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Fuzzy fractional hybrid differential equations

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This article is related to present and solve the theory of fractional hybrid differential equations with fuzzy initial values involving the fuzzy Riemann-Liouville fractional differential operators of order $0 < q < 1$. For the concerned presentation, we study the existence and uniqueness of a fuzzy solution are brought in detail basing on the concept of generalized division of fuzzy numbers. We have developed and investigated a fuzzy solution of a fuzzy fractional hybrid differential equation. At the end we have given an example is provided to illustrate the theory.

Key words and phrases: fuzzy fractional differential, fuzzy valued function, fuzzy fractional hybrid differential equation.

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Introduction

The differential equations involving Riemann-Liouville differential operators of fractional order $0 < q < 1$ are very important in modeling several physical phenomena [3, 8, 9, 11, 12, 22] and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

Receiving much attention in the recent literature are hybrid differential equations. Hybrid differential equations evolve in continuous time like differential equations. When the continuous-time dynamics of a hybrid equation is given by fuzzy differential equations, the equation is called a hybrid fuzzy differential equation. For analytical results on hybrid fuzzy differential equations see [1, 2, 4, 5, 10, 16, 19, 21]. In [7], it is discussed the following first-order hybrid fuzzy differential equation :

$$\begin{cases} \frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right] = g(t,u(t)), \quad t \in J, \\ u(t_0) = u_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases}$$

where $f \in C(J \times \mathbb{R}_{\mathcal{F}}, \ \mathbb{R}_{\mathcal{F}} \backslash \{0\})$ and $g \in C(J \times \mathbb{R}_{\mathcal{F}}, \ \mathbb{R}_{\mathcal{F}})$. They established the existence results for hybrid fuzzy differential equations initiating the study of the theory of such systems and proved to utilize the theory of division of fuzzy numbers, its existence of solutions.

From the above works, we develop the theory of fractional hybrid differential equations with fuzzy initial conditions involving Riemann-Liouville differential operators of order $0 < q < 1$, their compact and convex level-cuts, and generalized division

$$\begin{cases} D^q\left[\frac{u(t)}{f(t,u(t))}\right] = g(t,u(t)), \quad t \in J, \\ u(t_0) = u_0 \in \mathbb{R}_{\mathcal{F}}. \end{cases}$$

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As we can see, a key point in our investigation is played by the division concepts for fuzzy numbers. A recent very promising concept, the G-division proposed by [7]. We observe that this division has a great advantage over peer concepts, namely that it always exists. In comparison with the paper [7], we study fuzzy fractional hybrid differential equations with fuzzy initial value and fuzzy forcing functions, we propose a new theorem for finding the fuzzy solutions, we prove some results and we discuss the fuzzy solution with an example.

1 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $\mathbb{R}_{\mathcal{F}} = \{u : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties (see [14]):

(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e. for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)],$$

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$, then from (i)–(iv) it follows that the α-cuts set $[u]^\alpha \in P_{\mathcal{K}}(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which we denote by $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$. Here $P_{\mathcal{K}}(\mathbb{R})$ denotes the family of all nonempty compact convex subsets of $\mathbb{R}$ with the addition and scalar multiplication in $P_{\mathcal{K}}(\mathbb{R})$ defined as usual.

The property of the fuzzy numbers is that the α-cuts $[u]^\alpha$ are closed sets for all $\alpha \in [0,1]$.

Definition 1 ([7, 14, 17])**.** *We represent an arbitrary fuzzy number by an ordered pair of functions* $[u]^\alpha = [u_1^\alpha, u_2^\alpha], \alpha \in [0,1]$*, which satisfy the following requirements:*

(a) u_1^α *is a bounded monotonic nondecreasing left-continuous function* $\forall \alpha \in \,]0,1]$ *and right-continuous for* $\alpha = 0$*,*

(b) u_2^α *is a bounded monotonic nonincreasing left-continuous function* $\forall \alpha \in]0,1]$ *and right-continuous for* $\alpha = 0$*,*

(c) $u_1^\alpha \leq u_2^\alpha$, $0 \leq \alpha \leq 1$.

A trapezoidal fuzzy number, denoted by $u = \langle a, b, c, d\rangle$, where $a \leq b \leq c \leq d$, has α-cuts

$$[u]^\alpha = [a + \alpha(b-a),\ d - \alpha(d-c)], \quad \alpha \in [0,1],$$

obtaining a triangular fuzzy number if $b = c$.

Theorem 1 ([14]). *Let $u \in \mathbb{R}_{\mathcal{F}}$. Denote $A_\alpha = [u]^\alpha$ for $\alpha \in [0,1]$. Then the following is true.*

1. *A_α is a nonempty compact convex set in $\mathbb{R}$ for each $\alpha \in [0,1]$.*
2. *$A_\beta \subseteq A_\alpha$ for $0 < \alpha \leq \beta \leq 1$.*
3. *$A_\alpha = \bigcap_{i=1}^{\infty} A_{\alpha_i}$ for any nondecreasing sequence $\alpha_i \to \alpha$ on $[0,1]$.*

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$ by the equation

$$D(u,v) = \sup_{\alpha \in [0,1]} D_H\left([u]^\alpha, [v]^\alpha\right),$$

for all $u, v \in \mathbb{R}_{\mathcal{F}}$, where D_H is the Hausdorff metric defined as

$$D_H\left([u]^\alpha, [v]^\alpha\right) = \max\left\{\left|u_1^\alpha - v_1^\alpha\right|, \left|u_2^\alpha - v_2^\alpha\right|\right\}.$$

It is well known that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space. The following properties of $D(u,v)$

$$\begin{aligned} &D(u+w, v+w) = D(u,v), \\ &D(ku, kv) = |k| D(u,v), \\ &D(u,v) \leq D(u,w) + D(w,v) \end{aligned}$$

hold for all $u, v, w \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$.

The addition $u + v$ and the scalar multiplication ku are defined as having the level cuts

$$[u+v]^\alpha = [u_1^\alpha + v_1^\alpha, u_2^\alpha + v_2^\alpha], \qquad k[u]^\alpha = \begin{cases} [ku_1^\alpha, ku_2^\alpha], & k \geq 0, \\ [ku_2^\alpha, ku_1^\alpha], & k < 0, \end{cases}$$

and

$$[u]^\alpha [v]^\alpha = \Big[\min\{u_1^\alpha.v_1^\alpha, u_1^\alpha.v_2^\alpha, u_2^\alpha.v_1^\alpha, u_2^\alpha.v_2^\alpha\},\ \max\{u_1^\alpha.v_1^\alpha, u_1^\alpha.v_2^\alpha, u_2^\alpha.v_1^\alpha, u_2^\alpha.v_2^\alpha\}\Big],$$

$$[u]^\alpha \div [v]^\alpha = \left[\min\left\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\right\}, \max\left\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\right\}\right].$$

Definition 2 ([7,20]). *Given two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ the division (g-division for short) is the fuzzy number w, if it exists, such that*

$$[u]^\alpha \div_g [v]^\alpha = [w]^\alpha \quad \Longleftrightarrow \quad [u]^\alpha = [v]^\alpha [w]^\alpha \text{ or } [v]^\alpha = [u]^\alpha ([w]^\alpha)^{-1},$$

where $([w]^\alpha)^{-1} = \left[1/w_2^\alpha, 1/w_1^\alpha\right]$.

Definition 3 ([7,20]). *The generalized division (G-division for short) of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ and $0 \notin [v]^\alpha$ $\forall \alpha \in [0,1]$, is given by its levels sets as*

$$[u \div_G v]^\alpha = cl \bigcup_{\beta \geq \alpha} \left([u]^\beta \div_g [v]^\beta\right), \quad \forall \alpha \in [0,1], \tag{1}$$

where the g-division $\div_g$ is with interval operands $[u]^\beta$ and $[v]^\beta$.

Proposition 1 ([7]). *The G-division* (1) *is given by the expression*

$$[u \div_G v]^\alpha = \left[\inf_{\beta \geq \alpha} \min \left\{ \frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta} \right\}, \sup_{\beta \geq \alpha} \max \left\{ \frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta} \right\} \right].$$

Let $J \subset \mathbb{R}$ be an interval. We denote by $C(J, \mathbb{R}_\mathcal{F})$ the space of all continuous fuzzy functions on J. Let $a > 0$, $J = (0, a]$ and $\xi \geq 0$. Before proceeding further, we need the following notation

$$C_\xi (J, \mathbb{R}_\mathcal{F}) := \left\{ u \in C(J, \mathbb{R}_\mathcal{F});\ t^\xi u \in C(J, \mathbb{R}_\mathcal{F}) \right\}.$$

Obviously, $C_\xi (J, \mathbb{R}_\mathcal{F})$ is a complete metric space with respect to the metric

$$H_\xi (u, v) := \max_{t \in J} t^\xi D (u(t), v(t)).$$

Evidently, $C_0 (J, \mathbb{R}_\mathcal{F}) = C (J, \mathbb{R}_\mathcal{F})$.

Also, we denote by $L^1(J,\ \mathbb{R}_\mathcal{F})$ the space of all fuzzy functions $f : J \to \mathbb{R}_\mathcal{F}$, which are Lebesgue integrable on the bounded interval J of $\mathbb{R}$.

Let $u : J \to \mathbb{R}_\mathcal{F}$ be a fuzzy function. We denote

$$[u(t)]^\alpha = \left[u_1^\alpha(t), u_2^\alpha(t) \right], \quad t \in J,\ \alpha \in [0, 1].$$

The derivative $u'(t)$ of a fuzzy function u is defined by [18]

$$[u'(t)]^\alpha = \left[(u_1^\alpha)'(t),\ (u_2^\alpha)'(t) \right], \quad \alpha \in [0, 1],$$

provided this equation defines a fuzzy number $u'(t) \in \mathbb{R}_\mathcal{F}$. The fuzzy integral

$$\int_a^b u(t)\, dt, \quad a, b \in J,$$

is defined by [15]

$$\left[\int_a^b u(t)\, dt \right]^\alpha = \left[\int_a^b u_1^\alpha(t)\, dt,\ \int_a^b u_2^\alpha(t)\, dt \right],$$

provided that the Lebesgue integrals on the right exist. Moreover, we know [18] that the fuzzy integral is a fuzzy number.

2 Fuzzy fractional integral and fuzzy fractional derivative

Let $u : J \longrightarrow \mathbb{R}_\mathcal{F}$ be such that $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$ for all $t \in J$ and $q \in \mathbb{R}_+^*$. Suppose that $u_1^\alpha, u_2^\alpha \in C(J,\ \mathbb{R}) \cap L^1(J,\ \mathbb{R})$ for all $\alpha \in [0, 1]$ and let

$$A_\alpha := \frac{1}{\Gamma(q)} \left[\int_0^t (t - s)^{q-1} u_1^\alpha(s)\, ds,\ \int_0^t (t - s)^{q-1} u_2^\alpha(s)\, ds \right], \quad t \in J. \tag{2}$$

Lemma 1 ([6]). *The family* $\{A_\alpha : \alpha \in [0, 1]\}$*, given by* (2)*, define a fuzzy number* $u \in \mathbb{R}_\mathcal{F}$ *such that* $[u]^\alpha = A_\alpha$.

Let $u \in C(J,\ \mathbb{R}_{\mathcal{F}}) \cap L^1(J,\ \mathbb{R}_{\mathcal{F}})$. Define the fuzzy fractional primitive of order $q > 0$ of u

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s)\, ds, \quad t \in J, \tag{3}$$

by

$$[I^q u(t)]^\alpha = \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} u_1^\alpha(s)\, ds,\ \int_0^t (t-s)^{q-1} u_2^\alpha(s)\, ds \right], \quad t \in J.$$

For $q = 1$, we obtain $I^1 u(t) = \int_a^t u(s)\, ds$, $t \in J$, that is, the integral operator. Also, the following properties are obvious:

(i) $I^q(cu)(t) = cI^q(u)(t)$ for each constant $c \in \mathbb{R}_{\mathcal{F}}$,

(ii) $I^q(u+v)(t) = I^q(u)(t) + I^q(v)(t)$.

Proposition 2 ([6]). *If $u \in C(J,\ \mathbb{R}_{\mathcal{F}}) \cap L^1(J,\ \mathbb{R}_{\mathcal{F}})$ and $p,\ q > 0$, then we have*

$$I^p I^q u = I^{p+q} u. \tag{4}$$

Example 1. *Let $u : J \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function given by $u(t) = \lambda t$, where $\lambda \in \mathbb{R}_{\mathcal{F}}$. If $[\lambda]^\alpha = [1+\alpha, 3-\alpha]$, then*

$$\begin{aligned} [I^q u(t)]^\alpha &= \frac{1}{\Gamma(q)} \left[\int_0^t (1+\alpha)(t-s)^{q-1} s\, ds,\ \int_0^t (3-\alpha)(t-s)^{q-1} s\, ds \right] \\ &= \frac{t^{q+1}}{\Gamma(q+2)} [1+\alpha, 3-\alpha] \\ &= \frac{t^{q+1}}{\Gamma(q+2)} [\lambda]^\alpha. \end{aligned}$$

Definition 4. *Let $u \in C(J, \mathbb{R}_{\mathcal{F}}) \cap L^1(J, \mathbb{R}_{\mathcal{F}})$ be a given function such that $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$ for all $t \in J$ and $\alpha \in [0,1]$. The fuzzy fractional differential operator in the Riemann-Liouville sense is defined*

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} u(s)\, ds$$

by

$$\left[D^q u(t)\right]^\alpha = \frac{1}{\Gamma(1-q)} \left[\frac{d}{dt} \int_0^t (t-s)^{-q} u_1^\alpha(s)\, ds,\ \frac{d}{dt} \int_0^t (t-s)^{-q} u_2^\alpha(s)\, ds \right]$$

provided that the equation defines a fuzzy number $D^q u(t) \in \mathbb{R}_{\mathcal{F}}$.

In fact,

$$[D^q u(t)]^\alpha = [D^q u_1^\alpha(t),\ D^q u_2^\alpha(t)]$$

for all $t \in J$ and $\alpha \in [0,1]$.

Proposition 3. *If $u \in C(J,\ \mathbb{R}_{\mathcal{F}}) \cap L^1(J,\ \mathbb{R}_{\mathcal{F}})$ and $0 < q < 1$, then $D^q I^q u(t) = u(t)$.*

Proof. Indeed, using (4), we have

$$D^q I^q u(t) = D I^{1-q} I^q u(t) = D I u(t) = u(t).$$

□

Proposition 4. *There exists a function $\varphi \in L^1(J,\, \mathbb{R}_{\mathcal{F}})$ such that $u(t) = u(0) + I^q\varphi(t)$. Then*

$$I^qD^qu(t) = u(t) \ominus u(0).$$

Proof. Indeed, by Proposition 3 we have that

$$I^qD^qu(t) = I^qDI^{1-q}u(t) = I^qD(I^{1-q}u(0) + I^{1-q}I^q\varphi(t)) = I^qDI^{1-q}u(0) + I^qDI\varphi(t) = I^q\varphi(t)$$

or $u(t) \ominus u(0) = I^q\varphi(t)$, therefore $I^qD^qu(t) = u(t) \ominus u(0)$. □

Example 2. *Let $u : J \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function given by $u(t) = \lambda t$, where $\lambda \in \mathbb{R}_{\mathcal{F}}$. If $[\lambda]^\alpha = [1+\alpha, 3-\alpha]$, then*

$$\begin{aligned} [D^qu(t)]^\alpha &= \frac{1}{\Gamma(1-q)}\left[\frac{d}{dt}\int_0^t (1+\alpha)(t-s)^{-q}s\,ds, \frac{d}{dt}\int_0^t (3-\alpha)(t-s)^{-q}s\,ds\right] \\ &= \frac{t^{1-q}}{\Gamma(2-q)}\,[1+\alpha, 3-\alpha] = \frac{t^{1-q}}{\Gamma(2-q)}[\lambda]^\alpha, \end{aligned}$$

that is, $D^q\lambda t = \frac{t^{1-q}}{\Gamma(2-q)}\lambda$ for every $\lambda \in \mathbb{R}_{\mathcal{F}}$.

3 Fuzzy fractional hybrid differential equations

Were call the result which establishes the existence of solution for fractional hybrid differential equations (FHDEs) involving Riemann-Liouville differential operators of order $0 < q < 1$. This result will be useful in the study of the corresponding fuzzy problem.

We consider the initial value problem

$$\begin{cases} D^q\left[\frac{u(t)}{f(t,u(t))}\right] = g(t,u(t)), \quad t \in J, \\ u(t_0) = u_0 \in \mathbb{R}, \end{cases} \tag{5}$$

where $0 < q < 1$, $f \in C(J \times \mathbb{R},\, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R},\, \mathbb{R})$.

By a solution of the FHDE (5) we mean a function $u \in C(J,\, \mathbb{R})$ such that

(i) the function $t \mapsto \dfrac{u}{f(t,u)}$ is continuous for each $u \in \mathbb{R}$,

(ii) u satisfies the equation in (5),

where $C(J,\, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

Theorem 2 ([13]). *Let S be a non-empty closed convex and bounded subset of a Banach algebra X and let $A : X \to X$, $B : S \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant γ,*

(b) *B is completely continuous,*

(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$,*

(d) *$M\gamma(r) < r$, where $M = \|B(S)\| = \sup\{\|Bx\| \,:\, x \in S\}$.*

Then the operator equation $AxBx = x$ has a solution in S.

In what follows, we consider the following hypotheses.

(A_0) The function $x \mapsto \dfrac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(A_1) There exists a constant $L > 0$ such that

$$|f(t,\, x) - f(t,\, y)| \leq L|x - y| \tag{6}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) There exists a function $h \in L^1(J,\, \mathbb{R})$ such that

$$|g(t,\, x)| \leq h(t), \quad t \in J.$$

In the following section, we consider a fuzzy differential equation which is a fuzzy analogue to (5).

4 Some results for fractional hybrid differential equations

We shall consider the initial value problem

$$\begin{aligned} &D^q\Big[\frac{u(t)}{f(t,u(t))}\Big] = g(t,u(t)), \quad t \in J,\ q \in (0,1), \\ &u(0) = u_0 \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \tag{7}$$

The extension principle of Zadeh leads to the following definition of $f(t,u)$ and $g(t,u)$, when are fuzzy numbers

$$\begin{aligned} f(t,u)(y) &= \sup\{u(x) : y = f(t,\, x),\ x \in \mathbb{R}\}, \\ g(t,u)(y) &= \sup\{u(x) : y = g(t,\, x),\ x \in \mathbb{R}\}. \end{aligned}$$

It follows

$$\begin{aligned} [f(t,u)]^\alpha &= \Big[\min\{f(t,\, x)\ :\ x \in [u_1^\alpha, u_2^\alpha]\}, \max\{f(t,\, x) :\ x \in [u_1^\alpha, u_2^\alpha]\}\Big], \\ [g(t,u)]^\alpha &= \Big[\min\{g(t,\, x)\ :\ x \in [u_1^\alpha, u_2^\alpha]\}, \max\{g(t,\, x) :\ x \in [u_1^\alpha, u_2^\alpha]\}\Big], \end{aligned}$$

for $u \in \mathbb{R}_{\mathcal{F}}$ with α-level sets $[u]^\alpha = [u_1^\alpha, u_2^\alpha], 0 < \alpha \leq 1$. We call $u : J \to \mathbb{R}_{\mathcal{F}}$ a fuzzy solution of (7) if

$$\Bigg[D^q\Big[u(t) \div_G f(t,u(t))\Big]\Bigg]^\alpha = \Big[g(t,u(t))\Big]^\alpha \quad \text{and} \quad [u(0)]^\alpha = [u_0]^\alpha$$

for all $t \in J$, $q \in (0,1]$ and $\alpha \in [0,1]$. Denote $\tilde{f} = (f_1,\, f_2)$ and $\tilde{g} = (g_1,\, g_2)$,

$$f_1(t,u) = \min\{f(t,\, x) :\ x \in [u_1, u_2]\}, \quad f_2(t,u) = \max\{f(t,\, x)\ :\ x \in [u_1, u_2]\}$$

and

$$g_1(t,u) = \min\{g(t,\, x) :\ x \in [u_1, u_2]\}, \quad g_2(t,u) = \max\{g(t,\, x)\ :\ x \in [u_1, u_2]\},$$

where $u = (u_1, u_2) \in \mathbb{R}^2$.

Thus for fixed α we have an initial value problems in $\mathbb{R}^2$

$$D^q\left[\frac{u_1^\alpha(t)}{\tilde{f}(t,u_1^\alpha(t),u_2^\alpha(t))}\right]=\tilde{g}(t,u_1^\alpha(t),u_2^\alpha(t)),\qquad u_1^\alpha(0)=u_{01}^\alpha \tag{8}$$

and

$$D^q\left[\frac{u_2^\alpha(t)}{\tilde{f}(t,u_1^\alpha(t),u_2^\alpha(t))}\right]=\tilde{g}(t,u_1^\alpha(t),u_2^\alpha(t)),\qquad u_2^\alpha(0)=u_{02}^\alpha. \tag{9}$$

If we can solve them (uniquely) we have only to verify that the intervals $[u_1^\alpha(t),u_2^\alpha(t)],\alpha\in[0,1]$, define a fuzzy number $u(t)$ in $\mathbb{R}_{\mathcal{F}}$. Since f and g are assumed continue and Caratheodory (resp.), the initial value problems (8), (9) are equivalent to the following nonlinear fractional hybrid integral equation (FHIE)

$$u(t)=\tilde{f}(t,u(t))\left(\frac{u_0}{\tilde{f}(0,u(0))}+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\tilde{g}(s,u(s))\,ds\right). \tag{10}$$

Theorem 3. *Assume* $\operatorname{sign}(u(0))=\operatorname{sign}(u(t))$ *for all* $t\in J,\ q\in(0,1]$.
Let $z(t)=u(t)\div_G f(t,u(t))$, $0\notin[f(t,u)]^\alpha$, $\alpha\in[0,1]$ *and*

$$r(t)=f(t,u(t))\div_G\left(z(0)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}g(s,u(s))\,ds\right),$$

$$0\notin\left[z(0)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}g(s,u(s))\,ds\right]^\alpha.$$

1. *If* $z(0)*g(t,u)>0$*, then the function* $u(t)\in C((0,\ J],\ \mathbb{R}_{\mathcal{F}})$ *is a fuzzy solution of* (7).
2. *If* $z(t)$ *is* G_i*-division and* $r(t)$ *is* G_i*-division or* $z_1^\alpha(0)\leq 0\leq z_2^\alpha(0)$*, then* $u(t)$ *is a fuzzy solution.*

Proof. We solve the initial value problems in $\mathbb{R}^2$

$$D^q z_1^\alpha=\min\Big\{g(t,\ x):x\in\big[u_1^\alpha(t),u_2^\alpha(t)\big]\Big\},\qquad u_1^\alpha(0)=u_{01}^\alpha,$$
$$D^q z_2^\alpha=\max\Big\{g(t,\ x):x\in\big[u_1^\alpha(t),u_2^\alpha(t)\big]\Big\},\qquad u_2^\alpha(0)=u_{02}^\alpha,$$

where $q\in(0,1]$.

Step 1. It can be assumed that (6) implies

$$\|f(t,\ x)-f(t,\ y)\|\leq L\|x-y\|\quad\text{for all } t\in J,\ x,y\in\mathbb{R}, \tag{11}$$

where the $\|\cdot\|$ is defined by $\|u\|=\max\{|u_1|,\ |u_2|\}$. It is well known that (11) and the assumptions on g guarantee the existence and continuous dependence on initial of a solution to

$$\begin{cases} D^q\left[\frac{u(t)}{\tilde{f}(t,u(t))}\right]=\tilde{g}(t,u(t)),\\ u(0)=u_0\end{cases} \tag{12}$$

and that for any continuous function $u_0 \in \mathbb{R}^2$ we have (10).

By choosing $u_0 = (u_{01}^{\alpha}, u_{02}^{\alpha})$ in (12) we get a solution $u^{\alpha}(t) = (u_1^{\alpha}(t), u_2^{\alpha}(t))$ to (3) for all $\alpha \in (0,1]$.

Step 2. We will show that the intervals $[u_1^{\alpha}(t), u_2^{\alpha}(t)]$, $\alpha \in [0,1]$, define a fuzzy number $u(t) \in \mathbb{R}_{\mathcal{F}}$. For simplicity assume $[u(0)]^{\alpha} \leq 0$, $[f(t,u(t))]^{\alpha} > 0$ and $\big[g(t,u(t))\big]^{\alpha} < 0$ for all $\alpha \in [0,1]$. The proof for other cases is similar and omitted. So, we have two cases.

Case I. By equations (8), (9) we have the two following FHDE with initial conditions

$$\begin{cases} D^q\Big[\frac{u_1^{\alpha}(t)}{f_2^{\alpha}(t,u(t))}\Big] = g_1^{\alpha}(t,u(t)), \\ u_1^{\alpha}(0) = u_{01}^{\alpha} \end{cases} \tag{13}$$

and

$$\begin{cases} D^q\Big[\frac{u_2^{\alpha}(t)}{f_1^{\alpha}(t,u(t))}\Big] = g_2^{\alpha}(t,u(t)), \\ u_2^{\alpha}(0) = u_{02}^{\alpha}. \end{cases} \tag{14}$$

In the consequence by Step 1, we deduce that for every $\alpha \in [0,1]$ the solution to problems (13)–(14) are respectively

$$\begin{aligned} u_1^{\alpha}(t) &= f_2^{\alpha}(t,u(t))\left[\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_2^{\alpha}(s,u(s))\,ds\right], \\ u_2^{\alpha}(t) &= f_1^{\alpha}(t,u(t))\left[\frac{u_2^{\alpha}(0)}{f_1^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^{\alpha}(s,u(s))\,ds\right]. \end{aligned}$$

By applying the stacking Theorem 1, we check that $\{[u_1^{\alpha}(t), u_2^{\alpha}(t)],\ \alpha \in [0,1]\}$ represent the level set of a fuzzy set $u(t)$ in $\mathbb{R}_{\mathcal{F}}$ for each fixed $t \in J$. Indeed, we fix $t \in J$ and check the validity of the three conditions.

(1) First, we check that $u_1^{\alpha}(t) \leq u_2^{\alpha}(t)$ for every $\alpha \in [0,1]$ and $t \in J$. Indeed, for each $\alpha \in [0,1]$ and $t \in J$ we have that $f_1^{\alpha}(t,u(t)) \leq f_2^{\alpha}(t,u(t))$ and

$$\begin{aligned} &\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^{\alpha}(s,u(s))\,ds \\ &\qquad\qquad \leq \frac{u_2^{\alpha}(0)}{f_1^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_2^{\alpha}(s,u(s))\,ds \end{aligned}$$

and by classical arithmetic we have

$$\begin{aligned} u_1^{\alpha}(t) = f_2^{\alpha}(t,u(t))&\left[\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^{\alpha}(s,u(s))\,ds\right] \\ \leq f_1^{\alpha}(t,u(t))&\left[\frac{u_2^{\alpha}(0)}{f_1^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_2^{\alpha}(s,u(s))\,ds\right] = u_2^{\alpha}(t). \end{aligned}$$

(2) Let $0 \leq \alpha \leq \beta \leq 1$. Since $u_0 \in \mathbb{R}_{\mathcal{F}}$, we have that $f_2^{\beta}(t,u(t)) \leq f_2^{\alpha}(t,u(t))$ and

$$\begin{aligned} &\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^{\alpha}(s,u(s))\,ds \\ &\qquad\qquad \leq \frac{u_1^{\beta}(0)}{f_2^{\beta}(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^{\beta}(s,u(s))\,ds. \end{aligned}$$

We deduce that

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,u(t)) \left[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_1^\alpha(s,u(s))\, ds \right] \\ &\le f_2^\beta(t,u(t)) \left[\frac{u_1^\beta(0)}{f_2^\beta(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_1^\beta(s,u(s))\, ds \right] = u_1^\beta(t). \end{aligned}$$

Similarly, $f_1^\alpha(t,u(t)) \le f_1^\beta(t,u(t))$ and

$$\begin{aligned} &\frac{u_2^\beta(0)}{f_1^\beta(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_2^\beta(s,u(s))\, ds \\ &\qquad\qquad \le \frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_2^\alpha(s,u(s))\, ds, \end{aligned}$$

so

$$\begin{aligned} u_2^\beta(t) &= f_1^\beta(t,u(t)) \left[\frac{u_2^\beta(0)}{f_1^\beta(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_2^\beta(s,u(s))\, ds \right] \\ &\le f_1^\alpha(t,u(t)) \left[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_2^\alpha(s,u(s))\, ds \right] = u_2^\alpha(t), \end{aligned}$$

which proves that $[u_1^\beta(t), u_2^\beta(t)] \subseteq [u_1^\alpha(t), u_2^\alpha(t)]$.

(3) Given a nondecreasing sequence $\{\alpha_i\}$ in $(0,1]$ such that $\alpha_i \uparrow \alpha \in (0,1]$, we prove that $[u_1^\alpha(t), u_2^\alpha(t)] = \bigcap_{i=1}^{\infty} [u_1^{\alpha_i}(t), u_2^{\alpha_i}(t)]$. Indeed, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{\alpha_i \uparrow \alpha} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_1^{\alpha_i}(s,u(s))\, ds &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \lim_{\alpha_i \uparrow \alpha} g_1^{\alpha_i}(s,u(s))\, ds \\ &= \int_0^t g_1^\alpha(s,u(s))\, ds \end{aligned}$$

and, hence,

$$\begin{aligned} \lim_{\alpha_i \uparrow \alpha} u_1^{\alpha_i}(t) &= \lim_{\alpha_i \uparrow \alpha} \left(f_2^{\alpha_i}(t,u(t)) \left[\frac{u_1^{\alpha_i}(0)}{f_2^{\alpha_i}(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_1^{\alpha_i}(s,u(s))\, ds \right] \right) \\ &= \lim_{\alpha_i \uparrow \alpha} \left(f_2^{\alpha}(t,u(t)) \left[\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_1^{\alpha}(s,u(s))\, ds \right] \right) = u_1^\alpha(t). \end{aligned}$$

Hence, $u(t) \in \mathbb{R}_{\mathcal{F}}$.

Case II. By equations (8), (9) we have the two following FHDE with initial conditions

$$\begin{cases} D^q \left[\frac{u_2^\alpha(t)}{f_1^\alpha(t,u(t))} \right] = g_1^\alpha(t,u(t)), \\ u_2^\alpha(0) = u_{02}^\alpha \end{cases} \tag{15}$$

and

$$\begin{cases} D^q\left[\frac{u_1^\alpha(t)}{f_2^\alpha(t,u(t))}\right] = g_2^\alpha(t,u(t)), \\ u_1^\alpha(0) = u_{01}^\alpha. \end{cases} \tag{16}$$

The solution to problems (15)–(16) are respectively

$$u_2^\alpha(t) = f_1^\alpha(t,u(t))\left[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^\alpha(s,u(s))\,ds\right],$$

$$u_1^\alpha(t) = f_2^\alpha(t,u(t))\left[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_2^\alpha(s,u(s))\,ds\right].$$

By applying Step 1, we consider the situation, where $0 \notin \left[z(0) + \int_0^t \tilde{g}(s,u(s))\,ds\right]^\alpha$,

$$\frac{f_2^\alpha(t,u(t))}{z_1(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_1^\alpha(s,u(s))\,ds} \leq \frac{f_1^\alpha(t,u(t))}{z_2(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g_2^\alpha(s,u(s))\,ds}, \tag{17}$$

i.e. $(u_1^\alpha(t) \leq u_2^\alpha(t))$. Similarly, by applying Theorem 1, the details for the Case I are analogous. If the situation (17) does not hold, i.e. $(u_2^\alpha(t) \leq u_1^\alpha(t))$, then by Theorem 1 $u(t)$ is not a fuzzy solution of (12). □

Example 3. *Consider the fuzzy fractional hybrid differential equation*

$$\begin{cases} D^{\frac{1}{2}}\left[\frac{u(t)}{1+\left(\frac{\sin(t)}{16}\right)|u(t)|}\right] = \frac{tu(t)}{1+|u(t)|}, \quad t \in J = [0,\ \pi], \\ u(0) = u_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases}$$

where $[u_0]^\alpha = [\alpha,\ 2-\alpha]$, $\forall \alpha \in [0,1]$.

It is easy to see that all hypotheses of Theorem 3 are satisfied with

$$f(t,u(t)) = 1 + \left(\frac{\sin(t)}{16}\right)|u(t)| \geq 0,$$

$$g(t,u(t)) = \frac{tu(t)}{1+|u(t)|} \geq 0 \ \text{ and } \ u(0) \geq 0,$$

so $z(0) * g(t,u(t)) > 0$.

We conclude that

$$\begin{cases} D^{\frac{1}{2}}\left[\left[\frac{u(t)}{1+\left(\frac{\sin(t)}{16}\right)|u(t)|}\right]^\alpha\right] = \left[\frac{tu(t)}{1+|u(t)|}\right]^\alpha, \\ [u(0)]^\alpha = [\alpha,\ 2-\alpha]. \end{cases} \tag{18}$$

Hence (18) *has a fuzzy solution* $u(t) \in \mathbb{R}_{\mathcal{F}}$.

5 Conclusions

We have successfully studied fractional hybrid differential equations with a fuzzy initial value, using the Riemann-Liouville fuzzy fractional derivative of order $q \in (0,1)$. The obtained results have been testified by an interesting examples. Also, we have provided some sufficient conditions guaranteeing the existence of fuzzy solutions for a class of fuzzy hybrid fractional differential equations. Our results rely on a generalized division for fuzzy numbers, we prove some results and we have obtained fuzzy solutions to fuzzy fractional hybrid differential equations. We will apply this generalized division to more dynamics of the problems involving fuzzy with future scope.

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Харір А., Мелліані С., Чадлі Л.С. *Нечіткі дробові гібридні диференціальні рівняння* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 332–344.

Ця стаття стосується представлення та розвитку теорії дробових гібридних диференціальних рівнянь з нечіткими початковими даними, що включають нечіткі дробові диференціальні оператори Рімана-Ліувіля порядку $0 < q < 1$. Ми детально вивчаємо існування та єдиність нечіткого розв'язку на базі концепції узагальненого розподілу нечітких чисел. Ми побудували та дослідили нечіткий розв'язок нечіткого дробового гібридного диференціального рівняння. Наприкінці ми навели приклад, що ілюструє теорію.

Ключові слова і фрази: нечіткий дробовий диференціал, функція з нечітким значенням, нечітке дробове гібридне диференціальне рівняння.