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Existence and stability of traveling waves in parabolic systems of differential equations with weak diffusion

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The aim of the present paper is to investigate of some properties of periodic solutions of a nonlinear autonomous parabolic systems with a periodic condition. We investigate parabolic systems of differential equations using an integral manifolds method of the theory of nonlinear oscillations. We prove the existence of periodic solutions in an autonomous parabolic system of differential equations with weak diffusion on the circle. We study the existence and stability of an arbitrarily large finite number of cycles for a parabolic system with weak diffusion. The periodic solution of parabolic equation is sought in the form of traveling wave. A representation of the integral manifold is obtained. We seek a solution of parabolic system with the periodic condition in the form of a Fourier series in the complex form and introduce a norm in the space of the coefficients in the Fourier expansion. We use the normal forms method in the general parabolic system of differential equations with retarded argument and weak diffusion. We use bifurcation theory for delay differential equations and quasilinear parabolic equations. The existence of periodic solutions in an autonomous parabolic system of differential equations on the circle with retarded argument and small diffusion is proved. The problems of existence and stability of traveling waves in the parabolic system with retarded argument and weak diffusion are investigated.

Key words and phrases: bifurcation, stability, functional differential equation, integral manifold, traveling wave.

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Introduction

The approach used here follows the method of integral manifolds introduced by N.N. Bogolyubov and Yu.A. Mitropol'skii [2]. Generalizations of the method of integral manifolds and the averaging methods for functional differential equations the reader can find in the book by J.K. Hale [6]. The book by S.D. Eidel'man [3] is devoted exclusively to the study of parabolic systems. The books by D. Henry [8] and B.D. Hassard et al. [7] deal with the qualitative theory of quasilinear parabolic equations, and are devoted to searching answers to the following questions: do there exist integral manifolds or special solutions (periodic solutions etc.)? What about the stability and asymptotic behavior of these solutions? In [7], there are chapters involving ordinary differential equations and delay differential equations. The main subject of book [17] is the qualitative behavior of the solutions of semilinear partial differential equations with time delay and its applications. Book [5] is unique by its focus on the fundamental mathematical aspects of bifurcation theory of functional differential equations. The problems of stability and bifurcation of the solutions of functional-differential equations were

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considered, e.g., in [4,6,9,10,14]. The existence of countably many cycles in hyperbolic systems of differential equations with transformed argument were considered in [12]. The existence and stability of an arbitrarily large finite number of cycles for the equation of spin combustion with delay were considered in [1,13,16].

In the present paper, we study the existence and stability of an arbitrarily large finite number of cycles for a parabolic system with delay and weak diffusion. The existence of periodic solutions in this parabolic system is reduced to the similar problem for delay differential equations. We seek a solution of parabolic system with the periodic condition in the form of a Fourier series in the complex form and introduce the norm in the space of the coefficients in the Fourier expansion (see, e.g., [11]). Similar problems for partial differential equations were studied in numerous works (see, e.g., [1,13,15–17]).

1 Traveling waves for parabolic equations with weak diffusion

Consider the following system

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \varepsilon\gamma\frac{\partial^2 u_1}{\partial x^2} - \varepsilon\delta\frac{\partial^2 u_2}{\partial x^2} - \omega_0 u_2 + \varepsilon(\alpha u_1 - \beta u_2) + (d_0 u_1 - c_0 u_2)(u_1^2 + u_2^2),\\ \frac{\partial u_2}{\partial t} &= \varepsilon\gamma\frac{\partial^2 u_2}{\partial x^2} + \varepsilon\delta\frac{\partial^2 u_1}{\partial x^2} + \omega_0 u_1 + \varepsilon(\alpha u_2 + \beta u_1) + (d_0 u_2 + c_0 u_1)(u_1^2 + u_2^2)\end{aligned} \tag{1}$$

with the periodic condition

$$u_1(t, x + 2\pi) = u_1(t, x), \qquad u_2(t, x + 2\pi) = u_2(t, x), \tag{2}$$

where ε is a small positive parameter, $\omega_0 > 0$, $\alpha > 0$, $\gamma > 0$, $d_0 < 0$.

Passing to the complex variables $u = u_1 + iu_2$ and $\bar{u} = u_1 - iu_2$, we arrive at the equation

$$\frac{\partial u}{\partial t} = i\omega_0 u + \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 u}{\partial x^2} + (\alpha + i\beta)u\right] + (d_0 + ic_0)u^2\overline{u}. \tag{3}$$

In the present paper, we investigate the existence and stability of the wave solutions of problem (1), (2). The solution of equation (3) is sought in the form of traveling wave $u = \theta(y)$, $y = \sigma t + x$, where the function $\theta(y)$ is periodic with period 2π. We arrive at the equation

$$\sigma\frac{d\theta}{dy} = i\omega_0\theta + \varepsilon\left[(\gamma + i\delta)\frac{d^2\theta}{dy^2} + (\alpha + i\beta)\theta\right] + (d_0 + ic_0)\theta^2\overline{\theta}.$$

By the substitution $\dfrac{d\theta}{dy} = \theta_1$, this equation is reduced to the following system

$$\frac{d\theta}{dy} = \theta_1, \quad \sigma\theta_1 = i\omega_0\theta + \varepsilon\left[(\gamma + i\delta)\frac{d\theta_1}{dy} + (\alpha + i\beta)\theta\right] + (d_0 + ic_0)\theta^2\overline{\theta}. \tag{4}$$

The integral manifold of system (4) can be represented in the form

$$\theta_1 = \frac{i\omega_0}{\sigma}\theta + \varepsilon\left[\frac{\alpha + i\beta}{\sigma}\theta - \frac{\omega_0^2}{\sigma^3}(\gamma + i\delta)\theta\right] + \frac{d_0 + ic_0}{\sigma}\theta^2\bar{\theta} + \dots .$$

Here, we keep the terms of order $O(\varepsilon)$ in the linear terms and the terms of order $O(1)$ in the nonlinear terms. The equation on this manifold takes the form

$$\frac{d\theta}{dy} = \frac{i\omega_0}{\sigma}\theta + \varepsilon\left[\frac{\alpha + i\beta}{\sigma}\theta - \frac{\omega_0^2}{\sigma^3}(\gamma + i\delta)\theta\right] + \frac{d_0 + ic_0}{\sigma}\theta^2\bar{\theta} + \dots . \tag{5}$$

Passing to the polar coordinates $\theta = r\exp(i\varphi)$ in equation (5), we get

$$\frac{dr}{dy} = \varepsilon\left(\frac{\alpha}{\sigma} - \frac{\gamma}{\sigma^3}\omega_0^2\right)r + \frac{d_0}{\sigma}r^3. \tag{6}$$

Let $d_0 < 0$ and let the inequality $\alpha > \frac{\gamma}{\sigma^2}\omega_0^2$ be true. Then equation (6) possesses the stationary solution

$$r = \sqrt{\varepsilon}R_0, \quad R_0 = \sqrt{\left(\alpha - \frac{\gamma}{\sigma^2}\omega_0^2\right)|d_0|^{-1}},$$

hence, the periodic solution of equation (5) takes the form

$$\theta = \sqrt{\varepsilon}R_0\exp\left(\frac{i\omega_0}{\sigma}y\right) + O(\varepsilon).$$

Since the function θ is periodic with period 2π, we obtain

$$\sigma = \frac{\omega_0}{n} + O(\varepsilon), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Thus, the periodic solution of equation (3) takes the form

$$u_n = u_n(t,x) = \sqrt{\varepsilon}r_n\exp(i(\chi_n(\varepsilon)t + nx)) + O(\varepsilon), \tag{7}$$

where

$$r_n = \sqrt{(\alpha - n^2\gamma)|d_0|^{-1}}, \quad \chi_n(\varepsilon) = \omega_0 + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2, \quad n \in \mathbb{Z}.$$

Thus, the periodic solution of problem (1), (2) takes the form

$$u_1 = \sqrt{\varepsilon}r_n\cos(\chi_n(\varepsilon)t + nx), \quad u_2 = \sqrt{\varepsilon}r_n\sin(\chi_n(\varepsilon)t + nx), \quad n \in \mathbb{Z}. \tag{8}$$

The following statement is true.

Theorem 1. *Let $\omega_0 > 0$, $\alpha > 0$, $\gamma > 0$, $d_0 < 0$ and let the inequality*

$$\alpha > \gamma n^2$$

be true for some integer n. Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, problem (1), (2) *has solutions* (8) *periodic in t.*

2 Stability of periodic solutions

The equation in variations in the vicinity of solution (7) of equation (3) takes the form

$$\frac{\partial v}{\partial t} = i\omega_0 v + \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 v}{\partial x^2} + (\alpha + i\beta)v\right] + \varepsilon(d_0 + ic_0)(2r_n^2 v + w_n^2\bar{v}), \tag{9}$$

where $w_n = r_n \exp(i(\chi_n(\varepsilon)t + nx))$, $\chi_n(\varepsilon) = \omega_0 + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2$.

By the substitution $v = w\exp(i\chi_n(\varepsilon)t)$ in equation (9), we find

$$\frac{\partial w}{\partial t} = \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 w}{\partial x^2} + (\alpha + i\delta n^2 + d_0 r_n^2)w + (d_0 + ic_0)r_n^2(w + \overline{w}\exp(2inx))\right]. \tag{10}$$

We seek the solution of equation (10) in the form of Fourier series in the complex form

$$w(t,x) = \sum_{k=-\infty}^{\infty} y_k(t)\exp(ikx), \quad \overline{w}(t,x) = \sum_{k=-\infty}^{\infty} v_k(t)\exp(ikx). \tag{11}$$

Substituting (11) in (10) and equating the coefficients of $\exp(ikx)$, we obtain the equations for the coefficients of the Fourier series

$$\frac{dy_{k+n}}{dt} = \varepsilon[(\alpha + i\delta n^2 + d_0 r_n^2)y_{k+n} - (\gamma + i\delta)(k+n)^2 y_{k+n} + (d_0 + ic_0)r_n^2(y_{k+n} + v_{k-n})]. \tag{12}$$

Similarly, substituting (11) in the equation adjoint to (10), we get

$$\frac{dv_{k-n}}{dt} = \varepsilon[(\alpha - i\delta n^2 + d_0 r_n^2)v_{k-n} - (\gamma - i\delta)(k-n)^2 v_{k-n} + (d_0 - ic_0)r_n^2(v_{k-n} + y_{k+n})]. \tag{13}$$

The stability of the wave solutions of problem (1), (2) is determined by the stability of system (12), (13) with the parameter $k \in \mathbb{Z}$. By the substitution

$$y_{k+n} = z_{k+n}\exp(-2i\varepsilon\delta kn) \quad \textit{and} \quad v_{k-n} = w_{k-n}\exp(-2i\varepsilon\delta kn)$$

in system (12), (13), we get a linear system with the matrix

$$\varepsilon A = \begin{pmatrix} \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & \varepsilon a_{22} \end{pmatrix}.$$

The matrix A has an eigenvalue equal to zero for $k = 0$. Since the sum of diagonal elements of the matrix A is negative, $a = a_{11} + a_{22} < 0$, for the orbital exponential stability of the periodic solution $u_n(t,x)$, it is necessary and sufficient that the condition $a^2 c > f^2$, where

$$c = \mathrm{Re}(\det(A)), \quad f = \mathrm{Im}(\det(A)), \quad \text{and} \quad f = 4\gamma kn(c_0 r_n^2 - \delta k^2),$$

be satisfied for $k \neq 0$, i.e.

$$(d_0 r_n^2 - \gamma k^2)^2(\gamma^2 k^2 + \delta^2 k^2 - 2\gamma d_0 r_n^2 - 4\gamma^2 n^2 - 2\delta c_0 r_n^2) > 4\gamma^2 n^2 (c_0 r_n^2 - \delta k^2)^2, \tag{14}$$

where $r_n^2 = (\gamma n^2 - \alpha)/d_0$.

Theorem 2. *The traveling waves $u_n(t,x)$ of problem* (1), (2) *are exponentially orbitally stable if and only if condition* (14) *is satisfied for all $k \in \mathbb{Z}\backslash\{0\}$.*

Example 1. *We consider system* (1), *where $\delta = 0$, $c_0 = 0$. Hence, Theorem 1 implies that the periodic solution*

$$u_n = \sqrt{\varepsilon(\alpha - \gamma n^2)|d_0|^{-1}}\begin{pmatrix} \cos((\omega_0 + \varepsilon\beta)t + nx) \\ \sin((\omega_0 + \varepsilon\beta)t + nx) \end{pmatrix}$$

exists for $d_0 < 0$ and $\gamma n^2 < \alpha$. By Theorem 2, the traveling waves $u_n(t,x)$ are exponentially orbitally stable if and only if

$$n^2 < \frac{1}{6\gamma}(\gamma + 2\alpha).$$

3 Bifurcation of self-excited vibrations for parabolic systems with retarded argument and weak diffusion

Let $\mathbb{R}^n$ be the n-dimensional space with the norm $|u| = \sqrt{u_1^2 + ... + u_n^2}$, $\mathbb{C} = \mathbb{C}[-\Delta, 0]$ be the space of functions, continuous on $[-\Delta, 0]$ with values in $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup\limits_{-\Delta \le \theta \le 0} |\varphi(\theta)|$. We denote by u_t the element of the space $\mathbb{C}$ defined by the function

$$u_t(\theta, x) = u(t + \theta, x) - \Delta \le \theta \le 0.$$

We consider the following parabolic system with delay and weak diffusion

$$\frac{\partial u}{\partial t} = \varepsilon D \frac{\partial^2 u}{\partial x^2} + L(\varepsilon) u_t + f(u_t, \varepsilon) \tag{15}$$

with periodic condition

$$u(t, x + 2\pi) = u(t, x), \tag{16}$$

where ε is a small positive parameter, $u \in \mathbb{R}^n$, $L(\varepsilon) : \mathbb{C} \to \mathbb{R}^n$ is a continuous linear operator, $f : \mathbb{C} \times [0, \varepsilon_0) \to \mathbb{R}^n$, $f(\varphi, \varepsilon) = O(\|\varphi\|^2)$ as $\|\varphi\| \to 0$, the operator $f(\varphi, \varepsilon)$ is continuous in ε and four times continuously differentiable in φ. Let us assume that the zero solution of (15) for $\varepsilon = 0$ is asymptotically stable.

Along with (15) we consider the linear equations

$$\frac{\partial \tilde{u}}{\partial t} = L(\varepsilon) \tilde{u}_t, \tag{17}$$

$$\frac{\partial \tilde{u}}{\partial t} = L(0) \tilde{u}_t. \tag{18}$$

According to Riesz theorem, the operator $L(\varepsilon)$ can be represented in the form of a Stieltjes integral

$$L(\varepsilon)\varphi = \int_{-\Delta}^{0} [d\eta(\theta, \varepsilon)] \varphi(\theta),$$

where the matrix $\eta(\theta, \varepsilon)$ has bounded variation in θ. Let $\eta(\theta, \varepsilon)$ be twice continuously differentiable in ε. The characteristic equation for (17) has the form

$$\det \Lambda_\varepsilon(\lambda) = 0, \ \Lambda_\varepsilon(\lambda) = \lambda I - \int_{-\Delta}^{0} e^{\lambda\theta} d\eta(\theta, \varepsilon), \tag{19}$$

where I denotes the identity matrix. Let us assume that (19) has one pair of roots of the form $\xi(\varepsilon) \pm i\zeta(\varepsilon)$, $\xi(0) = 0$, $\xi'(0) > 0$, $\zeta(0) > 0$, and the other roots lie in the half-plane $\operatorname{Re} \lambda \le \lambda_0 < 0$.

One can represent (15) in the form

$$\frac{\partial u}{\partial t} = L(0) u_t + F(u_t, \varepsilon), \tag{20}$$

where $F(u_t,\varepsilon) = \varepsilon D\dfrac{\partial^2 u}{\partial x^2} + L(\varepsilon)u_t - L(0)u_t + f(u_t,\varepsilon)$. We denote by $\tilde{u}_t(\varphi)$ the solution of (18) with initial function $\varphi \in \mathbb{C}$. We define a translation operator with respect to the solution of (18) by $T(t)\varphi = \tilde{u}_t(\varphi)$. The family $\{T(t), t \geq 0\}$ forms a strongly continuous semigroup. A generating operator of this semigroup is the differentiation operator $A\varphi(\theta) = \dfrac{d\varphi(\theta)}{d\theta}$, $-\Delta \leq \theta \leq 0$, with domain defined as follows

$$D(A) = \left\{\varphi \in \mathbb{C}, \frac{d\varphi}{d\theta} \in \mathbb{C}, \frac{d\varphi(0)}{d\theta} = L(0)\varphi\right\}.$$

We denote by $\mathbb{P}$ the eigensubspace of $\mathbb{C}$, generated by solutions of (18), corresponding to the roots $\pm i\zeta(0)$. We decompose the space $\mathbb{C}$ into a direct sum: $\mathbb{C} = \mathbb{P} \oplus \mathbb{Q}$. Let $\Phi = \Phi(\theta)$ be a basis in $\mathbb{P}$. Considering the adjoint equation to (18), one can define a function $\Psi = \Psi(\theta)$, $0 \leq \theta \leq \Delta$ analogously. Then each element $u_t \in \mathbb{C}$ can be represented in the form $u_t = \Phi y(t) + z_t$, where $y(t) = (\Psi, u_t)$, $z_t = u_t - \Phi y(t)$, $y(t) \in \mathbb{R}^2$, $z_t \in \mathbb{Q}$, (Ψ, u_t) is some bilinear functional. Equation (15) is equivalent to the system of equations [4, 6]:

$$\frac{\partial y}{\partial t} = By + \Psi(0)F(\Phi y + z_t, \varepsilon), \quad z_t = T(t-\sigma)z_\sigma + \int_\sigma^t T(t-s)X_0^Q F(\Phi y(s) + z_s, \varepsilon)ds.$$

Here X_0^Q is the projection onto the subspace Q of the function $X_0(\theta) = 0$, $-\Delta \leq \theta < 0$, $X_0(0) = I$,

$$B = \begin{pmatrix} 0 & \zeta(0) \\ -\zeta(0) & 0 \end{pmatrix}.$$

Analogously to [4], one can prove the existence of a function $g : \mathbb{R}^2 \times [0, \varepsilon_0) \to \mathbb{Q}$ satisfying the conditions $g(0,\varepsilon) = 0$, $\|g(y,\varepsilon) - g(y',\varepsilon)\| \leq \dfrac{1}{2}|y - y'|$ and such that the set

$$S_\varepsilon = \{(\varphi, \varepsilon) | \varepsilon \in [0, \varepsilon_0), \varphi = \Phi y + \vartheta, y \in \mathbb{R}^2, \vartheta = g(y,\varepsilon), \vartheta \in \mathbb{Q}\}$$

is a local integral manifold of (20). The function $g(y,\varepsilon)$ will be four times continuously differentiable in y. The behavior of the solutions of (20) on the integral manifold S_ε is described by the equation

$$\frac{\partial v}{\partial t} = Bv + \Psi(0)F(\Phi v + g(v,\varepsilon), \varepsilon), \tag{21}$$

where $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. For any solution $u_t = \Phi y(t) + z_t$ of (20) there exists a solution $\chi_t = \Phi v(t) + g(v(t), \varepsilon)$ belonging to S_ε and such that

$$\|u_t - \chi_t\| \leq Ke^{-\nu t}, \quad K > 0, \quad \nu > 0.$$

In equation (21), we keep the terms of order $O(\varepsilon)$ in the linear terms. We arrive at the equation

$$\frac{\partial v}{\partial t} = Bv + \varepsilon\Psi(0)D\Phi(0)\frac{\partial^2 v}{\partial x^2} + \varepsilon\Psi(0)L'(0)\Phi v + \Psi(0)f(\Phi v + g(v,\varepsilon), \varepsilon).$$

Passing to the complex variables $w = v_1 + iv_2$, $\overline{w} = v_1 - iv_2$, we arrive at the equation

$$\frac{\partial w}{\partial t} = \varepsilon(\gamma + i\delta)\frac{\partial^2 w}{\partial x^2} + \varepsilon(\gamma_1 + i\delta_1)\frac{\partial^2 \overline{w}}{\partial x^2} - i\zeta(0)w + \varepsilon(\alpha + i\beta)w + \varepsilon(\alpha_1 + i\beta_1)\overline{w} + W(w, \overline{w}, \varepsilon), \tag{22}$$

where

$$(\gamma + i\delta)w + (\gamma_1 + i\delta_1)\overline{w} = (1,i)\Psi(0)D\Phi(0)v,\ (\alpha + i\beta)w + (\alpha_1 + i\beta_1)\overline{w} = (1,i)\Psi(0)L'(0)\Phi v,$$
$$\alpha = \xi'(0),\ \beta = \zeta'(0),\ W(w,\overline{w},\varepsilon) = (1,i)\Psi(0)f(\Phi v + g(v,\varepsilon),\varepsilon).$$

We transform (22) with the help of the substitution

$$w = s + V_2(s,\overline{s}) + V_3(s,\overline{s}), \tag{23}$$

where V_2 and V_3 are forms of strictly second and third order. One can choose a transformation (23) so that the equation for s assumes the form

$$\frac{\partial s}{\partial t} = \varepsilon(\gamma + i\delta)\frac{\partial^2 s}{\partial x^2} + \varepsilon(\gamma_1 + i\delta_1)\frac{\partial^2 \overline{s}}{\partial x^2} - i\zeta(0)s + \varepsilon(\alpha + i\beta)s + \varepsilon(\alpha_1 + i\beta_1)\overline{s} + (d_0 + ic_0)s^2\overline{s} + \dots. \tag{24}$$

Here, we keep the terms of order $O(\varepsilon)$ in the linear terms and the terms of order $O(1)$ in the nonlinear terms.

We investigate the existence and stability of the wave solutions of problem (15), (16). A solution of equation (24) is sought in the form of traveling wave $s = \theta(y)$, $y = \sigma t + x$, where the function $\theta(y)$ is periodic with period 2π. We arrive at the equation

$$\sigma\frac{d\theta}{dy} = \varepsilon(\gamma + i\delta)\frac{d^2\theta}{dy^2} + \varepsilon(\gamma_1 + i\delta_1)\frac{d^2\overline{\theta}}{dy^2} - i\zeta(0)\theta + \varepsilon(\alpha + i\beta)\theta + \varepsilon(\alpha_1 + i\beta_1)\overline{\theta} + (d_0 + ic_0)\theta^2\overline{\theta} + \dots.$$

By the substitution $\dfrac{d\theta}{dy} = \theta_1$, this equation is reduced to the following system

$$\begin{aligned}\frac{d\theta}{dy} = \theta_1,\ \sigma\theta_1 = \varepsilon(\gamma + i\delta)\frac{d\theta_1}{dy} + \varepsilon(\gamma_1 + i\delta_1)\frac{d\overline{\theta}_1}{dy} - i\zeta(0)\theta \\ + \varepsilon(\alpha + i\beta)\theta + \varepsilon(\alpha_1 + i\beta_1)\overline{\theta} + (d_0 + ic_0)\theta^2\overline{\theta} + \dots.\end{aligned} \tag{25}$$

The integral manifold of system (25) can be represented in the form

$$\theta_1 = -\frac{\zeta(0)}{\sigma}i\theta + \frac{\varepsilon}{\sigma}\left[-\frac{\zeta^2(0)}{\sigma^2}(\gamma + i\delta)\theta - \frac{\zeta^2(0)}{\sigma^2}(\gamma_1 + i\delta_1)\overline{\theta} + (\alpha + i\beta)\theta + (\alpha_1 + i\beta_1)\overline{\theta}\right] + \frac{d_0 + ic_0}{\sigma}\theta^2\overline{\theta} + \dots.$$

Here, we keep the terms of order $O(\varepsilon)$ in the linear terms and the terms of order $O(1)$ in the nonlinear terms. The equation on this manifold takes the form

$$\frac{d\theta}{dy} = -\frac{\zeta(0)}{\sigma}i\theta + \frac{\varepsilon}{\sigma}\left[-\frac{\zeta^2(0)}{\sigma^2}(\gamma + i\delta)\theta - \frac{\zeta^2(0)}{\sigma^2}(\gamma_1 + i\delta_1)\overline{\theta} + (\alpha + i\beta)\theta + (\alpha_1 + i\beta_1)\overline{\theta}\right] + \frac{d_0 + ic_0}{\sigma}\theta^2\overline{\theta} + \dots.$$

In this equation, we perform the substitution $\theta = p\exp\left(-\dfrac{\zeta(0)}{\sigma}iy\right)$ and apply the averaging method [2]. We arrive at the autonomous equation

$$\frac{dp}{dy} = \frac{\varepsilon}{\sigma}\left[-\frac{\zeta^2(0)}{\sigma^2}(\gamma + i\delta)p + (\alpha + i\beta)p\right] + \frac{d_0 + ic_0}{\sigma}p^2\overline{p}. \tag{26}$$

Passing to the polar coordinates $p = r\exp(i\varphi)$ in equation (26), we get

$$\frac{dr}{dy} = -\varepsilon\frac{\zeta^2(0)\gamma}{\sigma^3}r + \varepsilon\frac{\alpha}{\sigma}r + \frac{d_0}{\sigma}r^3. \tag{27}$$

Let the inequalities $\gamma > 0$, $d_0 < 0$, $\alpha\sigma^2 > \zeta^2(0)\gamma$ be satisfied. Then equation (27) possesses the stationary solution

$$r = \sqrt{\varepsilon}R_0, \quad R_0 = \sqrt{\left(\alpha - \frac{\zeta^2(0)\gamma}{\sigma^2}\right)|d_0|^{-1}},$$

hence, the periodic solution of system (25) takes the form

$$\theta = \sqrt{\varepsilon}R_0\exp\left(-\frac{\zeta(0)}{\sigma}iy\right) + O(\varepsilon),\ \theta_1 = \frac{d\theta}{dy}.$$

Since the function θ is periodic with period 2π, we get

$$\sigma = -\frac{\zeta(0)}{n} + O(\varepsilon), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Thus, the periodic solution of equation (24) takes the form

$$s_n = s_n(t,x) = \sqrt{\varepsilon}r_n\exp(i(\chi_n(\varepsilon)t + nx)), \tag{28}$$

where $r_n = \sqrt{(\alpha - n^2\gamma)\,|d_0|^{-1}}$, $\chi_n(\varepsilon) = -\zeta(0) + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2$. Thus, the periodic solution of problem (15), (16) takes the form

$$u_t = \sqrt{\varepsilon}r_n\Phi\begin{pmatrix}\cos(\chi_n(\varepsilon)t + nx)\\ \sin(\chi_n(\varepsilon)t + nx)\end{pmatrix}. \tag{29}$$

The equation in variations in the vicinity of solution (28) of equation (24) takes the form

$$\frac{\partial v}{\partial t} = -i\zeta(0)v + \varepsilon(\gamma + i\delta)\frac{\partial^2 v}{\partial x^2} + \varepsilon(\gamma_1 + i\delta_1)\frac{\partial^2\overline{v}}{\partial x^2} + \varepsilon(\alpha + i\beta)v + \varepsilon(\alpha_1 + i\beta_1)\overline{v}$$
$$+ (d_0 + ic_0)(2\varepsilon r_n^2 v + s_n^2\overline{v}).$$

In this equation, we perform the substitution $v = w\exp(i\chi_n(\varepsilon)t)$ and apply the averaging method [2]. We arrive at the equation

$$\frac{\partial w}{\partial t} = \varepsilon\left[(\gamma + i\delta)\frac{\partial^2 w}{\partial x^2} + (\alpha + i\delta n^2 + d_0 r_n^2)w + (d_0 + ic_0)r_n^2(w + \overline{w}\exp(2inx))\right]. \tag{30}$$

We seek the solution of equation (30) in the form of Fourier series in the complex form

$$w(t,x) = \sum_{k=-\infty}^{\infty} y_k(t)\exp(ikx), \quad \overline{w}(t,x) = \sum_{k=-\infty}^{\infty} v_k(t)\exp(ikx). \tag{31}$$

Substituting (31) in (30) and equating the coefficients of $\exp(ikx)$, we obtain the equations for the coefficients of the Fourier series

$$\frac{dy_{k+n}}{dt} = \varepsilon[(\alpha + i\delta n^2 + d_0 r_n^2)y_{k+n} - (\gamma + i\delta)(k+n)^2 y_{k+n} + (d_0 + ic_0)r_n^2(y_{k+n} + v_{k-n})]. \tag{32}$$

Similarly, substituting (31) in the equation adjoint to (30), we get

$$\frac{dv_{k-n}}{dt} = \varepsilon[(\alpha - i\delta n^2 + d_0 r_n^2)v_{k-n} - (\gamma - i\delta)(k-n)^2 v_{k-n} + (d_0 - ic_0)r_n^2(v_{k-n} + y_{k+n})]. \quad (33)$$

The stability of the wave solutions of problem (15), (16) is determined by the stability of system (32), (33) with a parameter $k \in \mathbb{Z}$. By the substitution

$$y_{k+n} = z_{k+n}\exp(-2i\varepsilon\delta kn) \quad \textit{and} \quad v_{k-n} = w_{k-n}\exp(-2i\varepsilon\delta kn)$$

in system (32), (33), we get a linear system with the matrix

$$\varepsilon A = \begin{pmatrix} \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & \varepsilon a_{22} \end{pmatrix}.$$

Since $\alpha - \gamma n^2 = -d_0 r_n^2$, the matrix A has an eigenvalue equal to zero for $k = 0$. Since the sum of diagonal elements of the matrix A is negative, $a = a_{11} + a_{22} < 0$, for the orbital exponential stability of the periodic solution u_t, it is necessary and sufficient that the condition $a^2 c > f^2$, where

$$c = \mathrm{Re}(\det(A)), \quad f = \mathrm{Im}(\det(A)), \quad \textit{and} \quad f = 4\gamma kn(c_0 r_n^2 - \delta k^2),$$

be satisfied for $k \neq 0$, i.e.

$$(d_0 r_n^2 - \gamma k^2)^2(\gamma^2 k^2 + \delta^2 k^2 - 2\gamma d_0 r_n^2 - 4\gamma^2 n^2 - 2\delta c_0 r_n^2) > 4\gamma^2 n^2 (c_0 r_n^2 - \delta k^2)^2. \quad (34)$$

Theorem 3. *Let $\gamma > 0$, $d_0 < 0$ and let the inequality*

$$\alpha > \gamma n^2$$

be true for some integer n. Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, problem (15), (16) *has solutions* (29) *periodic in t. These solutions are exponentially orbitally stable if and only if condition* (34) *is satisfied for all $k \in \mathbb{Z}\backslash\{0\}$.*

To find periodic solutions of problem (15), (16), approximately, it suffices to restrict oneself to terms of the second and third order in the expansion of the function $\Psi(0)f(\Phi v + g(v,0),0)$ in a power series in v. But for this, it suffices to know the second order terms in the expansion of the function $g(v,0)$. The first approximation of the function $g(v,0)$ has the form

$$g_1(v,0) = \int_{-\infty}^{0} T(-s)X_0^Q f(\Phi e^{Bs}v,0)ds.$$

We represent the function $f(\Phi y,0)$ in the form $f(\Phi y,0) = c_1 y_1^2 + c_2 y_1 y_2 + c_3 y_2^2 + O(|y|^3)$. Then the determination of the function $g_1(v,0)$ is reduced to the calculation of the integral

$$z = \int_{-\infty}^{0} T(-s)X_0^Q e^{i\omega s}ds,$$

where $\omega \in \{0, 2\zeta(0), -2\zeta(0)\}$. We note that the integral z converges and is uniformly bounded in ω.

Theorem 4. *For any real ω the function $z(\theta)$ belongs to $\mathbb{Q} \bigcap D(A)$ and one has [14]*

$$i\omega z - Az = X_0^Q. \tag{35}$$

Thus, to find z it is necessary to solve (35) with respect to z. This equation is equivalent to the following system:

$$\frac{dz(\theta)}{d\theta} - i\omega z(\theta) = -X_0^Q(\theta), \ -\Delta \leq \theta < 0, \tag{36}$$

$$\int_{-\Delta}^{0} [d\eta(\theta, 0)]z(\theta) - i\omega z(0) = -X_0^Q(0). \tag{37}$$

Analogously to [14], there exists a unique solution of system of equations (36), (37).

4 Conclusions

The methods of this article can be used to study the existence and stability of traveling waves in the Brusselator equations with weak diffusion.

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Клевчук І.І. *Існування та стійкість біжучих хвиль у параболічних системах із малою дифузією* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 493–503.

Досліджено деякі властивості періодичних розв'язків автономної параболічної системи з періодичною умовою. Для дослідження параболічних систем диференціальних рівнянь використовується метод інтегральних многовидів теорії нелінійних коливань. Доведено існування періодичних розв'язків автономної параболічної системи диференціальних рів-нянь з малою дифузією на колі. Вивчено питання існування та стійкості як завгодно великого скінченного числа циклів параболічної системи із малою дифузією. Періодичні розв'язки параболічної системи шукаються у вигляді біжучої хвилі. Одержано зображен-ня інтегральних многовидів. Ми шукаємо розв'язок параболічної системи з періодичною умовою у вигляді ряду Фур'є в комплексній формі і вводимо норму в просторі коефіцієнтів розкладу в ряд Фур'є. Використано метод нормальних форм для загальної параболічної системи диференціальних рівнянь із запізненням аргументу та малою дифузією. Також використовуються методи теорії біфуркацій для диференціальних рівнянь із запізненням та квазілінійних параболічних рівнянь. Доведено існування періодичних розв'язків авто-номної параболічної системи диференціальних рівнянь на колі із запізненням аргументу та малою дифузією. Досліджено існування та стійкість хвильових розв'язків параболічної системи із запізненням аргументу та малою дифузією.

Ключові слова і фрази: біфуркація, стійкість, диференціально-функціональне рівняння, інтегральний многовид, біжуча хвиля.