big(r_{2m}^2 + t_{m-1} r_m^2 t_{m+1}\big)}{t_{m-1} r_{m-1}^2 t_m^2 r_m^2 t_{m+1} r_{m+1}^2} = \frac{1}{p^2 b^2 (pb-a)^2}.$$

P r o o f. Write (3) as

$$p w_{m+r}^2 + pq u_{r+1} u_{r-1} w_m^2 = u_r u_{r+1} w_{m+1}^2 + q^3 u_r u_{r-1} w_{m-1}^2$$

and divide both sides by $u_{r-1} u_r^2 u_{r+1} w_{m-1}^2 w_m^2 w_{m+1}^2$ to get

$$\frac{p w_{m+r}^2}{u_{r-1} u_r^2 u_{r+1} w_{m-1}^2 w_m^2 w_{m+1}^2} + \frac{pq}{u_r^2 w_{m-1}^2 w_{m+1}^2} = \frac{1}{u_{r-1} u_r w_{m-1}^2 w_m^2} + \frac{q^3}{u_r u_{r+1} w_m^2 w_{m+1}^2}.$$

Now, setting $q = -1$ and $r = m$, and summing over m we recognize that the right-hand side telescopes. This completes the proof of the first part.

Similarly, with $q = 1$ and $r = m$ we identify the telescoping behavior of the alternating sum. $\square$

If we put in (7) $a = 0$, $b = 1$ and $p = 1$, then $h_m = s_m = F_m$ and using $F_{2m} = F_m L_m$ we have

$$\sum_{m=2}^{n} \frac{L_m^2 - F_{m-1}F_{m+1}}{F_{m-1}^3 F_m^2 F_{m+1}^3} = 1 - \frac{1}{F_n^3 F_{n+1}^3}\,.$$

Similarly, putting in (7) $a = 2$, $b = 1$ and $p = 1$ yields $h_m = L_m$, $s_m = F_m$ and, therefore,

$$\sum_{m=2}^{n} \frac{L_{2m}^2 - F_{m-1}L_m^2 F_{m+1}}{F_{m-1}L_{m-1}^2 F_m^2 L_m^2 F_{m+1} L_{m+1}^2} = \frac{1}{9} - \frac{1}{F_n L_n^2 F_{n+1} L_{n+1}^2}.$$

Focusing on Chebyshev polynomials, from (8) at $p = 2x$, $a = 1$, $b = x$ and $p = 2x$, $a = 1$, $b = 2x$ we have, respectively, the following examples (the argument x in Chebyshev polynomials is dropped to simplify notation):

$$\sum_{m=2}^{n} \frac{(-1)^m (T_{2m}^2 + U_{m-2} T_m^2 U_m)}{U_{m-2} T_{m-1}^2 U_{m-1}^2 T_m^2 U_m T_{m+1}^2} = \frac{1}{2x} \left(\frac{1}{2x^3(2x^2-1)^2} + \frac{(-1)^n}{U_{n-1} T_n^2 U_n T_{n+1}^2} \right),$$

$$\sum_{m=2}^{n} \frac{(-1)^m (U_{2m}^2 + U_{m-2} U_m^3)}{U_{m-2} U_{m-1}^4 U_m^3 U_{m+1}^2} = \frac{1}{2x} \left(\frac{1}{8x^3(4x^2-1)^2} + \frac{(-1)^n}{U_{n-1} U_n^3 U_{n+1}^2} \right).$$

From (4) we can deduce the following reciprocal sum identities.

Theorem 3.2. *For $n \geq 0$, we have*

$$\sum_{m=1}^{n} (-1)^m \frac{h_{m+2}^2 + h_{m-1}^2}{h_m^2 h_{m+1}^2} = (p^2+1) \left(\frac{(-1)^n}{h_{n+1}^2} - \frac{1}{b^2} \right), \tag{9}$$

$$\sum_{m=1}^{\infty} (-1)^{m-1} \frac{h_{m+2}^2 + h_{m-1}^2}{h_m^2 h_{m+1}^2} = \frac{p^2+1}{b^2} \tag{10}$$

and

$$\sum_{m=1}^{n} \frac{r_{m+2}^2 - r_{m-1}^2}{r_m^2 r_{m+1}^2} = (p^2-1) \left(\frac{1}{b^2} - \frac{1}{r_{n+1}^2} \right), \tag{11}$$

$$\sum_{m=1}^{\infty} \frac{r_{m+2}^2 - r_{m-1}^2}{r_m^2 r_{m+1}^2} = \frac{p^2-1}{b^2}.$$

P r o o f. We will prove only identities (9) and (10). The proof of the second part of the theorem, which we omit, is similar.

Replace m by $m + r$ in (4) to get

$$w_{m+2r}^2 - q^{3r} w_{m-r}^2 = \left(v_r^2 - q^r\right)\left(w_{m+r}^2 - q^r w_m^2\right),$$

which has an equivalent form as

$$\left(v_r^2 - q^r\right) \left(\frac{1}{w_m^2} - \frac{q^r}{w_{m+r}^2} \right) = \frac{w_{m+2r}^2 - q^{3r} w_{m-r}^2}{w_m^2 w_{m+r}^2}.$$

Now, setting $r = 1$ the telescoping nature of the left-hand side can be deduced. This completes the proof of (9). Upon letting $n \to +\infty$ in (9) yields (10). $\square$

We conclude this section with the gibonacci version of (9) and the Chebyshev version of (11). The first is

$$\sum_{m=1}^{n}(-1)^m \frac{G_{m+2}^2 + G_{m-1}^2}{G_m^2 G_{m+1}^2} = \frac{2(-1)^n}{G_{n+1}^2} - \frac{2}{b^2}$$

or, equivalently,

$$\sum_{m=1}^{n}(-1)^m \frac{G_m^2 + G_{m+1}^2}{G_m^2 G_{m+1}^2} = \frac{(-1)^n}{G_{n+1}^2} - \frac{1}{b^2}.$$

The Chebyshev identities are stated as

$$\sum_{m=1}^{n} \frac{T_{m+2}^2(x) - T_{m-1}^2(x)}{T_m^2(x) T_{m+1}^2(x)} = (4x^2 - 1)\left(\frac{1}{x^2} - \frac{1}{T_{n+1}^2(x)}\right)$$

and

$$\sum_{m=1}^{n} \frac{U_{m+2}^2(x) - U_{m-1}^2(x)}{U_m^2(x) U_{m+1}^2(x)} = (4x^2 - 1)\left(\frac{1}{4x^2} - \frac{1}{U_{n+1}^2(x)}\right).$$

4. Partial sum of the squares of Horadam numbers

Lemma 4.1 ([2, Lemma 2.3]). *Let X_j be any arbitrary sequence defined by the recurrence relation*

$$X_j = f_1 X_{j-c_1} + f_2 X_{j-c_2} + \cdots + f_n X_{j-c_n},$$

where $f_1, f_2, \ldots, f_n$ are arbitrary non-vanishing complex functions, not dependent on j, and $c_1, c_2, \ldots, c_n$ are fixed integers. Then the following identity holds for arbitrary x and $k \geq 0$:

$$\sum_{j=0}^{k} X_j z^j = \frac{\sum_{m=1}^{n} z^{c_m} f_m \left(\sum_{j=1}^{c_m} X_{-j} z^{-j} - \sum_{j=k-c_m+1}^{k} X_j z^j \right)}{1 - \sum_{m=1}^{n} z^{c_m} f_m}.$$

Theorem 4.2. *For $k \geq 0$, we have*

$$\sum_{j=0}^{k} w_j^2 x^j = \frac{E_k(x) + F_k(x) + G_k(x) + H_k(x)}{(1 - qx)\big(1 + (2q - p^2)x + q^2 x^2\big)},$$

where

$$E_k(x) = \big(p^2 - q\big)\big(x^{k+2} w_{k+1}^2 - a^2 x\big), \qquad F_k(x) = -q^3 x^2 \big(x^{k+1} w_k^2 - w_{-1}^2\big),$$
$$G_k(x) = -x^{k+1} w_{k+1}^2 + a^2, \qquad H_k(x) = -x^{k+2} w_{k+2}^2 + b^2 x,$$

with $w_{-1} = \frac{ap-b}{q}$.

Proof. Set $r = 2$ in identity (3) and rearrange to obtain

$$q(p^2-q)w_m^2 = (p^2-q)w_{m+1}^2 + q^3 w_{m-1}^2 - w_{m+2}^2 .$$

Now use Lemma 4.1 with

$$(c_1, c_2, c_3) = (-1, 1, -2)$$

and

$$(f_1, f_2, f_3) = \left(\frac{1}{q}, \frac{q^2}{p^2-q}, \frac{1}{q(p^2-q)}\right).$$

This yields

$$\sum_{j=0}^{k} w_j^2 x^j = \frac{E^*(x;k) + F^*(x;k) + G^*(x;k) + H^*(x;k)}{1 - \frac{1}{qx} - \frac{q^2 x}{p^2-q} + \frac{1}{qx^2(p^2-q)}},$$

where

$$E^*(x;k) = \frac{x^{k+1}w_{k+1}^2 - w_0^2}{qx}, \qquad F^*(x;k) = -\frac{q^2(x^{k+1}w_k^2 - w_{-1}^2)}{p^2-q},$$

$$G^*(x;k) = -\frac{x^{k+1}w_{k+1}^2 - w_0^2}{qx^2(p^2-q)}, \qquad H^*(x;k) = -\frac{x^{k+1}w_{k+2}^2 - w_1^2}{qx(p^2-q)}.$$

Simplify and the proof is completed. □

Corollary 4.3. *For $k \geq 0$,*

$$\sum_{j=0}^{k} G_j^2 x^j = \frac{1}{(1+x)(1-3x+x^2)} \times \big(-x^{k+2}G_{k+2}^2 + (2x-1)x^{k+1}G_{k+1}^2 + x^{k+3}G_k^2 + a^2 + (b^2-2a^2)x - (b-a)^2x^2\big). \quad (12)$$

Proof. Insert $w_0 = a$, $w_1 = b$, $p = 1$ and $q = -1$ in Theorem 4.2 and simplify. □

Three particular examples of (12) are

$$\sum_{j=0}^{k} G_j^2 = G_k G_{k+1} + a(a-b), \quad (13)$$

$$\sum_{j=0}^{k} G_j^2 L_j = \frac{1}{2}\big(L_{k+1}G_{k+2}^2 - 5F_k G_{k+1}^2 - L_{k+2}G_k^2 + 6a^2 - 2ab - b^2\big),$$

$$\sum_{j=0}^{k} G_j^2 F_j = \frac{1}{2}\big(F_{k+1}G_{k+2}^2 - L_k G_{k+1}^2 - F_{k+2}G_k^2 + b^2 - 2ab\big).$$

Note that formula (13) above is well-known (for example, see [11, p. 144]).

Another interesting example is the evaluation of the alternating sum $\sum_{j=0}^{k}(-1)^j G_j^2$. Inserting $x = -1$ in (12), we see that the sum is expressed as a fraction with denominator 0. For the numerator on the right-hand side we get

$$\begin{aligned}&(-1)^{k+1}(G_{k+2}^2 - 3G_{k+1}^2 + G_k^2) + a^2 - b^2 + 2a^2 - (b-a)^2\\ &= (-1)^{k+1}\big((G_{k+1}+G_k)^2 - 3G_{k+1}^2 + G_k^2\big) + a^2 - b^2 + 2a^2 - (b-a)^2\\ &= 2(-1)^{k+2}(G_{k+1}G_{k-1} - G_k^2) + 2a^2 - 2b^2 + 2ab.\end{aligned}$$

Now, using the Catalan identity $w_{m-r}w_{m+r} - w_n^2 = eq^{m-r}u_r^2$ [7], we see that

$$G_{k+1}G_{k-1} - G_k^2 = (-1)^{k+1}(a^2 - b^2 + ab)$$

and hence that the sum is an expression of the form $\frac{0}{0}$. Therefore, we apply L'Hospital rule and get after several steps of manipulations the closed form

$$5\sum_{j=0}^{k}(-1)^j G_j^2 = (-1)^k G_k(G_{k+1} + 2G_k) + (2k+3)(a^2 + ab - b^2) + 3b^2 - 4ab.$$

5. Double binomial sums involving the squares of Horadam numbers

In this section we present double sum identities with two binomial coefficients and the squares of the Horadam numbers in the summand.

Double sums with one binomial coefficient, as well as with two binomial coefficients, evaluating to Fibonacci and Lucas numbers, have been reported in existing literature. Examples of the former are

$$\sum_{0\le i,j\le n}\binom{n+i}{j-i} = F_{2n+3} - 2^n, \qquad \sum_{0\le i,j\le n}(-1)^j\binom{n+i}{j-i} = (-1)^n F_{2n},$$

while examples of the latter include

$$\sum_{0\le i,j\le n}(-1)^i\binom{n-i}{j}\binom{i+j}{j} = F_{n+1}, \qquad \sum_{0\le i,j\le n}\binom{n+i}{2j}\binom{j}{i}2^j = F_{3n+1}.$$

The above results were obtained by Kiliç and Arıkan [10]. Recently, attention has shifted to the study of double sums and triple sums with Fibonacci and Lucas numbers in the summand. For example, Taşdemir and Toska [20] presented the identity

$$\sum_{0\le i,j\le n}\binom{i}{j}L_{(4t-2)i+j} = \frac{L_{2nt}F_{2(n+1)t}}{F_{2t}}, \qquad t \ne 0,$$

while Ömür and Duran [15] studied triple sums with two binomial coefficients of the form

$$\sum_{0\le i,j,k\le n}\binom{i}{j}\binom{j}{k}F_{ri+j+k}.$$

Taşdemir [18,19] studied triple sums with two binomial coefficients, including Lucas numbers in the summand, as well as triple sums involving three binomial coefficients and Fibonacci numbers.

Lemma 5.1 ([2, Lemma 4.1]). *Let (X_n) be any arbitrary sequence, satisfying a four-term recurrence relation*

$$hX_n = f_1 X_{n-c_1} + f_2 X_{n-c_2} + f_3 X_{n-c_3},$$

where h, f_1, f_2 and f_3 are arbitrary non-vanishing functions and c_1, c_2 and c_3 are integers. Then the following identities hold:

$$\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\left(\frac{f_1}{f_2}\right)^i\left(\frac{f_2}{f_3}\right)^j X_{n-c_3k+(c_3-c_2)j+(c_2-c_1)i} = \left(\frac{h}{f_3}\right)^k X_n,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\left(\frac{f_1}{f_3}\right)^i\left(\frac{f_3}{f_2}\right)^j X_{n-c_2k+(c_2-c_3)j+(c_3-c_1)i} = \left(\frac{h}{f_2}\right)^k X_n, \tag{14}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\left(\frac{f_2}{f_3}\right)^i\left(\frac{f_3}{f_1}\right)^j X_{n-c_1k+(c_1-c_3)j+(c_3-c_2)i} = \left(\frac{h}{f_1}\right)^k X_n, \tag{15}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k-i}\binom{k}{j}\binom{j}{i}\left(\frac{h}{f_2}\right)^i\left(\frac{f_2}{f_3}\right)^j X_{n-(c_3-c_1)k+(c_3-c_2)j+c_2i} = \left(\frac{f_1}{f_3}\right)^k X_n,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k-i}\binom{k}{j}\binom{j}{i}\left(\frac{h}{f_1}\right)^i\left(\frac{f_1}{f_3}\right)^j X_{n-(c_3-c_2)k+(c_3-c_1)j+c_1i} = \left(\frac{f_2}{f_3}\right)^k X_n,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k-i}\binom{k}{j}\binom{j}{i}\left(\frac{h}{f_1}\right)^i\left(\frac{f_1}{f_2}\right)^j X_{n-(c_2-c_3)k+(c_2-c_1)j+c_1i} = \left(\frac{f_3}{f_2}\right)^k X_n.$$

Theorem 5.2. *Let r, n be integers and k be a non-negative integer. Then*

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k-j}\binom{k}{j}\binom{j}{i}\left(\frac{u_{r+1}}{q^3u_{r-1}}\right)^i\left(\frac{q^3u_{r-1}u_r}{p}\right)^j w^2_{n+rk-(r+1)j+2i} = \left(qu_{r-1}u_{r+1}\right)^k w_n^2, \tag{16}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\left(\frac{u_ru_{r+1}}{p}\right)^i\left(\frac{p}{q^3u_{r-1}u_r}\right)^j w^2_{n-k+(r+1)j-(r-1)i} = \left(\frac{pu_{r+1}}{q^2u_r}\right)^k w_n^2, \tag{17}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\left(\frac{q^3u_{r-1}u_r}{p}\right)^i\left(\frac{p}{u_ru_{r+1}}\right)^j w^2_{n+k+(r-1)j-(r+1)i} = \left(\frac{pqu_{r-1}}{u_r}\right)^k w_n^2, \tag{18}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\left(\frac{pu_{r+1}}{q^2u_r}\right)^i\left(\frac{q^3u_{r-1}u_r}{p}\right)^j w^2_{n+(r-1)k-(r+1)j+i} = \left(\frac{u_ru_{r+1}}{p}\right)^k w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\left(\frac{pqu_{r-1}}{u_r}\right)^i\left(\frac{u_ru_{r+1}}{p}\right)^j w^2_{n+(r+1)k-(r-1)j-i} = \left(\frac{q^3u_{r-1}u_r}{p}\right)^k w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\binom{k}{j}\binom{j}{i}\left(\frac{pqu_{r-1}}{u_r}\right)^i\left(\frac{u_{r+1}}{q^3u_{r-1}}\right)^j w^2_{n-(r+1)k+2j-i} = \left(\frac{p}{q^3u_{r-1}u_r}\right)^k w_n^2.$$

P r o o f. To prove (16), rearrange identity (3) as

$$pqu_{r+1}u_{r-1}w_n^2 = u_ru_{r+1}w_{n+1}^2 + q^3u_ru_{r-1}w_{n-1}^2 - pw_{n+r}^2.$$

Make the following identifications:

$$\begin{aligned} &X = w^2, && f_1 = u_ru_{r+1}, && f_2 = q^3u_ru_{r-1}, && f_3 = -p,\\ &c_1 = -1, && c_2 = 1, && c_3 = -r, && h = pqu_{r-1}u_{r+1}, \end{aligned} \tag{19}$$

and use these in Lemma 5.1.

To prove (17), use (19) in (14). To prove (18), use (19) in (15), and so on. □

Theorem 5.3. *Let r and n be integers. Let k be a non-negative integer. Then*

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\frac{q^{(k+2j-3i)r}}{(v_r^2-q^r)^j}w^2_{n-r(k-3i+j)} = w_n^2, \tag{20}$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k-j}\binom{k}{j}\binom{j}{i}\frac{q^{(3k-2j-i)r}}{(v_r^2-q^r)^{k-j+i}}w^2_{n-r(2k-j-2i)} = w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\binom{k}{j}\binom{j}{i}\frac{q^{(2i+j)r}}{(v_r^2-q^r)^{k-j+i}}w^2_{n+r(k-2j-i)} = w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\binom{k}{j}\binom{j}{i}q^{(3j-2i)r}(v_r^2-q^r)^{k-j+i}w^2_{n-r(k+2j-i)} = w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\binom{k}{j}\binom{j}{i}\frac{(v_r^2-q^r)^{k-j+i}}{q^{(j+2k)r}}w^2_{n+r(k+2j-i)} = w_n^2,$$

$$\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{j}\binom{j}{i}\frac{q^{(2k-3j)r}}{(v_r^2-q^r)^{k-i}}w^2_{n-r(k-3j+i)} = w_n^2.$$

Proof. To prove (20), write identity (4) as

$$\left(v_r^2 - q^r\right)w_n^2 = w_{n+r}^2 - q^{3r}w_{n-2r}^2 + q^r\left(v_r^2 - q^r\right)w_{n-r}^2\,,$$

then make the identifications

$$(f_1, f_2, f_3) = \left(1, -q^{3r}, q^r(v_r^2 - q^r)\right), \quad (c_1, c_2, c_3) = (-r, 2r, r), \quad h = v_r^2 - q^r$$

and use these in Lemma 5.1. $\square$

The special case of Theorem 5.3 when $w_n = F_n$, $v_n = L_n$ and $r = 1$ can be stated as ($n \geq 0$):

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\left(-\frac{1}{2}\right)^{j}\binom{n}{j}\binom{j}{i}F_{3i-j}^2 = (-1)^n F_n^2,$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\left(-\frac{1}{2}\right)^{n-j+i}\binom{n}{j}\binom{j}{i}F_{2i+j}^2 = (-1)^n F_{2n}^2,$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\left(-\frac{1}{2}\right)^{n-j+i}\binom{n}{j}\binom{j}{i}F_{2n-i-2j}^2 = (-1)^n F_n^2,$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-2)^{n-j+i}\binom{n}{j}\binom{j}{i}F_{i-2j}^2 = (-1)^n F_n^2,$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-2)^{n-j+i}\binom{n}{j}\binom{j}{i}F_{2n-i+2j}^2 = (-1)^n F_n^2$$

and

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\left(-\frac{1}{2}\right)^{n-i}\binom{n}{j}\binom{j}{i}F_{3j-i}^2 = (-1)^n F_n^2.$$

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