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RECONSTRUCTION OF ENERGY-DEPENDENT STURM-LIOUVILLE EQUATIONS FROM TWO SPECTRA. II

We study the problem of reconstruction of singular energy-dependent Sturm-Liouville equation from two spectra. We suggest a new method of solving this inverse problem by establishing its connection with the problem of reconstruction from one spectrum and the set of norming constants.

Key words and phrases: inverse problems, Sturm-Liouville equations, energy-dependent potentials, singular potentials.

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INTRODUCTION

The main object of our study is energy-dependent Sturm-Liouville equation

$$-y'' + qy + 2\lambda py = \lambda^2 y \tag{1}$$

on $(0, 1)$; here $\lambda \in \mathbb{C}$ is the spectral parameter, p is a real-valued function in $L_2(0, 1)$ and q is a real-valued distribution in the Sobolev space $W_2^{-1}(0, 1)$, i.e. $q = r'$ with a real-valued $r \in L_2(0, 1)$. We consider this equation under two types of boundary conditions: the Dirichlet ones

$$y(0) = y(1) = 0 \tag{2}$$

and the so-called mixed conditions

$$y(0) = y^{[1]}(1) + Hy(1) = 0,$$

where $H \in \mathbb{R}$ is some constant and $y^{[1]} := y' - ry$ is a quasi-derivative of the function y used in the regularization procedure due to Savchuk and Shkalikov (see [19, 20] and the next section for details). Since primitive of q is defined only up to an additive constant, by replacing r with $r - H$ we reduce the above mixed boundary conditions to the following ones:

$$y(0) = y^{[1]}(1) = 0. \tag{3}$$

In what follows, we shall denote by $\mathscr{L}_D(p, r)$ and $\mathscr{L}_M(p, r)$ the spectral problems (1), (2) and (1), (3) respectively. Our main aim in this paper is to solve the inverse problem of reconstructing the potentials p and r given the spectra of $\mathscr{L}_D(p, r)$ and $\mathscr{L}_M(p, r)$.

The spectral problem under study often arise in classical and quantum mechanics. In particular, the equations of the form (1) are used in modelling of the motion of relativistic massless particles, in describing the interactions of colliding spinless particles, in modelling of the mechanical system vibrations in viscous media etc.

The spectral equation (1) was considered on the line and studied in the context of inverse scattering problems (see, e.g. [1, 7, 9, 10, 12, 18, 21], and [5] for a more extensive reference list). The inverse spectral problems for (1) with $p \in W_2^1(0,1)$ and $q \in L_2(0,1)$ and with Robin boundary conditions were discussed by M. Gasymov and G. Guseinov in their short paper [3] of 1981 containing no proofs. Such problems were also considered in [2, 4, 13, 14, 22], but only Borg-type uniqueness results were obtained therein.

We studied the inverse problems of reconstruction of (1) with potentials $p \in L_2(0,1)$ and $q \in W_2^{-1}(0,1)$ from the spectra of $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ in [17] and from one spectrum and the set of norming constants in [5]. In this paper, we suggest another method of reconstructing (1) from two spectra that exploits connection of this problem with the problem of reconstructing (1) from one spectrum and the set of norming constants.

Namely, given two sequences λ and μ, which are supposed to be the spectra of $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ with the sought potentials p and $q = r'$, we construct another sequence, which turns out to consist of the norming constants for $\mathscr{L}_D(p,r)$. Then, using the results of [5], we reconstruct the potentials p and q of (1) such that λ is the spectrum of (1), (2) with these p and q. Next we show that the primitive r of q can be chosen uniquely so that the spectrum of $\mathscr{L}_M(p,r)$ coincides with the given sequence μ. The main result of the paper is the existence and uniqueness theorem giving a complete characterisation of the spectra of the problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ as well as the reconstruction algorithm.

1   Preliminaries and main results

In this section we introduce the necessary definitions and formulate the main results of the paper. To start with, consider the differential expression

$$\ell(y) = -y'' + qy$$

and recall that $q = r'$ is a real-valued distribution from $W_2^{-1}(0,1)$. Therefore we need to define the action of $\ell(y)$ more rigourously. To do this we use the regularization procedure due to Savchuk and Shkalikov (see [19, 20]) based on the notion of quasi-derivatives. Namely, for every absolutely continuous function y we denote by $y^{[1]} := y' - ry$ its quasi-derivative and define $\ell(y)$ as

$$\ell(y) = -\big(y^{[1]}\big)' - ry^{[1]} - r^2 y$$

on the domain

$$\operatorname{dom}\ell = \{y \in AC(0,1) \mid y^{[1]} \in AC[0,1],\ \ell(y) \in L_2(0,1)\}.$$

It is straightforward to see that so defined $\ell(y)$ coincides with $-y'' + qy$ in the distributional sense.

Now we can recast the spectral equation (1) as

$$\ell(y) + 2\lambda p y = \lambda^2 y. \tag{4}$$

Then a number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the problem $\mathscr{L}_D(p,r)$ (resp. $\mathscr{L}_M(p,r)$) if equation (4) possesses a nontrivial solution satisfying the boundary conditions (2) (resp. (3)). This solution is then called an *eigenfunction* of the problem $\mathscr{L}_D(p,r)$ (resp. $\mathscr{L}_M(p,r)$) corresponding to λ.

In this paper we study the following inverse spectral problem:

(IP1) Given the spectra of the problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$, determine the potentials p and r.

A complete solution of this problem is only possible under some extra assumption, which we formulate further. Denote by T_j, $j = 1,2$, the operator pencils defined via

$$T_j(\lambda)y := \ell(y) + 2\lambda p - \lambda^2 y$$

on the λ-independent domains

$$\operatorname{dom} T_1 := \{y \in \operatorname{dom} \ell \mid y(0) = y(1) = 0\},$$
$$\operatorname{dom} T_2 := \{y \in \operatorname{dom} \ell \mid y(0) = y^{[1]}(1) = 0\}.$$

Note that the spectra of the problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ coincide with those of the pencils T_1 and T_2 respectively. Our standing assumption is the following:

(A) there is a $\mu_* \in \mathbb{R}$ such that the operator $T_2(\mu_*)$ is positive.

Under this assumption all the eigenvalues of both problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ are real and simple (see [16]). Moreover, they can be enumerated in increasing order as λ_n and μ_n so that the pair of sequences $((\lambda_n),(\mu_n))$ forms an element of the set SD_1 defined below (see [17]).

Definition 1. *We denote by SD_1 the family of all pairs $(\boldsymbol{\lambda},\boldsymbol{\mu})$ of increasing sequences $\boldsymbol{\lambda} := (\lambda_n)_{n\in\mathbb{Z}^*}$, $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$, and $\boldsymbol{\mu} := (\mu_n)_{n\in\mathbb{Z}}$ of real numbers satisfying the following conditions:*

(i) asymptotics: *there is an $h \in \mathbb{R}$ such that*

$$\lambda_n = \pi n + h + \tilde{\lambda}_n, \quad \mu_n = \pi\left(n - \tfrac{1}{2}\right) + h + \tilde{\mu}_n, \tag{5}$$

where $(\tilde{\lambda}_n)$ is a sequence in $\ell_2(\mathbb{Z}^*)$ and $(\tilde{\mu}_n)$ is from $\ell_2(\mathbb{Z})$;

(ii) almost interlacing:
$$\mu_k < \lambda_k < \mu_{k+1} \quad \text{for every } k \in \mathbb{Z}^*. \tag{6}$$

Remark 1. (a) *If the eigenvalues λ_n of $\mathscr{L}_D(p,r)$ and μ_n of $\mathscr{L}_M(p,r)$ are ordered so that (i) and (ii) of the above definition hold, then the number μ_* in assumption (A) satisfies the inequalities $\mu_0 < \mu_* < \mu_1$, see [16]. Moreover, then assumption (A) holds with every μ_* from (μ_0, μ_1).*

(b) *For the most of the paper, it will be convenient to assume that μ_* in (A) is zero. If this does not hold, we can shift the spectral parameter via $\lambda = \hat{\lambda} + \mu_*$; then the spectral equation (1) can be recast as*

$$-y'' + \hat{q}y + 2\hat{\lambda}\hat{p}y = \hat{\lambda}^2 y$$

with the new potentials $\hat{p} := p - \mu_*$ and $\hat{q} := q + 2\mu_* p - \mu_*^2$. Moreover, choosing the primitive $\hat{r} := r - \int_x^1 (2\mu_* p - \mu_*^2)$ of $\hat{q}$ so that $(\hat{r} - r)(1) = 0$ and introducing the corresponding quasi-derivative $y^{[1]} := y' - \hat{r}y$, we see that the boundary conditions (2) and (3) remain unchanged. Now if λ_n (resp. μ_n) are eigenvalues of the problem $\mathscr{L}_D(p, r)$ (resp. $\mathscr{L}_M(p, r)$), then $\hat{\lambda}_n := \lambda_n - \mu_*$ (resp. $\hat{\mu}_n := \mu_n - \mu_*$) are eigenvalues of the problem $\mathscr{L}_D(\hat{p}, \hat{r})$ (resp. $\mathscr{L}_M(\hat{p}, \hat{r})$), while the eigenfunctions for the corresponding eigenvalues are the same. In particular, the problems $\mathscr{L}_D(\hat{p}, \hat{r})$ and $\mathscr{L}_M(\hat{p}, \hat{r})$ satisfy assumption (A) with $\mu_* = 0$. Having $\hat{p}, \hat{q}$ and $\hat{r}$ we can find p, q and r by formulae

$$p = \hat{p} + \mu_*, \quad q = \hat{q} - 2\mu_* p - \mu_*^2, \quad r = \hat{r} + \int_x^1 (2\mu_* \hat{p} + \mu_*^2). \tag{7}$$

In view of the above remark, without loss of generality we can work under a simplifying assumption

(A0) the operator $T_2(0)$ is positive.

However, the main results of the paper will be proved under the general assumption (A).

Clearly, under assumption (A0) the eigenvalues of $\mathscr{L}_D(p, r)$ and $\mathscr{L}_M(p, r)$ can be enumerated in increasing order as λ_n and μ_n so that the pair of sequences $((\lambda_n), (\mu_n))$ forms an element of the set SD_1 with $\mu_0 < 0 < \mu_1$.

In this paper we establish connection between the inverse problem (IP1) and the inverse problem (IP2) formulated below; it was already studied in [5]. Namely, for an eigenvalue λ of the problem $\mathscr{L}_D(p, r)$, denote by y the corresponding eigenfunction normalized by the initial conditions $y(0) = 0$ and $y^{[1]}(0) = 1$. The quantity

$$\alpha := 2\lambda^2 \int_0^1 y^2(t)dt - 2\lambda \int_0^1 p(t)y^2(t)dt \tag{8}$$

is called the *norming constant* corresponding to the eigenvalue λ. Then (λ, α) is called the *(spectral) eigenpair* of $\mathscr{L}_D(p, r)$. The *spectral data* $\mathbf{sd}(p, r)$ of the problem $\mathscr{L}_D(p, r)$ is the set of all eigenpairs (λ, α) of $\mathscr{L}_D(p, r)$.

The inverse spectral problem (IP2) reads as follows:

(IP2) Given the spectral data $\mathbf{sd}(p, r)$ of $\mathscr{L}_D(p, r)$, determine the potentials p and r.

The results of [5] imply that under assumption (A0) the spectral data $\mathbf{sd}(p, r)$ form an element of the set SD_2 defined below.

Definition 2. *We denote by SD_2 the family of all sets $\{(\lambda_n, \alpha_n)\}_{n \in \mathbb{Z}^*}$, which consist of pairs (λ_n, α_n) of real numbers satisfying the following properties:*

(i) *λ_n are nonzero, strictly increase with $n \in \mathbb{Z}^*$, and have the representation $\lambda_n = \pi n + h + \tilde{\lambda}_n$ for some $h \in \mathbb{R}$ and a sequence $(\tilde{\lambda}_n)$ in $\ell_2(\mathbb{Z}^*)$;*

(ii) *$\alpha_n > 0$ for all $n \in \mathbb{Z}^*$ and the numbers $\tilde{\alpha}_n := \alpha_n - 1$ form an $\ell_2(\mathbb{Z}^*)$-sequence.*

The main results of [5] are the following:

Theorem A (Uniqueness). *Under assumption (A0), the potentials p and $q = r'$ of equation (1) are uniquely determined by its spectral data $\mathbf{sd}(p, r)$.*

Theorem B (Existence). *For every* **sd** $\in SD_2$, *there exist real-valued* p *from* $L_2(0,1)$ *and* q *from* $W_2^{-1}(0,1)$ *such that* **sd** *is the spectral data for the problem* $\mathscr{L}_D(p,r)$ *with the potentials* p *and with* r *a primitive of* q, *i.e.* **sd** $=$ **sd**(p,r).

Note that neither the spectrum of $\mathscr{L}_D(p,r)$ nor the set of norming constants depend on the particular choice of the primitive r of q. That is why the results of [5] guarantee unique reconstruction of q but leave r determined up to an additive constant. However, the boundary conditions (3) for the problem $\mathscr{L}_M(p,r)$ do depend on the choice of r, and we shall show that r is determined uniquely in the inverse problem (IP1).

To investigate the connection between (IP1) and (IP2) we use the characteristic functions of the problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$. Denote by $y(x,z)$ the solution of (4) with z instead of λ and subject to the initial conditions $y(0) = 0$, $y^{[1]}(0) = 1$. Then λ is an eigenvalue of the problem $\mathscr{L}_D(p,r)$ if and only if it is a zero of its *characteristic function* $\varphi(z) := y(1,z)$. Analogously a number μ is an eigenvalue of the problem $\mathscr{L}_M(p,r)$ if and only if μ is a zero of the corresponding *characteristic function* $\psi(z) := y^{[1]}(1,z)$. It was shown in [15] that the functions φ and ψ can be written in factorized form in terms of their zeros, namely

$$\varphi(\lambda) = \begin{cases} \text{V.p.} \prod_{n\in\mathbb{Z}^*} \dfrac{\lambda_n - \lambda}{\pi n}, & \text{if } p_0 \neq \pi l, \ l \in \mathbb{Z}, \\[3mm] (-1)^l \text{ V.p.} \prod_{n\in\mathbb{Z}^*} \dfrac{\lambda_n - \lambda}{\pi n}, & \text{if } p_0 = \pi l, \ l \in \mathbb{Z}, \end{cases} \tag{9}$$

$$\psi(\mu) = \begin{cases} -\text{V.p.} \prod_{n\in\mathbb{Z}} \dfrac{\mu_n - \mu}{\pi (n + 1/2)}, & \text{if } p_0 \neq \dfrac{\pi}{2} + \pi l, \ l \in \mathbb{Z}, \\[3mm] (-1)^{l+1}(\mu_0 - \mu)\text{V.p.} \prod_{n\in\mathbb{Z}^*} \dfrac{\mu_n - \mu}{\pi n}, & \text{if } p_0 = \dfrac{\pi}{2} + \pi l, \ l \in \mathbb{Z}, \end{cases} \tag{10}$$

where $p_0 = \int_0^1 p$. The link between (IP1) and (IP2) is given by (9), (10) and the following formula, which relates the characteristic functions (and so the spectra) of $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ and the norming constants of $\mathscr{L}_D(p,r)$ (see [16]):

$$\alpha_n = \lambda_n \dot{\varphi}(\lambda_n)\psi(\lambda_n). \tag{11}$$

In the next section we shall prove the following theorem:

Theorem 1. *Given the pair of sequences* $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *from* SD_1 *with* $\mu_0 < 0 < \mu_1$, *construct* $(\alpha_n)_{n\in\mathbb{Z}^*}$ *via (9), (10) and (11). Then the* α_n *are positive and the sequence* $(\alpha_n - 1)_{n\in\mathbb{Z}^*}$ *belongs to* $\ell_2(\mathbb{Z}^*)$.

As a result, the set of pairs $\{(\lambda_n, \alpha_n)\}_{n\in\mathbb{Z}^*}$, with given numbers λ_n and the numbers α_n constructed as in (11), forms an element of SD_2. Therefore by Theorems A and B there exist unique real-valued $p \in L_2(0,1)$ and $q \in W_2^{-1}(0,1)$ such that $\{(\lambda_n, \alpha_n)\}_{n\in\mathbb{Z}^*}$ coincides with the spectral data **sd**(p,r), with every primitive r of q. Then we show that this primitive r can be uniquely chosen as to make μ_n the eigenvalues of $\mathscr{L}_M(p,r)$. This will lead to the main result of the paper (cf. [17]):

Theorem 2. *Assume that a pair* $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *of sequences of real numbers is an element of* SD_1. *Then there exist unique real-valued* $p, r \in L_2(0,1)$ *such that* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$ *are the spectra of the problems* $\mathscr{L}_D(p,r)$ *and* $\mathscr{L}_M(p,r)$. *In particular, the singular potential* q *in (1) is equal to* r'.

2   CONNECTION BETWEEN (IP1) AND (IP2)

2.1   Proof of Theorem 1

This subsection is devoted to the proof of Theorem 1, which establishes connection between (IP1) and (IP2).

Suppose we have two sequences $\lambda := (\lambda_n)_{n \in \mathbb{Z}^*}$ and $\mu := (\mu_n)_{n \in \mathbb{Z}}$ with $\mu_0 < 0 < \mu_1$, which form an element (λ, μ) of the set SD_1. Set $\lambda_0 := 0$, and denote by λ^* the sequence $(\lambda_n)_{n \in \mathbb{Z}}$, which is λ augmented with λ_0. Then λ^* and μ strictly interlace. By means of these sequences we construct the functions

$$
s_1(z) := \begin{cases} z \mathrm{V.p.} \prod\limits_{n \in \mathbb{Z}^*} \dfrac{\lambda_n - z}{\pi n}, & \text{if } h \neq \pi l,\ l \in \mathbb{Z}, \\[2mm] (-1)^l z\ \mathrm{V.p.} \prod\limits_{n \in \mathbb{Z}^*} \dfrac{\lambda_n - z}{\pi n}, & \text{if } h = \pi l,\ l \in \mathbb{Z}, \end{cases}
\tag{12}
$$

$$
c(z) := \begin{cases} -\mathrm{V.p.} \prod\limits_{n \in \mathbb{Z}} \dfrac{\mu_n - z}{\pi\,(n + 1/2)}, & \text{if } h \neq \dfrac{\pi}{2} + \pi l,\ l \in \mathbb{Z}, \\[2mm] (-1)^{l+1}(\mu_0 - z)\mathrm{V.p.} \prod\limits_{n \in \mathbb{Z}^*} \dfrac{\mu_n - z}{\pi n}, & \text{if } h = \dfrac{\pi}{2} + \pi l,\ l \in \mathbb{Z}, \end{cases}
\tag{13}
$$

where h is the number in the asymptotics (i) of Definition 1.

Observe that λ^* is the sequence of zeros of s_1 and μ is that of c. The results of [6] imply that there exist functions f and g from $L_2(0, 1)$ such that

$$
s_1(z) = \sin(z - h) + \int_0^1 f(t)e^{iz(1-2t)}dt \quad \text{and} \quad c(z) = \cos(z - h) + \int_0^1 g(t)e^{iz(1-2t)}dt.
\tag{14}
$$

Note that

$$
\dot{s}_1(\lambda_n) = \cos(\lambda_n - h) + \int_0^1 f(t)i(1 - 2t)e^{i\lambda_n(1-2t)}dt.
$$

Next put $s(z) := \frac{s_1(z)}{z}$; since $\lambda_n, n \in \mathbb{Z}$, are zeros of s_1 we have $\dot{s}(\lambda_n) = \frac{\dot{s}_1(\lambda_n)}{\lambda_n}$ for every $n \in \mathbb{Z}^*$.

Now we shall prove the following auxiliary lemma.

Lemma 1. *Let F be a function from $L_2(0, 1)$. Then the sequence $(f_n)_{n \in \mathbb{Z}}$ with*

$$
f_n := \int_0^1 F(t)e^{i\lambda_n(1-2t)}dt
\tag{15}
$$

belongs to $\ell_2(\mathbb{Z})$.

Proof. Let us firstly make a change of variables $u := 1 - 2t$ in the integral of the righthand side of (15). We obtain

$$
f_n := \int_{-1}^1 G(u)e^{i\omega_n u}du,
$$

where $G(u) = \frac{1}{2}F(\frac{1-u}{2})e^{ihu}$ is the function from $L_2(-1, 1)$ and $\omega_n = \pi n + \tilde{\lambda}_n$. To complete the proof it is enough to show that the system $e^{i\omega_n u}$ forms a Riesz basis in $L_2(-1, 1)$; then f_n are the Fourier coefficients of G relative to the system $e^{i\omega_n u}$ and so form a sequence from $\ell_2(\mathbb{Z})$ (see e.g. [23, Ch. 1]).

Note firstly that the system $\{e^{i\pi n u}\}$ is an orthogonal basis in $L_2(-1, 1)$. One can find a constant $L < \pi/4$ and a sufficiently large N such that $|\tilde{\lambda}_n| < L$ for all n, $|n| > N$. Then Kadec's 1/4-Theorem (see [23, Ch. 1], [8]) yields that the system $\{e^{i\tilde{\omega}_n u}\}$ with

$$\tilde{\omega}_n = \begin{cases} \pi n + \tilde{\lambda}_n, & |n| > N, \\ \pi n, & |n| \le N, \end{cases}$$

forms a Riesz basis. It remains to observe that the sequence (ω_n) is obtained from $(\tilde{\omega}_n)$ by changing a finite number of elements. Theorems 3.11 and 1.12 of [23] imply that the system $\{e^{i\omega_n u}\}$ is a Riesz basis. $\qquad\square$

The above lemma yields that

$$\dot{s}_1(\lambda_n) = (-1)^n \cos \tilde{\lambda}_n + \int_0^1 f(t)i(1-2t)e^{i\lambda_n(1-2t)}dt = (-1)^n(1+s_n),$$

$$c(\lambda_n) = (-1)^n \cos \tilde{\lambda}_n + \int_0^1 g(t)e^{i\lambda_n(1-2t)}dt = (-1)^n(1+c_n) \tag{16}$$

with ℓ_2-sequences $(s_n)_{n\in\mathbb{Z}}$ and $(c_n)_{n\in\mathbb{Z}}$. Define the sequence $(\alpha_n)_{n\in\mathbb{Z}^*}$ as follows

$$\alpha_n := \lambda_n \dot{s}(\lambda_n)c(\lambda_n) = \dot{s}_1(\lambda_n)c(\lambda_n). \tag{17}$$

Then (16) implies that $\alpha_n = (-1)^n(1+s_n)(-1)^n(1+c_n) = 1 + \tilde{\alpha}_n$ with ℓ_2-sequence $(\tilde{\alpha}_n)$.

Since the sequences $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}$ interlace a straightforward analysis of definitions (12), (13) and formula (17) gives that all α_n, $n \in \mathbb{Z}^*$, are of the same sign and thus are positive thus finishing the proof of Theorem 1.

2.2 Solution of (IP2)

Theorem 1 together with Theorems A and B yields that for the given sequence $(\lambda_n)_{n\in\mathbb{Z}^*}$ and the constructed $(\alpha_n)_{n\in\mathbb{Z}^*}$ we can uniquely determine potentials p and q such that the problem $\mathscr{L}_D(p,r)$ with r an arbitrary primitive of q has $(\lambda_n)_{n\in\mathbb{Z}^*}$ as its spectrum and $(\alpha_n)_{n\in\mathbb{Z}^*}$ as the corresponding norming constants.

From [15] we know that the shift h in the asymptotics (5) of eigenvalues of $\mathscr{L}_D(p,r)$ equals to $p_0 = \int_0^1 p$.

2.3 Solution of (IP1)

Now we show that the potentials p and q constructed in the previous subsection also provide a solution to the (IP2). Namely, we shall show that there exists a primitive r of q such that $\mu_n, n \in \mathbb{Z}$, are all the eigenvalues of the problem $\mathscr{L}_M(p,r)$.

To start with, note that a primitive r of q is determined up to an additive constant. We choose r in the following way. Let $y(x, \mu_0)$ be the solution of the equation (1) with μ_0 instead of λ satisfying the initial condition $y(0, \mu_0) = 0$. Then $y(1, \mu_0)$ is not equal to 0 as μ_0 is not in the spectrum of $\mathscr{L}_D(p,r)$. This allows us to choose r uniquely so that $y^{[1]}(1, \mu_0) = 0$. Then μ_0 is an eigenvalue of the problem $\mathscr{L}_M(p,r)$ with this fixed r. Denote by $\nu_n, n \in \mathbb{Z}$, the eigenvalues of $\mathscr{L}_M(p,r)$ enumerated in increasing order so that $\nu_0 = \mu_0$.

Lemma 2. $\nu_n = \mu_n$ for all $n \in \mathbb{Z}$.

Proof. Recall that the eigenvalues ν_n of $\mathscr{L}_M(p,r)$ satisfy the asymptotics $\nu_n = \pi(n - 1/2) + p_0 + \tilde{\nu}_n$ with an ℓ_2-sequence $(\tilde{\nu}_n)$ and that the corresponding characteristic function ψ is given by (10) with ν_n instead of μ_n. The function ψ can be represented in an integral form, (see [6, 15])

$$\psi(z) = \cos(z - p_0) + \int_0^1 g_1(t)e^{iz(1-2t)}dt \tag{18}$$

with some $g_1 \in L_2(0, 1)$. We are going to show that ψ coincides with the function c of (13). Since $\nu_n, n \in \mathbb{Z}$, are zeros of ψ and $\mu_n, n \in \mathbb{Z}$, are those of c, this will finish the proof.

Suppose, on the contrary, that $\psi \neq c$, i.e., that the function $\hat{\psi} := \psi - c$ is not identically zero. On account of the equality $h = p_0$ the representations (14) and (18) for the functions c and ψ give that

$$\hat{\psi}(z) = \int_0^1 (g_1(t) - g(t)) e^{iz(1-2t)} dt$$

and so, by a refined version of the Riemann-Lebesgue lemma [11, Lemma 1.3.1],

$$\hat{\psi}(z) = o(e^{|\mathrm{Im}z|}), \quad |z| \to \infty. \tag{19}$$

Taking (9) and the equality $h = p_0$ into account, we observe that $s(z)$ defined as $s_1(z)/z$ with s_1 of (12) coincides with the characteristic function φ of the problem $\mathscr{L}_D(p, r)$. Comparing the construction (17) of α_n and the relation (11) for the norming constants of $\mathscr{L}_D(p, r)$, we conclude that $\dot{s}(\lambda_n)\psi(\lambda_n) = \dot{s}(\lambda_n)c(\lambda_n)$, $n \in \mathbb{Z}^*$. As the sequence λ strictly increases, each zero of s is simple and so $\dot{s}(\lambda_n) \neq 0, n \in \mathbb{Z}^*$. Therefore $c(\lambda_n) = \psi(\lambda_n)$ or equivalently $\hat{\psi}(\lambda_n) = 0$ for every $n \in \mathbb{Z}^*$. Clearly, $c(\mu_0) = \psi(\mu_0) = 0$ giving that $\hat{\psi}(\mu_0) = 0$. Hence $\{\lambda_n\}_{n \in \mathbb{Z}^*} \cup \{\mu_0\}$ are zeros of the function $\hat{\psi}(z)$.

Let us show that $\hat{\psi}$ possesses no other zeros. Denote by $n(t)$ the number of zeros of $\hat{\psi}$ in the disk $|z| \leq t$; then, in view of (19), the Jensen's formula gives

$$\int_0^r \frac{n(t)}{t} dt \leq \frac{2r}{\pi} + C_1 \tag{20}$$

with some constant $C_1 \in \mathbb{R}$. If $\hat{\psi}$ possessed other zeros apart from $\{\lambda_n\}_{n \in \mathbb{Z}^*} \cup \{\mu_0\}$, then the asymptotics of λ_n would guarantee that there exists $\varepsilon \in (0, \frac{1}{2})$ and N sufficiently large such that for every $l \geq N$ $n(\pi(l + \varepsilon)) \geq 2l + 2$. Put $t_l := \pi(l + \varepsilon)$ and use Stirling's approximation of the Euler gamma-function to obtain

$$\int_{t_m}^{t_{n+1}} \frac{n(t)}{t} dt \geq \sum_{l=m}^n (2l+2) \log \frac{t_{l+1}}{t_l} = (2n+2) \log t_{n+1} - 2 \sum_{l=m}^n \log t_l - 2m \log t_m$$

$$\geq (2n+2) \log \frac{t_{n+1}}{\pi} - 2 \log \Gamma \left(\frac{t_{n+1}}{\pi} \right) + C_2 \geq \frac{2t_{n+1}}{\pi} + (1 - 2\varepsilon) \log \frac{t_{n+1}}{\pi} + C_3$$

with some constants C_2 and C_3. This estimate contradicts (20) and thus shows that $\hat{\psi}$ has no other zeros besides $\{\lambda_n\}_{n \in \mathbb{Z}^*} \cup \{\mu_0\}$.

The function $\hat{\psi}$ is of exponential type less than or equal to 1. Using this and the Hadamard factorization theorem, we obtain that

$$\hat{\psi}(z) = e^{Az+B} \left(1 - \frac{z}{\mu_0} \right) \mathrm{V.p.} \prod_{n \in \mathbb{Z}^*} \left(1 - \frac{z}{\lambda_n} \right) e^{\frac{z}{\lambda_n}}$$

with some constants A and B. Since

$$\mathrm{V.p.} \sum_{n \in \mathbb{Z}^*} \frac{1}{\lambda_n} = \sum_{n=1}^\infty \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_{-n}} \right) = \sum_{n=1}^\infty \frac{\pi^2 n^2}{\lambda_n \lambda_{-n}} \cdot \frac{\lambda_n + \lambda_{-n}}{\pi^2 n^2},$$

with absolutely convergent series $\sum_{n=1}^\infty \frac{\lambda_n + \lambda_{-n}}{\pi^2 n^2}$ and uniformly bounded sequence $\left(\frac{\pi^2 n^2}{\lambda_n \lambda_{-n}} \right)$, the series $\sum \frac{z}{\lambda_n}$ is convergent. Therefore,

$$\hat{\psi}(z) = e^{A'z+B} \left(1 - \frac{z}{\mu_0} \right) \mathrm{V.p.} \prod_{n \in \mathbb{Z}^*} \left(1 - \frac{z}{\lambda_n} \right)$$

with a suitable constant A'.

Let us now fix $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and take z of the form $z = \rho e^{i\theta}$, $\rho > 0$. By (19),

$$\frac{\hat{\psi}(z)}{\sin(z-h)} \to 0, \quad \rho \to \infty. \tag{21}$$

Recall (see e.g. [23, Ch.2]) that the function $\sin(z - h)$ can be factorized as follows

$$\sin(z-h) = (z-h)\text{V.p.} \prod_{n \in \mathbb{Z}^*} \frac{\pi n + h - z}{\pi n},$$

so that

$$\frac{\hat{\psi}(z)}{\sin(z-h)} = e^{A'z+B} \frac{\mu_0 - z}{(z-h)\mu_0} \text{V.p.} \prod_{n \in \mathbb{Z}^*} \frac{\pi n}{\lambda_n} \cdot \frac{\lambda_n - z}{\pi n + h - z}.$$

By Lemma 3 of [15], the product $\text{V.p.} \prod_{n \in \mathbb{Z}^*} \frac{\pi n}{\lambda_n}$ is convergent and, by Lemma 4 of [15], the product $\text{V.p.} \prod_{n \in \mathbb{Z}^*} \frac{\lambda_n - z}{\pi n + h - z}$ converges to 1 as $\rho \to \infty$ and $\theta \neq 0, \pi$. In view of (21), this means that $e^{A'z+B}$ converges to 0 as $\rho \to \infty$. But this is impossible; the contradiction derived shows that our assumption that $\hat{\psi} \not\equiv 0$ is false. Therefore $\hat{\psi} \equiv 0$ and $\nu_n = \mu_n$ for all $n \in \mathbb{Z}$. The proof is complete. $\qquad\qquad\square$

3   PROOF OF MAIN RESULTS

In this section we turn to assumption (A) and proof Theorem 2 in the case of arbitrary $\mu_* \in \mathbb{R}$. Then we formulate a reconstruction algorithm.

3.1   Proof of Theorem 2

Given $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in SD_1$, we firstly put $\mu_* := (\mu_0 + \mu_1)/2$ and shift the sequences $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ by $-\mu_*$ to obtain new sequences $\hat{\boldsymbol{\lambda}} := (\hat{\lambda}_n)_{n \in \mathbb{Z}^*}$ and $\hat{\boldsymbol{\mu}} := (\hat{\mu}_n)_{n \in \mathbb{Z}}$ from SD_1 with $\hat{\mu}_0 < 0 < \hat{\mu}_1$. To prove the theorem it is enough to show that for the sequences $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$ there exist unique real-valued $\hat{p}$ and $\hat{r}$ from $L_2(0,1)$ such that $\hat{\boldsymbol{\lambda}}$ is the spectrum of $\mathscr{L}_D(\hat{p}, \hat{r})$ and $\hat{\boldsymbol{\mu}}$ is that of $\mathscr{L}_M(\hat{p}, \hat{r})$. Then by formulae (7) with $\mu_* = (\mu_0 + \mu_1)/2$ we can uniquely determine potentials p and r from $\hat{p}$ and $\hat{r}$ such that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the spectra of problems $\mathscr{L}_D(p,r)$ and $\mathscr{L}_M(p,r)$ respectively.

By means of sequences $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$ construct the functions s_1 and c by formulae (12) and (13) and then the sequence (α_n) by (17). Due to Theorem 1, the set of pairs $\mathbf{sd} = \{(\hat{\lambda}_n, \alpha_n)\}_{n \in \mathbb{Z}^*}$ with the given $\hat{\lambda}_n$ and the constructed α_n belongs to SD_2. Then, using Theorem B, we construct potentials $\hat{p}$ and $\hat{q}$ such that $\hat{\boldsymbol{\lambda}}$ is the spectrum of $\mathscr{L}_D(\hat{p}, \hat{r})$ with the constructed $\hat{p}$ and any primitive $\hat{r}$ of $\hat{q}$. Next we fix the primitive $\hat{r}$ of $\hat{q}$ as explained in Subsection 2.3; then $\hat{\boldsymbol{\mu}}$ is the spectrum of $\mathscr{L}_M(\hat{p}, \hat{r})$ by Lemma 2. This establishes the existence part.

To prove uniqueness we assume that there are two pairs of potentials $\hat{p}_1, \hat{r}_1$ and $\hat{p}_2, \hat{r}_2$ such that the sequence $\hat{\boldsymbol{\lambda}}$ is the spectrum of both problems $\mathscr{L}_D(\hat{p}_1, \hat{r}_1)$ and $\mathscr{L}_D(\hat{p}_2, \hat{r}_2)$ and the sequence $\hat{\boldsymbol{\mu}}$ is the spectrum of both $\mathscr{L}_M(\hat{p}_1, \hat{r}_1)$ and $\mathscr{L}_M(\hat{p}_2, \hat{r}_2)$. This means that the norming constants of the problems $\mathscr{L}_D(\hat{p}_1, \hat{r}_1)$ and $\mathscr{L}_D(\hat{p}_2, \hat{r}_2)$ coincide as they are uniquely determined by two spectra $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\mu}}$ (see (11)). Then Theorem A implies that $\hat{p}_1 = \hat{p}_2$ and $\hat{r}_1' = \hat{r}_2'$; in particular, $\hat{r}_1 - \hat{r}_2 = H$ with some constant H. To complete the proof it is enough to show that $H = 0$.

Observe that equations (1) for the problems $\mathscr{L}_M(\hat{p}_1, \hat{r}_1)$ and $\mathscr{L}_M(\hat{p}_2, \hat{r}_2)$ are the same. As a result, eigenfunctions for the both problems corresponding to the common eigenvalue $\hat{\mu}_0$ coincide as well; denote it by y. Then $(y' - \hat{r}_1 y)(1) = (y' - \hat{r}_2 y)(1) = 0$ or equivalently $Hy(1) = 0$. However $\hat{\mu}_0$ is not in the spectra of $\mathscr{L}_D(\hat{p}_1, \hat{r}_1)$ and $\mathscr{L}_D(\hat{p}_2, \hat{r}_2)$, hence $y(1) \neq 0$. Therefore $H = 0$ thus finishing the proof.

3.2 Reconstruction algorithm

To sum up we formulate the reconstruction algorithm.

Suppose we have a pair of sequences $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ from SD_1. Then we

1) put $\mu_* := (\mu_0 + \mu_1)/2$ and consider a new pair of sequences $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}})$ such that $\hat{\boldsymbol{\lambda}} := (\hat{\lambda}_n)_{n \in \mathbb{Z}^*}$ with $\hat{\lambda}_n := \lambda_n - \mu_*$ and $\hat{\boldsymbol{\mu}} := (\hat{\mu}_n)_{n \in \mathbb{Z}}$ with $\hat{\mu}_n := \mu_n - \mu_*$;

2) augment $\hat{\boldsymbol{\lambda}}$ with $\hat{\lambda}_0 := 0$ and denote the new sequence by $\hat{\boldsymbol{\lambda}}^*$;

3) by means of sequences $\hat{\boldsymbol{\lambda}}^*$ and $\hat{\boldsymbol{\mu}}$ construct the functions s_1 and c by formulae (12) and (13) with $\hat{\lambda}_n$ and $\hat{\mu}_n$ instead of λ_n and μ_n respectively;

4) construct the sequence (α_n) by (17) with $\hat{\lambda}_n$ instead of λ_n;

5) having the set of pairs $\mathbf{sd} = \{(\hat{\lambda}_n, \alpha_n)\}_{n \in \mathbb{Z}^*}$, which, due to Theorem 1, belongs to SD_2, construct potentials $\hat{p}$ and $\hat{q}$ using the procedure of [5];

6) choose the primitive $\hat{r}$ of $\hat{q}$ as to make $\hat{\mu}_0$ an eigenvalue of $\mathscr{L}_M(\hat{p}, \hat{r})$;

7) determine potentials p and r from $\hat{p}$ and $\hat{r}$ using formulae (7).

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Пронська Н.І. *Відновлення енергозалежних рівнянь Штурма-Ліувілля за двома спектрами. II //* Карпатські математичні публікації. — 2013. — Т.5, №2. — C. 315–325.

Вивчається задача відновлення сингулярних енергозалежних рівнянь Штурма-Ліувілля за двома спектрами. Ми пропонуємо новий метод розв'язання цієї задачі, досліджуючи її зв'язок із задачею відновлення за одним спектром і множиною нормівних множників.

Ключові слова і фрази: обернені задачі, рівняння Штурма-Ліувілля, енергозалежні потенціали, сингулярні потенціали.

Пронська Н.И. *Восстановление уравнений Штурма-Лиувилля зависящих от энергии по двум спектрам. II. //* Карпатские математические публикации. — 2013. — Т.5, №2. — C. 315–325.

Изучается задача восстановление сингулярного уравнения Штурма-Лиувилля зависящего от энергии по двум спектрам. Предложен новый метод решения этой обратной задачи, который использует ее связь с задачей восстановления по одному спектру и множеством нормирующих множителей.

Ключевые слова и фразы: обратные задачи, уравнения Штурма-Лиувилля, энергозависимые потенциалы, сингулярные потенциалы.