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JOINT FUNCTIONAL CALCULUS IN ALGEBRA OF POLYNOMIAL TEMPERED DISTRIBUTIONS

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Abstract. In this paper we develop a functional calculus for a countable system of generators of contraction strongly continuous semigroups. As a symbol class of such calculus we use the algebra of polynomial tempered distributions. We prove a differential property of constructed calculus and describe its image with the help of the commutant of polynomial shift semigroup. As an application, we consider a function of countable set of second derivative operators.

Introduction

A functional calculus is a theory that studies how to construct functions depending on operators (roughly speaking, how to "substitute" an operator instead of the variable in a function). Also it is said that the functional calculus for some (not necessary bounded) operator A on a Banach space is a method of associating an operator $f(A)$ to a function f belonging to a topological algebra $\mathcal{A}$ of functions. If we have such a method then, actually, we have a continuous homomorphism from the algebra $\mathcal{A}$ to a topological algebra of operators. So, in this terminology the functional calculus can be identified with the above-mentioned homomorphism (but as a theory the functional calculus studies such homomorphisms).

There are many different approaches to construct a functional calculus for one operator acting on a Banach space. For Riesz-Dunford functional calculus, based on the Cauchy formula, we refer the reader to the book [11]. Such a functional calculus has applications, in particular, in the spectral theory of elliptic differential equations and maximal regularity of parabolic evolution equations (see e.g. [14, 18]). Another method, based on the Laplace transformation, was developed in [13]. This method is known as the Hille-Phillips functional calculus. It has many helpful applications, in particular, in hydrology (see [2] and the references given there). Such type of calculus is the main object of investigation in this article.

The Hille-Phillips functional calculus for functions of several variables is well developed (see e.g. [19, 21]). The case of functions of infinitely many variables is less studied. We mention the book [23] that is devoted to spectral questions (among them there is a functional calculus) of countable families of self-adjoint operators on a Hilbert space. The main goal of this article is the construction of Hille-Phillips type functional calculus for countable set of generators of contraction strongly continuous semigroups, acting on a Banach space.

The Borchers-Uhlmann algebra, i.e. the tensor algebra over the space of rapidly decreasing functions with tensor product as a multiplication was effectively used in quantum field theory (see e.g. [6, 7, 27]). Such algebras have an equivalent structure of

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polynomials with pointwise multiplication [9]. It was an incitement to research the problems connected with the polynomially extended cross-correlation of (ultra)distributions, differentiations and the corresponding functional calculus [20]. Elements of the Borchers-Uhlmann algebra can be treated as functionals on spaces of smooth functions of infinite many variables. So, we can understand this algebra as space of polynomial distributions with tensor structure. In this paper we would like to consider this structure for a special case.

A Fréchet-Schwartz space (briefly, (FS) space) is one that is Fréchet and Schwartz simultaneously (see [29]). Let $\mathcal{S}_+$ be the space of rapidly decreasing functions on $[0,+\infty)$ and $\mathcal{S}'_+$ be its dual. It is known (see e.g. [16, 26]) that these spaces are nuclear Fréchet-Schwartz and dual Fréchet-Schwartz spaces ((DFS) for short), respectively. These facts are crucial for our investigation. The main objects of investigation are the algebras $\mathcal{P}(\mathcal{S}'_+)$ and $\mathcal{P}'(\mathcal{S}'_+)$ of polynomial test and generalized functions, which have the tensor structures of the forms $\bigoplus_{\mathrm{fin}} \mathcal{S}_+^{\hat{\otimes} n}$ and $\times\, \mathcal{S}_+'^{\hat{\otimes} n}$, respectively.

Using the Grothendieck technique [10], we introduce the polynomial extension of cross-correlation and prove the Theorem 1.3 about isomorphic representation of the algebra of polynomial distributions onto the commutant of polynomial shift semigroup (see (3)) in the space of linear continuous operators on $\bigoplus_{\mathrm{fin}} \mathcal{S}_+^{\hat{\otimes} n}$. In Proposition 1.4 we prove the differential property of polynomial cross-correlation, which is essentially used in main Theorem 3.2.

In the section 2 we extend the generalized Fourier transformation onto the spaces of polynomial test and generalized functions. Images of this map we understand as functions and functionals of infinite many variables (see Remarks 2.1 and 2.2), respectively.

The constructed polynomial test and generalized functions we apply to an operator semigroup with infinitely many parameters. Namely, we construct the functional calculus for countable system of generators of contraction C_0-semigroups and prove its properties (see Remark 3.1 and Theorem 3.2). This calculus is an infinite-dimensional analogue of the one constructed in [19]. As an example we consider the infinite-dimensional Gaussian semigroup, which is generated by a countable set of second derivative operators.

Finally we note that there are other known and widely used infinite-dimensional generalizations of classical spaces of distributions [3, 4]. For example, white noise analysis is based on an infinite dimensional analogue of the Schwartz distribution theory (see e.g. [12, 15, 17, 22]).

1. Background on polynomial tempered distributions

In what follows $L(X,Y)$ denotes the space of all continuous linear operators from a locally convex space X into another such space Y, endowed with the topology of uniform convergence on bounded subsets of X. Let $L(X) := L(X,X)$ and I_X denotes the identity operator in $L(X)$. The dual space $X' := L(X,\mathbb{C})$ is endowed with strong topology. The pairing between elements of X' and X we denote by $\langle\cdot,\cdot\rangle$.

Let $X^{\hat{\otimes} n}$, $n\in\mathbb{N}$, be the symmetric nth tensor degree of X, completed in the projective tensor topology. For any $x\in X$ we denote $x^{\otimes n} := \underbrace{x\otimes\cdots\otimes x}_{n} \in X^{\hat{\otimes} n}$, $n\in\mathbb{N}$. Set $X^{\hat{\otimes} 0} := \mathbb{C}$, $x^{\otimes 0} := 1\in\mathbb{C}$.

For any $A\in L(X)$ its tensor power $A^{\otimes n}\in L(X^{\hat{\otimes} n})$, $n\in\mathbb{N}$, is defined as a linear continuous extension of the map $x^{\otimes n}\longmapsto (Ax)^{\otimes n}$, where $x\in X$. It follows from results of [5] that such an extension exists if X is a projective or inductive limit of separable Hilbert spaces. In this article we consider only such spaces.

Let $\mathcal{S}_+$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}_+ := [0,+\infty)$ and $\mathcal{S}'_+$ be its dual space of tempered distributions supported by $\mathbb{R}_+$. Note that strong topology on $\mathcal{S}'_+$ coincides with the Mackey topology and topology of inductive limit (see [24, IV.4, IV.5]). Let δ_t be the Dirac delta functional concentrated at a point $t \in \mathbb{R}_+$. It is known [28] that $\mathcal{S}'_+$ is a topological algebra with the unit $\delta := \delta_0$ under the convolution, defined as

$$\langle f * g, \varphi\rangle = \langle f(s), \langle g(t), \varphi(s+t)\rangle\rangle, \quad f, g \in \mathcal{S}'_+, \quad \varphi \in \mathcal{S}_+.$$

Note that here and everywhere the notation $f(t)$ shows that a functional f acts on a test function in the variable t.

From the duality theory as well as from the theory of nuclear spaces it follows that $\mathcal{S}_+$ is a nuclear (FS) space, and $\mathcal{S}'_+$ is a nuclear (DFS) space.

To define the locally convex space $\mathcal{P}({}^n\mathcal{S}'_+)$ of n-homogeneous polynomials on $\mathcal{S}'_+$ we use the canonical topological linear isomorphism

$$\mathcal{P}({}^n\mathcal{S}'_+) \simeq (\mathcal{S}'^{\hat{\otimes} n}_+)'$$

described in [9]. Namely, given a functional $p_n \in (\mathcal{S}'^{\hat{\otimes} n}_+)'$, we define an n-homogeneous polynomial $P_n \in \mathcal{P}({}^n\mathcal{S}'_+)$ by $P_n(f) := p_n(f^{\otimes n})$, $f \in \mathcal{S}'_+$. We equip $\mathcal{P}({}^n\mathcal{S}'_+)$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{S}'_+$. Set $\mathcal{P}({}^0\mathcal{S}'_+) := \mathbb{C}$. The space $\mathcal{P}(\mathcal{S}'_+)$ of all continuous polynomials on $\mathcal{S}'_+$ is defined to be the complex linear span of all $\mathcal{P}({}^n\mathcal{S}'_+)$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{S}'_+)$ mean the strong dual of $\mathcal{P}(\mathcal{S}'_+)$ and

$$\Gamma(\mathcal{S}_+) := \bigoplus_{n\in\mathbb{Z}_+}{}_{\mathrm{fin}} \mathcal{S}^{\hat{\otimes} n}_+ \subset \bigoplus_{n\in\mathbb{Z}_+} \mathcal{S}^{\hat{\otimes} n}_+ \quad \text{and} \quad \Gamma(\mathcal{S}'_+) := \bigtimes_{n\in\mathbb{Z}_+} \mathcal{S}'^{\hat{\otimes} n}_+.$$

Note that we consider only the case when elements of the direct sum consist of a finite but not fixed number of terms. For simplicity of notation we write $\Gamma(\mathcal{S}_+)$ instead of commonly used $\Gamma_{\mathrm{fin}}(\mathcal{S}_+)$.

We have the following assertion (see also [20, Proposition 2.1]).

Proposition 1.1. *There exist linear topological isomorphisms*

$$\Upsilon : \mathcal{P}(\mathcal{S}'_+) \longrightarrow \Gamma(\mathcal{S}_+), \quad \Psi : \mathcal{P}'(\mathcal{S}'_+) \longrightarrow \Gamma(\mathcal{S}'_+).$$

Elements of the spaces $\mathcal{P}(\mathcal{S}'_+)$ and $\mathcal{P}'(\mathcal{S}'_+)$ we call the polynomial test functions and polynomial distributions, respectively. In what follows elements of the spaces $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$ will be written as

$$\bigoplus_{n=0}^{m} p_n = (p_0, p_1, \dots, p_m, 0, \dots) \in \Gamma(\mathcal{S}_+) \quad \text{and} \quad \bigtimes_{n\in\mathbb{Z}_+} f_n = (f_0, f_1, \dots, f_n, \dots) \in \Gamma(\mathcal{S}'_+)$$

for some $m \in \mathbb{N}$, where $p_n \in \mathcal{S}^{\hat{\otimes} n}_+$ and $f_n \in \mathcal{S}'^{\hat{\otimes} n}_+$ for all $n \in \mathbb{Z}_+$. To simplify, we write $\big(p_n\big)$ and $\big(f_n\big)$ instead of $\bigoplus_{n=0}^m p_n$ and $\bigtimes_{n\in\mathbb{Z}_+} f_n$, respectively.

Note that the following systems of elements

$$\big\{\big(\varphi^{\otimes n}\big) : \varphi \in \mathcal{S}_+\big\}, \quad \big\{\big(f^{\otimes n}\big) : f \in \mathcal{S}'_+\big\} \tag{1}$$

are total in $\Gamma(\mathcal{S}_+)$ and $\Gamma(\mathcal{S}'_+)$, respectively.

Let us define the operation

$$\big(f^{\otimes n}\big) \circledast \big(g^{\otimes n}\big) := \big((f * g)^{\otimes n}\big)$$

for elements from the total subset (1) of the space $\Gamma(\mathcal{S}'_+)$ and extend it to the whole space by linearity and continuity. It is obvious that $\Gamma(\mathcal{S}'_+)$ is an algebra relative to the operation $\circledast$ with the unit element $\big(\delta^{\otimes n}\big)$. Since $\Gamma(\mathcal{S}_+)$ is continuously and densely embedded into $\Gamma(\mathcal{S}'_+)$ (see [20]) and the space $\mathcal{S}_+$ is a convolution algebra (see [28]), the space $\Gamma(\mathcal{S}_+)$ becomes an algebra with respect to $\circledast$.

For any $K \in L(\mathcal{S}_+)$ let us define an operator $K^{\otimes} \in L\big(\Gamma(\mathcal{S}_+)\big)$ as follows:

$$K^{\otimes} := \big(K^{\otimes n}\big) : \boldsymbol{p} = (p_n) \quad \longmapsto \quad K^{\otimes}\boldsymbol{p} := \big(K^{\otimes n} p_n\big), \tag{2}$$

where $K^{\otimes 0} := I_{\mathbb{C}}$ and each operator $K^{\otimes n} \in L(\mathcal{S}_+^{\hat{\otimes} n})$ is defined as a linear continuous extension of the map $\varphi^{\otimes n} \longmapsto (K\varphi)^{\otimes n}$, with $\varphi \in \mathcal{S}_+$, $n \in \mathbb{N}$.

Consider the one-parameter C_0-semigroup of shifts,

$$T\colon \mathbb{R}_+ \ni s \longmapsto T_s \in L(\mathcal{S}_+), \quad T_s\varphi(t) := \varphi(t+s), \quad t \in \mathbb{R}_+, \quad \varphi \in \mathcal{S}_+.$$

Hence, the map $T^{\otimes n} : \mathbb{R}_+ \ni s \longmapsto T_s^{\otimes n} \in L(\mathcal{S}_+^{\hat{\otimes} n})$ is well defined. It easy to check that $T^{\otimes n}$ is a one-parameter semigroup of operators. Denote $T_s^{\otimes} := \big(T_s^{\otimes n}\big)$, $s \in \mathbb{R}_+$. The mapping

$$T^{\otimes} : \mathbb{R}_+ \ni s \longmapsto T_s^{\otimes} \in L\big(\Gamma(\mathcal{S}_+)\big) \tag{3}$$

is called the polynomial shift semigroup.

The cross-correlation of a distribution $f \in \mathcal{S}'_+$ and a function $\varphi \in \mathcal{S}_+$ is defined to be the function

$$(f \star \varphi)(s) := \langle f, T_s\varphi\rangle = \langle f(t), \varphi(t+s)\rangle.$$

Similarly to [25] it is easy to prove that

$$f \star \varphi \in \mathcal{S}_+, \quad f \star T_s\varphi = T_s(f \star \varphi) \quad \text{and} \quad (f * g) \star \varphi = f \star (g \star \varphi) \tag{4}$$

for any $s \in \mathbb{R}_+$, $f, g \in \mathcal{S}'_+$ and $\varphi \in \mathcal{S}_+$. It follows that the cross-correlation operator defined by

$$K_f \colon \varphi \longmapsto f \star \varphi$$

belongs to $L(\mathcal{S}_+)$ for any $f \in \mathcal{S}'_+$. From (2) it follows that

$$K_{\boldsymbol{f}}^{\otimes} := \big(K_{f_n}^{\otimes n}\big) \in L\big(\Gamma(\mathcal{S}_+)\big) \quad \text{and} \quad K_{f_n}^{\otimes n} \in L(\mathcal{S}_+^{\hat{\otimes} n}), \tag{5}$$

where $\boldsymbol{f} := (f_n) \in \Gamma(\mathcal{S}'_+)$ with $f_n \in \mathcal{S}'^{\hat{\otimes} n}_+$, $n \in \mathbb{Z}_+$.

The cross-correlation of a polynomial distribution $\boldsymbol{f} = (f_n) \in \Gamma(\mathcal{S}'_+)$ and a polynomial test function $\boldsymbol{p} = (p_n) \in \Gamma(\mathcal{S}_+)$ is given by

$$\boldsymbol{f} \star \boldsymbol{p} := K_{\boldsymbol{f}}^{\otimes} \boldsymbol{p} = \big(K_{f_n}^{\otimes n} p_n\big).$$

Proposition 1.2. *For any $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$ and $\boldsymbol{p} \in \Gamma(\mathcal{S}_+)$ the cross-correlation $\boldsymbol{f} \star \boldsymbol{p}$ is a polynomial test function belonging to $\Gamma(\mathcal{S}_+)$.*

Proof. Let $\boldsymbol{f} = (f_n) \in \Gamma(\mathcal{S}'_+)$ and $\boldsymbol{p} = (p_n) \in \Gamma(\mathcal{S}_+)$. Since $\boldsymbol{f} \star \boldsymbol{p} = \big(K_{f_n}^{\otimes n} p_n\big)$ by definition, we only need to check that $K_{f_n}^{\otimes n} p_n \in \mathcal{S}_+^{\hat{\otimes} n}$ for all $n \in \mathbb{Z}_+$. In the case $n = 0$ this is obvious. If $n = 1$ we obtain that $\langle f_1, T_s p_1\rangle = (f_1 \star p_1)(s)$ belongs to $\mathcal{S}_+$ (see (4)). Consider the case $n > 1$. Since the operators $K_{f_n}^{\otimes n}$ are linear and continuous, it is sufficient to prove the statement for $f_n = f^{\otimes n}$ and $p_n = \varphi^{\otimes n}$ with $f \in \mathcal{S}'_+$ and $\varphi \in \mathcal{S}_+$. Then the function

$$K_{f_n}^{\otimes n} p_n = \big\langle f^{\otimes n}, T_s^{\otimes n}\varphi^{\otimes n}\big\rangle = \big\langle f^{\otimes n}, (T_s\varphi)^{\otimes n}\big\rangle = \big\langle f, T_s\varphi\big\rangle^{\otimes n} = (f \star \varphi)^{\otimes n}$$

belongs to $\mathcal{S}_+^{\hat{\otimes} n}$ as the n-th tensor power of a function from $\mathcal{S}_+$. $\square$

The commutant $\big[T^{\otimes}\big]^c \subset L\big(\Gamma(\mathcal{S}_+)\big)$ of the polynomial shift semigroup $T^{\otimes}$ is defined to be the set

$$\big[T^{\otimes}\big]^c := \big\{K^{\otimes} \in L\big(\Gamma(\mathcal{S}_+)\big) : K^{\otimes} \circ T_s^{\otimes} = T_s^{\otimes} \circ K^{\otimes}, \forall\, s \in \mathbb{R}_+\big\},$$

where $K^{\otimes}$ is defined by (2).

Theorem 1.3. *The mapping*

$$\Gamma(\mathcal{S}'_+) \ni \boldsymbol{f} \longmapsto K_{\boldsymbol{f}}^{\otimes} \in L\big(\Gamma(\mathcal{S}_+)\big)$$

is an algebraic isomorphism from the algebra $\{\Gamma(\mathcal{S}'_+), \circledast\}$ *onto the commutant* $\left[T^{\otimes}\right]^c$ *of the semigroup* $T^{\otimes}$ *in the algebra* $\{L\big(\Gamma(\mathcal{S}_+)\big), \circ\}$*. In particular, the following relation holds:*

$$K^{\otimes}_{\boldsymbol{f}\circledast\boldsymbol{g}} = K^{\otimes}_{\boldsymbol{f}} \circ K^{\otimes}_{\boldsymbol{g}}, \quad \boldsymbol{f}, \boldsymbol{g} \in \Gamma(\mathcal{S}'_+).$$

Proof. Since the operator $K^{\otimes}_{\boldsymbol{f}}$ is linear and continuous, it is sufficient to consider only elements from the total subsets (1). Let $\boldsymbol{p} = (\varphi^{\otimes n}) \in \Gamma(\mathcal{S}_+)$ with $\varphi \in \mathcal{S}_+$ and $\boldsymbol{f} = (f^{\otimes n}), \boldsymbol{g} = (g^{\otimes n}) \in \Gamma(\mathcal{S}'_+)$ with $f, g \in \mathcal{S}'_+$ be given. From definitions of operations $\circledast$ and $\star$, as well as from (4), we obtain

$$\begin{aligned} K^{\otimes}_{\boldsymbol{f}\circledast\boldsymbol{g}}\boldsymbol{p} =& \Big(\big\langle (f * g)^{\otimes n}, T_s^{\otimes n}\varphi^{\otimes n}\big\rangle\Big) = \Big(\big((f * g) \star \varphi\big)^{\otimes n}\Big) \\ =& \Big(\big(f \star (g \star \varphi)\big)^{\otimes n}\Big) = \Big(\big\langle f, T_s(g \star \varphi)\big\rangle^{\otimes n}\Big) = \Big(\big\langle f^{\otimes n}, T_s^{\otimes n}(g \star \varphi)^{\otimes n}\big\rangle\Big) = K^{\otimes}_{\boldsymbol{f}} K^{\otimes}_{\boldsymbol{g}} \boldsymbol{p}. \end{aligned}$$

Using (4), we obtain

$$\begin{aligned} K^{\otimes}_{\boldsymbol{f}} T^{\otimes}_s \boldsymbol{p} =& \big(f^{\otimes n}\big) \star \big(T_s^{\otimes n}\varphi^{\otimes n}\big) = \big(f^{\otimes n}\big) \star \big((T_s\varphi)^{\otimes n}\big) \\ =& \big((f \star T_s\varphi)^{\otimes n}\big) = \big((T_s(f \star \varphi))^{\otimes n}\big) = \big(T_s^{\otimes n}(f \star \varphi)^{\otimes n}\big) \\ =& \big(T_s^{\otimes n} K^{\otimes n}_{f_n} \varphi^{\otimes n}\big) = T^{\otimes}_s K^{\otimes}_{\boldsymbol{f}} \boldsymbol{p} \end{aligned}$$

for all $s \in \mathbb{R}_+$. Hence, the operator $K^{\otimes}_{\boldsymbol{f}}$ belongs to $\left[T^{\otimes}\right]^c$ for all $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$.

Conversely, let $K \in L(\mathcal{S}_+)$ be an operator such that $K^{\otimes} \in \left[T^{\otimes}\right]^c$. Let us show that there exists $\boldsymbol{h} \in \Gamma(\mathcal{S}'_+)$ such that $K^{\otimes} = K^{\otimes}_{\boldsymbol{h}}$. Such an element is $\boldsymbol{h} := (1, h, \dots, h^{\otimes n}, \dots)$, where the distribution $h \in \mathcal{S}'_+$ is defined by the relation $\langle h, \varphi\rangle := (K\varphi)(0)$, $\varphi \in \mathcal{S}_+$. Since $(h \star \varphi)(s) = \langle h, T_s\varphi\rangle = (KT_s\varphi)(0) = (K\varphi)(s)$, we obtain

$$K^{\otimes}_{\boldsymbol{h}}\boldsymbol{p} = \big((h \star \varphi)^{\otimes n}\big) = \big((K\varphi)^{\otimes n}\big) = \big(K^{\otimes n}\varphi^{\otimes n}\big) = K^{\otimes}\boldsymbol{p}.$$

Thus, $K^{\otimes} = K^{\otimes}_{\boldsymbol{h}}$. So, the range of the mapping $\Gamma(\mathcal{S}'_+) \ni \boldsymbol{f} \longmapsto K^{\otimes}_{\boldsymbol{f}} \in L\big(\Gamma(\mathcal{S}_+)\big)$ coincides with the commutant $\left[T^{\otimes}\right]^c$. □

Let D mean the differential operator on $\mathcal{S}_+$. We use the same letter D to denote the operator of generalized differentiation on $\mathcal{S}'_+$, i.e. $\langle Df, \varphi\rangle = -\langle f, D\varphi\rangle$.

Let us define the operator $\mathbb{D} \in L\big(\Gamma(\mathcal{S}'_+)\big)$ as follows

$$\begin{array}{llcl} \mathbb{D}: & \Gamma(\mathcal{S}'_+) & \longrightarrow & \Gamma(\mathcal{S}'_+) \\ & (1, f, \dots, f^{\otimes n}, \dots) & \longmapsto & \Big(0, Df, \dots, \sum_{j=1}^{n} f^{\otimes(j-1)} \hat{\otimes}\, Df \,\hat{\otimes}\, f^{\otimes(n-j)}, \dots\Big). \end{array}$$

Its restriction onto $\Gamma(\mathcal{S}_+)$ acts as

$$\begin{array}{llcl} \mathbb{D}: & \Gamma(\mathcal{S}_+) & \longrightarrow & \Gamma(\mathcal{S}_+) \\ & (1, \varphi, \dots, \varphi^{\otimes n}, \dots) & \longmapsto & \Big(0, D\varphi, \dots, \sum_{j=1}^{n} \varphi^{\otimes(j-1)} \hat{\otimes}\, D\varphi \,\hat{\otimes}\, \varphi^{\otimes(n-j)}, \dots\Big). \end{array}$$

Analogically as in [20] it is easy to prove that $\mathbb{D}$ is a continuous derivative.

Proposition 1.4. *For any* $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$ *and* $\boldsymbol{p} \in \Gamma(\mathcal{S}_+)$ *the following equality holds:*

$$(\mathbb{D}\boldsymbol{f}) \star \boldsymbol{p} = -\boldsymbol{f} \star (\mathbb{D}\boldsymbol{p}).$$

Proof. For any $\boldsymbol{f} = \big(f^{\otimes n}\big) \in \Gamma(\mathcal{S}'_+)$ with $f \in \mathcal{S}'_+$ and $\boldsymbol{p} = \big(\varphi^{\otimes n}\big) \in \Gamma(\mathcal{S}_+)$ with $\varphi \in \mathcal{S}_+$ we have

$$\begin{aligned}
(\mathbb{D}\boldsymbol{f}) \star \boldsymbol{p} =& \Big(0, Df \star \varphi, \dots, \sum_{j=1}^{n} (f \star \varphi)^{\otimes(j-1)} \hat{\otimes} (Df \star \varphi) \hat{\otimes} (f \star \varphi)^{\otimes(n-j)}, \dots\Big) \\
=& - \Big(0, f \star D\varphi, \dots, \sum_{j=1}^{n} (f \star \varphi)^{\otimes(j-1)} \hat{\otimes} (f \star D\varphi) \hat{\otimes} (f \star \varphi)^{\otimes(n-j)}, \dots\Big) \\
=& - \Big(0, \big\langle f, T_s D\varphi\big\rangle, \dots, \big\langle f^{\otimes n}, \sum_{j=1}^{n} (T_s\varphi)^{\otimes(j-1)} \hat{\otimes} (T_s D\varphi) \hat{\otimes} (T_s\varphi)^{\otimes(n-j)}\big\rangle, \dots\Big) \\
=& - \Big(0, \big\langle f, T_s D\varphi\big\rangle, \dots, \big\langle f^{\otimes n}, T_s^{\otimes n} \sum_{j=1}^{n} \varphi^{\otimes(j-1)} \hat{\otimes} D\varphi \hat{\otimes} \varphi^{\otimes(n-j)}\big\rangle, \dots\Big) \\
=& - \boldsymbol{f} \star (\mathbb{D}\boldsymbol{p}).
\end{aligned}$$

The proposition is proved. □

2. Fourier transform of polynomial tempered distributions

Since each element of the space $\mathcal{S}_+$ may be considered as a function $\varphi \in L^1(0,\infty) \cap L^2(0,\infty)$, we define the Fourier transform and its inverse, as follows:

$$\begin{aligned}
&\mathcal{F}_+ : \mathcal{S}_+ \ni \varphi \longmapsto \widehat{\varphi}(\xi) := \int_{\mathbb{R}_+} e^{-\mathrm{i}t\xi} \varphi(t)\, dt, \qquad \xi \in \mathbb{R}, \\
&\mathcal{F}_+^{-1} : \widehat{\varphi} \longmapsto \varphi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}t\xi} \widehat{\varphi}(\xi)\, d\xi, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

Let $\widehat{\mathcal{S}}_+ := \mathcal{F}_+[\mathcal{S}_+]$ stand for the range of $\mathcal{S}_+$ under the map $\mathcal{F}_+$. It is known [1] that $\widehat{\mathcal{S}}_+ \subset L^2(\mathbb{R})$. Using the injectivity of $\mathcal{F}_+$, we endow the space $\widehat{\mathcal{S}}_+$ with a topology induced by the topology in $\mathcal{S}_+$. Therefore, $\widehat{\mathcal{S}}_+$ is a nuclear (F) space (see [24]). For the strong duals the appropriate adjoint transform $(\mathcal{F}_+^{-1})' \colon \mathcal{S}'_+ \longmapsto \widehat{\mathcal{S}}'_+$ is well defined. The mapping

$$\mathcal{F}'_+ := 2\pi(\mathcal{F}_+^{-1})' \colon \mathcal{S}'_+ \ni f \longmapsto \widehat{f} \in \widehat{\mathcal{S}}'_+$$

is called the generalized Fourier transform of distributions from $\mathcal{S}'_+$. The space $\widehat{\mathcal{S}}'_+$ is a nuclear (DFS) space as a strong dual of the nuclear (FS) space $\widehat{\mathcal{S}}_+$ (see [24]).

Since delta functional is a unit element in the convolution algebra $\mathcal{S}'_+$ (see [28]), we obtain $\widehat{\delta * f} = \widehat{f} = \widehat{f * \delta}$ and the space $\widehat{\mathcal{S}}'_+$ is a commutative multiplicative algebra with the unit $\widehat{\delta}$ with respect to the multiplication $\widehat{f} \cdot \widehat{h} := \widehat{f * h}$, $f, h \in \mathcal{S}'_+$. The following bilinear form

$$\langle \mathcal{F}'_+ f, \mathcal{F}_+ \varphi\rangle = \langle 2\pi(\mathcal{F}_+^{-1})' f, \mathcal{F}_+\varphi\rangle = 2\pi\langle f, \mathcal{F}_+^{-1}\mathcal{F}_+\varphi\rangle = 2\pi\langle f, \varphi\rangle,$$

with $f \in \mathcal{S}'_+$, $\varphi \in \mathcal{S}_+$, defines the new duality $\langle \widehat{\mathcal{S}}'_+, \widehat{\mathcal{S}}_+\rangle$.

Denote $\Gamma(\widehat{\mathcal{S}}_+) := \bigoplus_{n\in\mathbb{Z}_+}^{\mathrm{fin}} \widehat{\mathcal{S}}_+^{\hat{\otimes} n}$ and $\Gamma(\widehat{\mathcal{S}}'_+) := \bigtimes_{n\in\mathbb{Z}_+} \widehat{\mathcal{S}}_+'^{\hat{\otimes} n}$. For any elements $\widehat{\boldsymbol{f}} = \big(\widehat{f}^{\otimes n}\big)$, $\widehat{\boldsymbol{h}} = \big(\widehat{h}^{\otimes n}\big) \in \Gamma(\widehat{\mathcal{S}}'_+)$ with $f, h \in \mathcal{S}'_+$ we define the operation

$$\widehat{\boldsymbol{f}} \mathbin{\widehat{\circledast}} \widehat{\boldsymbol{h}} := \big((\widehat{f} \cdot \widehat{h})^{\otimes n}\big)$$

and extend it to the whole space $\Gamma(\widehat{\mathcal{S}}'_+)$ by linearity and continuity. It is obvious, that $\Gamma(\widehat{\mathcal{S}}'_+)$ is an algebra relative to the operation $\widehat{\circledast}$. Similarly as above, we can induce this operation on the space $\Gamma(\widehat{\mathcal{S}}_+)$ that becomes an algebra too. From [20, Proposition 2.1] it follows that there exist the linear topological isomorphisms of algebras

$$\widehat{\Upsilon} : \mathcal{P}(\widehat{\mathcal{S}}'_+) \longrightarrow \Gamma(\widehat{\mathcal{S}}_+) \quad \text{and} \quad \widehat{\Psi} : \mathcal{P}'(\widehat{\mathcal{S}}'_+) \longrightarrow \Gamma(\widehat{\mathcal{S}}'_+).$$

Using Proposition 1.1 we can extend the map $\mathcal{F}_+$ onto the space $\Gamma(\mathcal{S}_+)$ as follows. First of all, for any $\varphi^{\otimes n} \in \mathcal{S}_+^{\hat{\otimes} n}$ with $\varphi \in \mathcal{S}_+$ we define the operation $\mathcal{F}_+^{\otimes_n}$ by the relations

$$\mathcal{F}_+^{\otimes_n} : \varphi^{\otimes n} \longmapsto \widehat{\varphi}^{\otimes n} \quad \text{and} \quad \mathcal{F}_+^{\otimes_0} := I_{\mathbb{C}}.$$

Next, we extend the mapping $\mathcal{F}_+^{\otimes_n}$ to the whole space $\mathcal{S}_+^{\hat{\otimes} n}$ by linearity and continuity, so $\mathcal{F}_+^{\otimes_n} \in L\big(\mathcal{S}_+^{\hat{\otimes} n}, \widehat{\mathcal{S}}_+^{\hat{\otimes} n}\big)$. Finally, $\mathcal{F}_+^{\otimes}$ is defined to be the mapping

$$\mathcal{F}_+^{\otimes} = \big(\mathcal{F}_+^{\otimes_n}\big) : \Gamma(\mathcal{S}_+) \ni \boldsymbol{p} = \big(p_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{p}} := \big(\widehat{p}_n\big) \in \Gamma(\widehat{\mathcal{S}}_+),$$

where $\widehat{p}_n := \mathcal{F}_+^{\otimes_n} p_n$. It is easy to check that $\mathcal{F}_+^{\otimes}$ is a homomorphism of the corresponding algebras.

Remark 2.1. Note that $\widehat{\varphi}^{\otimes n}$ for any $n \in \mathbb{N}$ may be treated as a function of n variables $\mathbb{R}^n \ni (\xi_1, \dots, \xi_n) \longmapsto \widehat{\varphi}(\xi_1) \cdot \ldots \cdot \widehat{\varphi}(\xi_n) \in \mathbb{C}$ and may be written in the following way:

$$\widehat{\varphi}^{\otimes n}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}_+^n} e^{-\mathrm{i}(t,\xi)_n} \varphi(t_1) \cdot \ldots \cdot \varphi(t_n)\, dt,$$

where $(t,\xi)_n := t_1\xi_1 + \cdots + t_n\xi_n$, $dt := dt_1 \dots dt_n$. So, elements of $\Gamma(\widehat{\mathcal{S}}_+)$ can be considered as functions of infinitely many variables

$$\widehat{\boldsymbol{p}} : (\xi_1, \dots, \xi_n, \dots) \longmapsto \big(\widehat{p}_0, \widehat{p}_1(\xi_1), \widehat{p}_2(\xi_2, \xi_3), \dots, \widehat{p}_n(\xi_{\mathfrak{b}_n}, \dots, \xi_{\mathfrak{e}_n}), \dots\big), \tag{6}$$

where $\mathfrak{b}_n := \frac{n(n-1)}{2} + 1$, $\mathfrak{e}_n := \frac{n(n+1)}{2}$. But we note that actually each $\widehat{\boldsymbol{p}} \in \Gamma(\widehat{\mathcal{S}}_+)$ depends on a finite (depending on $\widehat{\boldsymbol{p}}$) number of variables, because for each $\widehat{\boldsymbol{p}}$ the sequence in the right-hand side of (6) is finite.

Define the operator $\mathcal{F}_+'^{\otimes}$ as follows

$$\mathcal{F}_+'^{\otimes} := (\mathcal{F}_+'^{\otimes_n}) : \Gamma(\mathcal{S}'_+) \ni \boldsymbol{f} = \big(f_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{f}} := \big(\widehat{f}_n\big) \in \Gamma(\widehat{\mathcal{S}}'_+),$$

where $\widehat{f}_n := \mathcal{F}_+'^{\otimes_n} f_n \in \widehat{\mathcal{S}}_+'^{\hat{\otimes} n}$, $\mathcal{F}_+'^{\otimes_0} := I_{\mathbb{C}}$, and each operator $\mathcal{F}_+'^{\otimes_n} : \mathcal{S}_+'^{\hat{\otimes} n} \longrightarrow \widehat{\mathcal{S}}_+'^{\hat{\otimes} n}$, $n \in \mathbb{N}$, is defined as a linear and continuous extension of the map $f^{\otimes n} \longmapsto (\mathcal{F}'_+ f)^{\otimes n}$ with $f \in \mathcal{S}'_+$.

Remark 2.2. From Remark 2.1 it follows that $\widehat{f}_n \in \widehat{\mathcal{S}}_+'^{\hat{\otimes} n} \simeq (\widehat{\mathcal{S}}_+^{\hat{\otimes} n})'$ is a functional of n "variables"

$$\widehat{p}_n(\xi_1, \dots, \xi_n) \longmapsto \langle \widehat{f}_n, \widehat{p}_n \rangle := \langle \widehat{f}_n(\xi_1, \dots, \xi_n), \widehat{p}_n(\xi_1, \dots, \xi_n) \rangle \in \mathbb{C}$$

with $\widehat{p}_n \in \widehat{\mathcal{S}}_+^{\hat{\otimes} n}$. So, any $\widehat{\boldsymbol{f}} = \big(\widehat{f}_n\big) \in \Gamma(\widehat{\mathcal{S}}'_+)$ we consider as a functional of infinitely many "variables" in the following sense (cf. (6)):

$$\begin{array}{rccc} \widehat{\boldsymbol{f}} : & \Gamma(\widehat{\mathcal{S}}_+) & \longrightarrow & \mathbb{C} \\ & \widehat{\boldsymbol{p}}(\xi_1, \dots, \xi_n, \dots) = \big(\widehat{p}_n(\xi_{\mathfrak{b}_n}, \dots, \xi_{\mathfrak{e}_n})\big) & \longmapsto & \langle \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{p}} \rangle := \displaystyle\sum_{n \in \mathbb{Z}_+} \langle \widehat{f}_n, \widehat{p}_n \rangle. \end{array}$$

3. Infinite parameter operator semigroups

Let E be a complex Banach space. Let $\mathbf{A} := (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \dots)$ be a countable system of operators, acting on E. It is convenient for us to rewrite this system as follows. Denote $A_n := (\mathbf{A}_{\mathfrak{b}_n}, \dots, \mathbf{A}_{\mathfrak{e}_n})$, where $\mathfrak{b}_n := \frac{n(n-1)}{2} + 1$, $\mathfrak{e}_n := \frac{n(n+1)}{2}$. Let by definition $A_0 := \emptyset$. Then the countable system of operators $\mathbf{A}$ can be represented as $\mathbf{A} = (A_0, A_1, A_2, \dots, A_n, \dots)$ or $\mathbf{A} = \big(A_n\big)$ for short.

For any $t \in \mathbb{R}_+^n$ let us denote $t \cdot A_n := t_1 \mathbf{A}_{\mathfrak{b}_n} + \dots + t_n \mathbf{A}_{\mathfrak{e}_n}$. Let A_n be a generator (see [8, 13]) of n-parameter C_0-semigroup $\mathbb{R}_+^n \ni t \longmapsto e^{-\mathrm{i}t\cdot A_n} \in L(E)$, satisfying the condition

$$\sup_{t\in\mathbb{R}_+^n} \|e^{-\mathrm{i}t\cdot A_n}\|_{L(E)} \le 1. \tag{7}$$

In what follows we assume that operators of the set A_n for all $n \in \mathbb{N}$ commute with each other. Note that in this case the semigroup can be represented (see [8, 13]) as a composition of commuting one-parameter marginal semigroups

$$e^{-\mathrm{i}t\cdot A_n} = e^{-\mathrm{i}t_1\mathbf{A}_{\mathfrak{b}_n}} \circ \dots \circ e^{-\mathrm{i}t_n\mathbf{A}_{\mathfrak{e}_n}}.$$

Let $\mathcal{G}$ be the set of countable systems of such generators. For all $n \in \mathbb{N}$ let $\mathcal{G}_n$ be a set of collections of some n generators of one-parameter C_0-semigroups satisfying the condition (7), and let $\mathcal{G}_0 := \{\emptyset\}$ by definition.

Define the mapping

$$\mathcal{L} := (\mathcal{L}_n)\colon \Gamma(\mathcal{S}_+) \ni \boldsymbol{p} = (p_n) \quad\longmapsto\quad \widetilde{\boldsymbol{p}} := \sum_{n\in\mathbb{Z}_+} \widetilde{p}_n \in \widetilde{\mathcal{S}}, \tag{8}$$

where $\widetilde{\mathcal{S}} := \sum_{n\in\mathbb{Z}_+} \widetilde{\mathcal{S}}_n$. Here each $\widetilde{\mathcal{S}}_n$, $n \in \mathbb{Z}_+$, is defined to be the space of functions

$$\widetilde{p}_n : \mathcal{G}_n \ni A_n \longmapsto \widetilde{p}_n(A_n) := \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} p_n(t)\,dt \in L(E) \tag{9}$$

for $n \in \mathbb{N}$, and $\widetilde{p}_0\colon \mathcal{G}_0 \ni A_0 \longmapsto \widetilde{p}_0(A_0) := p_0 I_E \in L(E)$, where the integral is understood in the sense of Bochner.

If the assumption (7) holds, then all the mappings $\mathcal{L}_n : p_n \longmapsto \widetilde{p}_n$, $n \in \mathbb{Z}_+$, are isomorphisms by virtue of [13, Theorem 15.2.1]. Indeed, the semigroups $\{e^{-\mathrm{i}(\lambda,t)} I_E : t \in \mathbb{R}_+^n\}$ with $-\operatorname{Im}\lambda \in \operatorname{int}\mathbb{R}_+^n$ satisfy the condition (7). Therefore, their generators $(-\mathrm{i}\lambda_1 I_E, \dots, -\mathrm{i}\lambda_n I_E)$ belong to $\mathcal{G}_n$. Note that

$$\widetilde{p}_n(-\mathrm{i}\lambda_1 I_E, \dots, -\mathrm{i}\lambda_n I_E) = \int_{\mathbb{R}_+^n} e^{-\lambda\cdot t} p_n(t)\,dt$$

is the Laplace transform of a function $p_n \in \mathcal{S}_+^{\hat{\otimes} n}$. Particularly, it follows that if $\widetilde{p}_n \equiv 0$, then $p_n \equiv 0$, i.e., $\operatorname{Ker}\mathcal{L}_n = \{0\}$, $n \in \mathbb{N}$. Hence, $\operatorname{Ker}\mathcal{L} = \{0\}$ and the map $\mathcal{L}$ is an isomorphism.

Remark 3.1. The mapping $\mathcal{L}\colon \Gamma(\mathcal{S}_+) \longrightarrow \widetilde{\mathcal{S}}$ is a homomorphism of the algebra $\{\Gamma(\mathcal{S}_+), \circledast\}$ and an algebra of operator valued functions defined on $\mathcal{G}$. On the other hand, the map $\mathcal{F}_+^{\otimes} : \Gamma(\mathcal{S}_+) \longrightarrow \Gamma(\widehat{\mathcal{S}}_+)$ is a homomorphism too. So, we can treat the mapping

$$\mathcal{L} \circ (\mathcal{F}_+^{\otimes})^{-1}\colon \Gamma(\widehat{\mathcal{S}}_+) \longrightarrow \widetilde{\mathcal{S}}$$

as an "elementary" functional calculus. In other words, we understand the operator $\widetilde{\boldsymbol{p}}(\mathbf{A}) = \sum_{n\in\mathbb{Z}_+} \widetilde{p}_n(A_n) \in L(E)$ as a "value" of a function $\widehat{\boldsymbol{p}}$ of infinitely many variables (see (6)) at a countable system $\mathbf{A} := (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \dots) \in \mathcal{G}$ of generators of contraction C_0-semigroups.

Consider the one-parameter semigroup $\widetilde{T}^{\otimes} : \mathbb{R}_+ \ni s \longmapsto \widetilde{T}_s^{\otimes} \in L(\widetilde{\mathcal{S}})$ on the space $\widetilde{\mathcal{S}}$, where

$$\widetilde{T}_s^{\otimes} := (\widetilde{T}_s^{\otimes n})\colon \widetilde{\boldsymbol{p}} = \sum_{n\in\mathbb{Z}_+} \widetilde{p}_n \quad\longmapsto\quad \widetilde{T}_s^{\otimes}\widetilde{\boldsymbol{p}} := \sum_{n\in\mathbb{Z}_+} \widetilde{T}_s^{\otimes n}\widetilde{p}_n.$$

The function $\widetilde{T}_s^{\otimes n}\widetilde{p}_n \in \widetilde{\mathcal{S}}_n$ is defined to be the map

$$\widetilde{T}_s^{\otimes n}\widetilde{p}_n\colon A_n \longmapsto \widetilde{T}_s^{\otimes n}\widetilde{p}_n(A_n) := \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} p_n(t+s)\,dt.$$

Here the function $\widetilde{p}_n$ of operator argument is defined by (9).

Using Bochner's integral properties (see [13, 3.7]), we obtain that for any $\boldsymbol{p} = (p_n) \in \Gamma(\mathcal{S}_+)$ with $p_n = \varphi^{\otimes n} \in \mathcal{S}_+^{\hat{\otimes} n}$, $\varphi \in \mathcal{S}_+$, the following equalities

$$\begin{aligned}\widetilde{T_s^{\otimes}\boldsymbol{p}}(\mathbf{A}) = \mathcal{L}\big[\big(T_s^{\otimes n} p_n\big)\big](\mathbf{A}) &= \mathcal{L}\big[\big((T_s\varphi)^{\otimes n}\big)\big](\mathbf{A}) \\ &= I_E + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} \varphi(t_1+s)\cdot\ldots\cdot\varphi(t_n+s)\,dt \\ &= \widetilde{p}_0(A_0) + \sum_{n\in\mathbb{N}} \widetilde{T}_s^{\otimes n}\widetilde{p}_n(A_n) = \widetilde{T}_s^{\otimes}\widetilde{\boldsymbol{p}}(\mathbf{A})\end{aligned}$$

hold for all $s \in \mathbb{R}_+$ and $\mathbf{A} := (A_n) \in \mathcal{G}$.

Hence, the operator $\widetilde{T}_s^{\otimes}$ can be represented as follows: $\widetilde{T}_s^{\otimes} = \mathcal{L}\circ T_s^{\otimes}\circ\mathcal{L}^{-1}$. Continuity of the mappings $T_s^{\otimes}$ and $\mathcal{L}$ as well as openness of $\mathcal{L}$ imply that the semigroup $\widetilde{T}^{\otimes} : \mathbb{R}_+ \ni s \longmapsto \widetilde{T}_s^{\otimes} \in L(\widetilde{\mathcal{S}})$ has the C_0-property.

We define commutant of the semigroup $\widetilde{T}^{\otimes}$ to be the set

$$\big[\widetilde{T}^{\otimes}\big]^c := \big\{\widetilde{T} \in L(\widetilde{\mathcal{S}}) : \widetilde{T}\circ\widetilde{T}_s^{\otimes} = \widetilde{T}_s^{\otimes}\circ\widetilde{T}, \forall s \in \mathbb{R}_+\big\}.$$

Define the mapping

$$\mathit{\Phi} := (\mathit{\Phi}_n) : \Gamma(\mathcal{S}'_+) \ni \boldsymbol{f} = (f_n) \quad\longmapsto\quad \mathit{\Phi}_{\boldsymbol{f}} := \sum_{n\in\mathbb{Z}_+} \mathit{\Phi}_{f_n} \in L(\widetilde{\mathcal{S}}), \tag{10}$$

where $f_n := f^{\otimes n} \in \mathcal{S}'^{\hat{\otimes} n}_+$, $f \in \mathcal{S}'_+$. Here $\mathit{\Phi}_{f_n} \in L(\widetilde{\mathcal{S}}_n)$, $n \in \mathbb{Z}_+$, is defined by the following formulas: $(\mathit{\Phi}_{f_0}\widetilde{p}_0)(A_0) := I_E$ and

$$\mathit{\Phi}_{f_n} : \widetilde{p}_n \longmapsto \widetilde{q}_n := \mathit{\Phi}_{f_n}\widetilde{p}_n, \quad \text{where} \quad \widetilde{q}_n(A_n) := \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} K_f^{\otimes n} p_n(t)\,dt, \quad n \in \mathbb{N}.$$

Here the function $\widetilde{p}_n$ of operator argument is defined by (9), and the operator $K_f^{\otimes n}$ is defined by (2) and (5).

Theorem 3.2. *The map $\mathit{\Phi}$ defined by* (10) *is an algebraic isomorphism of the algebra $\{\Gamma(\mathcal{S}'_+), \circledast\}$ and the subalgebra in the commutant $\big[\widetilde{T}^{\otimes}\big]^c$ of operators of the form $\widetilde{K}^{\otimes} = \mathcal{L}\circ K^{\otimes}\circ\mathcal{L}^{-1} \in L(\widetilde{\mathcal{S}})$, where $K \in L(\mathcal{S}_+)$. In particular, the equality $\mathit{\Phi}_{\boldsymbol{f}\circledast\boldsymbol{g}} = \mathit{\Phi}_{\boldsymbol{f}}\circ\mathit{\Phi}_{\boldsymbol{g}}$ holds for all $\boldsymbol{f}, \boldsymbol{g} \in \Gamma(\mathcal{S}'_+)$ and $\mathit{\Phi}_{\boldsymbol{\delta}}$ is the identity in $L(\widetilde{\mathcal{S}})$, where $\boldsymbol{\delta} = (\delta^{\otimes n})$.*

Moreover, differential the property

$$\mathit{\Phi}_{\mathbb{D}\boldsymbol{f}}\widetilde{\boldsymbol{p}} = -\mathit{\Phi}_{\boldsymbol{f}}\widetilde{\mathbb{D}\boldsymbol{p}} \tag{11}$$

holds for any $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$ and $\boldsymbol{p} \in \Gamma(\mathcal{S}_+)$.

Proof. For any $\boldsymbol{f} = (f_n) \in \Gamma(\mathcal{S}'_+)$, where $f_n := f^{\otimes n}$ with $f \in \mathcal{S}'_+$, and $\boldsymbol{p} = (p_n) \in \Gamma(\mathcal{S}_+)$ the equalities

$$\begin{aligned}(\mathit{\Phi}_{\boldsymbol{f}}\widetilde{\boldsymbol{p}})(\mathbf{A}) = \sum_{n\in\mathbb{Z}_+} (\mathit{\Phi}_{f_n}\widetilde{p}_n)(A_n) &= I_E + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} K_f^{\otimes n} p_n(t)\,dt \\ &= \mathcal{L}\big[\big(K_f^{\otimes n} p_n\big)\big](\mathbf{A}) = \widetilde{K_{\boldsymbol{f}}^{\otimes}\boldsymbol{p}}(\mathbf{A})\end{aligned} \tag{12}$$

are valid for all $\mathbf{A} := (A_n) \in \mathcal{G}$. It follows that the map $\mathit{\Phi}$ can be represented in the form $\mathit{\Phi}_{\boldsymbol{f}} = \mathcal{L}\circ K_{\boldsymbol{f}}^{\otimes}\circ\mathcal{L}^{-1}$. Continuity of the mappings $K_{\boldsymbol{f}}^{\otimes}$ and $\mathcal{L}$ as well as openness of

$\mathcal{L}$ imply that $\mathit{\Phi}_{\boldsymbol{f}} \in L\big(\widetilde{\mathcal{S}}\big)$ for all $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$. It follows that the equalities

$$\begin{aligned}(\mathit{\Phi}_{\boldsymbol{f}} \widetilde{T}_s^{\otimes} \widetilde{\boldsymbol{p}})(\mathbf{A}) = \sum_{n\in\mathbb{Z}_+} (\mathit{\Phi}_{f_n} \widetilde{T}_s^{\otimes n} \widetilde{p}_n)(A_n) &= I_E + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} K_f^{\otimes n} T_s^{\otimes n} p_n(t)\,dt \\ &= I_E + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} T_s^{\otimes n} K_f^{\otimes n} p_n(t)\,dt \\ &= I_E + \sum_{n\in\mathbb{N}} \widetilde{T}_s^{\otimes n} \int_{\mathbb{R}_+^n} e^{-\mathrm{i}t\cdot A_n} K_f^{\otimes n} p_n(t)\,dt \\ &= \sum_{n\in\mathbb{Z}_+} (\widetilde{T}_s^{\otimes n} \mathit{\Phi}_{f_n} \widetilde{p}_n)(A_n) = (\widetilde{T}_s^{\otimes} \mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}})(\mathbf{A})\end{aligned}$$

hold for all $s \in \mathbb{R}_+$, $\widetilde{\boldsymbol{p}} = \sum_{n\in\mathbb{Z}_+} \widetilde{p}_n \in \widetilde{\mathcal{S}}$ and $\mathbf{A} := \big(A_n\big) \in \mathcal{G}$. Hence, for all $\boldsymbol{f} \in \Gamma(\mathcal{S}'_+)$ the operator $\mathit{\Phi}_{\boldsymbol{f}}$ belongs to the commutant $\big[\widetilde{T}^{\otimes}\big]^c$.

Conversely, let $\widetilde{K}^{\otimes} = \mathcal{L} \circ K^{\otimes} \circ \mathcal{L}^{-1} \in L\big(\widetilde{\mathcal{S}}\big)$ with $K \in L(\mathcal{S}_+)$ belong to the commutant $\big[\widetilde{T}^{\otimes}\big]^c$. Then

$$\begin{aligned}\mathcal{L} \circ K^{\otimes} \circ T_s^{\otimes} \circ \mathcal{L}^{-1} =& \mathcal{L} \circ K^{\otimes} \circ \mathcal{L}^{-1} \circ \mathcal{L} \circ T_s^{\otimes} \circ \mathcal{L}^{-1} = \widetilde{K}^{\otimes} \circ \widetilde{T}_s^{\otimes} = \widetilde{T}_s^{\otimes} \circ \widetilde{K}^{\otimes} \\ =& \mathcal{L} \circ T_s^{\otimes} \circ \mathcal{L}^{-1} \circ \mathcal{L} \circ K^{\otimes} \circ \mathcal{L}^{-1} = \mathcal{L} \circ T_s^{\otimes} \circ K^{\otimes} \circ \mathcal{L}^{-1},\end{aligned}$$

therefore the operator $K^{\otimes}$ belongs to the commutant of the semigroup $T_s^{\otimes}$. From the proof of Theorem 1.3 it follows that there exists a unique $f \in \mathcal{S}'_+$ such that $K = K_f$ and $K^{\otimes} = K_{\boldsymbol{f}}^{\otimes}$ with $\boldsymbol{f} = \big(f^{\otimes n}\big)$. Hence, $\widetilde{K}^{\otimes} = \widetilde{K}_{\boldsymbol{f}}^{\otimes}$.

The proved above property, $K_{\boldsymbol{f}\circledast\boldsymbol{g}} = K_{\boldsymbol{f}} \circ K_{\boldsymbol{g}}$, implies the equality $K_{\boldsymbol{f}\circledast\boldsymbol{g}}^{\otimes} = K_{\boldsymbol{f}}^{\otimes} \circ K_{\boldsymbol{g}}^{\otimes}$. Therefore,

$$\begin{aligned}\mathit{\Phi}_{\boldsymbol{f}\circledast\boldsymbol{g}} = \mathcal{L} \circ K_{\boldsymbol{f}\circledast\boldsymbol{g}}^{\otimes} \circ \mathcal{L}^{-1} &= \mathcal{L} \circ K_{\boldsymbol{f}}^{\otimes} \circ K_{\boldsymbol{g}}^{\otimes} \circ \mathcal{L}^{-1} \\ &= \mathcal{L} \circ K_{\boldsymbol{f}}^{\otimes} \circ \mathcal{L}^{-1} \circ \mathcal{L} \circ K_{\boldsymbol{g}}^{\otimes} \circ \mathcal{L}^{-1} = \mathit{\Phi}_{\boldsymbol{f}} \circ \mathit{\Phi}_{\boldsymbol{g}}.\end{aligned}$$

As a consequence, we obtain the equalities $\mathit{\Phi}_{\boldsymbol{\delta}} \circ \mathit{\Phi}_{\boldsymbol{f}} = \mathit{\Phi}_{\boldsymbol{\delta}\circledast\boldsymbol{f}} = \mathit{\Phi}_{\boldsymbol{f}} = \mathit{\Phi}_{\boldsymbol{f}\circledast\boldsymbol{\delta}} = \mathit{\Phi}_{\boldsymbol{f}} \circ \mathit{\Phi}_{\boldsymbol{\delta}}$, so, $\mathit{\Phi}_{\boldsymbol{\delta}} \in L\big(\widetilde{\mathcal{S}}\big)$ acts as the identity operator.

It remains to prove the differential property (11). From (12) it follows $\mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}} = \widetilde{\boldsymbol{f} \star \boldsymbol{p}}$. So, using the Proposition 1.4, we obtain

$$\mathit{\Phi}_{\mathbb{D}\boldsymbol{f}} \widetilde{\boldsymbol{p}} = \widetilde{(\mathbb{D}\boldsymbol{f}) \star \boldsymbol{p}} = -\widetilde{\boldsymbol{f} \star (\mathbb{D}\boldsymbol{p})} = -\mathit{\Phi}_{\boldsymbol{f}} \widetilde{\mathbb{D}\boldsymbol{p}}.$$

Thus, the theorem is proved. □

Remark 3.3. For any fixed $\boldsymbol{p} \in \Gamma(\mathcal{S}_+)$ the map $\Gamma(\mathcal{S}'_+) \ni \boldsymbol{f} \longmapsto \mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{S}}$ is a homomorphism of the algebra $\big\{\Gamma(\mathcal{S}'_+), \circledast\big\}$ and the algebra of operator-valued functions defined on $\mathcal{G}$. Therefore we can treat this map as a functional calculus in the algebra of polynomial tempered distributions. It is easy to see that a function $\mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}}$ of operator argument can be represented as $\mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}} = \widetilde{\boldsymbol{f} \star \boldsymbol{p}}$ (see (8)). From (12) it follows that the operator $\mathit{\Phi}_{\boldsymbol{f}} \widetilde{\boldsymbol{p}}(\mathbf{A}) = \widetilde{\boldsymbol{f} \star \boldsymbol{p}}(\mathbf{A}) \in L(E)$ can be understood as a "value" of a function $\widehat{\boldsymbol{f} \star \boldsymbol{p}}$ of infinite many variables at a countable system $\mathbf{A} := (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n, \ldots) \in \mathcal{G}$ of generators of contraction C_0-semigroups.

Example. Let us consider the case of a countable set of second derivative operators. Let $H_n := L^2_{sym}(\mathbb{R}^n) \simeq L^2(\mathbb{R})^{\hat{\otimes} n}$, $n \in \mathbb{N}$, be the space of complex valued square integrable symmetric functions $y(\xi) = y(\xi_1, \ldots, \xi_n)$. Set $H_0 := \mathbb{C}$. It is known that the symmetric Fock space $H := \bigoplus_{n\in\mathbb{Z}_+} H_n$ is a Hilbert space (see e.g. [22]). As above, let $\mathfrak{b}_n = \frac{n(n-1)}{2} + 1$,

$\mathfrak{e}_n = \frac{n(n+1)}{2}$. Define the operators $\mathbf{D}^2_{n,m} : H \longrightarrow H$, $n \in \mathbb{N}$, $\mathfrak{b}_n \le m \le \mathfrak{e}_n$, as follows

$$\mathbf{D}^2_{n,m} := 0_{H_0} \otimes \cdots \otimes 0_{H_{n-1}} \otimes \frac{\partial^2}{\partial \xi_m^2} \otimes 0_{H_{n+1}} \otimes \ldots,$$

where 0_{H_n}, $n \in \mathbb{Z}_+$, denote zero operators of the corresponding spaces.

Let us define an "elementary" functional calculus in the algebra of polynomial test functions for the countable set of operators

$$\mathbf{D}^2 := (\mathbf{D}^2_{1,1}, \mathbf{D}^2_{2,1}, \mathbf{D}^2_{2,2}, \ldots, \mathbf{D}^2_{n,\mathfrak{b}_n}, \ldots, \mathbf{D}^2_{n,\mathfrak{e}_n} \ldots).$$

Let $D_n^2 := (\mathbf{D}^2_{n,\mathfrak{b}_n}, \ldots, \mathbf{D}^2_{n,\mathfrak{e}_n})$. It is easy to see that D_n^2, $n \in \mathbb{N}$, generates the semigroup

$$\mathbb{R}^n_+ \ni t = (t_1, \ldots, t_n) \longmapsto e^{-\mathrm{i}t \cdot D_n^2} \in L(H),$$

where

$$e^{-\mathrm{i}t\cdot D_n^2} := I_{H_0} \otimes \ldots \otimes I_{H_{\mathfrak{b}_n - 1}} \otimes e^{-\mathrm{i}t_1 \frac{\partial^2}{\partial \xi^2_{\mathfrak{b}_n}}} \circ \ldots \circ e^{-\mathrm{i}t_n \frac{\partial^2}{\partial \xi^2_{\mathfrak{e}_n}}} \otimes I_{H_{\mathfrak{e}_n+1}} \otimes \ldots$$

Denote

$$\mathfrak{g}_n(t, \zeta) := \prod_{j=1}^n \frac{1}{\sqrt{4\pi t_j}} e^{-\frac{\zeta_j^2}{4t_j}}.$$

From [19, Example 2] it follows that the semigroup $e^{-\mathrm{i}t\cdot D_n^2}$ acts as

$$e^{-\mathrm{i}t\cdot D_n^2} y = \big(y_0, \ldots, y_{n-1}, \mathfrak{g}_n(-\mathrm{i}t, \cdot) * y_n, y_{n+1}, \ldots\big)$$

for any $y = (y_0, y_1, \ldots, y_n, \ldots) \in H$.

Let $\boldsymbol{p} = \big(p_n\big) \in \Gamma(\mathcal{S}_+)$ be given. If we "substitute" the countable set $\mathbf{D}^2$ of operators instead of variables of a function $\widehat{\boldsymbol{p}}$ (see (6)) we obtain the operator acting as

$$\begin{aligned}\widetilde{\boldsymbol{p}}(\mathbf{D}^2) y(\xi_1, \xi_2, \ldots) &= y_0 + \sum_{n\in\mathbb{N}} \widetilde{p}_n(D_n^2) y_n(\xi_{\mathfrak{b}_n}, \ldots, \xi_{\mathfrak{e}_n}) \\ &= y_0 + \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n_+} (\mathfrak{g}_n(-\mathrm{i}t, \cdot) * y_n)(\xi_{\mathfrak{b}_n}, \ldots, \xi_{\mathfrak{e}_n}) p_n(t)\, dt,\end{aligned}$$

where $y(\xi_1, \xi_2, \ldots) = \big(y_0, y_1(\xi_1), y_2(\xi_2, \xi_3), \ldots, y_n(\xi_{\mathfrak{b}_n}, \ldots, \xi_{\mathfrak{e}_n}), \ldots\big) \in H$ is a function of infinite many variables.

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