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On $*$-measure monads on the category of ultrametric spaces

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The functor of $*$-measures of compact support on the category of ultrametric spaces and non-expanding maps is introduced in the previous publication of the authors. In the present note, we prove that this functor determines a monad on this category. The monad structure allows us to define the tensor product of $*$-measures. We consider some applications of this notion to equilibrium theory.

Key words and phrases: ultrametric space, non-expanding map, $*$-measure, monad, equilibrium.

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Introduction

In [6], the functors of $*$-measures on the category **Ultr** of ultrametric spaces and non-expanding maps are defined and some of their fundamental properties are established.

The present note is devoted to the monad structure determined by these functors. In particular, this structure allows us to define the tensor product of the $*$-measures in the category **Ultr**. In turn, we define the games in $*$-measure-valued strategies and prove the continuity of the payoff functions for these games.

Finally, it is proved that any equilibrium for games in $*$-measure-valued strategies can be approximated by almost equilibria consisting of $*$-measures of finite support.

1 Preliminaries

Recall that a metric d on a set X is called an ultrametric (a non-Archimedean metric) if d satisfies the strong triangle inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}, \quad x,y,z \in X.$$

By $\mathbb{I}$ we denote the unit segment $[0,1]$. Recall that a triangular norm (a t-norm) is a continuous function $\mathbb{I} \times \mathbb{I} \ni (a,b) \mapsto a * b \in \mathbb{I}$ satisfying the following conditions:

1) $*$ is associative;

2) $*$ is commutative;

3) $*$ is monotone, i.e. $a \leq a'$ and $b \leq b'$ both imply $a * b \leq a' * b'$ for all a, a', b, $b' \in \mathbb{I}$;

4) 1 is a unit.

See, e.g., [3] for the details. The following are examples of t-norms: $\cdot$ (multiplication), min, $(a,b) \mapsto \max\{a+b-1, 0\}$ (Łukasiewicz t-norm).

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Let us recall the notion of $*$-measure (see [5] for the details). Given topological spaces X, Y, by $C(X,Y)$ we denote the set of continuous functions from X to Y. By $\vee$ we denote the operation of maximum of numbers as well as pointwise maximum of real-valued functions.

Definition 1. *Let $*$ be a t-norm. A functional $\mu\colon C(X,\mathbb{I}) \to \mathbb{I}$ is called a $*$-measure on a compact Hausdorff space X if the following conditions hold:*

1) $\mu(c_X) = c$, where c_X denotes the constant function on X taking value c;

2) $\mu(\lambda * \varphi) = \lambda * \mu(\varphi)$;

3) $\mu(\varphi \vee \psi) = \mu(\varphi) \vee \mu(\psi)$.

The notion of $*$-measure can be also formulated for Tychonov spaces X. We additionally require that there exists a compact subset A in X satisfying the condition: for every $\varphi, \psi \in C(X,\mathbb{I})$, if $\varphi|A = \psi|A$, then $\mu(\varphi) = \mu(\psi)$. The minimal (with respect to inclusion) set A satisfying such a condition is called the support of μ and is denoted by $\mathrm{supp}(\mu)$.

By $M^*(X)$ we denote the set of all $*$-measures of compact support on a Tychonov space X.

Let $x_1, \ldots, x_n \in X$ and let $\alpha_1, \ldots, \alpha_n \in \mathbb{I}$ be such that $\vee_{i=1}^n \alpha_i = 1$. The following is an example of $*$-measure: $\mu = \vee_{i=1}^n \alpha_i * \delta_{x_i}$. One can easily see that $\mathrm{supp}(\mu) = \{x_i : \alpha_i > 0\}$.

If $f\colon X \to Y$ is a continuous map of Tychonov spaces, then one can define a map $M^*(X) \to M^*(Y)$ by the condition

$$M^*(f)(\mu)(\varphi) = \mu(\varphi f), \quad \mu \in M^*(X), \quad \varphi \in C(X,\mathbb{I}).$$

Given an ultrametric space X, by $\mathcal{F}_r(X)$ we denote the set of functions from $C(X,\mathbb{I})$ that are constant on all balls of radius r in X.

Let X be an ultrametric space and $M^*(X)$ is a set of all $*$-measures with compact support on X. Recall that the distance $\tilde{d}(\mu,\nu)$ between $\mu, \nu \in M^*(X)$ is defined by the formula

$$\tilde{d}(\mu,\nu) = \inf\{r > 0 : \mu(\varphi) = \nu(\varphi) \ \text{ for all } \ \varphi \in \mathcal{F}_r(X)\}.$$

One can provide an alternative description of the ultrametric $\tilde{d}$ as follows. Given $r > 0$, denote by X_r the quotient space of X with respect to decomposition of X whose elements are the balls of radius r. Let $q_r\colon X \to X_r$ denote the quotient map. Clearly, the quotient metric on X_r is an ultrametric. Then it is not difficult to show that

$$\tilde{d}(\mu,\nu) = \inf\{r > 0 : M^*(q_r)(\mu) = M^*(q_r)(\nu)\}.$$

By $\exp X$ we denote the hyperspace of X, i.e. the set of all nonempty compact subsets of X endowed with the Hausdorff metric. Actually, in the ultrametric case, the Hausdorff metric can be defined by the condition

$$d_H(A_1, A_2) = \inf\{r > 0 : \text{ for every } x \in X,\ B_r(x) \cap A_1 \neq \varnothing \iff B_r(x) \cap A_2 \neq \varnothing\}.$$

In [6], it is proved that the map $\mathrm{supp}\colon M^*(X) \to \exp X$ is non-expanding.

Proposition 1. *Let $\mu \in M^*(X)$. The map $\mu\colon C(X,\mathbb{I}) \to \mathbb{I}$ is uniformly continuous.*

Proof. Actually, this is a part of the proof of [6, Theorem 2.9]. $\square$

Given a function $\varphi \in C(X, [0,1])$, define $\bar{\varphi}\colon M^*(X) \longrightarrow [0,1]$ by the formula $\bar{\varphi}(\mu) = \mu(\varphi)$.

Proposition 2. *The map $\bar{\varphi}$ is continuous.*

Proof. We first assume that X is compact. Let $\mu_0 \in M^*(X)$. Given $\varepsilon > 0$, by Proposition 1 there exists $\eta > 0$ such that, for every $\chi, \psi \in C(X, \mathbb{I})$, $\|\chi - \psi\| < \eta$ implies $|\mu_0(\chi) - \mu_0(\psi)| < \varepsilon$.

There exists $\delta > 0$ such that the oscillation of φ on every ball of radius δ intersecting $\mathrm{supp}(\mu_0)$ does not exceed η. There is a finite cover $\{B_\delta(x_1), \dots, B_\delta(x_n)\}$ of $\mathrm{supp}(\mu_0)$.

Suppose that $\tilde{d}(\mu, \mu_0) < \delta$. Then clearly $\mathrm{supp}(\mu) \subset \cup_{i=1}^n B_\delta(x_i)$. There exist $\psi_1, \psi_2 \in \mathcal{F}_\delta(X)$ such that $\psi_1 \le \varphi \le \psi_2$ and $\|\psi_1 - \psi_2\| < \eta$.

It follows that

$$\mu_0(\psi_1) = \mu(\psi_1) \le \mu_0(\varphi) \le \mu_0(\psi_2) = \mu(\psi_2)$$

and

$$\mu_0(\psi_1) = \mu(\psi_1) \le \mu(\varphi) \le \mu_0(\psi_2) = \mu(\psi_2).$$

By the choice of η, $|\mu_0(\varphi) - \mu(\varphi)| < \varepsilon$ and therefore $|\bar{\varphi}(\mu_0) - \bar{\varphi}(\mu)| < \varepsilon$.

Let us now pass to the general case, i.e. of arbitrary ultrametric space X. Suppose that a sequence $(\mu_i)_{i=1}^\infty$ converges to μ_0 in $M^*(X)$. Since the map supp is non-expanding, without loss of generality one may assume that

$$X = \mathrm{supp}(\mu_0) \cup \cup_{i=1}^\infty \mathrm{supp}(\mu_i),$$

i.e. X is compact. □

Given an ultrametric space X, we define a map $\zeta_X\colon M^{*2}(X) \to M^*(X)$ by the formula

$$\zeta_X(\mathcal{M})(\varphi) = \mathcal{M}(\bar{\varphi}).$$

We are going to show that $\mathcal{M}(\bar{\varphi}) \in M^*(X)$. Clearly, $\mathcal{M}(\bar{c_X}) = c$, because $\overline{c_X} = c_{M^*(X)}$. Given $\varphi \in C(X, \mathbb{I})$ and $c \in \mathbb{I}$, we obtain

$$\zeta_X(\mathcal{M})(c * \varphi) = \mathcal{M}(\overline{c * \varphi}) = \mathcal{M}(c * \bar{\varphi}) = c * \mathcal{M}(\bar{\varphi}) = c * \zeta_X(\mathcal{M})$$

(we used the equality $\overline{c * \varphi} = c * \bar{\varphi}$: indeed, for any $\mu \in M^*(X)$, $\overline{c * \varphi}(\mu) = \mu(c * \varphi) = c * \mu(\varphi) = c * \bar{\varphi}(\mu)$).

Next, for any $\varphi, \psi \in C(X, \mathbb{I})$, we see that, clearly, $\overline{\varphi \vee \psi} = \bar{\varphi} \vee \bar{\psi}$, therefore

$$\zeta_X(\mathcal{M})(\varphi \vee \psi) = \mathcal{M}(\overline{\varphi \vee \psi}) = \mathcal{M}(\bar{\varphi}) \vee \mathcal{M}(\bar{\psi}) = \zeta_X(\mathcal{M})(\varphi) = \zeta_X(\mathcal{M})(\psi).$$

We are going to show that the support of $\zeta_X(\mathcal{M})$ is compact. Note that, since the map $\mathrm{supp}(\mu)$ is non-expanding, the set $A = \bigcup\{\mathrm{supp}(\mu) : \mu \in \mathrm{supp}(\mathcal{M})\}$ is compact. Given $\varphi, \psi \in C(X, [0,1])$ such that $\varphi|A = \psi|A$, we see that $\mu(\varphi) = \mu(\psi)$ for every $\mu \in \mathrm{supp}(\mathcal{M})$. We conclude that $\bar{\varphi}|\mathrm{supp}(\mathcal{M}) \equiv \bar{\psi}|\mathrm{supp}(\mathcal{M})$ and therefore

$$\zeta_X(\mathcal{M})(\varphi) = \mathcal{M}(\bar{\varphi}) = \mathcal{M}(\bar{\psi}) = \zeta_X(\mathcal{M})(\psi).$$

Thus, $\zeta_X(\mathcal{M}) \in M^*(X)$.

Lemma 1. *If $\varphi \in \mathcal{F}_r(X)$, then $\bar{\varphi} \in \mathcal{F}_r(M^*(X))$.*

Proof. For $\mu, \mu' \in M^*(X)$ we have that $\tilde{d}(\mu, \mu_0) < r$ if and only if $\mu(\varphi) = \mu'(\varphi)$ for all $\varphi \in \mathcal{F}_r(X)$. By the definition $\bar{\varphi}(\mu) = \bar{\varphi}(\mu')$ and it follows that $\bar{\varphi} \in \mathcal{F}_r(M^*(X))$. □

Proposition 3. *The map ζ_X is nonexpanding.*

Proof. Suppose that $\mathcal{M}, \mathcal{M}' \in M^{*2}(X)$. Let $\tilde{\tilde{d}}(\mathcal{M}, \mathcal{M}') < r$ and $\varphi \in \mathcal{F}_r(X)$ for some $r > 0$. By the definition,

$$\zeta_X(\mathcal{M})(\varphi) = \mathcal{M}(\bar{\varphi}) = \mathcal{M}'(\bar{\varphi}) = \zeta_X(\mathcal{M}')(\varphi)$$

and we see that $\tilde{d}(\zeta_X(\mathcal{M}), \zeta_X(\mathcal{M}')) < r$. □

2 Monads

Proposition 4. $\zeta = (\zeta(X))$ *is a natural transformation of the functor M^{*2} to the functor M^*.*

Proof. Let $f\colon X \longrightarrow Y$ be a nonexpanding map. We need to show that the diagram

$$\begin{tikzcd} M^{*2}(X) \arrow[r, "M^{*2}(f)"] \arrow[d, "\zeta_X"'] & M^{*2}(X) \arrow[d, "\zeta_Y"] \\ M^*(X) \arrow[r, "M^*(f)"'] & M^*(Y) \end{tikzcd}$$

is commutative.

Let $\mathcal{M} \in M^{*2}(X)$ and $\varphi \in C(X, [0,1])$. First remark that $\overline{\varphi f} = \bar{\varphi} M^*(f)$. Using this we obtain

$$\begin{aligned} M^*(f)(\zeta_X(\mathcal{M}))(\varphi) = \zeta_X(\mathcal{M})(\varphi f) = \mathcal{M}(\overline{\varphi f}) &= \mathcal{M}(\bar{\varphi} M^*(f)) \\ &= M^{*2}(f)(\mathcal{M})(\bar{\varphi}) = \zeta_Y M^{*2}(f)(\mathcal{M})(\varphi). \end{aligned}$$

□

Recall that a monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor, $\eta\colon 1_{\mathcal{C}} \to T$, $\mu\colon T^2 \to T$ are natural transformations such that $\mu T\eta = \mu\eta_T = 1_T$ and $\mu\mu_T = \mu T\mu$ (see, e.g., [1] for the details).

Given monads $\mathbb{T}_i = (T_i, \eta_i, \mu_i)$, $i = 1, 2$, on a category $\mathcal{C}$, we say that a natural transformation $\gamma\colon T_1 \to T_2$ is a morphism of $\mathbb{T}_1$ to $\mathbb{T}_2$ if $\gamma\eta_1 = \eta_2$ and $\mu_2\gamma_{T_2}T_1(\gamma) = \gamma\mu_1$.

Let $\mathbb{H} = (\exp, s, u)$ be the hyperspace monad on the category **Ultr**. Recall that the singleton map s_X acts as follows: $s_X(x) = \{x\}$, $x \in X$. Also, $u_X\colon \exp^2 X \to \exp X$ is the union map.

Theorem 1. *The triple $\mathbb{M}^*$ is a monad on the category* **Ultr**.

Proof. Let us first prove that the diagram

$$\begin{tikzcd} M^*(X) \arrow[r, "\delta_{M^*(X)}"] \arrow[dr, equal] & M^{*2}(X) \arrow[d, "\zeta_X"] & M^*(X) \arrow[l, "M^*(\delta_X)"'] \arrow[dl, equal] \\ & M^*(X) & \end{tikzcd}$$

is commutative.

First note that $\bar{\psi}\delta_X(x) = \bar{\psi}\delta_x = \delta_x(\psi) = \psi(x)$, for any $\psi \in C(X, [0,1])$.

Let $\mu \in M^*(X)$, then

$$\zeta_X M^*(\delta_X)(\mu)(\psi) = M^*(\delta_X)(\mu)(\bar{\psi}) = \mu(\bar{\psi}\delta_X) = \mu(\psi),$$

i.e. $\zeta_X M^*(\delta_X) = 1_{M^*(X)}$.

Also,

$$\zeta_X \delta_{M^*(X)}(\mu)(\psi) = \delta_{M^*(X)}(\mu)(\bar{\psi}) = \bar{\psi}(\mu) = \mu(\psi),$$

i.e. $\zeta_X \delta_{M^*(X)} = 1_{M^*(X)}$.

Let us prove now that the diagram

$$\begin{tikzcd} M^{*3}(X) \arrow[r, "M^*(\zeta_X)"] \arrow[d, "\zeta_{M^*_X}"'] & M^{*2}(X) \arrow[d, "\zeta_X"] \\ M^{*2}(X) \arrow[r, "\zeta_X"'] & M^*(X) \end{tikzcd}$$

is commutative.

Let $\mathfrak{M} \in M^{*3}(X)$, then

$$\zeta_X \zeta_{M^*_X}(\mathfrak{M})(\varphi) = \zeta_{M^*_X}(\mathfrak{M})(\bar{\varphi}) = \mathfrak{M}(\bar{\bar{\varphi}}).$$

On the other hand,

$$\zeta_X M^*(\zeta_X)(\mathfrak{M})(\varphi) = M^*(\zeta_X)(\mathfrak{M})(\bar{\varphi}) = \mathfrak{M}(\bar{\varphi} \cdot \zeta_X).$$

We need to show that $\bar{\bar{\varphi}} = \bar{\varphi} \cdot \zeta_X$. Indeed,

$$\bar{\bar{\varphi}}(\mathcal{M}) = \mathcal{M}(\bar{\varphi}) = \zeta_X(\mathcal{M})(\varphi) = \bar{\varphi}\zeta_X(\mathcal{M}).$$

□

We say that a t-norm $*$ does not have zero divisors, if $a * b = 0$ implies $a \wedge b = 0$.

Theorem 2. *Suppose that a t-norm $*$ does not have zero divisors. Then the natural transformation* supp *is a morphism of the monad $\mathbb{M}^*$ to the monad $\mathbb{H}$.*

Proof. Clearly, $\mathrm{supp}(\delta_x) = \{x\} = s_X(x)$, $x \in X$. We therefore have to show that the diagram

$$\begin{tikzcd} M^{*2}(X) \arrow[r, "\mathrm{supp}_{M^*(X)}"] \arrow[d, "\zeta_X"'] & \exp M^*(X) \arrow[r, "\exp(\mathrm{supp}_X)"] & \exp^2 X \arrow[d, "u_X"] \\ M^* \arrow[rr, "\mathrm{supp}"'] & & \exp X \end{tikzcd} \tag{1}$$

is commutative.

Let $\mathcal{M} \in M^{*2}(X)$, $\mathcal{M} = \vee_{i=1}^n \alpha_i * \delta_{\mu_i}$, where $\mu_i = \vee_{j=1}^{m_i} \beta_{ij} * \delta_{x_{ij}}$, $x_{ij} \in X$. Without loss of generality, one may assume that $\alpha_i > 0$, $\beta_{ij} > 0$ for all i, j. Then

$$\mathrm{supp}_{M^*(X)}(\mathcal{M}) = \{\mu_1, \dots, \mu_n\}, \quad \mathrm{supp}_X(\mu_i) = \{x_{i1}, \dots, x_{im_i}\},$$

where $i = 1, \dots, n$.

Then $\zeta_X(\mathcal{M}) = \vee_{i=1}^n \vee_{j=1}^{m_i} \alpha_i * \beta_{ij} * \delta_{x_{ij}}$ and, since $*$ has no zero divisors,

$$\mathrm{supp}(\zeta_X(\mathcal{M})) = \{x_{ij} : i = 1, \dots, n,\ j = 1, \dots, m_j\}.$$

This shows that the diagram (1) is commutative for $\mathcal{M}$ as above. Since, by [6, Proposition 2.7], such measures are dense in $M^{*2}(X)$, we conclude that diagram (1) is commutative. □

The following example shows that the requirement of absence of zero divisors is essential. Suppose that $*$ is the Łukasiewicz t-norm. Let $X = \{a, b\}$, $a \neq b$, $\mu = \delta_a$, $\nu = \delta_a \vee \frac{1}{2} * \delta_b$, and $\mathcal{M} = \delta_\mu \vee \frac{1}{2} * \delta_\nu$. Then

$$\zeta_X(\mathcal{M}) = \delta_a \vee \frac{1}{2} * \delta_b \vee \frac{1}{2} * \frac{1}{2} * \delta_b = \delta_a$$

and

$$\mathrm{supp}(\zeta_X(\mathcal{M})) = \{a\} \neq \{a, b\} = \bigcup_{\tau \in \mathrm{supp}(\mathcal{M})} \mathrm{supp}(\tau).$$

3 Tensor products

Let X, Y be ultrametric spaces, $\mu \in M^*(X)$, $\nu \in M^*(Y)$. We recall the definition of tensor product (see, e.g., [7] for the general case).

In the sequel, we consider the max ultrametric on the product of ultrametric spaces. Given $x \in X$, define the map $i_x \colon Y \to X \times Y$ by the formula $i_x(y) = (x, y)$, $y \in Y$. Clearly, i_x is an isometric embedding.

Given $\nu \in M^*(Y)$, define the map $j_\nu \colon X \to M^*(X \times Y)$ by the formula: $j_\nu(x) = M^*(i_x)(\nu)$, $x \in X$.

Lemma 2. *The map j_ν is an isometric embedding.*

Proof. Suppose that $x, y \in X$ and $d(x, y) < r$. Note that, for any $s \geq r$,

$$i_x^{-1}(B_s^{X\times Y}(a, b)) = i_x^{-1}(B_s^{X\times Y}(a, b)) = B_s^Y(b).$$

Therefore, for any $\varphi \in \mathcal{F}_s(X \times Y)$, we have $\varphi i_x = \varphi i_y$. Thus, for any $\varphi \in \mathcal{F}_s(X \times Y)$,

$$j_\nu(x)(\varphi) = \nu(\varphi i_x) = \nu(\varphi i_y) = j_\nu(y)(\varphi)$$

and consequently $\tilde{d}(j_\nu(x), j_\nu(y)) < r$. We conclude that j_ν is a non-expanding map.

Now, assume that $d(x, y) \geq r$. Then clearly, $d_H\big(\mathrm{supp}(j_\nu(x)), \mathrm{supp}(j_\nu(y))\big) \geq r$ and therefore, by [6, Proposition 2.5], $\tilde{d}(j_\nu(x), j_\nu(y)) \geq r$. $\square$

Finally, define the tensor product of μ and ν as follows:

$$\mu \otimes \nu = \zeta_{X\times Y} M^*(j_\nu)(\mu) \in M^*(X \times Y).$$

Proposition 5. *The map*

$$\otimes \colon M^*(X) \times M^*(Y) \to M^*(X \times Y)$$

is non-expanding.

Proof. Let $\mu, \mu' \in M^*(X)$, $\nu, \nu' \in M^*(Y)$. First note that clearly $\tilde{d}(j_\nu, j_{\nu'}) \leq \tilde{d}(\nu, \nu')$, and, since the functor M^* is locally non-expanding, $\tilde{d}(M^*(j_\nu), M^*(j_{\nu'})) \leq \tilde{d}(\nu, \nu')$.

Finally,

$$\begin{aligned}
\tilde{d}(\mu \otimes \nu, \mu' \otimes \nu') &= \tilde{d}(\zeta_{X\times Y} M^*(j_\nu)(\mu), \zeta_{X\times Y} M^*(j_{\nu'})(\mu')) \\
&\leq \tilde{d}(M^*(j_\nu)(\mu), M^*(j_{\nu'})(\mu')) \\
&\leq \max\{\tilde{d}(M^*(j_\nu)(\mu), M^*(j_\nu)(\mu')), \tilde{d}(M^*(j_\nu)(\mu'), M^*(j_{\nu'})(\mu'))\} \\
&\leq \max\{\tilde{d}(\mu, \mu'), \tilde{d}(\nu, \nu')\} \\
&= \tilde{d}((\mu, \nu), (\mu', \nu')).
\end{aligned}$$

$\square$

Proposition 6. *Let* $\mu = \vee_{i=1}^{n}\alpha_i * \delta_{x_i} \in M^*(X)$, $\nu = \vee_{j=1}^{m}\beta_j * \delta_{y_j} \in M^*(Y)$. *Then*

$$\mu \otimes \nu = \vee_{i=1}^{n} \vee_{j=1}^{m} \alpha_i * \beta_j * \delta_{(x_i,y_j)} \in M^*(X \times Y).$$

Proof. Note that

$$M^*(j_\nu)(\mu) = \vee_{i=1}^{n}\alpha_i * \delta(j_\nu(x_i)) = \vee_{i=1}^{n}\alpha_i * \delta(\vee_{j=1}^{m}\beta_j * \delta(x_i, y_j))$$

and therefore

$$\mu \otimes \nu = \zeta_{X\times Y}(\vee_{i=1}^{n}\alpha_i * \delta(\vee_{j=1}^{m}\beta_j * \delta(x_i, y_j)))$$

whence the statement follows. □

4 Applications

We will consider a two-person game on ultrametric spaces of strategies X_i, $i = 1,2$. The payoff functions are denoted by $u_i \colon X_1 \times X_2 \to [0,1]$, $i = 1,2$. A $*$-measure-valued strategy of player i is an element of the space $M^*(X_i)$, $i = 1,2$.

Define $U_i \colon M^*(X_1) \times M^*(X_2) \to [0,1]$ as follows:

$$U_i(\mu_1, \mu_2) = (\mu_1 \otimes \mu_2)(u_i),\ i = 1,2.$$

Proposition 7. *The map U_i, $i = 1,2$, is continuous.*

Proof. Since $(\mu_1 \otimes \mu_2)(u_i) = \bar{u}_i(\mu_1 \otimes \mu_2)$, $i = 1,2$, this is a combination of Propositions 5 and 2. □

We say that $(\mu_1^0, \mu_2^0) \in M^*(X_1) \times M^*(X_2)$ is an equilibrium in $*$-measure-valued strategies if $U_2(\mu_1^0, \mu_2) \le U_2(\mu_1^0, \mu_2^0)$ for every $\mu_2 \in M^*(X_2)$, and $U_1(\mu_1, \mu_2^0) \le U_2(\mu_1^0, \mu_2^0)$ for every $\mu_1 \in M^*(X_1)$.

One can also define the notion of ε-equilibrium for games in $*$-measure-valued strategies. Given $\varepsilon > 0$, we say that $(\mu_1^0, \mu_2^0) \in M^*(X_1) \times M^*(X_2)$ is an equilibrium in $*$-measure-valued strategies if

$$U_2(\mu_1^0, \mu_2) \le U_2(\mu_1^0, \mu_2^0) + \varepsilon$$

for every $\mu_2 \in M^*(X_2)$, and

$$U_1(\mu_1, \mu_2^0) \le U_2(\mu_1^0, \mu_2^0) + \varepsilon$$

for every $\mu_1 \in M^*(X_1)$.

Proposition 8. *For every equilibrium $(\mu_1^0, \mu_2^0) \in M^*(X_1) \times M^*(X_2)$ and every $\varepsilon > 0$ there exists an ε-equilibrium in $M^*_\omega(X)$.*

Proof. This follows from Proposition 7 and the fact that the set of $*$-measures of finite support is dense in the space $M^*(X)$. □

Note that an approximation theorem for games in mixed strategies is considered in [2].

The question of existence of equilibrium for games in $*$-measure-valued strategies remains open. The main problem consists in the lack of convex structure in the spaces of the form $M^*(X)$ in the ultrametric case (see [4] for the case of compact metric spaces).

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У попередній публікації авторів введено функтор $*$-мір з компактним носієм у категорії ультраметричних просторів і нерозтягуючих відображень. У цій публікації ми доведемо, що цей функтор визначає монаду у цій категорії. Монадна структура дозволяє визначити тензорний добуток $*$-мір. Розглянемо деякі застосування цього поняття до теорії рівноваги.

Ключові слова і фрази: ультраметричний простір, нерозтягуюче відображення, $*$-міра, монада, рівновага.