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Fibonacci–Lucas–Pell–Jacobsthal relations

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Abstract. In this paper, we prove several identities involving linear combinations of convolutions of the generalized Fibonacci and Lucas sequences. Our results apply more generally to broader classes of second-order linearly recurrent sequences with constant coefficients. As a consequence, we obtain as special cases many identities relating exactly four sequences amongst the Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas number sequences. We make use of algebraic arguments to establish our results, frequently employing the Binet-like formulas and generating functions of the corresponding sequences. Finally, our identities above may be extended so that they include only terms whose subscripts belong to a given arithmetic progression of the non-negative integers.

Keywords: Generalized Fibonacci sequence, generalized Lucas sequence, Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, generating function

AMS Subject Classification: 11B39, 11B37

1. Introduction

Let $U_n = U_n(p, q)$ denote the sequence defined recursively by

$$U_0 = 0,\ U_1 = 1, \quad U_n = pU_{n-1} + qU_{n-2},\ n \geq 2,$$

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and let $V_n = V_n(p,q)$ be given by

$$V_0 = 2,\ V_1 = p, \quad V_n = pV_{n-1} + qV_{n-2},\ n \geq 2.$$

Note that U_n and V_n correspond to special cases of the Horadam sequence and will be referred to here as *generalized* Fibonacci and Lucas sequences, respectively.

We note the special cases $F_n = U_n(1,1)$, $P_n = U_n(2,1)$, and $J_n = U_n(1,2)$ corresponding to the Fibonacci, Pell, and Jacobsthal number sequences, respectively, as well as $L_n = V_n(1,1)$, $Q_n = V_n(2,1)$, and $j_n = V_n(1,2)$ corresponding to the Lucas, Pell–Lucas, and Jacobsthal–Lucas numbers. In addition, we note that the balancing numbers $B_n = U_n(6,-1)$ also belong to the class of generalized Fibonacci sequences, while Lucas-balancing numbers C_n, usually defined by the initial values $C_0 = 1$ and $C_1 = 3$, do not belong to the class V_n.

The sequences F_n, L_n, P_n, Q_n, J_n, j_n, and B_n are indexed in the On-Line Encyclopedia of Integer Sequences [14], the first few terms of which are stated below:

n	0	1	2	3	4	5	6	7	8	9	Sequence in [14]
F_n	0	1	1	2	3	5	8	13	21	34	A000045
L_n	2	1	3	4	7	11	18	29	47	76	A000032
P_n	0	1	2	5	12	29	70	169	408	985	A000129
Q_n	2	2	6	14	34	82	198	478	1154	2786	A002203
J_n	0	1	1	3	5	11	21	43	85	171	A001045
j_n	2	1	5	7	17	31	65	127	257	511	A014551
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	A001109

In this paper, we adopt a unifying approach to identities involving various combinations of these sequences. In this direction, Adegoke [2] derived several identities for arbitrary homogeneous second order recursive sequences with constant coefficients and applied these results to present a unified study of the sequences above. Later in [1], he found binomial and ordinary summation formulas arising from an identity connecting any two second-order linearly recurrent sequences having the same recurrence but whose initial terms may differ. Illustrative examples were drawn from the aforementioned sequences and their generalizations.

Further, some isolated results in this direction have also occurred. For example, in [12], the author asked to express P_n in terms of F_n and L_n. One possible solution is to express this relationship as [9]

$$\sum_{s=0}^{n} F_s P_{n-s} = P_n - F_n.$$

A generalization of this identity was given by Seiffert in [13]:

$$\sum_{s=0}^{n} F_{k(s+1)} P_{k(n+1-s)} = \frac{F_k P_{k(n+2)} - P_k F_{k(n+2)}}{2Q_k - L_k}, \quad k \geq 1.$$

Moreover, similar convolution identities involving Fibonacci, Lucas, and generalized balancing numbers can be found in [5], whereas new convolution relations between Fibonacci, Lucas, tribonacci, and tribonacci-Lucas numbers were derived by the second author in [7]. A short time later, in [6], these results were extended to generalized Fibonacci and tribonacci sequences defined, respectively, by the recurrences $u_n = u_{n-1} + u_{n-2}$ and $v_n = v_{n-1} + v_{n-2} + v_{n-3}$ with arbitrary initial values.

The first and second authors [8] have established connection formulas between the Mersenne numbers $M_n = 2^n - 1$ and Horadam numbers w_n defined by $w_n = pw_{n-1} + qw_{n-2}$ for $n \geq 2$ with $w_0 = a$ and $w_1 = b$ and stated several explicit examples involving Fibonacci, Lucas, Pell, and Jacobsthal numbers to highlight the results. In [3], some special families of finite sums with squared Horadam numbers were found, which yield formulas involving squared Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas, and tribonacci numbers as particular cases. In [11], Koshy and Griffiths developed convolution formulas linking the Fibonacci, Lucas, Jacobsthal, and Jacobsthal–Lucas polynomials, and then deduced the corresponding results for Fibonacci–Jacobsthal–Lucas, Lucas–Jacobsthal, and Lucas–Jacobsthal–Lucas convolutions. Bramham and Griffiths in [4] obtained, using combinatorial arguments, a number of convolution identities involving the Jacobsthal and Jacobsthal–Lucas numbers as well as various generalizations of the Fibonacci numbers. Using generating functions, Koshy [10] developed a number of properties for sums of products of Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers. In [15, 16], Szakács dealt with convolutions of second order recursive sequences and gave some special convolutions involving the Fibonacci, Pell, Jacobsthal, and Mersenne sequences and their associated sequences.

In the next section, we prove several general formulas involving linear combinations of certain convolutions of U_n and V_n. These results in turn are obtained as special cases of more general identities involving second-order linearly recurrent sequences with constant coefficients and arbitrary initial values meeting at times certain auxiliary conditions. As a consequence of our formulas for U_n and V_n, we obtain several identities for F_n, L_n, P_n, Q_n, J_n, and j_n, each involving exactly four of these sequences. In the third section, it is demonstrated that the aforementioned formulas for U_n and V_n may be extended so that the subscript of each summand term belongs to a given arithmetic progression. Finally, some further general results are given in which it is required that the sequences appearing in the convolutions meet certain conditions with regard to their initial values and recurrence coefficients.

2. Main results

Let $T_n = T_n(a, b, p, q)$ denote the sequence defined recursively by

$$T_n = pT_{n-1} + qT_{n-2}, \quad n \geq 2,$$

with $T_0 = a$ and $T_1 = b$, where a, b, p, and q are arbitrary and $p^2 + 4q \neq 0$. Note that T_n reduces to U_n when $a = 0$, $b = 1$ and to V_n when $a = 2$, $b = p$. It can be shown that $T_n = \alpha r_1^n + \beta r_2^n$ for $n \geq 0$, where

$$\alpha = \frac{2b - ap + a\Delta}{2\Delta}, \quad \beta = \frac{ap - 2b + a\Delta}{2\Delta}, \quad r_1 = \frac{p + \Delta}{2}, \quad r_2 = \frac{p - \Delta}{2},$$

and $\Delta = \sqrt{p^2 + 4q}$. Note that

$$\alpha r_2 + \beta r_1 = \frac{(2b - ap + a\Delta)(p - \Delta) + (ap - 2b + a\Delta)(p + \Delta)}{4\Delta} = ap - b. \tag{2.1}$$

Thus, we get

$$\begin{aligned}\sum_{n\geq 0} T_n x^n = \sum_{n\geq 0} \big(\alpha r_1^n + \beta r_2^n\big) x^n &= \frac{\alpha}{1 - r_1 x} + \frac{\beta}{1 - r_2 x} \\ &= \frac{\alpha + \beta - (\alpha r_2 + \beta r_1)x}{(1 - r_1 x)(1 - r_2 x)} = \frac{a - (ap - b)x}{1 - px - qx^2}.\end{aligned} \tag{2.2}$$

Let $T_n^{(i)} = T_n(a_i, b_i, p_i, q_i)$, where i is fixed and (a_i, b_i, p_i, q_i) is arbitrary for each i. We will make frequent use of the following generating function formula for the product $T_n^{(1)} T_n^{(2)}$.

Lemma 2.1. *We have*

$$\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} x^n = \frac{G_1(x)}{G_2(x)}, \tag{2.3}$$

where

$$\begin{aligned}G_1(x) = a_1 a_2 &+ (b_1 b_2 - a_1 a_2 p_1 p_2)x \\ &+ \big(a_1 b_2 p_2 q_1 + a_2 b_1 p_1 q_2 - a_1 a_2 (p_1^2 q_2 + p_2^2 q_1 + q_1 q_2)\big)x^2 \\ &- q_1 q_2 (b_1 - a_1 p_1)(b_2 - a_2 p_2)x^3, \\ G_2(x) = 1 &- p_1 p_2 x - (p_1^2 q_2 + p_2^2 q_1 + 2q_1 q_2)x^2 - p_1 p_2 q_1 q_2 x^3 + q_1^2 q_2^2 x^4.\end{aligned}$$

Proof. For $i = 1, 2$, let

$$\begin{aligned}\Delta_i = \sqrt{p_i^2 + 4q_i}, \quad r_1^{(i)} &= \frac{p_i + \Delta_i}{2}, \quad r_2^{(i)} = \frac{p_i - \Delta_i}{2}, \\ \alpha_i = \frac{2b_i - a_i p_i + a_i \Delta_i}{2\Delta_i}, &\quad \beta_i = \frac{a_i p_i - 2b_i + a_i \Delta_i}{2\Delta_i}.\end{aligned}$$

Then

$$T_n^{(1)} T_n^{(2)} = \big(\alpha_1 (r_1^{(1)})^n + \beta_1 (r_2^{(1)})^n\big)\big(\alpha_2 (r_1^{(2)})^n + \beta_2 (r_2^{(2)})^n\big)$$

implies

$$\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} x^n = \frac{\alpha_1 \alpha_2}{1 - r_1^{(1)} r_1^{(2)} x} + \frac{\alpha_1 \beta_2}{1 - r_1^{(1)} r_2^{(2)} x} + \frac{\alpha_2 \beta_1}{1 - r_1^{(2)} r_2^{(1)} x} + \frac{\beta_1 \beta_2}{1 - r_2^{(1)} r_2^{(2)} x}$$

$$= \alpha_2\left(\frac{\alpha_1}{1-r_1^{(1)}(r_1^{(2)}x)}+\frac{\beta_1}{1-r_2^{(1)}(r_1^{(2)}x)}\right)+\beta_2\left(\frac{\alpha_1}{1-r_1^{(1)}(r_2^{(2)}x)}+\frac{\beta_1}{1-r_2^{(1)}(r_2^{(2)}x)}\right)$$
$$= \alpha_2\cdot\frac{a_1-(a_1p_1-b_1)(r_1^{(2)}x)}{1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2}+\beta_2\cdot\frac{a_1-(a_1p_1-b_1)(r_2^{(2)}x)}{1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2},$$

where we have used (2.2) (with x replaced by $r_1^{(2)}x$ and by $r_2^{(2)}x$) in the last equality.

Thus we have

$$\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n = \frac{H_1(x)}{H_2(x)},$$

where

$$\begin{aligned} H_1(x) &= \alpha_2\big(a_1-(a_1p_1-b_1)r_1^{(2)}x\big)\big(1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2\big)\\ &\quad+\beta_2\big(a_1-(a_1p_1-b_1)r_2^{(2)}x\big)\big(1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2\big),\\ H_2(x) &= \big(1-p_1r_1^{(2)}x-q_1(r_1^{(2)})^2x^2\big)\big(1-p_1r_2^{(2)}x-q_1(r_2^{(2)})^2x^2\big). \end{aligned}$$

We now work separately on the numerator and denominator of the last expression, starting with the numerator. Expanding the numerator, and using the facts $\alpha_2+\beta_2=a_2$ and $r_1^{(2)}r_2^{(2)}=-q_2$, gives

$$\begin{aligned} H_1(x) &= a_1(\alpha_2+\beta_2)\\ &\quad-\Big(a_1\alpha_2p_1r_2^{(2)}+\alpha_2(a_1p_1-b_1)r_1^{(2)}+a_1\beta_2p_1r_1^{(2)}+\beta_2(a_1p_1-b_1)r_2^{(2)}\Big)x\\ &\quad+\Big(\alpha_2\big(p_1(a_1p_1-b_1)r_1^{(2)}r_2^{(2)}-a_1q_1(r_2^{(2)})^2\big)\\ &\quad+\beta_2\big(p_1(a_1p_1-b_1)r_1^{(2)}r_2^{(2)}-a_1q_1(r_1^{(2)})^2\big)\Big)x^2\\ &\quad+\Big(\alpha_2q_1(a_1p_1-b_1)r_1^{(2)}(r_2^{(2)})^2+\beta_2q_1(a_1p_1-b_1)(r_1^{(2)})^2r_2^{(2)}\Big)x^3\\ &= a_1a_2-\Big((a_1a_2p_1-b_1\alpha_2)r_1^{(2)}+(a_1a_2p_1-b_1\beta_2)r_2^{(2)}\Big)x\\ &\quad+\Big(a_2p_1q_2(b_1-a_1p_1)-a_1q_1\big(\alpha_2(r_2^{(2)})^2+\beta_2(r_1^{(2)})^2\big)\Big)x^2\\ &\quad+q_1q_2(b_1-a_1p_1)\Big(\alpha_2r_2^{(2)}+\beta_2r_1^{(2)}\Big)x^3. \end{aligned}$$

Concerning the coefficient of x in the last expression, note that

$$a_1a_2p_1\big(r_1^{(2)}+r_2^{(2)}\big)-b_1\big(\alpha_2r_1^{(2)}+\beta_2r_2^{(2)}\big)=a_1a_2p_1p_2-b_1b_2.$$

Also, observe that $\alpha r_2^2+\beta r_1^2$ (in the notation above) is given by

$$\frac{(2b-ap+a\Delta)(p^2+2q-p\Delta)+(ap-2b+a\Delta)(p^2+2q+p\Delta)}{4\Delta}=ap^2+aq-bp.$$

Thus, the coefficient of x^2 in the numerator equals

$$a_2p_1q_2(b_1-a_1p_1)+a_1q_1(b_2p_2-a_2q_2-a_2p_2^2).$$

Finally, note that $\alpha_2 r_2^{(2)} + \beta_2 r_1^{(2)} = a_2 p_2 - b_2$, by (2.1) (with a_2 and b_2 in place of a and b and p_2 and q_2 in place of p and q), which implies that the coefficient of x^3 is given by $q_1 q_2 (b_1 - a_1 p_1)(a_2 p_2 - b_2)$. Thus, the numerator of the generating function works out to

$$a_1 a_2 + (b_1 b_2 - a_1 a_2 p_1 p_2)x + \big(a_2 p_1 q_2 (b_1 - a_1 p_1) + a_1 q_1 (b_2 p_2 - a_2 q_2 - a_2 p_2^2)\big)x^2 \\ - q_1 q_2 (b_1 - a_1 p_1)(b_2 - a_2 p_2)x^3.$$

In the denominator, we have

$$\begin{aligned} H_2(x) &= \Big(1 - p_1 r_1^{(2)} x - q_1 (r_1^{(2)})^2 x^2\Big)\Big(1 - p_1 r_2^{(2)} x - q_1 (r_2^{(2)})^2 x^2\Big) \\ &= 1 - p_1\big(r_1^{(2)} + r_2^{(2)}\big)x - \Big(q_1\big((r_1^{(2)})^2 + (r_2^{(2)})^2\big) - p_1^2 r_1^{(2)} r_2^{(2)}\Big)x^2 \\ &\quad + p_1 q_1 r_1^{(2)} r_2^{(2)}\big(r_1^{(2)} + r_2^{(2)}\big)x^3 + q_1^2 (r_1^{(2)} r_2^{(2)})^2 x^4 \\ &= 1 - p_1 p_2 x - \big(q_1(p_2^2 + 2q_2) + p_1^2 q_2\big)x^2 - p_1 p_2 q_1 q_2 x^3 + q_1^2 q_2^2 x^4. \end{aligned}$$

Combining this expression with the one above for the numerator yields (2.3). □

We having the following general formula involving certain sums of convolutions of $T_n^{(1)} T_n^{(2)}$ with $T_n^{(3)} T_n^{(4)}$ where there are no restrictions on the parameters of the various $T_n^{(i)}$.

Theorem 2.2. *If $n \geq 4$, then*

$$\begin{aligned} &\sum_{s=0}^{n-4}\Big((p_1 p_2 - p_3 p_4) T_{n-1-s}^{(3)} T_{n-1-s}^{(4)} \\ &\quad + \big(p_1^2 q_2 + p_2^2 q_1 - p_3^2 q_4 - p_4^2 q_3 + 2q_1 q_2 - 2q_3 q_4\big) T_{n-2-s}^{(3)} T_{n-2-s}^{(4)} \\ &\quad + (p_1 p_2 q_1 q_2 - p_3 p_4 q_3 q_4) T_{n-3-s}^{(3)} T_{n-3-s}^{(4)} - (q_1^2 q_2^2 - q_3^2 q_4^2) T_{n-4-s}^{(3)} T_{n-4-s}^{(4)}\Big) T_s^{(1)} T_s^{(2)} \\ &= - a_1 a_2 T_n^{(3)} T_n^{(4)} + (a_1 a_2 p_1 p_2 - b_1 b_2) T_{n-1}^{(3)} T_{n-1}^{(4)} \\ &\quad + \big(a_1 a_2 (p_1^2 q_2 + p_2^2 q_1 + q_1 q_2) - a_1 b_2 p_2 q_1 - a_2 b_1 p_1 q_2\big) T_{n-2}^{(3)} T_{n-2}^{(4)} \\ &\quad + q_1 q_2 (b_1 - a_1 p_1)(b_2 - a_2 p_2) T_{n-3}^{(3)} T_{n-3}^{(4)} + a_3 a_4 T_n^{(1)} T_n^{(2)} \\ &\quad - (a_3 a_4 p_1 p_2 - b_3 b_4) T_{n-1}^{(1)} T_{n-1}^{(2)} \\ &\quad - \Big(a_3 a_4 (p_1^2 q_2 + p_2^2 q_1 + 2q_1 q_2 - q_3 q_4) + b_3 b_4 (p_1 p_2 - p_3 p_4) \\ &\quad - a_3 b_4 p_4 q_3 - a_4 b_3 p_3 q_4\Big) T_{n-2}^{(1)} T_{n-2}^{(2)} \\ &\quad - \Big(a_3 a_4 p_1 p_2 q_3 q_4 + a_3 b_4 p_1 p_2 p_4 q_3 + a_4 b_3 p_1 p_2 p_3 q_4 + b_3 b_4 p_1 p_2 p_3 p_4 - a_3 a_4 p_3 p_4 q_3 q_4 \\ &\quad - a_3 b_4 p_3 p_4^2 q_3 - a_4 b_3 p_3^2 p_4 q_4 - b_3 b_4 p_3^2 p_4^2 + a_3 a_4 p_1 p_2 q_1 q_2 - a_3 b_4 p_3 q_3 q_4 \\ &\quad - a_4 b_3 p_4 q_3 q_4 + b_3 b_4 \big(p_1^2 q_2 + p_2^2 q_1 - p_3^2 q_4 - p_4^2 q_3 + 2q_1 q_2 - q_3 q_4\big)\Big) T_{n-3}^{(1)} T_{n-3}^{(2)}. \end{aligned} \tag{2.4}$$

Proof. Consider the quantity

$$\begin{aligned}
&a_3a_4T_n^{(1)}T_n^{(2)} + (b_3b_4 - a_3a_4p_3p_4)T_{n-1}^{(1)}T_{n-1}^{(2)} \\
&\quad + \Big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)\Big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\
&\quad - q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)} \\
&\quad - \Big(a_1a_2T_n^{(3)}T_n^{(4)} + (b_1b_2 - a_1a_2p_1p_2)T_{n-1}^{(3)}T_{n-1}^{(4)} \\
&\quad + \big(a_1b_2p_2q_1 + a_2b_1p_1q_2 - a_1a_2(p_1^2q_2 + p_2^2q_1 + q_1q_2)\big)T_{n-2}^{(3)}T_{n-2}^{(4)} \\
&\quad - q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)T_{n-3}^{(3)}T_{n-3}^{(4)}\Big),
\end{aligned} \tag{2.5}$$

where $T_m^{(i)}$ is taken to be zero for all i if $m < 0$. By Lemma 2.1, the generating function of the quantity (2.5) for $n \geq 0$ is given by the product of $\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n$ and $\sum_{n\geq 0} T_n^{(3)}T_n^{(4)}x^n$ with

$$\begin{aligned}
&(p_1p_2 - p_3p_4)x + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)x^2 \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)x^3 - (q_1^2q_2^2 - q_3^2q_4^2)x^4.
\end{aligned}$$

Extracting the coefficient of x^n of this generating function gives for $n \geq 4$,

$$\begin{aligned}
&(p_1p_2 - p_3p_4)\sum_{s=0}^{n-1} T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)\sum_{s=0}^{n-2} T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)\sum_{s=0}^{n-3} T_{n-3-s}^{(3)}T_{n-3-s}^{(4)}T_s^{(1)}T_s^{(2)} \\
&\quad - (q_1^2q_2^2 - q_3^2q_4^2)\sum_{s=0}^{n-4} T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}T_s^{(1)}T_s^{(2)},
\end{aligned}$$

which holds also for $0 \leq n \leq 3$ since empty sums are zero by convention. Equating this last quantity with (2.5) above, and shifting summands to the other side so that each sum has upper index $n-4$, gives

$$\begin{aligned}
&\sum_{s=0}^{n-4}\Big((p_1p_2 - p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)} \\
&\quad + (p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)} \\
&\quad + (p_1p_2q_1q_2 - p_3p_4q_3q_4)T_{n-3-s}^{(3)}T_{n-3-s}^{(4)} - (q_1^2q_2^2 - q_3^2q_4^2)T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)} \\
&= -a_1a_2T_n^{(3)}T_n^{(4)} + (a_1a_2p_1p_2 - b_1b_2)T_{n-1}^{(3)}T_{n-1}^{(4)}
\end{aligned}$$

$$+ \big(a_1a_2(p_1^2q_2 + p_2^2q_1 + q_1q_2) - a_1b_2p_2q_1 - a_2b_1p_1q_2\big)T^{(3)}_{n-2}T^{(4)}_{n-2}$$
$$+ q_1q_2(b_1 - a_1p_1)(b_2 - a_2p_2)T^{(3)}_{n-3}T^{(4)}_{n-3} + a_3a_4T^{(1)}_nT^{(2)}_n$$
$$+ \big(b_3b_4 - a_3a_4p_3p_4 - a_3a_4(p_1p_2 - p_3p_4)\big)T^{(1)}_{n-1}T^{(2)}_{n-1}$$
$$+ \Big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4) - b_3b_4(p_1p_2 - p_3p_4)$$
$$- a_3a_4(p_1^2q_2 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)\Big)T^{(1)}_{n-2}T^{(2)}_{n-2}$$
$$-\Big(q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4) + b_3b_4(p_1^2q_1 + p_2^2q_1 - p_3^2q_4 - p_4^2q_3 + 2q_1q_2 - 2q_3q_4)$$
$$+ a_3a_4(p_1p_2q_1q_2 - p_3p_4q_3q_4)$$
$$+ (p_1p_2 - p_3p_4)(a_3q_3 + b_3p_3)(a_4q_4 + b_4p_4)\Big)T^{(1)}_{n-3}T^{(2)}_{n-3}.$$

Simplifying the right side of the last equality gives (2.4). □

Note that (2.4) also holds for $0 \le n \le 3$, by the convention for empty sums, with this applying comparably to subsequent results.

We now state some special cases of (2.4) involving the generalized Fibonacci and Lucas sequences. Let $U^{(i)}_n = T_n(0, 1, p_i, q_i)$, $V^{(i)}_n = T_n(2, p_i, p_i, q_i)$ for a fixed i. Equivalently, these are the specializations of $T^{(i)}_n$ when $a_i = 0$, $b_i = 1$ and when $a_i = 2$, $b_i = p_i$, respectively.

Letting (a_i, b_i, p_i, q_i) for $1 \le i \le 4$ be given by $(0, 1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_1, p_1, q_1)$, $(2, p_2, p_2, q_2)$, respectively, in (2.4) yields the following formula.

Corollary 2.3 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n U^{(2)}_n}\big)$ and $\big(\boldsymbol{V^{(1)}_n V^{(2)}_n}\big)$)**.** *For $n \ge 3$,*

$$V^{(1)}_{n-1}V^{(2)}_{n-1} - q_1q_2V^{(1)}_{n-3}V^{(2)}_{n-3} = 4U^{(1)}_nU^{(2)}_n - 3p_1p_2U^{(1)}_{n-1}U^{(2)}_{n-1}$$
$$- 2(p_1^2q_2 + p_2^2q_1 + 2q_1q_2)U^{(1)}_{n-2}U^{(2)}_{n-2} - p_1p_2q_1q_2U^{(1)}_{n-3}U^{(2)}_{n-3}.$$

Example 2.4.

$$\begin{aligned} &L_{n-1}Q_{n-1} - L_{n-3}Q_{n-3} \\ &= 2\big(2F_nP_n - 3F_{n-1}P_{n-1} - 7F_{n-2}P_{n-2} - F_{n-3}P_{n-3}\big), \\ &L_{n-1}j_{n-1} - 2L_{n-3}j_{n-3} = 4F_nJ_n - 3F_{n-1}J_{n-1} - 14F_{n-2}J_{n-2} - 2F_{n-3}J_{n-3}, \\ &Q_{n-1}j_{n-1} - 2Q_{n-3}j_{n-3} = 2\big(2P_nJ_n - 3P_{n-1}J_{n-1} - 13P_{n-2}J_{n-2} - 2P_{n-3}J_{n-3}\big). \end{aligned} \tag{2.6}$$

Letting (a_i, b_i, p_i, q_i) for $1 \le i \le 4$ be given by $(0, 1, p_1, q_1)$, $(2, p_2, p_2, q_2)$, $(0, 1, p_2, q_2)$, $(2, p_1, p_1, q_1)$, respectively, in (2.4), and replacing n by $n + 1$, yields the following result.

Corollary 2.5 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n V^{(2)}_n}\big)$ and $\big(\boldsymbol{U^{(2)}_n V^{(1)}_n}\big)$)**.** *For $n \ge 2$,*

$$p_1U^{(1)}_nV^{(2)}_n + 2p_2q_1U^{(1)}_{n-1}V^{(2)}_{n-1} + p_1q_1q_2U^{(1)}_{n-2}V^{(2)}_{n-2}$$
$$= p_2U^{(2)}_nV^{(1)}_n + 2p_1q_2U^{(2)}_{n-1}V^{(1)}_{n-1} + p_2q_1q_2U^{(2)}_{n-2}V^{(1)}_{n-2}.$$

Example 2.6.

$$F_nQ_n + 4F_{n-1}Q_{n-1} + F_{n-2}Q_{n-2} = 2(L_nP_n + L_{n-1}P_{n-1} + L_{n-2}P_{n-2}),$$
$$F_nj_n + 2F_{n-1}j_{n-1} + 2F_{n-2}j_{n-2} = L_nJ_n + 4L_{n-1}J_{n-1} + 2L_{n-2}J_{n-2},$$
$$2\big(P_nj_n + P_{n-1}j_{n-1} + 2P_{n-2}j_{n-2}\big) = Q_nJ_n + 8Q_{n-1}J_{n-1} + 2Q_{n-2}J_{n-2}.$$

Taking $(0,1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(2,p_2,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in (2.4) gives the following result.

Corollary 2.7 (Sequence pairs $(\boldsymbol{U_n^{(1)}U_n^{(2)}})$ and $(\boldsymbol{V_n^{(2)}V_n^{(3)}})$)**.** *For* $n \geq 4$,

$$\begin{aligned}
&\sum_{s=1}^{n-4}\Big(p_2(p_1-p_3)V_{n-1-s}^{(2)}V_{n-1-s}^{(3)}\\
&\quad + \big(p_1^2q_2 + p_2^2q_1 - p_2^2q_3 - p_3^2q_2 + 2q_1q_2 - 2q_2q_3\big)V_{n-2-s}^{(2)}V_{n-2-s}^{(3)}\\
&\quad + p_2q_2(p_1q_1 - p_3q_3)V_{n-3-s}^{(2)}V_{n-3-s}^{(3)} + q_2^2(q_3^2 - q_1^2)V_{n-4-s}^{(2)}V_{n-4-s}^{(3)}\Big)U_s^{(1)}U_s^{(2)}\\
&= -V_{n-1}^{(2)}V_{n-1}^{(3)} + q_1q_2V_{n-3}^{(2)}V_{n-3}^{(3)} + 4U_n^{(1)}U_n^{(2)} + p_2(p_3-4p_1)U_{n-1}^{(1)}U_{n-1}^{(2)}\\
&\quad + \Big(p_2^2p_3^2 - p_1p_2^2p_3 - 4p_1^2q_2 - 4p_2^2q_1 + 2p_2^2q_3 + 2p_3^2q_2 + 4q_2q_3 - 8q_1q_2\Big)U_{n-2}^{(1)}U_{n-2}^{(2)}\\
&\quad + p_2\Big(p_2^2p_3^3 + 3p_2^2p_3q_3 + 3p_3^3q_2 + 9p_3q_2q_3 - p_1p_2^2p_3^2 - 2p_1p_2^2q_3 - 2p_1p_3^2q_2\\
&\quad - 4p_1q_2q_3 - p_1^2p_3q_2 - p_2^2p_3q_1 - 2p_3q_1q_2 - 4p_1q_1q_2\Big)U_{n-3}^{(1)}U_{n-3}^{(2)}.
\end{aligned}$$

Example 2.8.

$$\begin{aligned}
&\sum_{s=1}^{n-4}\big(6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s} - 3Q_{n-4-s}j_{n-4-s}\big)F_sP_s\\
&= Q_{n-1}j_{n-1} - Q_{n-3}j_{n-3} - 4F_nP_n + 6F_{n-1}P_{n-1} + 2F_{n-2}P_{n-2} - 16F_{n-3}P_{n-3},\\
&\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} + 6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s}\big)F_sJ_s\\
&= Q_{n-1}j_{n-1} - 2Q_{n-3}j_{n-3} - 4F_nJ_n + 2F_{n-1}J_{n-1} - 46F_{n-3}J_{n-3},\\
&\sum_{s=1}^{n-4}\big(L_{n-1-s}j_{n-1-s} + 6L_{n-2-s}j_{n-2-s} + 2L_{n-3-s}j_{n-3-s}\big)P_sJ_s\\
&= -L_{n-1}j_{n-1} + 2L_{n-3}j_{n-3} + 4P_nJ_n - 7P_{n-1}J_{n-1} - 39P_{n-2}J_{n-2} - 31P_{n-3}J_{n-3}.
\end{aligned}$$

Taking $(0,1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(0,1,p_3,q_3)$, $(2,p_2,p_2,q_2)$ as the four sets of parameters in Theorem 2.2, and replacing n by $n+1$, gives the following.

Corollary 2.9 (Sequence pairs $(\boldsymbol{U_n^{(1)}U_n^{(2)}})$ and $(\boldsymbol{U_n^{(3)}V_n^{(2)}})$)**.** *For* $n \geq 3$,

$$\sum_{s=1}^{n-3}\Big(p_2(p_1-p_3)U_{n-s}^{(3)}V_{n-s}^{(2)}$$

$$+\big(p_1^2q_2+p_2^2q_1-p_2^2q_3-p_3^2q_2+2q_1q_2-2q_2q_3\big)U^{(3)}_{n-1-s}V^{(2)}_{n-1-s}$$
$$+p_2q_2(p_1q_1-p_3q_3)U^{(3)}_{n-2-s}V^{(2)}_{n-2-s}+q_2^2(q_3^2-q_1^2)U^{(3)}_{n-3-k}V^{(2)}_{n-3-k}\Big)U^{(1)}_sU^{(2)}_s$$
$$=-U^{(3)}_nV^{(2)}_n+q_1q_2U^{(3)}_{n-2}V^{(2)}_{n-2}+p_2U^{(1)}_nU^{(2)}_n+(p_2^2p_3-p_1p_2^2+2p_3q_2)U^{(1)}_{n-1}U^{(2)}_{n-1}$$
$$+p_2\Big(p_2^2p_3^2+3p_3^2q_2-p_1p_2^2p_3-2p_1p_3q_2+3q_2q_3-p_1^2q_2$$
$$-p_2^2q_1-2q_1q_2+p_2^2q_3\Big)U^{(1)}_{n-2}U^{(2)}_{n-2}.$$

Example 2.10.

$$\sum_{s=1}^{n-3}\big(6Q_{n-1-s}J_{n-1-s}+2Q_{n-2-s}J_{n-2-s}-3Q_{n-3-s}J_{n-3-s}\big)F_sP_s$$
$$=Q_nJ_n-Q_{n-2}J_{n-2}-2F_nP_n-2F_{n-1}P_{n-1}-16F_{n-2}P_{n-2},$$
$$\sum_{s=1}^{n-3}\big(P_{n-s}j_{n-s}+6P_{n-1-s}j_{n-1-s}+2P_{n-2-s}j_{n-2-s}\big)F_sJ_s$$
$$=P_nj_n-2P_{n-2}j_{n-2}-F_nJ_n-9F_{n-1}J_{n-1}-18F_{n-2}J_{n-2},$$
$$\sum_{s=1}^{n-3}\big(F_{n-s}j_{n-s}+6F_{n-1-s}j_{n-1-s}+2F_{n-2-s}j_{n-2-s}\big)P_sJ_s$$
$$=-F_nj_n+2F_{n-2}j_{n-2}+P_nJ_n+3P_{n-1}J_{n-1}-9P_{n-2}J_{n-2}.$$

Taking $(0,1,p_2,q_2)$, $(2,p_3,p_3,q_3)$, $(2,p_1,p_1,q_1)$, $(2,p_2,p_2,q_2)$ in Theorem 2.2 gives the following.

Corollary 2.11 (Sequence pairs $\big(\boldsymbol{U_n^{(2)}V_n^{(3)}}\big)$ and $\big(\boldsymbol{V_n^{(1)}V_n^{(2)}}\big)$). *For $n\ge 4$,*

$$\sum_{s=1}^{n-4}\Big(p_2(p_1-p_3)V^{(1)}_{n-1-s}V^{(2)}_{n-1-s}$$
$$+\big(p_1^2q_2+p_2^2q_1-p_2^2q_3-p_3^2q_2+2q_1q_2-2q_2q_3\big)V^{(1)}_{n-2-s}V^{(2)}_{n-2-s}$$
$$+p_2q_2(p_1q_1-p_3q_3)V^{(1)}_{n-3-s}V^{(2)}_{n-3-s}+q_2^2(q_3^2-q_1^2)V^{(1)}_{n-4-s}V^{(2)}_{n-4-s}\Big)U^{(2)}_sV^{(3)}_s$$
$$=p_3V^{(1)}_{n-1}V^{(2)}_{n-1}+2p_2q_3V^{(1)}_{n-2}V^{(2)}_{n-2}+p_3q_2q_3V^{(1)}_{n-3}V^{(2)}_{n-3}-4U^{(2)}_nV^{(3)}_n$$
$$+p_2(4p_3-p_1)U^{(2)}_{n-1}V^{(3)}_{n-1}+\big(p_1p_2^2p_3-p_1^2p_2^2-2p_1^2q_2-2p_2^2q_1+4p_2^2q_3$$
$$+4p_3^2q_2-4q_1q_2+8q_2q_3\big)U^{(2)}_{n-2}V^{(3)}_{n-2}$$
$$-p_2\Big(9p_1q_1q_2-4p_3q_2q_3-p_1p_2^2q_3-2p_1q_2q_3-p_1p_3^2q_2+p_1^3p_2^2-p_1^2p_2^2p_3$$
$$+3p_1^3q_2-2p_1^2p_3q_2+3p_1p_2^2q_1-2p_2^2p_3q_1-4p_3q_1q_2\Big)U^{(2)}_{n-3}V^{(3)}_{n-3}.$$

Example 2.12.

$$\sum_{s=1}^{n-4}\big(6L_{n-2-s}Q_{n-2-s}+2L_{n-3-s}Q_{n-3-s}-3L_{n-4-s}Q_{n-4-s}\big)P_sj_s$$

$$= -L_{n-1}Q_{n-1} - 8L_{n-2}Q_{n-2} - 2L_{n-3}Q_{n-3} + 4P_n j_n - 6P_{n-1}j_{n-1} - 38P_{n-2}j_{n-2} - 22P_{n-3}j_{n-3},$$

$$\sum_{s=1}^{n-4}\big(L_{n-1-s}j_{n-1-s} + 3L_{n-4-s}j_{n-4-s}\big)F_sQ_s = -2L_{n-1}j_{n-1} - 2L_{n-2}j_{n-2} - 2L_{n-3}j_{n-3} + 4F_nQ_n - 7F_{n-1}Q_{n-1} - 15F_{n-2}Q_{n-2} - 17F_{n-3}Q_{n-3},$$

$$\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} + 6Q_{n-2-s}j_{n-2-s} + 2Q_{n-3-s}j_{n-3-s}\big)L_sJ_s = Q_{n-1}j_{n-1} + 2Q_{n-2}j_{n-2} + 2Q_{n-3}j_{n-3} - 4L_nJ_n + 2L_{n-1}J_{n-1} - 46L_{n-3}J_{n-3}.$$

Taking $(0,1,p_1,q_1)$, $(2,p_2,p_2,q_2)$, $(0,1,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in Theorem 2.2, and replacing n by $n+1$, gives the following.

Corollary 2.13 (Sequence pairs $\big(\boldsymbol{U_n^{(1)}V_n^{(2)}}\big)$ and $\big(\boldsymbol{U_n^{(2)}V_n^{(3)}}\big)$)**.** *For $n \geq 3$,*

$$\begin{aligned}
&\sum_{s=1}^{n-3}\Big(p_2(p_1-p_3)U_{n-s}^{(2)}V_{n-s}^{(3)} \\
&\quad + \big(p_1^2q_2 + p_2^2q_1 - p_2^2q_3 - p_3^2q_2 + 2q_1q_2 - 2q_2q_3\big)U_{n-1-s}^{(2)}V_{n-1-s}^{(3)} \\
&\quad + p_2q_2(p_1q_1 - p_3q_3)U_{n-2-s}^{(2)}V_{n-2-s}^{(3)} + q_2^2(q_3^2 - q_1^2)U_{n-3-s}^{(2)}V_{n-3-s}^{(3)}\Big)U_s^{(1)}V_s^{(2)} \\
&= -p_2U_n^{(2)}V_n^{(3)} - 2p_1q_2U_{n-1}^{(2)}V_{n-1}^{(3)} - p_2q_1q_2U_{n-2}^{(2)}V_{n-2}^{(3)} + p_3U_n^{(1)}V_n^{(2)} \\
&\quad + p_2(p_3^2 - p_1p_3 + 2q_3)U_{n-1}^{(1)}V_{n-1}^{(2)} + \Big(p_2^2p_3^3 + 3p_2^2p_3q_3 - p_1p_2^2p_3^2 \\
&\quad - 2p_1p_2^2q_3 - p_1^2p_3q_2 - 2p_3q_1q_2 - p_2^2p_3q_1 + 3p_3q_2q_3 + p_3^3q_2\Big)U_{n-2}^{(1)}V_{n-2}^{(2)}.
\end{aligned}$$

Example 2.14.

$$\sum_{s=1}^{n-3}\big(6P_{n-1-s}j_{n-1-s} + 2P_{n-2-s}j_{n-2-s} - 3P_{n-3-s}j_{n-3-s}\big)F_sQ_s = 2P_nj_n + 2P_{n-1}j_{n-1} + 2P_{n-2}j_{n-2} - F_nQ_n - 8F_{n-1}Q_{n-1} - 8F_{n-2}Q_{n-2},$$

$$\sum_{s=1}^{n-3}\big(L_{n-s}J_{n-s} + 6L_{n-1-s}J_{n-1-s} + 2L_{n-2-s}J_{n-2-s}\big)P_sj_s = -L_nJ_n - 8L_{n-1}J_{n-1} - 2L_{n-2}J_{n-2} + P_nj_n + P_{n-1}j_{n-1} - 7P_{n-2}j_{n-2},$$

$$\sum_{s=1}^{n-3}\big(Q_{n-s}J_{n-s} + 6Q_{n-1-s}J_{n-1-s} + 2Q_{n-2-s}J_{n-2-s}\big)F_sj_s = Q_nJ_n + 4Q_{n-1}J_{n-1} + 2Q_{n-2}J_{n-2} - 2F_nj_n - 4F_{n-1}j_{n-1} - 22F_{n-2}j_{n-2}.$$

We now prove a general identity of a similar form to Theorem 2.2 above in which $T_n^{(1)}$ and $T_n^{(2)}$ are assumed to share p_i and q_i parameter values meeting a certain restriction.

Theorem 2.15. *Suppose $p_1 = p_2 = p$ and $q_1 = q_2 = q$, with p and q satisfying*

$$2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p.$$

Then for $n \geq 4$,

$$\begin{aligned}
&\sum_{s=0}^{n-4}\Big((p^2+2q-p_3p_4)T^{(3)}_{n-1-s}T^{(4)}_{n-1-s} - (p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)T^{(3)}_{n-2-s}T^{(4)}_{n-2-s}\\
&\quad - p_3p_4q_3q_4T^{(3)}_{n-3-s}T^{(4)}_{n-3-s} + q_3^2q_4^2T^{(3)}_{n-4-s}T^{(4)}_{n-4-s}\Big)T^{(1)}_sT^{(2)}_s\\
&= -a_1a_2T^{(3)}_nT^{(4)}_n + (b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}\\
&\quad + a_3a_4T^{(1)}_nT^{(2)}_n + \big(b_3b_4 - a_3a_4(p^2+2q)\big)T^{(1)}_{n-1}T^{(1)}_{n-1}\\
&\quad + \big(a_3b_4p_4q_3 + a_4b_3p_3q_4 + a_3a_4q_3q_4 + a_3a_4q^2 - b_3b_4(p^2+2q-p_3p_4)\big)T^{(1)}_{n-2}T^{(2)}_{n-2}\\
&\quad + \Big(b_3b_4(p_3^2q_4+p_4^2q_3+q_3q_4+q^2) + q_3q_4(a_4b_3p_4+a_3b_4p_3)\\
&\quad - (p^2+2q-p_3p_4)(a_3q_3+b_3p_3)(a_4q_4+b_4p_4)\Big)T^{(1)}_{n-3}T^{(2)}_{n-3}. \qquad (2.7)
\end{aligned}$$

Proof. Using $2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p$, one can show

$$2a_1a_2q - (b_1-a_1p)(b_2-a_2p) = b_1b_2 - a_1a_2p^2$$

and

$$a_1a_2q - 2(b_1-a_1p)(b_2-a_2p) = a_1b_2p + a_2b_1p - a_1a_2(2p^2+q),$$

from which it follows the factorization

$$\begin{aligned}
&a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2+q)\big)x^2\\
&\quad - q^2(b_1-a_1p)(b_2-a_2p)x^3 = \big(a_1a_2 - (b_1-a_1p)(b_2-a_2p)x\big)(1+2qx+q^2x^2).
\end{aligned}$$

Thus for $T^{(1)}_n$ and $T^{(2)}_n$ whose parameters meet the stated conditions, we have by (2.3) that the generating function $\sum_{n\geq 0} T^{(1)}_nT^{(2)}_nx^n$ is given by

$$\begin{aligned}
&\frac{a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2+q)\big)x^2}{1 - p^2x - 2q(p^2+q)x^2 - p^2q^2x^3 + q^4x^4}\\
&\quad - \frac{q^2(b_1-a_1p)(b_2-a_2p)x^3}{1 - p^2x - 2q(p^2+q)x^2 - p^2q^2x^3 + q^4x^4}\\
&= \frac{\big(a_1a_2 - (b_1-a_1p)(b_2-a_2p)x\big)(1+2qx+q^2x^2)}{(1-(p^2+2q)x+q^2x^2)(1+2qx+q^2x^2)}\\
&= \frac{a_1a_2 - (b_1-a_1p)(b_2-a_2p)x}{1-(p^2+2q)x+q^2x^2}.
\end{aligned}$$

Consider now the quantity

$$a_3a_4T^{(1)}_nT^{(2)}_n + (b_3b_4 - a_3a_4p_3p_4)T^{(1)}_{n-1}T^{(2)}_{n-1}$$

$$+\big(a_3b_4p_4q_3+a_4b_3p_3q_4-a_3a_4(p_3^2q_4+p_4^2q_3+q_3q_4)\big)T_{n-2}^{(1)}T_{n-2}^{(2)}$$
$$-q_3q_4(b_3-a_3p_3)(b_4-a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)}-a_1a_2T_n^{(3)}T_n^{(4)}$$
$$+(b_1-a_1p)(b_2-a_2p)T_{n-1}^{(3)}T_{n-1}^{(4)}. \tag{2.8}$$

Then the generating function of (2.8) for $n\geq 0$ is given by the product of

$$\sum_{n\geq 0}T_n^{(1)}T_n^{(2)}x^n \quad \text{and} \quad \sum_{n\geq 0}T_n^{(3)}T_n^{(4)}x^n$$

with

$$(p^2+2q-p_3p_4)x-(p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)x^2-p_3p_4q_3q_4x^3+q_3^2q_4^2x^4.$$

Extracting the coefficient of x^n then yields

$$(p^2+2q-p_3p_4)\sum_{s=0}^{n-1}T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}T_s^{(1)}T_s^{(2)}$$
$$-(p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)\sum_{s=0}^{n-2}T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}T_s^{(1)}T_s^{(2)}$$
$$-p_3p_4q_3q_4\sum_{s=0}^{n-3}T_{n-3-s}^{(3)}T_{n-3-s}^{(4)}T_s^{(1)}T_s^{(2)}+q_3^2q_4^2\sum_{s=0}^{n-4}T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}T_s^{(1)}T_s^{(2)}.$$

Equating this last expression with (2.8) above and shifting the appropriate summands gives

$$\sum_{s=0}^{n-4}\Big((p^2+2q-p_3p_4)T_{n-1-s}^{(3)}T_{n-1-s}^{(4)}-(p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)T_{n-2-s}^{(3)}T_{n-2-s}^{(4)}$$
$$-p_3p_4q_3q_4T_{n-3-s}^{(3)}T_{n-3-s}^{(4)}+q_3^2q_4^2T_{n-4-s}^{(3)}T_{n-4-s}^{(4)}\Big)T_s^{(1)}T_s^{(2)}$$
$$=-a_1a_2T_n^{(3)}T_n^{(4)}+(b_1-a_1p)(b_2-a_2p)T_{n-1}^{(3)}T_{n-1}^{(4)}$$
$$+a_3a_4T_n^{(1)}T_n^{(2)}+\big(b_3b_4-a_3a_4p_3p_4-a_3a_4(p^2+2q-p_4p_4)\big)T_{n-1}^{(1)}T_{n-1}^{(2)}$$
$$+\Big(a_3b_4p_4q_3+a_4b_3p_3q_4-a_3a_4(p_3^2q_4+p_4^2q_3+q_3q_4)$$
$$+a_3a_4(p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)-b_3b_4(p^2+2q-p_3p_4)\Big)T_{n-2}^{(1)}T_{n-2}^{(2)}$$
$$+\Big(a_3a_4p_3p_4q_3q_4+b_3b_4(p_3^2q_4+p_4^2q_3+2q_3q_4+q^2)$$
$$-(p^2+2q-p_3p_4)(a_3q_3+b_3p_3)(a_4q_4+b_4p_4)$$
$$-q_3q_4(b_3-a_3p_3)(b_4-a_4p_4)\Big)T_{n-3}^{(1)}T_{n-3}^{(2)},$$

which may be rewritten as (2.7). □

Note that taking $a_1 = 0$ in the condition on p and q in the preceding theorem reduces it to $2b_2 = a_2 p$ since it may be assumed $b_1 \neq 0$ (note $a_1 = b_1 = 0$ would result in a triviality). In particular, the condition is satisfied when $T_n^{(1)} = U_n(p,q)$ and $T_n^{(2)} = V_n(p,q)$, upon taking $a_1 = 0$, $b_1 = 1$ and $a_2 = 2$, $b_2 = p$.

Letting (a_i, b_i, p_i, q_i) be given by $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(0,1,p_2,q_2)$ and $(0,1,p_3,q_3)$ in (2.7), and replacing n by $n+1$, yields the following result.

Corollary 2.16 (Sequence pairs $\big(\boldsymbol{U_n^{(1)} V_n^{(1)}}\big)$ and $\big(\boldsymbol{U_n^{(2)} U_n^{(3)}}\big)$)**.** *For $n \geq 3$,*

$$\begin{aligned}
&p_1 U_n^{(2)} U_n^{(3)} + \sum_{s=1}^{n-3} \Big((p_1^2 - p_2 p_3 + 2q_1) U_{n-s}^{(2)} U_{n-s}^{(3)} \\
&\quad - (p_2^2 q_3 + p_3^2 q_2 + q_1^2 + 2q_2 q_3) U_{n-1-s}^{(2)} U_{n-1-s}^{(3)} \\
&\quad - p_2 p_3 q_2 q_3 U_{n-2-s}^{(2)} U_{n-2-s}^{(3)} + q_2^2 q_3^2 U_{n-3-s}^{(2)} U_{n-3-s}^{(3)} \Big) U_s^{(1)} V_s^{(1)} \\
&= U_n^{(1)} V_n^{(1)} - (p_1^2 - p_2 p_3 + 2q_1) U_{n-1}^{(1)} V_{n-1}^{(1)} \\
&\quad - \big(p_1^2 p_2 p_3 + 2p_2 p_3 q_1 - p_2^2 p_3^2 - q_2 q_3 - p_2^2 q_3 - p_3^2 q_2 - q_1^2\big) U_{n-2}^{(1)} V_{n-2}^{(1)}.
\end{aligned}$$

Example 2.17.

$$\begin{aligned}
&P_n J_n + \sum_{s=1}^{n-3} \big(P_{n-s} J_{n-s} - 14 P_{n-1-s} J_{n-1-s} \\
&\qquad - 4 P_{n-2-s} J_{n-2-s} + 4 P_{n-3-s} J_{n-3-s} \big) F_s L_s \\
&= F_n L_n - F_{n-1} L_{n-1} + 10 F_{n-2} L_{n-2}, \\
&F_n P_n + \sum_{s=1}^{n-3} \big(3F_{n-s} P_{n-s} - 11 F_{n-1-s} P_{n-1-s} \\
&\qquad - 2F_{n-2-s} P_{n-2-s} + F_{n-3-s} P_{n-3-s} \big) J_s j_s \\
&= J_n j_n - 3J_{n-1} j_{n-1} + 4 J_{n-2} j_{n-2}, \\
&2F_n J_n + \sum_{s=1}^{n-3} \big(5F_{n-s} J_{n-s} - 8F_{n-1-s} J_{n-1-s} \\
&\qquad - 2F_{n-2-s} J_{n-2-s} + 4F_{n-3-s} J_{n-3-s} \big) P_s Q_s \\
&= P_n Q_n - 5P_{n-1} Q_{n-1} + P_{n-2} Q_{n-2}.
\end{aligned}$$

Taking $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(2,p_2,p_2,q_2)$, $(2,p_3,p_3,q_3)$ in Theorem 2.15 yields the following result.

Corollary 2.18 (Sequence pairs $\big(\boldsymbol{U_n^{(1)} V_n^{(1)}}\big)$ and $\big(\boldsymbol{V_n^{(2)} V_n^{(3)}}\big)$)**.** *For $n \geq 4$,*

$$\sum_{s=1}^{n-4} \Big((p_1^2 - p_2 p_3 + 2q_1) V_{n-1-s}^{(2)} V_{n-1-s}^{(3)} - (p_2^2 q_3 + p_3^2 q_2 + q_1^2 + 2q_2 q_3) V_{n-2-s}^{(2)} V_{n-2-s}^{(3)}$$

$$- p_2p_3q_2q_3V^{(2)}_{n-3-s}V^{(3)}_{n-3-s} + q_2^2q_3^2V^{(2)}_{n-4-s}V^{(3)}_{n-4-s}\Big)U^{(1)}_sV^{(1)}_s$$

$$= - p_1V^{(2)}_{n-1}V^{(3)}_{n-1} + 4U^{(1)}_nV^{(1)}_n - (4p_1^2 - p_2p_3 + 8q_1)U^{(1)}_{n-1}V^{(1)}_{n-1}$$

$$- (p_1^2p_2p_3 + 2p_2p_3q_1 - p_2^2p_3^2 - 2p_2^2q_3 - 4q_1^2 - 4q_2q_3 - 2p_3^2q_2)U^{(1)}_{n-2}V^{(1)}_{n-2}$$

$$- \Big(p_1^2p_2^2p_3^2 + 2p_1^2p_2^2q_3 + 2p_1^2p_3^2q_2 + 4p_1^2q_2q_3 - p_2p_3q_1^2 + 2p_2^2p_3^2q_1 + 4p_2^2q_1q_3$$

$$- p_2^3p_3^3 - 3p_2^3p_3q_3 - 3p_2p_3^3q_2 - 9p_2p_3q_2q_3 + 4p_3^2q_1q_2 + 8q_1q_2q_3\Big)U^{(1)}_{n-3}V^{(1)}_{n-3}.$$

Example 2.19.

$$\sum_{s=1}^{n-4}\big(Q_{n-1-s}j_{n-1-s} - 14Q_{n-2-s}j_{n-2-s} - 4Q_{n-3-s}j_{n-3-s} + 4Q_{n-4-s}j_{n-4-s}\big)F_sL_s$$

$$= 4F_nL_n - 10F_{n-1}L_{n-1} + 28F_{n-2}L_{n-2} + 10F_{n-3}L_{n-3} - Q_{n-1}j_{n-1},$$

$$\sum_{s=1}^{n-4}\big(3L_{n-1-s}Q_{n-1-s} - 11L_{n-2-s}Q_{n-2-s}$$

$$- 2L_{n-3-s}Q_{n-3-s} + L_{n-4-s}Q_{n-4-s}\big)J_sj_s$$

$$= 4J_nj_n - 18J_{n-1}j_{n-1} + 24J_{n-2}j_{n-2} - 26J_{n-3}j_{n-3} - L_{n-1}Q_{n-1},$$

$$\sum_{s=1}^{n-4}\big(5L_{n-1-s}j_{n-1-s} - 8L_{n-2-s}j_{n-2-s} - 2L_{n-3-s}j_{n-3-s} + 4L_{n-4-s}j_{n-4-s}\big)P_sQ_s$$

$$= 4P_nQ_n - 23P_{n-1}Q_{n-1} + 13P_{n-2}Q_{n-2} - 61P_{n-3}Q_{n-3} - 2L_{n-1}j_{n-1}.$$

Taking $(0, 1, p_1, q_1)$, $(2, p_1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_3, p_3, q_3)$ in Theorem 2.15, and replacing n by $n + 1$, yields the following.

Corollary 2.20 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n V^{(1)}_n}\big)$ and $\big(\boldsymbol{U^{(2)}_n V^{(3)}_n}\big)$)**.** *For $n \geq 3$,*

$$\sum_{s=1}^{n-3}\Big((-p_1^2 + p_2p_3 - 2q_1)U^{(2)}_{n-s}V^{(3)}_{n-s} + (p_2^2q_3 + p_3^2q_2 + q_1^2 + 2q_2q_3)U^{(2)}_{n-1-s}V^{(3)}_{n-1-s}$$

$$+ p_2p_3q_2q_3U^{(2)}_{n-2-s}V^{(3)}_{n-2-s} - q_2^2q_3^2U^{(2)}_{n-3-s}V^{(3)}_{n-3-s}\Big)U^{(1)}_sV^{(1)}_s$$

$$= p_1U^{(2)}_nV^{(3)}_n - p_3U^{(1)}_nV^{(1)}_n + (p_1^2p_3 - p_2p_3^2 - 2p_2q_3 + 2p_3q_1)U^{(1)}_{n-1}V^{(1)}_{n-1}$$

$$+ \Big(p_1^2p_2p_3^2 + 2p_1^2p_2q_3 - p_2^2p_3^3 + 2p_2p_3^2q_1 + 4p_2q_1q_3 - 3p_2^2p_3q_3$$

$$- p_3^3q_2 - p_3q_1^2 - 3p_3q_2q_3\Big)U^{(1)}_{n-2}V^{(1)}_{n-2}.$$

Example 2.21.

$$\sum_{s=1}^{n-3}\big(P_{n-s}j_{n-s} - 14P_{n-1-s}j_{n-1-s} - 4P_{n-2-s}j_{n-2-s} + 4P_{n-3-s}j_{n-3-s}\big)F_sL_s$$

$$= F_nL_n + 7F_{n-1}L_{n-1} + 6F_{n-2}L_{n-2} - P_nj_n,$$

$$\sum_{s=1}^{n-3}\big(5F_{n-s}j_{n-s}-8F_{n-1-s}j_{n-1-s}-2F_{n-2-s}j_{n-2-s}+4F_{n-3-s}j_{n-3-s}\big)P_sQ_s$$
$$=P_nQ_n-P_{n-1}Q_{n-1}-15P_{n-2}Q_{n-2}-2F_nj_n,$$
$$\sum_{s=1}^{n-3}\big(3P_{n-s}L_{n-s}-11P_{n-1-s}L_{n-1-s}-2P_{n-2-s}L_{n-2-s}+P_{n-3-s}L_{n-3-s}\big)J_sj_s$$
$$=J_nj_n+J_{n-1}j_{n-1}-6J_{n-2}j_{n-2}-P_nL_n.$$

Remark 2.22. Taking $(0,1,p_1,q_1)$, $(2,p_1,p_1,q_1)$, $(0,1,p_2,q_2)$, $(0,1,p_3,q_3)$ in (2.4) instead of (2.7) yields a more complicated variant of Corollary 2.16 which we will not state here. Similar remarks apply to the identities in Corollaries 2.18 and 2.20.

We now prove a general result in the case when the p and q parameters are the same in both function pairs.

Theorem 2.23. *Suppose $p_1=p_2=p$, $q_1=q_2=q$, $p_3=p_4=y$, and $q_3=q_4=z$. Further, assume that p,q and y,z satisfy $2b_1b_2-2a_1a_2q=(a_1b_2+a_2b_1)p$ and $2b_3b_4-2a_3a_4z=(a_3b_4+a_4b_3)y$. Then for $n\geq 2$,*

$$\sum_{s=0}^{n-2}\Big((p^2-y^2+2q-2z)T^{(3)}_{n-1-s}T^{(4)}_{n-1-s}+(z^2-q^2)T^{(3)}_{n-2-s}T^{(4)}_{n-2-s}\Big)T^{(1)}_sT^{(2)}_s$$
$$=-a_1a_2T^{(3)}_nT^{(4)}_n+(b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}+a_3a_4T^{(1)}_nT^{(2)}_n$$
$$+\big(a_3b_4y+a_4b_3y-b_3b_4-a_3a_4(p^2+2q-2z)\big)T^{(1)}_{n-1}T^{(2)}_{n-1}. \tag{2.9}$$

Proof. By the assumptions on the parameters, we have

$$\sum_{n\geq 0}T^{(1)}_nT^{(2)}_nx^n=\frac{a_1a_2-(b_1-a_1p)(b_2-a_2p)x}{1-(p^2+2q)x+q^2x^2}$$

and

$$\sum_{n\geq 0}T^{(3)}_nT^{(4)}_nx^n=\frac{a_3a_4-(b_3-a_3y)(b_4-a_4y)x}{1-(y^2+2z)x+z^2x^2}.$$

Then the quantity

$$a_3a_4T^{(1)}_nT^{(2)}_n-(b_3-a_3y)(b_4-a_4y)T^{(1)}_{n-1}T^{(2)}_{n-1}-a_1a_2T^{(3)}_nT^{(4)}_n$$
$$+(b_1-a_1p)(b_2-a_2p)T^{(3)}_{n-1}T^{(4)}_{n-1}$$

has generating function given by

$$\sum_{n\geq 0}T^{(1)}_nT^{(2)}_nx^n\cdot\sum_{n\geq 0}T^{(3)}_nT^{(4)}_nx^n\cdot\big((p^2-y^2+2q-2z)x+(z^2-q^2)x^2\big).$$

Extracting the coefficient of x^n gives

$$(p^2-y^2+2q-2z)\sum_{s=0}^{n-1}T^{(3)}_{n-1-s}T^{(4)}_{n-1-s}T^{(1)}_sT^{(2)}_s$$

$$+ (z^2 - q^2) \sum_{s=0}^{n-2} T^{(3)}_{n-2-s} T^{(4)}_{n-2-s} T^{(1)}_{s} T^{(2)}_{s},$$

and equating this with the original quantity leads to (2.9). □

Taking $(0, 1, p_1, q_1)$, $(2, p_1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_2, p_2, q_2)$ in Theorem 2.23, and replacing n by $n + 1$, implies the following result.

Corollary 2.24 (Sequence pairs $\big(\boldsymbol{U^{(1)}_n V^{(1)}_n}\big)$ and $\big(\boldsymbol{U^{(2)}_n V^{(2)}_n}\big)$)**.** *For $n \geq 1$,*

$$\sum_{s=1}^{n-1} \Big((p_2^2 - p_1^2 - 2q_1 + 2q_2) U^{(2)}_{n-s} V^{(2)}_{n-s} + (q_1^2 - q_2^2) U^{(2)}_{n-1-s} V^{(2)}_{n-1-s} \Big) U^{(1)}_s V^{(1)}_s$$
$$= p_1 U^{(2)}_n V^{(2)}_n - p_2 U^{(1)}_n V^{(1)}_n.$$

Example 2.25.

$$3 \sum_{s=1}^{n-1} P_{n-s} Q_{n-s} F_s L_s = P_n Q_n - 2F_n L_n,$$
$$\sum_{s=1}^{n-1} \big(P_{n-s} Q_{n-s} + 3P_{n-1-s} Q_{n-1-s}\big) J_s j_s = P_n Q_n - 2J_n j_n, \qquad (2.10)$$
$$\sum_{s=1}^{n-1} \big(2J_{n-s} j_{n-s} - 3J_{n-1-s} j_{n-1-s}\big) F_s L_s = J_n j_n - F_n L_n.$$

Remark 2.26. Taking $(0, 1, p_1, q_1)$, $(2, p_1, p_1, q_1)$, $(0, 1, p_2, q_2)$, $(2, p_2, p_2, q_2)$ in either (2.4) or (2.7) above instead of (2.9) leads to more complicated variants of Corollary 2.24.

3. Further remarks

In this section, we point out some further extensions of the prior results. We first allow for the indices of the sequences whose terms appear in the identities above to come from an arbitrary arithmetic sequence. Let $k \geq 1$ be fixed and $0 \leq i \leq k - 1$. Then we have the recurrence $U_{nk+i} = V_k U_{(n-1)k+i} - (-q)^k U_{(n-2)k+i}$ for $n \geq 2$, which can be shown using the Binet formulas for U_n and V_n. The same recurrence is seen to hold also for the sequence V_{nk+i}. Thus, taking $a = U_i$, $b = U_{i+k}$, $p = V_k$, $q = -(-q)^k$ or $a = V_i$, $b = V_{i+k}$, $p = V_k$, $q = -(-q)^k$ in Theorem 2.2 gives various formulas involving products of terms derived from the U_{nk+i} and/or the V_{nk+i} sequences.

For example, taking (a_j, b_j, p_j, q_j) for $1 \leq j \leq 4$ to be $\big(0, U^{(1)}_k, V^{(1)}_k, -(-q_1)^k\big)$, $\big(0, U^{(2)}_k, V^{(2)}_k, -(-q_2)^k\big)$, $\big(2, V^{(1)}_k, V^{(1)}_k, -(-q_1)^k\big)$, $\big(2, V^{(2)}_k, V^{(2)}_k, -(-q_2)^k\big)$, respectively, in (2.4) gives

$$U^{(1)}_k U^{(2)}_k V^{(1)}_{(n-1)k} V^{(2)}_{(n-1)k} - (q_1 q_2)^k U^{(1)}_k U^{(2)}_k V^{(1)}_{(n-3)k} V^{(2)}_{(n-3)k}$$

$$= 4U_{nk}^{(1)}U_{nk}^{(2)} - 3V_k^{(1)}V_k^{(2)}U_{(n-1)k}^{(1)}U_{(n-1)k}^{(2)}$$
$$+ 2\Big((-q_1)^k\big(V_k^{(2)}\big)^2 + (-q_2)^k\big(V_k^{(1)}\big)^2 - 2(q_1q_2)^k\Big)U_{(n-2)k}^{(1)}U_{(n-2)k}^{(2)}$$
$$- (q_1q_2)^kV_k^{(1)}V_k^{(2)}U_{(n-3)k}^{(1)}U_{(n-3)k}^{(2)}. \tag{3.1}$$

Note that (3.1) reduces to Corollary 2.3 when $k = 1$. Taking, for instance, $U_k^{(1)} = F_k$, $U_k^{(2)} = P_k$, and $q_1 = q_2 = 1$ in (3.1) gives

$$F_kP_kL_{(n-1)k}Q_{(n-1)k} - L_{(n-3)k}Q_{(n-3)k} = 4F_{nk}P_{nk} - 3L_kQ_kF_{(n-1)k}P_{(n-1)k}$$
$$+ 2\big((-1)^k(L_k^2 + Q_k^2) - 2\big)F_{(n-2)k}P_{(n-2)k} - L_kQ_kF_{(n-3)k}P_{(n-3)k},$$

which reduces to (2.6) when $k = 1$. Further, formula (3.1) represents only the $i = 0$ case of a more general identity, though it is a bit more complex, which involves products of terms from the sequences $U_{nk+i}^{(1)}$, $U_{nk+i}^{(2)}$, $V_{nk+i}^{(1)}$, $V_{nk+i}^{(2)}$ for any $0 \le i \le k-1$.

As another example, letting $\big(0, U_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$, $\big(2, V_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$, $\big(0, U_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$, $\big(2, V_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$ in Theorem 2.2, and replacing n by $n+1$, gives

$$U_k^{(1)}V_k^{(2)}U_{nk}^{(2)}V_{nk}^{(1)} - 2(-q_2)^kU_k^{(1)}V_k^{(1)}U_{(n-1)k}^{(2)}V_{(n-1)k}^{(1)}$$
$$+ (q_1q_2)^kU_k^{(1)}V_k^{(2)}U_{(n-2)k}^{(2)}V_{(n-2)k}^{(1)}$$
$$= U_k^{(2)}V_k^{(1)}U_{nk}^{(1)}V_{nk}^{(2)} - 2(-q_1)^kU_k^{(2)}V_k^{(2)}U_{(n-1)k}^{(1)}V_{(n-1)k}^{(2)}$$
$$+ (q_1q_2)^kU_k^{(2)}V_k^{(1)}U_{(n-2)k}^{(1)}V_{(n-2)k}^{(2)}, \tag{3.2}$$

which reduces to Corollary 2.5 when $k = 1$. From (3.2), one can obtain such identities as

$$L_kJ_kF_{nk}j_{nk} - 2(-1)^kJ_kj_kF_{(n-1)k}j_{(n-1)k} + 2^kL_kJ_kF_{(n-2)k}j_{(n-2)k}$$
$$= F_kj_kL_{nk}J_{nk} + (-2)^{k+1}F_kL_kL_{(n-1)k}J_{(n-1)k} + 2^kF_kj_kL_{(n-2)k}J_{(n-2)k}.$$

Next, observe the identity

$$2U_{i+k}V_{i+k} + 2(-q)^kU_iV_i = (U_iV_{i+k} + V_iU_{i+k})V_k, \tag{3.3}$$

which follows from combining the formulas

$$U_iV_{i+k} + V_iU_{i+k} = 2U_{2i+k} \quad \text{and} \quad U_{i+k}V_{i+k} + (-q)^kU_iV_i = U_{2i+k}V_k,$$

which can be shown using the Binet formulas for U_n and V_n. Thus, by (3.3), the condition $2b_1b_2 - 2a_1a_2q = (a_1b_2 + a_2b_1)p$ in Theorem 2.15 remains satisfied when one considers generalized Fibonacci or Lucas sequences whose indices come from an arbitrary arithmetic progression. Hence, one may apply Theorems 2.15 and 2.23 to obtain analogous identities involving products of terms derived from the sequences U_{nk+i} and V_{nk+i} wherein members of at least one sequence pair share p and q parameter values.

For example, letting

$$\big(0, U_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big), \quad \big(2, V_k^{(1)}, V_k^{(1)}, -(-q_1)^k\big),$$
$$\big(0, U_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big), \quad \big(2, V_k^{(2)}, V_k^{(2)}, -(-q_2)^k\big)$$

in (2.9) gives

$$\begin{aligned}
&\sum_{s=0}^{n-1}\Big(\Big(\big(V_k^{(2)}\big)^2 - \big(V_k^{(1)}\big)^2 + 2(-q_1)^k - 2(-q_2)^k\Big) U_{(n-s)k}^{(2)} V_{(n-s)k}^{(2)} \\
&\quad + (q_1^{2k} - q_2^{2k}) U_{(n-1-s)k}^{(2)} V_{(n-1-s)k}^{(2)}\Big) U_{sk}^{(1)} V_{sk}^{(1)} \\
&= U_k^{(1)} V_k^{(1)} U_{nk}^{(2)} V_{nk}^{(2)} - U_k^{(2)} V_k^{(2)} U_{nk}^{(1)} V_{nk}^{(1)},
\end{aligned} \tag{3.4}$$

which reduces to Corollary 2.24 when $k = 1$. Letting, for instance, $U_k^{(1)} = J_k$, $U_k^{(2)} = P_k$, $q_1 = 2$, and $q_2 = 1$ in (3.4) yields

$$\begin{aligned}
&\sum_{s=0}^{n-1}\Big((Q_k^2 - j_k^2 - (-2)^{k+1} - 2(-1)^k) P_{(n-s)k} Q_{(n-s)k} \\
&\quad + (4^k - 1) P_{(n-1-s)k} Q_{(n-1-s)k}\Big) J_{sk} j_{sk} = J_k j_k P_{nk} Q_{nk} - P_k Q_k J_{nk} j_{nk},
\end{aligned}$$

which reduces to (2.10) when $k = 1$. Formula (3.4) may be generalized to U_{nk+i} and V_{nk+i} for any $0 \le i \le k-1$ by taking $\big(U_i^{(1)}, U_{i+k}^{(1)}, V_k^{(1)}, -(-q_1)^k\big)$ for the first 4-tuple and the analogous quantities for the other three. Generalizations comparable to (3.4) may be given for the identities in Corollaries 2.16, 2.18, and 2.20.

We have the following further general result in the case when $T_n^{(1)}$ and $T_n^{(2)}$ share p and q parameter values.

Theorem 3.1. *Suppose $p_1 = p_2 = p$ and $q_1 = q_2 = q$, with $a_1 a_2 p = a_1 b_2 + a_2 b_1$ and $a_1 a_2 q = -b_1 b_2$. Then for $n \ge 4$,*

$$\begin{aligned}
&\sum_{s=0}^{n-4}\Big((q + p_3 p_4) T_{n-1-s}^{(3)} T_{n-1-s}^{(4)} + (p_3^2 q_4 + p_4^2 q_3 + 2 q_3 q_4) T_{n-2-s}^{(3)} T_{n-2-s}^{(4)} \\
&\quad + p_3 p_4 q_3 q_4 T_{n-3-s}^{(3)} T_{n-3-s}^{(4)} - q_3^2 q_4^2 T_{n-4-s}^{(3)} T_{n-4-s}^{(4)}\Big) T_s^{(1)} T_s^{(2)} \\
&= a_1 a_2 T_n^{(3)} T_n^{(4)} - a_3 a_4 T_n^{(1)} T_n^{(2)} - (b_3 b_4 + a_3 a_4 q) T_{n-1}^{(1)} T_{n-1}^{(2)} \\
&\quad - \big(b_3 b_4 q + a_3 a_4 q_3 q_4 + a_3 b_4 p_4 q_3 + a_4 b_3 p_3 q_4 + b_3 b_4 p_3 p_4\big) T_{n-2}^{(1)} T_{n-2}^{(2)} \\
&\quad - \Big(b_3 b_4 (p_3^2 q_4 + p_4^2 q_3 + q_3 q_4) + q_3 q_4 (a_3 b_4 p_3 + a_4 b_3 p_4) \\
&\quad + (q + p_3 p_4)(a_3 q_3 + b_3 p_3)(a_4 q_4 + b_4 p_4)\Big) T_{n-3}^{(1)} T_{n-3}^{(2)}.
\end{aligned}$$

Proof. Using $a_1a_2p = a_1b_2 + a_2b_1$ and $a_1a_2q = -b_1b_2$, one can show

$$a_1a_2p^2 - b_1b_2 = a_1a_2(p^2 + 2q) + (b_1 - a_1p)(b_2 - a_2p)$$

and

$$q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big) = a_1a_2q^2 + (p^2 + 2q)(b_1 - a_1p)(b_2 - a_2p),$$

from which it follows the factorization

$$\begin{aligned} &a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big)x^2 \\ &\quad - q^2(b_1 - a_1p)(b_2 - a_2p)x^3 \\ &= \big(a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x\big)\big(1 - (p^2 + 2q)x + q^2x^2\big). \end{aligned}$$

Thus if $T_n^{(1)}$ and $T_n^{(2)}$ are such that their parameters satisfy the required conditions, then by (2.3) the generating function $\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}x^n$ is given by

$$\begin{aligned} &\frac{a_1a_2 + (b_1b_2 - a_1a_2p^2)x + q\big(a_1b_2p + a_2b_1p - a_1a_2(2p^2 + q)\big)x^2}{1 - p^2x - 2q(p^2 + q)x^2 - p^2q^2x^3 + q^4x^4} \\ &\quad - \frac{q^2(b_1 - a_1p)(b_2 - a_2p)x^3}{1 - p^2x - 2q(p^2 + q)x^2 - p^2q^2x^3 + q^4x^4} \\ &= \frac{\big(a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x\big)\big(1 - (p^2 + 2q)x + q^2x^2\big)}{(1 + 2qx + q^2x^2)\big(1 - (p^2 + 2q)x + q^2x^2\big)} \\ &= \frac{a_1a_2 - (b_1 - a_1p)(b_2 - a_2p)x}{1 + 2qx + q^2x^2} = \frac{a_1a_2 - b_1b_2x}{1 + 2qx + q^2x^2} \\ &= \frac{a_1a_2(1 + qx)}{(1 + qx)^2} = \frac{a_1a_2}{1 + qx}. \end{aligned}$$

The proof is completed in a similar manner as before upon considering the generating function of the quantity

$$\begin{aligned} &a_1a_2T_n^{(3)}T_n^{(4)} - a_3a_4T_n^{(1)}T_n^{(2)} - (b_3b_4 - a_3a_4p_3p_4)T_{n-1}^{(1)}T_{n-1}^{(2)} \\ &\quad - \big(a_3b_4p_4q_3 + a_4b_3p_3q_4 - a_3a_4(p_3^2q_4 + p_4^2q_3 + q_3q_4)\big)T_{n-2}^{(1)}T_{n-2}^{(2)} \\ &\quad + q_3q_4(b_3 - a_3p_3)(b_4 - a_4p_4)T_{n-3}^{(1)}T_{n-3}^{(2)}. \end{aligned}$$ □

Remark 3.2. If $p = q = 1$ in the prior theorem, then the a_i and b_i satisfy $a_1a_2 = a_1b_2 + a_2b_1 = -b_1b_2$. Replacing b_2 with $-b_2$, we then have $a_1a_2 = a_1b_2 - a_2b_1 = b_1b_2$. If all variables are positive in the last system, then eliminating b_2 leads to the equality $a_1b_1 = a_1^2 - b_1^2$, where a_2 can be chosen arbitrarily and $b_2 = \frac{a_1a_2}{b_1}$. This essentially covers all the cases when a_1a_2 is non-zero, upon considering separately when a_1a_2 is positive or negative and renaming quantities as needed. Note that the case when a_1a_2 is zero is trivial since one (or both) of $T_n^{(1)}$ and $T_n^{(2)}$ is seen to be the sequence of all zeros in that case.

In analogy with Theorem 2.23 above, we have the following further result when $T_n^{(3)}$ and $T_n^{(4)}$ also share p and q parameter values.

Theorem 3.3. *Suppose* $p_1 = p_2 = p$*,* $q_1 = q_2 = q$*,* $p_3 = p_4 = y$*, and* $q_3 = q_4 = z$*. Further, assume that* p, q *and* y, z *satisfy* $a_1a_2p = a_1b_2 + a_2b_1$*,* $a_1a_2q = -b_1b_2$ *and* $a_3a_4y = a_3b_4 + a_4b_3$*,* $a_3a_4z = -b_3b_4$*. Then for* $n \geq 1$*,*

$$(z-q)\sum_{s=0}^{n-1} T_{n-1-s}^{(3)} T_{n-1-s}^{(4)} T_s^{(1)} T_s^{(2)} = a_3a_4T_n^{(1)}T_n^{(2)} - a_1a_2T_n^{(3)}T_n^{(4)}.$$

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