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ON OPERATORS OF STOCHASTIC DIFFERENTIATION ON SPACES OF REGULAR TEST AND GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

The operators of stochastic differentiation, which are closely related with the extended Skorohod stochastic integral and with the Hida stochastic derivative, play an important role in the classical (Gaussian) white noise analysis. In particular, these operators can be used in order to study properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

In this paper we introduce and study bounded and unbounded operators of stochastic differentiation in the Lévy white noise analysis. More exactly, we consider these operators on spaces from parametrized regular rigging of the space of square integrable with respect to the measure of a Lévy white noise functions, using the Lytvynov's generalization of the chaotic representation property. This gives a possibility to extend to the Lévy white noise analysis and to deepen the corresponding results of the classical white noise analysis.

Key words and phrases: operator of stochastic differentiation, stochastic derivative, extended stochastic integral, Lévy process.

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INTRODUCTION

Let $L = (L_t)_{t\in[0,+\infty)}$ be a Lévy process, i.e., a random process on $[0, +\infty)$ with stationary independent increments and such that $L_0 = 0$ (see, e.g., [4, 25, 26] for detailed information about Lévy processes). In particular cases, when L is a Wiener or Poisson process, any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L. This property of L is called the *chaotic representation property* (CRP), see, e.g., [23] for detailed information. The CRP plays a very important role in the stochastic analysis (in particular, it can be used in order to construct extended stochastic integrals [14, 29, 13], stochastic derivatives and operators of stochastic differentiation, e.g., [32, 1]), but, unfortunately, for a general Lévy process this property does not hold (e.g., [31]).

There are different generalizations of the CRP for Lévy processes: one can use the Itô's approach [12], the Nualart-Schoutens' approach [24, 27], the Lytvynov's approach [22], the Oksendal's approach [6, 5] etc. The interconnection between these generalizations of the CRP is described in, e.g., [22, 2, 6, 30, 5, 21].

Let from now L be a Lévy process without Gaussian part and drift (it is comparatively simply to consider such processes from technical point of view). In the paper [21] the extended

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Skorohod stochastic integral with respect to L and the corresponding Hida stochastic derivative, in terms of the Lytvynov's generalization of the CRP, on the space of square integrable random variables (L^2) were constructed; some properties of these operators were established; and it was shown that the extended stochastic integrals constructed with use of the above-mentioned generalizations of the CRP coincide. In the papers [19, 8] the stochastic integral and derivative were extended to spaces of test and generalized functions from riggings of (L^2), this gives a possibility to extend an area of their possible applications (in particular, now it is possible to define the stochastic integral and derivative as linear *continuous* operators). But together with the mentioned operators, it is natural to introduce and to study *operators of stochastic differentiation* in the Lévy white noise analysis, by analogy with the Gaussian analysis [32, 1], the Gamma-analysis [15, 16], and the Meixner analysis [17, 18]. These operators are closely related with the extended Skorohod stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative and, by analogy with the "classical case", can be used, in particular, in order to study properties of the extended stochastic integral and properties of solutions of normally ordered stochastic equations (stochastic equations with Wick-type nonlinearities in another terminology). So, the aims of the present paper are to introduce the operators of stochastic differentiation on spaces of the so-called regular parametrized rigging of (L^2) (e.g., [19, 8, 7]) and to study some properties of these operators. In the next papers we'll consider elements of the so-called Wick calculus in the Lévy white noise analysis, this will give us the possibility to continue the study of properties and applications of the mentioned operators. Note that some results of the present paper were announced without detailed proofs in the short paper [7].

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L; then, following [21, 19], we describe in details the Lytvynov's generalization of the CRP, the extended stochastic integral with respect to L, and the corresponding Hida stochastic derivative, on the spaces of the regular parametrized rigging of (L^2). In the second section we deal with the operators of stochastic differentiation.

1 PRELIMINARIES

1.1 Lévy processes

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t \in \mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is well known (e.g., [6]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp\left[t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x)\nu(dx)\right], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra, $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}} x^2 e^{\varepsilon |x|}\nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2 \nu(dx) = 1. \tag{2}$$

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [3]). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$. The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of spaces.

Definition 1. *A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x)du\nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{3}$$

is called the measure of a Lévy white noise.

The existence of μ from the Bochner–Minlos theorem (e.g., [11]) follows. Below we will reckon that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is complete with respect to μ*, i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) real-valued square integrable with respect to μ functions on $\mathcal{D}'$; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (3) $\varphi = t\psi$, $t \in \mathbb{R}$, $\psi \in \mathcal{D}$, and using the Taylor decomposition by t and (2), one can show that

$$\int_{\mathcal{D}'} \langle \omega, \psi \rangle^2 \mu(d\omega) = \int_{\mathbb{R}_+} \big(\psi(u)\big)^2 du \tag{4}$$

(this statement follows also from results of [22] and [6]). Let $f \in \mathcal{H}$ and $\mathcal{D} \ni \varphi_k \to f$ in $\mathcal{H}$ as $k \to \infty$. It follows from (4) that $\{\langle \circ, \varphi_k \rangle\}_{k \geq 1}$ is a Cauchy sequence in (L^2), therefore one can define $\langle \circ, f \rangle := (L^2) - \lim_{k \to \infty} \langle \circ, \varphi_k \rangle$. It is easy to show (by the method of "mixed sequences") that $\langle \circ, f \rangle$ does not depend on a choice of an approximating sequence for f and therefore is well defined in (L^2).

Let us consider $\langle \circ, 1_{[0,t)} \rangle \in (L^2)$, $t \in \mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1) and (3) that $\big(\langle \circ, 1_{[0,t)} \rangle\big)_{t \in \mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$, i.e., one can write $L_t = \langle \circ, 1_{[0,t)} \rangle \in (L^2)$.

Remark 1. *Note that one can understand the Lévy white noise as a generalized random process (in the sense of [9]) with trajectories from $\mathcal{D}'$: formally $L'_t(\omega) = \langle \omega, 1_{[0,t)} \rangle' = \langle \omega, \delta_t \rangle = \omega(t)$, where δ_t is the Dirac delta-function concentrated at t. Therefore μ is the measure of L' in the classical sense of this notion [10].*

1.2 Lytvynov's generalization of the CRP

Denote by $\widehat{\otimes}$ a symmetric tensor product and set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of continuous polynomials on $\mathcal{D}'$, i.e., $\mathcal{P}$ consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \ N_f \in \mathbb{Z}_+, \ f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, \ f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}^{\widehat{\otimes} 0} := \mathbb{R}$. Since the measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [22]), $\mathcal{P}$ is a dense set in (L^2) [28]. Denote by $\mathcal{P}_n$ the set of continuous polynomials of power $\le n$, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)} \rangle:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto $\mathbf{P}_n$. Let us define scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)} \rangle::\langle \omega^{\otimes n}, g^{(n)} \rangle: \mu(d\omega),$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{(f^{(n)}, f^{(n)})_{ext}}$. Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)} \rangle: \stackrel{\text{def}}{=} (L^2) - \lim_{k\to\infty} :\langle \circ^{\otimes n}, f_k^{(n)} \rangle:$, where $\mathcal{D}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$ (well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)} \rangle: | f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}\}$ is a dense one in $\mathbf{P}_n$, $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)} \rangle: \tag{5}$$

and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{ext}^2 < \infty.$$

So, for $F, G \in (L^2)$ the scalar product has the form

$$(F,G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega)G(\omega)\mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n!(F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}\big(:\langle \circ^{\otimes n}, F^{(n)} \rangle:, :\langle \circ^{\otimes m}, G^{(m)} \rangle:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, F^{(n)} \rangle::\langle \omega^{\otimes m}, G^{(m)} \rangle: \mu(d\omega) \\ &= \mathbb{E}\big[:\langle \circ^{\otimes n}, F^{(n)} \rangle::\langle \circ^{\otimes m}, G^{(m)} \rangle:\big] = \delta_{n,m} n! (F^{(n)}, G^{(n)})_{ext}.\end{aligned}$$

Also we note that in the space (L^2) $:\langle \circ^{\otimes 0}, F^{(0)} \rangle: = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $:\langle \circ, F^{(1)} \rangle: = \langle \circ, F^{(1)} \rangle$ [22].

In order to work with spaces $\mathcal{H}_{ext}^{(n)}$, it is necessary to know the explicit formulas for the scalar products $(\cdot,\cdot)_{ext}$. Let us write out these formulas. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R}, \nu)$ of (classes of) square integrable with respect to ν real-valued functions on $\mathbb{R}$. Let

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R},\ j \in \{1, \ldots, n-1\},\ n \in \mathbb{N}, \tag{6}$$

be orthogonal in $L^2(\mathbb{R}, \nu)$ polynomials, i.e., for natural numbers n, m such that $n \ne m$,

$\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then for $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, we have [22]

$$
\begin{aligned}
(F^{(n)}, G^{(n)})_{ext} = & \sum_{\substack{k,l_j,s_j \in \mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k, \\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k} \\
& \times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} F^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k}) \\
& \times G^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})du_1\cdots du_{s_1+\dots+s_k}.
\end{aligned}
\tag{7}
$$

In particular, for $n = 1$ $(F^{(1)}, G^{(1)})_{ext} = \|p_1\|_\nu^2\langle F^{(1)}, G^{(1)}\rangle$; if $n = 2$ then $(F^{(2)}, G^{(2)})_{ext} = \|p_1\|_\nu^4\langle F^{(2)}, G^{(2)}\rangle + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} f^{(2)}(u,u)g^{(2)}(u,u)du$, etc.

It follows from (7) that $\mathcal{H}_{ext}^{(1)} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by (6) $p_1(x) = x$ and therefore by (2) $\|p_1\|_\nu = 1$; and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k, j \in \{1,\dots,n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the *space* $\mathcal{H}_{ext}^{(n)}$ *is an extension of* $\mathcal{H}^{\widehat{\otimes} n}$ (this explains why we use the subscript *ext* in the designations $\mathcal{H}_{ext}^{(n)}$, $(\cdot,\cdot)_{ext}$ and $|\cdot|_{ext}$).

1.3 A regular rigging of (L^2)

Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle\circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Accept on default $\beta \in [0,1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0,1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Define scalar products $(\cdot,\cdot)_{q,\beta}$ on $\mathcal{P}_W$ by setting for

$$
f = \sum_{n=0}^{N_f} :\langle\circ^{\otimes n}, f^{(n)}\rangle:,\ g = \sum_{n=0}^{N_g} :\langle\circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W
$$

$$
(f,g)_{q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta}2^{qn}(f^{(n)}, g^{(n)})_{ext}.
$$

Let $\|\cdot\|_{q,\beta}$ be the corresponding norms, i.e., $\|f\|_{q,\beta} = \sqrt{(f,f)_{q,\beta}}$.

Definition 2. *We define parametrized Kondratiev-type spaces of test functions* $(L^2)_q^\beta$ *as completions of* $\mathcal{P}_W$ *with respect to the norms* $\|\cdot\|_{q,\beta}$*; and set* $(L^2)^\beta := \mathrm{pr}\lim_{q\to+\infty}(L^2)_q^\beta$ *(the projective limit of spaces).*

As is easy to see, $F \in (L^2)_q^\beta$ if and only if F can be presented in form (5) with $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and

$$
\|F\|_{q,\beta}^2 := \|F\|_{(L^2)_q^\beta}^2 = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}|F^{(n)}|_{ext}^2 < \infty. \tag{8}
$$

For $F, G \in (L^2)_q^\beta$ the scalar product has a form $(F,G)_{(L^2)_q^\beta} = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}(F^{(n)}, G^{(n)})_{ext}$, where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G correspondingly. Further, $F \in (L^2)^\beta$ if and only if F can be presented in form (5) and norm (8) is finite for each $q \in \mathbb{Z}_+$.

Proposition 1 ([19]). *For any* $\beta \in (0,1]$ *and* $q \in \mathbb{Z}$ *(in the same way as for* $\beta = 0$ *and any* $q \in \mathbb{Z}_+$*) the space* $(L^2)_q^\beta$ *is densely and continuously embedded into* (L^2).

In view of this proposition, one can consider a chain

$$(L^2)^{-\beta} \supset (L^2)^{-\beta}_{-q} \supset (L^2) \supset (L^2)^{\beta}_{q} \supset (L^2)^{\beta}, \tag{9}$$

where $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta} = \operatorname{ind}\lim_{q\to+\infty}(L^2)^{-\beta}_{-q}$ (the inductive limit of spaces) are the spaces dual of $(L^2)^{\beta}_{q}$, $(L^2)^{\beta}$ correspondingly with respect to (L^2).

Definition 3. *The spaces $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta}$ are called parametrized Kondratiev-type spaces of regular generalized functions.*

The next statement from the definition of the spaces $(L^2)^{-\beta}_{-q}$ and the general duality theory follows.

Proposition 2. *1) Any regular generalized function $F \in (L^2)^{-\beta}_{-q}$ can be presented as formal series* (5) *(with coefficients $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$) that converges in $(L^2)^{-\beta}_{-q}$, i.e.,*

$$\|F\|^2_{-q,-\beta} := \|F\|^2_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}|F^{(n)}|^2_{ext} < \infty, \tag{10}$$

and, vice versa, any formal series (5) *with finite norm* (10) *is a regular generalized function from $(L^2)^{-\beta}_{-q}$;*

2) for $F, G \in (L^2)^{-\beta}_{-q}$ the scalar product has a form

$$(F,G)_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}(F^{(n)},G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (5) *for F and G respectively;*

3) the dual pairing between $F \in (L^2)^{-\beta}_{-q}$ and $f \in (L^2)^{\beta}_{q}$ that is generated by the scalar product in (L^2), has a form

$$\langle\langle F,f\rangle\rangle_{(L^2)} = \sum_{n=0}^{\infty} n!(F^{(n)},f^{(n)})_{ext}, \tag{11}$$

where $F^{(n)}, f^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (5) *for F and f respectively;*

4) $F \in (L^2)^{-\beta}$ if and only if F can be presented in form (5) *and norm* (10) *is finite for some $q \in \mathbb{Z}_+$.*

Remark 2. *We use the term "regular generalized functions" for elements of $(L^2)^{-\beta}_{-q}$ and of $(L^2)^{-\beta}$ because the kernels from decompositions* (5) *of these elements and the kernels from decompositions* (5) *of test functions belong to the same spaces.*

In what follows, it will be convenient to denote the spaces $(L^2)^{\beta}_{q}$, $(L^2) = (L^2)^{0}_{0}$, $(L^2)^{-\beta}_{-q}$ from chain (9) by $(L^2)^{\beta}_{q}$, $\beta \in [-1,1]$, $q \in \mathbb{Z}$ (we accept this on default). The norms in these spaces are given, obviously, by formula (8).

1.4 Stochastic integrals and derivatives

Let $F \in (L^2)^{\beta}_{q} \otimes \mathcal{H}$. It follows from representation (5) for elements of $(L^2)^{\beta}_{q}$ that F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)}_{\cdot}\rangle:, \quad F^{(n)}_{\cdot} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}. \tag{12}$$

Let us describe the construction of an extended stochastic integral that is based on this decomposition and correlated with the structure of the spaces $\mathcal{H}_{ext}^{(n)}$ (a detailed presentation is given in [21, 19]; in the case when L is a process of Meixner type (e.g., [22]), such an integral is constructed and studied in [20]).

Let $F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}_\cdot^{(n)} \in F_\cdot^{(n)}$ such that

$$\dot{f}_u^{(n)}(u_1, \ldots, u_n) = 0 \text{ if for some k} \in \{1, \ldots, \text{n}\}\ u = u_k. \tag{13}$$

Accept on default $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. Let $\widehat{f}_{[t_1,t_2)}^{(n)}$ be the symmetrization of a function $\dot{f}_\cdot^{(n)} 1_{[t_1,t_2)}(\cdot)$ by $n+1$ variables. Define $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{ext}^{(n+1)}$ generated by $\widehat{f}_{[t_1,t_2)}^{(n)}$ (i.e., $\widehat{f}_{[t_1,t_2)}^{(n)} \in \widehat{F}_{[t_1,t_2)}^{(n)}$).

Lemma 1 ([19, 21]). *For each $F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$, the element $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ is well defined (in particular, $\widehat{F}_{[t_1,t_2)}^{(n)}$ does not depend on a choice of a representative $\dot{f}_\cdot^{(n)} \in F_\cdot^{(n)}$ satisfying* (13)*) and*

$$|\widehat{F}_{[t_1,t_2)}^{(n)}|_{ext} \le |F_\cdot^{(n)} 1_{[t_1,t_2)}(\cdot)|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}} \le |F_\cdot^{(n)}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}. \tag{14}$$

Definition 4. *We define the extended stochastic integral*

$$\int_{t_1}^{t_2} \circ(u) \widehat{d}L_u : (L^2)_q^\beta \otimes \mathcal{H} \to (L^2)_{q-1}^\beta \tag{15}$$

by the formula

$$\int_{t_1}^{t_2} F(u) \widehat{d}L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}_{[t_1,t_2)}^{(n)} \rangle :, \tag{16}$$

where $\widehat{F}_{[t_1,t_2)}^{(0)} := F_\cdot^{(0)} 1_{[t_1,t_2)}(\cdot) \in \mathcal{H} = \mathcal{H}_{ext}^{(1)}$, and $\widehat{F}_{[t_1,t_2)}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{N}$, are constructed by the kernels $F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ from decomposition (12) *for F.*

As it is shown in [19, 8], this integral is a linear *continuous* operator. Moreover, if F is integrable by Itô (i.e., $F \in (L^2) \otimes \mathcal{H}$ and is adapted with respect to the flow of σ-algebras generated by the Lévy process L) then F is integrable in the extended sense and $\int_{t_1}^{t_2} F(u) \widehat{d}L_u = \int_{t_1}^{t_2} F(u) dL_u \in (L^2)$, where $\int_{t_1}^{t_2} F(u) dL_u$ is the Itô stochastic integral [21] (this explains why the integral $\int_{t_1}^{t_2} \circ(u) \widehat{d}L_u$ is called the *extended* one).

Sometimes it can be convenient to define the extended stochastic integral by formula (16) as a linear operator

$$\int_{t_1}^{t_2} \circ(u) \widehat{d}L_u : (L^2)_q^\beta \otimes \mathcal{H} \to (L^2)_q^\beta. \tag{17}$$

If $\beta = -1$ then this operator is continuous [19], for $\beta \in (-1, 1]$ this is not the case. But if we accept the set

$$\Big\{ F \in (L^2)_q^\beta \otimes \mathcal{H} : \Big\| \int_{t_1}^{t_2} F(u) \widehat{d}L_u \Big\|_{q,\beta}^2 = \sum_{n=0}^{\infty} ((n+1)!)^{1+\beta} 2^{q(n+1)} |\widehat{F}_{[t_1,t_2)}^{(n)}|_{ext}^2 < \infty \Big\}$$

as the domain of integral (17) then the last will be a *closed* operator [19, 8]. Also we note that the extended stochastic integral can be defined by formula (16) as a linear continuous operator acting from $(L^2)^\beta \otimes \mathcal{H} := \operatorname{pr} \lim_{q \to +\infty} (L^2)_q^\beta \otimes \mathcal{H}$ to $(L^2)^\beta$, or from $(L^2)^{-\beta} \otimes \mathcal{H} := \operatorname{ind} \lim_{q \to +\infty} (L^2)_{-q}^{-\beta} \otimes \mathcal{H}$ to $(L^2)^{-\beta}$, here $\beta \in [0, 1]$.

At last, we recall briefly the notion of a Hida stochastic derivative in the Lévy white noise analysis in terms of the Lytvynov's CRP ([21, 19, 8]).

Definition 5. *We define a Hida stochastic derivative* $1_{[t_1,t_2)}(\cdot)\partial_\cdot : (L^2)^{-\beta}_{1-q} \to (L^2)^{-\beta}_{-q} \otimes \mathcal{H}$ *as a linear continuous operator adjoint to extended stochastic integral* (15)*, i.e., for all* $F \in (L^2)^{\beta}_{q} \otimes \mathcal{H}$ *and* $G \in (L^2)^{-\beta}_{1-q}$

$$\langle\langle F(\cdot), 1_{[t_1,t_2)}(\cdot)\partial_\cdot G\rangle\rangle_{(L^2)\otimes\mathcal{H}} = \langle\langle \int_{t_1}^{t_2} F(u)\widehat{d}L_u, G\rangle\rangle_{(L^2)},$$

here $\langle\langle\cdot,\cdot\rangle\rangle_{(L^2)\otimes\mathcal{H}}$ *denotes the dual pairing generated by the scalar product in* $(L^2)\otimes\mathcal{H}$.

If instead of integral (15) one uses integral (17), the corresponding Hida stochastic derivative will be a linear unbounded (except the case $\beta = -1$), but closed operator acting from $(L^2)^{-\beta}_{-q}$ to $(L^2)^{-\beta}_{-q} \otimes \mathcal{H}$ [8]. It is clear also that the Hida stochastic derivative can be defined as a linear continuous operator acting from $(L^2)^{\beta}$ to $(L^2)^{\beta} \otimes \mathcal{H}$ ($\beta \in [-1, 1]$) that is adjoint to the corresponding extended stochastic integral.

In order to write out an explicit formula for the Hida stochastic derivative in terms of decompositions by the Wick monomials, we need some preparation. Let $G^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in G^{(n)}$ be a representative of $G^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate one argument of $\dot{g}^{(n)}$, and define $G^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ as the equivalence class in $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ generated by $\dot{g}^{(n)}(\cdot)$ (i.e., $\dot{g}^{(n)}(\cdot) \in G^{(n)}(\cdot)$).

Lemma 2 ([21]). *For each* $G^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}$, *the element* $G^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ *is well defined (in particular,* $G^{(n)}(\cdot)$ *does not depend on a choice of a representative* $\dot{g}^{(n)} \in G^{(n)}$*) and*

$$|G^{(n)}(\cdot)|_{\mathcal{H}^{(n-1)}_{ext}\otimes\mathcal{H}} \le |G^{(n)}|_{ext}. \tag{18}$$

Note that, in spite of estimate (18), the space $\mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{N}\backslash\{1\}$, can not be considered as a subspace of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$ because different elements of $\mathcal{H}^{(n)}_{ext}$ can coincide as elements of $\mathcal{H}^{(n-1)}_{ext} \otimes \mathcal{H}$.

The next statement easily follows from results of [21, 19, 8].

Proposition 3. *For a test or square integrable or generalized function* G *of form* (5)

$$1_{[t_1,t_2)}(\cdot)\partial_\cdot G = \sum_{n=1}^{\infty} n{:}\langle \circ^{\otimes n-1}, G^{(n)}(\cdot)1_{[t_1,t_2)}(\cdot)\rangle{:} \equiv \sum_{n=0}^{\infty}(n+1){:}\langle \circ^{\otimes n}, G^{(n+1)}(\cdot)1_{[t_1,t_2)}(\cdot)\rangle{:}.$$

Finally, we note that the extended stochastic integral and the Hida stochastic derivative are mutually adjoint operators [21, 19, 8].

2 OPERATORS OF STOCHASTIC DIFFERENTIATION

2.1 The case of bounded operators

In order to define operators of stochastic differentiation on spaces $(L^2)^{\beta}_{q}$, we need some preparation. Let $n, m \in \mathbb{Z}_+$. Consider a function $H : \mathbb{R}^{n+m}_+ \to \mathbb{R}$. Denote

$$\widetilde{H}(u_1, \ldots, u_n; u_{n+1}, \ldots, u_{n+m}) := \begin{cases} H(u_1, \ldots, u_{n+m}), & \text{if for all } i \in \{1, \ldots, n\}, j \in \{n+1, \ldots, n+m\}\ u_i \ne u_j, \\ 0, & \text{in other cases.} \end{cases}$$

Let $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$. We select representatives (functions) $\dot{f}^{(n)} \in F^{(n)}$, $\dot{g}^{(m)} \in G^{(m)}$ from the equivalence classes $F^{(n)}$, $G^{(m)}$, and set $\widetilde{f^{(n)}g^{(m)}} := \dot{f}^{(n)} \cdot \dot{g}^{(m)}$. Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of $\widetilde{f^{(n)}g^{(m)}}$ by all variables, $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ be the equivalence class in $\mathcal{H}_{ext}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$ (i.e., $\widehat{f^{(n)}g^{(m)}} \in F^{(n)} \diamond G^{(m)}$). The next statement in a sense is a generalization of Lemma 1.

Lemma 3. *The element $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ is well defined (in particular, $F^{(n)} \diamond G^{(m)}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}$) and*

$$|F^{(n)} \diamond G^{(m)}|_{ext} \leq |F^{(n)}|_{ext}|G^{(m)}|_{ext}. \tag{19}$$

Remark 3. *Not strictly speaking, $F^{(n)} \diamond G^{(m)}$ is the symmetrization of a function*

$$\begin{cases} F^{(n)}(u_1, \dots, u_n)G^{(m)}(u_{n+1}, \dots, u_{n+m}), & \textit{if} \, \forall i \in \{1, \dots, n\}, j \in \{n+1, \dots, n+m\} u_i \neq u_j, \\ 0, & \textit{in other cases} \end{cases}$$

by all arguments.

Proof. For $n = 0$ or $m = 0$ the statement of the lemma is, obviously, true. Let $n, m \in \mathbb{N}$, $\dot{f}^{(n)} \in F^{(n)}$, $\dot{g}^{(m)} \in G^{(m)}$. It is clear that

$$\widehat{f^{(n)}g^{(m)}}(u_1, \dots, u_{n+m}) = \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} \widetilde{f^{(n)}g^{(m)}}(u_{\pi(1)}, \dots, u_{\pi(n+m)}), \tag{20}$$

where S_{n+m} is a family of all permutations of numbers $1, \dots, n+m$. Without loss of generality we can think that $\dot{f}^{(n)}$, $\dot{g}^{(m)}$ are symmetric functions and $m \geq n$. For each collection of arguments $u_{\pi(1)}, \dots, u_{\pi(n)}$ we consider all summands from sum (20) with such a collection (it is clear that there are $m!$ such summands). Taking into consideration the symmetric property of $\dot{g}^{(m)}$, one can conclude that all these summands are equal inter se, therefore one can replace the mentioned summands by a representative multiplied by $m!$. After it one can use by analogy the symmetric property of $\dot{f}^{(n)}$ and rewrite equality (20) in the form

$$\begin{aligned} &\widehat{f^{(n)}g^{(m)}}(u_1, \dots, u_{n+m}) \\ &\quad = \frac{n!m!}{(n+m)!} \sum_{\substack{1 \leq p_1, \dots, p_n \leq n, n+1 \leq q_1, \dots, q_m \leq n+m, 0 \leq r \leq n \\ p_1 < \dots < p_r, p_{r+1} < \dots < p_n, q_1 < \dots < q_{n-r}, q_{n-r+1} < \dots < q_m}} \widetilde{f^{(n)}g^{(m)}}(u_{p_1}, \dots, u_{p_r}, u_{q_1}, \dots, u_{q_{n-r}}; \\ &\qquad\qquad u_{p_{r+1}}, \dots, u_{p_n}, u_{q_{n-r+1}}, \dots, u_{q_m}) \end{aligned} \tag{21}$$

(here for $r = n$ the argument in the right hand side of (21) is $(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m})$; for $r = 0$ this argument is $(u_{q_1}, \dots, u_{q_n}; u_1, \dots, u_n, u_{q_{n+1}}, \dots, u_{q_m})$). To put it in another way, the arguments of $\widetilde{f^{(n)}g^{(m)}}$ in sum (21) are u_j, $j \in \{1, \dots, n+m\}$, where the indexes of n first and m last arguments (before and after ';') are (independently) ordered in ascending. (Note that we selected arrangement in ascending when we used the symmetric property of $\dot{f}^{(n)}$ and $\dot{g}^{(m)}$ because this is convenient for a consequent calculation.)

Let us estimate $|\widehat{f^{(n)}g^{(m)}}|_{ext}$. Substituting (21) in the expression for $|\cdot|_{ext}$ (see (7)) we obtain

$$|\widehat{f^{(n)}g^{(m)}}|_{ext}^2 = \sum_{\substack{k,l_j,s_j\in\mathbb{N}:j=1,\dots,k,l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{(n+m)!}{s_1!\dots s_k!}\left(\frac{\|p_{l_1}\|_\nu}{l_1!}\right)^{2s_1}\dots\left(\frac{\|p_{l_k}\|_\nu}{l_k!}\right)^{2s_k}$$

$$\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}|\widehat{f^{(n)}g^{(m)}}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})|^2du_1\dots du_{s_1+\dots+s_k}$$

$$\begin{aligned}
&\le \sum_{\substack{k,l_j,s_j\in\mathbb{N}:j=1,\dots,k,l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{(n+m)!}{s_1!\dots s_k!}\left(\frac{\|p_{l_1}\|_\nu}{l_1!}\right)^{2s_1}\dots\left(\frac{\|p_{l_k}\|_\nu}{l_k!}\right)^{2s_k}\left(\frac{n!m!}{(n+m)!}\right)^2\frac{(n+m)!}{n!m!}\\
&\times\left[\int_{\mathbb{R}_+^{s_1+\dots+s_k}}|\widetilde{f^{(n)}g^{(m)}}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})|^2du_1\dots du_{s_1+\dots+s_k}+\cdots\right]\\
&= \sum_{\substack{k,l_j,s_j\in\mathbb{N}:j=1,\dots,k,l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n+m}} \frac{n!m!}{s_1!\dots s_k!}\left(\frac{\|p_{l_1}\|_\nu}{l_1!}\right)^{2s_1}\dots\left(\frac{\|p_{l_k}\|_\nu}{l_k!}\right)^{2s_k}\\
&\times\left[\int_{\mathbb{R}_+^{s_1+\dots+s_k}}|\widetilde{f^{(n)}g^{(m)}}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})|^2du_1\dots du_{s_1+\dots+s_k}+\cdots\right]
\end{aligned} \tag{22}$$

(here we used the inequality $|\sum_{l=1}^p a_l|^2 \le p\sum_{l=1}^p|a_l|^2$ and the fact that the sum in the right hand side of (21) contains $\frac{(n+m)!}{n!m!}$ summands). We say that a collection of equal inter se arguments (e.g., $(\underbrace{u_1,\dots,u_1}_{l_1})$) is called a *procession*. It follows from the ordering in ascending of indexes in (21) and in the statement for $|\cdot|_{ext}$ (see (22)) that in summands in interior sums $[\cdots]$ from (22) processions can "tear" only so that different parts of a "torn" procession will be for different parties from ';'; processions being for one side from ';' do not switch places; and elements in processions do not switch places. Further, it follows from a construction of $\widetilde{f^{(n)}g^{(m)}}$ that summands in interior sums $[\cdots]$ from (22), in which a procession is divided by ';', are equal to zero. Another summands (if there exist for a collection k, l_j, s_j) disintegrate on groups of equal inter se integrals. These groups arise by means of transpositions of processions with equal quantity of members, which are placed before ';' and after ';'. It is clear that if there are s' processions of length l before ';' and s'' processions of length l after ';' then by means of mutual transpositions of these processions one can obtain $\frac{(s'+s'')!}{s'!s''!}$ equal inter se summands.

So, nonzero summands in the last sum from (22) are "connected" with the expressions

$$l_1s_1+\dots+l_ks_k=n+m \tag{23}$$

that can be presented in the form

$$\begin{aligned}
&l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m,\\
&k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N},\\
&l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''}
\end{aligned} \tag{24}$$

(the first sum in (24) corresponds to first n arguments of $\widetilde{f^{(n)}g^{(m)}}$, the second one corresponds to last m arguments). Now for each s_j from (23) either there exists $s'_i=s_j$ ($l'_i=l_j$) or there exists

$s''_i = s_j$ ($l''_i = l_j$) or there exist s'_i and s''_w such that $s'_i + s''_w = s_j$ ($l'_i = l''_w = l_j$). Inequalities for $l'_\cdot$, $l''_\cdot$ in (24) follow from inequalities $l_1 > \cdots > l_k$ and ordering of indexes in (21) (more long processions have smaller indexes of arguments).

We will replace each group of equal inter se summands in the last expression from (22) by a representative multiplied by a quantity of summands in the group. Now, taking into account that $w^{s'+s''} = w^{s'}w^{s''}$, one can rewrite the last expression from (22) in the form

$$\begin{aligned}
&\sum_{\substack{k',k'',l'_1,\dots,l'_{k'},s'_1,\dots,s'_{k'},l''_1,\dots,l''_{k''},s''_1,\dots,s''_{k''}\in\mathbb{N},\\ l'_1>\dots>l'_{k'},\ l''_1>\dots>l''_{k''},\\ l'_1s'_1+\dots+l'_{k'}s'_{k'}=n,\ l''_1s''_1+\dots+l''_{k''}s''_{k''}=m}} \frac{n!m!}{s'_1!\dots s'_{k'}!s''_1!\dots s''_{k''}!}\\
&\times \left(\frac{\|p_{l'_1}\|_\nu}{l'_1!}\right)^{2s'_1}\dots\left(\frac{\|p_{l'_{k'}}\|_\nu}{l'_{k'}!}\right)^{2s'_{k'}}\left(\frac{\|p_{l''_1}\|_\nu}{l''_1!}\right)^{2s''_1}\dots\left(\frac{\|p_{l''_{k''}}\|_\nu}{l''_{k''}!}\right)^{2s''_{k''}}\\
&\times\int_{\mathbb{R}_+^{s'_1+\dots+s'_{k'}+s''_1+\dots+s''_{k''}}} |\widetilde{f^{(n)}g^{(m)}}(\underbrace{u_1,\dots,u_1}_{l'_1},\dots,\underbrace{u_{s'_1+\dots+s'_{k'}},\dots,u_{s'_1+\dots+s'_{k'}}}_{l'_{k'}};\\
&\underbrace{u_{n+1},\dots,u_{n+1}}_{l''_1},\dots,\underbrace{u_{n+s''_1+\dots+s''_{k''}},\dots,u_{n+s''_1+\dots+s''_{k''}}}_{l''_{k''}})|^2\\
&\times du_1\dots du_{s'_1+\dots+s'_{k'}}du_{n+1}\dots du_{n+s''_1+\dots+s''_{k''}}.
\end{aligned} \tag{25}$$

Since the Lebesgue measure is a non-atomic one, we can replace $\widetilde{f^{(n)}g^{(m)}}$ in this expression by $\dot f^{(n)}\dot g^{(m)}$, therefore (25) is equal to $|\dot f^{(n)}|^2_{ext}|\dot g^{(m)}|^2_{ext}$, whence

$$|\widehat{f^{(n)}g^{(m)}}|_{ext} \le |\dot f^{(n)}|_{ext}|\dot g^{(m)}|_{ext}. \tag{26}$$

It follows from this inequality that $\widehat{f^{(n)}g^{(m)}}$ generates an element $F^{(n)} \diamond G^{(m)}$ of $\mathcal{H}^{(n+m)}_{ext}$ and estimate (19) is fulfilled.

Let $\dot f^{(n)}_1 \in F^{(n)}$ and $\dot g^{(m)}_1 \in G^{(m)}$ be another representatives of $F^{(n)}$ and $G^{(m)}$, $F^{(n)}_1 \diamond G^{(m)}_1$ be the corresponding element of $\mathcal{H}^{(n+m)}_{ext}$. Using obvious properties of the operation $\widehat{\circ}$ and estimate (26) we obtain

$$\begin{aligned}
|F^{(n)}\diamond G^{(m)} - F^{(n)}_1\diamond G^{(m)}_1|_{ext} &= |\widehat{f^{(n)}g^{(m)}} - \widehat{f^{(n)}_1 g^{(m)}_1}|_{ext}\\
&\le |\widehat{f^{(n)}g^{(m)}} - \widehat{f^{(n)}g^{(m)}_1}|_{ext} + |\widehat{f^{(n)}g^{(m)}_1} - \widehat{f^{(n)}_1 g^{(m)}_1}|_{ext}\\
&= |f^{(n)}\widehat{(g^{(m)} - g^{(m)}_1)}|_{ext} + |\widehat{(f^{(n)} - f^{(n)}_1)}g^{(m)}_1|_{ext}\\
&\le |\dot f^{(n)}|_{ext}|\dot g^{(m)} - \dot g^{(m)}_1|_{ext} + |\dot f^{(n)} - \dot f^{(n)}_1|_{ext}|\dot g^{(m)}_1|_{ext} = 0,
\end{aligned}$$

therefore $F^{(n)} \diamond G^{(m)}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}$. $\square$

Let $F^{(m)} \in \mathcal{H}^{(m)}_{ext}$, $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $m > n$. We define a "product" $(F^{(m)}, f^{(n)})_{ext} \in \mathcal{H}^{(m-n)}_{ext}$ by setting for each $g^{(m-n)} \in \mathcal{H}^{(m-n)}_{ext}$

$$((F^{(m)}, f^{(n)})_{ext}, g^{(m-n)})_{ext} = (F^{(m)}, g^{(m-n)} \diamond f^{(n)})_{ext}. \tag{27}$$

Since (see (19))

$$|(F^{(m)}, g^{(m-n)} \diamond f^{(n)})_{ext}| \le |F^{(m)}|_{ext}|g^{(m-n)} \diamond f^{(n)}|_{ext} \le |F^{(m)}|_{ext}|g^{(m-n)}|_{ext}|f^{(n)}|_{ext},$$

this definition is correct and

$$|(F^{(m)}, f^{(n)})_{ext}|_{ext} \leq |F^{(m)}|_{ext}|f^{(n)}|_{ext}. \tag{28}$$

Definition 6. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$. We define an operator of stochastic differentiation*

$$(D^n\circ)(f^{(n)}) : (L^2)_q^\beta \to (L^2)_{q-1}^\beta \tag{29}$$

by setting

$$\begin{aligned}(D^nF)(f^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, (F^{(m)}, f^{(n)})_{ext}\rangle: \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, (F^{(m+n)}, f^{(n)})_{ext}\rangle:,\end{aligned} \tag{30}$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) *for F. Also we denote $D := D^1$.*

Since (see (8), (28))

$$\begin{aligned}\|(D^nF)(f^{(n)})\|_{q-1,\beta}^2 &= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{(q-1)m} \Big(\frac{(m+n)!}{m!}\Big)^2 |(F^{(m+n)}, f^{(n)})_{ext}|_{ext}^2 \\ &= 2^{-qn} \sum_{m=0}^{\infty} ((m+n)!)^{1+\beta} 2^{q(m+n)} \Big[2^{-m}\Big(\frac{(m+n)!}{m!}\Big)^{1-\beta}\Big] |(F^{(m+n)}, f^{(n)})_{ext}|_{ext}^2 \\ &\leq 2^{-qn}|f^{(n)}|_{ext}^2 c \sum_{m=0}^{\infty} ((m+n)!)^{1+\beta} 2^{q(m+n)} |F^{(m+n)}|_{ext}^2 \leq 2^{-qn}|f^{(n)}|_{ext}^2 c\|F\|_{q,\beta}^2,\end{aligned} \tag{31}$$

where $c = \max_{m\in\mathbb{Z}_+}\Big[2^{-m}\Big(\frac{(m+n)!}{m!}\Big)^{1-\beta}\Big]$, this definition is correct and operator (29) is a linear continuous one. Moreover, for each $F \in (L^2)_q^\beta$ one can understand $(D^nF)(\circ)$ as a linear continuous operator acting from $\mathcal{H}_{ext}^{(n)}$ to $(L^2)_{q-1}^\beta$.

Remark 4. *As is easily seen, for each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, $(D^n\circ)(f^{(n)})$ can be defined by formula* (30) *as a linear continuous operator on $(L^2)^\beta$, $\beta \in [-1, 1]$. In the case $\beta = 1$ formula* (30) *defines a linear continuous operator $(D^n\circ)(f^{(n)})$ on $(L^2)_q^1$, $q \in \mathbb{Z}$, this can be proved by analogy with calculation* (31).

Let us consider main properties of the operators D^n.

Theorem 1. *1) For $k_1, \ldots, k_m \in \mathbb{N}$, $f_j^{(k_j)} \in \mathcal{H}_{ext}^{(k_j)}$, $j \in \{1, \ldots, m\}$,*

$$(D^{k_m}(\cdots(D^{k_2}((D^{k_1}\circ)(f_1^{(k_1)})))(f_2^{(k_2)})\cdots))(f_m^{(k_m)}) = (D^{k_1+\cdots+k_m}\circ)(f_1^{(k_1)} \diamond \cdots \diamond f_m^{(k_m)}).$$

2) For each $F \in (L^2)_q^\beta$ the kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, from decomposition (5) *can be presented in the form*

$$F^{(n)} = \frac{1}{n!}\mathbb{E}(D^nF),$$

i.e., for each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ $(F^{(n)}, f^{(n)})_{ext} = \frac{1}{n!}\mathbb{E}((D^nF)(f^{(n)}))$, here $\mathbb{E}\circ := \langle\!\langle \circ, 1\rangle\!\rangle_{(L^2)}$ is a generalized expectation.

3) The adjoint to D^n operator has the form

$$(D^nG)(f^{(n)})^* = \sum_{m=0}^{\infty} :\langle \circ^{m+n}, G^{(m)} \diamond f^{(n)}\rangle: \in (L^2)_{-q}^{-\beta}, \tag{32}$$

where $G \in (L^2)_{1-q}^{-\beta}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) *for G.*

Proof. 1) The proof consists in the application of the mathematical induction method.

2) Using (30) and (11), for each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ we obtain

$$\mathbb{E}((D^n F)(f^{(n)})) = \langle\langle (D^n F)(f^{(n)}), 1\rangle\rangle_{(L^2)} = n!(F^{(n)}, f^{(n)})_{ext}.$$

3) Let $F \in (L^2)_q^\beta$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G \in (L^2)_{1-q}^{-\beta}$. Using (30), (5), (11) and (27), we obtain

$$\begin{aligned}\langle\langle (D^n F)(f^{(n)}), G\rangle\rangle_{(L^2)} &= \langle\langle \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, (F^{(m+n)}, f^{(n)})_{ext}\rangle :, \sum_{k=0}^{\infty} :\langle \circ^{\otimes k}, G^{(k)}\rangle :\rangle\rangle_{(L^2)} \\ &= \sum_{m=0}^{\infty} (m+n)!((F^{(m+n)}, f^{(n)})_{ext}, G^{(m)})_{ext} = \sum_{m=0}^{\infty} (m+n)!(F^{(m+n)}, G^{(m)} \diamond f^{(n)})_{ext} \\ &= \langle\langle \sum_{k=0}^{\infty} :\langle \circ^{\otimes k}, F^{(k)}\rangle :, \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+n}, G^{(m)} \diamond f^{(n)}\rangle :\rangle\rangle_{(L^2)} = \langle\langle F, (D^n G)(f^{(n)})^*\rangle\rangle_{(L^2)},\end{aligned}$$

whence the result follows. □

Now we consider in more detail the case $n = 1$. Denote $\partial_\cdot := 1_{[0,+\infty)}(\cdot)\partial_\cdot$ (see Subsection 1.4).

Theorem 2. *1) For all $G \in (L^2)_{1-q}^{-\beta}$ and $f^{(1)} \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}$*

$$(DG)(f^{(1)})^* = \int_{\mathbb{R}_+} G \cdot f^{(1)}(u)\widehat{d}L_u \in (L^2)_{-q}^{-\beta}. \tag{33}$$

2) For all $F \in (L^2)_q^\beta$ and $f^{(1)} \in \mathcal{H}_{ext}^{(1)}$

$$(DF)(f^{(1)}) = \int_{\mathbb{R}_+} \partial_u F \cdot f^{(1)}(u)du \in (L^2)_{q-1}^{\beta}, \tag{34}$$

here the integral in the right hand side is a Pettis one (the weak integral).

3) Let $F \in (L^2)_q^\beta \otimes \mathcal{H}$. Then for all $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, and $f^{(1)} \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}$

$$(D\int_{t_1}^{t_2} F(u)\widehat{d}L_u)(f^{(1)}) = \int_{t_1}^{t_2} (DF(u))(f^{(1)})\widehat{d}L_u + \int_{t_1}^{t_2} F(u)f^{(1)}(u)du \in (L^2)_{q-1}^{\beta}, \tag{35}$$

here the last integral is a Pettis one.

Proof. 1) The result follows from representation (32) with $n = 1$: it is necessary to compare the construction of kernels of the extended stochastic integral (see Subsection 1.4) with the construction of a product $\diamond$.

2) Taking into account (33) and the definition of $\partial_\cdot$ (see Subsection 1.4), for all $G \in (L^2)_{1-q}^{-\beta}$ we obtain

$$\begin{aligned}\langle\langle (DF)(f^{(1)}), G\rangle\rangle_{(L^2)} &= \langle\langle F, \int_{\mathbb{R}_+} G \cdot f^{(1)}(u)\widehat{d}L_u\rangle\rangle_{(L^2)} \\ &= \langle\langle \partial_\cdot F, G \otimes f^{(1)}(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}} = \langle\langle \int_{\mathbb{R}_+} \partial_u F \cdot f^{(1)}(u)du, G\rangle\rangle_{(L^2)},\end{aligned}$$

whence the result follows.

3) Using (16) and (30), we obtain

$$(D\int_{t_1}^{t_2} F(u)\widehat{d}L_u)(f^{(1)}) = \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, (\widehat{F}_{[t_1,t_2)}^{(m)}, f^{(1)})_{ext}\rangle :, \tag{36}$$

where $\widehat{F}^{(m)}_{[t_1,t_2)} \in \mathcal{H}^{(m+1)}_{ext}$ are the kernels from decomposition (16) (which is decomposition (5) for the extended stochastic integral $\int_{t_1}^{t_2} F(u)\widehat{d}L_u$), these kernels are constructed by $F^{(m)}_{\cdot} \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}$ from decomposition (12) for F. On the other hand, by (30), (16) and (12)

$$\int_{t_1}^{t_2} (DF(u))(f^{(1)})\widehat{d}L_u = \sum_{m=1}^{\infty} m :\langle \circ^{\otimes m}, \widehat{(F^{(m)}_{\cdot}, f^{(1)})}_{ext\,[t_1,t_2)}\rangle :,$$

$$\int_{t_1}^{t_2} F(u)f^{(1)}(u)du = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \int_{t_1}^{t_2} F^{(m)}_u f^{(1)}(u)du\rangle :,$$

where the integrals $\int_{t_1}^{t_2} F^{(m)}_u f^{(1)}(u)du \in \mathcal{H}^{(m)}_{ext}$ are Pettis ones. Therefore, in order to prove equality (35) it is sufficient to show that for each $m \in \mathbb{Z}_+$

$$(m+1)(\widehat{F}^{(m)}_{[t_1,t_2)}, f^{(1)})_{ext} = m\widehat{(F^{(m)}_{\cdot}, f^{(1)})}_{ext\,[t_1,t_2)} + \int_{t_1}^{t_2} F^{(m)}_u f^{(1)}(u)du$$

in $\mathcal{H}^{(m)}_{ext}$. In turn, in order to prove this equality, it is sufficient to show that for each $g^{(m)} \in \mathcal{H}^{(m)}_{ext}$

$$\begin{aligned}(m+1)((\widehat{F}^{(m)}_{[t_1,t_2)}, f^{(1)})_{ext}, g^{(m)})_{ext} \\ = m(\widehat{(F^{(m)}_{\cdot}, f^{(1)})}_{ext\,[t_1,t_2)}, g^{(m)})_{ext} + \Big(\int_{t_1}^{t_2} F^{(m)}_u f^{(1)}(u)du, g^{(m)}\Big)_{ext}.\end{aligned} \tag{37}$$

Using (27), the equality $(\widehat{F}^{(m)}_{[t_1,t_2)}, h^{(m+1)})_{ext} = \int_{t_1}^{t_2} (F^{(m)}_u, h^{(m+1)}(u))_{ext}du$, $h^{(m+1)} \in \mathcal{H}^{(m+1)}_{ext}$, $h^{(m+1)}(\cdot) \in \mathcal{H}^{(m)}_{ext} \otimes \mathcal{H}$ (see Lemma 2), which is proved in [21], the symmetry of $g^{(m)}$, and the non-atomicity of the Lebesgue measure, we obtain

$$\begin{aligned}(m+1)((\widehat{F}^{(m)}_{[t_1,t_2)}, f^{(1)})_{ext}, g^{(m)})_{ext} &= (m+1)(\widehat{F}^{(m)}_{[t_1,t_2)}, g^{(m)} \diamond f^{(1)})_{ext} \\ = (m+1)\int_{t_1}^{t_2} \big(F^{(m)}_u, (g^{(m)} \diamond f^{(1)})(u)\big)_{ext}du &= \int_{t_1}^{t_2} \big(F^{(m)}_u, g^{(m)}(\cdot_1, \widetilde{\cdots, \cdot_m})f^{(1)}(u) \\ + g^{(m)}(\cdot_2, \widetilde{\cdots, u})f^{(1)}(\cdot_1) + \cdots + g^{(m)}(u, \widetilde{\cdots, \cdot_{m-1}})f^{(1)}(\cdot_m)\big)_{ext}du \\ = \Big(\int_{t_1}^{t_2} F^{(m)}_u f^{(1)}(u)du, g^{(m)}\Big)_{ext} \\ + \int_{t_1}^{t_2} \big(F^{(m)}_u, g^{(m)}(\cdot_2, \widetilde{\cdots, u})f^{(1)}(\cdot_1) + \cdots + g^{(m)}(u, \widetilde{\cdots, \cdot_{m-1}})f^{(1)}(\cdot_m)\big)_{ext}du.\end{aligned} \tag{38}$$

On the other hand, by analogy with (38) we obtain

$$\begin{aligned}m(\widehat{(F^{(m)}_{\cdot}, f^{(1)})}_{ext\,[t_1,t_2)}, g^{(m)})_{ext} &= m\int_{t_1}^{t_2} ((F^{(m)}_u, f^{(1)})_{ext}, g^{(m)}(u))_{ext}du \\ = m\int_{t_1}^{t_2} (F^{(m)}_u, (g^{(m)}(u)) \diamond f^{(1)})_{ext}du &= \int_{t_1}^{t_2} \big(F^{(m)}_u, g^{(m)}(\cdot_2, \widetilde{\cdots, u})f^{(1)}(\cdot_1) \\ + g^{(m)}(\cdot_3, \widetilde{\cdots, u, \cdot_1})f^{(1)}(\cdot_2) + \cdots + g^{(m)}(u, \widetilde{\cdots, \cdot_{m-1}})f^{(1)}(\cdot_m)\big)_{ext}du.\end{aligned} \tag{39}$$

Substituting (39) into (38), we obtain (37).

Now it remains to prove that $(D\int_{t_1}^{t_2} F(u)\widehat{d}L_u)(f^{(1)}) \in (L^2)^{\beta}_{q-1}$ if $F \in (L^2)^{\beta}_q \otimes \mathcal{H}$ (it follows directly from the definitions of the extended stochastic integral and of the operators of stochastic differentiation that $(D\int_{t_1}^{t_2} F(u)\widehat{d}L_u)(f^{(1)}) \in (L^2)^{\beta}_{q-2}$, but this statement can be amplified).

In fact, by (36), (8), (28) and (14)

$$\Big\|(D\int_{t_1}^{t_2}F(u)\widehat{d}L_u)(f^{(1)})\Big\|^2_{q-1,\beta}=\sum_{m=0}^{\infty}(m!)^{1+\beta}2^{(q-1)m}(m+1)^2|(\widehat{F}^{(m)}_{[t_1,t_2)},f^{(1)})_{ext}|^2_{ext}$$
$$\leq\sum_{m=0}^{\infty}(m!)^{1+\beta}2^{qm}[2^{-m}(m+1)^2]|\widehat{F}^{(m)}_{[t_1,t_2)}|^2_{ext}|f^{(1)}|^2_{ext}$$
$$\leq|f^{(1)}|^2_{ext}c\sum_{m=0}^{\infty}(m!)^{1+\beta}2^{qm}|F^{(m)}_{\cdot}|^2_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}}=|f^{(1)}|^2_{ext}c\|F\|^2_{(L^2)^{\beta}_q\otimes\mathcal{H}}<\infty,$$

where $c=\max_{m\in\mathbb{Z}_+}[2^{-m}(m+1)^2]$. ☐

Remark 5. *Taking into account equality* (34)*, one can write formally* $\partial_{\cdot}\circ=(D\circ)(\delta_{\cdot})$*, where* $\delta_{\cdot}$ *is the Dirac delta-function concentrated at* $\cdot$*. In order to give a nonformal sense to this equality, one can consider a stochastic differentiation on so-called spaces of nonregular generalized functions, it will be done in another paper.*

As is easily seen, the results of Theorems 1, 2 hold true (up to obvious modifications) if we consider the operators of stochastic differentiation on the spaces $(L^2)^{\beta}$, $\beta\in[-1,1]$.

Remark 6. *As is known ([17, 18]), in the Meixner white noise analysis the operator of stochastic differentiation* D *is a differentiation with respect to a Wick product. In the Lévy white noise analysis this result holds true, the detailed presentation will be given in another paper.*

2.2 The case of unbounded operators

Sometimes it can be useful to consider $(D^n\circ)(f^{(n)})$, $f^{(n)}\in\mathcal{H}^{(n)}_{ext}$, $n\in\mathbb{N}$, as an operator acting in $(L^2)^{\beta}_q$ (we remind that, for example, $(L^2)^0_0=(L^2)$). If $\beta=1$ then this operator can be defined by formula (30) as a linear continuous one (see Remark 4), but for $\beta\in[-1,1)$ this is not the case. Let us accept a corresponding definition.

Definition 7. *Let* $n\in\mathbb{N}$*,* $f^{(n)}\in\mathcal{H}^{(n)}_{ext}$*. We define an operator*

$$(D^n\circ)(f^{(n)}):(L^2)^{\beta}_q\to(L^2)^{\beta}_q \tag{40}$$

with the domain

$$\begin{aligned}\mathrm{dom}\big((D^n\circ)(f^{(n)})\big)&=\big\{F\in(L^2)^{\beta}_q:\|(D^nF)(f^{(n)})\|^2_{q,\beta}\\&=\sum_{m=0}^{\infty}(m!)^{1+\beta}2^{qm}\Big(\frac{(m+n)!}{m!}\Big)^2|(F^{(m+n)},f^{(n)})_{ext}|^2_{ext}<\infty\big\}\end{aligned} \tag{41}$$

(here $F^{(m)}\in\mathcal{H}^{(m)}_{ext}$ *are the kernels from decomposition* (5) *for* F*) by formula* (30).

Proposition 4. *Operator* (40) *with domain* (41) *is a closed one.*

Proof. Let us show that there exists a second adjoint to $(D^n\circ)(f^{(n)})$ operator $(D^n\circ)(f^{(n)})^{**}=(D^n\circ)(f^{(n)})$ (it is well known that an adjoint operator is a closed one). Since, obviously, the domain of operator (40) is a dense set in $(L^2)^{\beta}_q$, the adjoint operator $(D^n\circ)(f^{(n)})^*:(L^2)^{-\beta}_{-q}\to(L^2)^{-\beta}_{-q}$ is well defined. By definition, $G\in\mathrm{dom}\big((D^n\circ)(f^{(n)})^*\big)$ if and only if

$$(L^2)^{\beta}_q\supset\mathrm{dom}\big((D^n\circ)(f^{(n)})\big)\ni F\mapsto\langle\!\langle(D^nF)(f^{(n)}),G\rangle\!\rangle_{(L^2)}$$

is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $K \in (L^2)_{-q}^{-\beta}$ such that $\langle\langle (D^n F)(f^{(n)}), G\rangle\rangle_{(L^2)} = \langle\langle F, K\rangle\rangle_{(L^2)}$. But by the calculation in the proof of statement 3) in Theorem 1 K has form (32), therefore

$$\operatorname{dom}\big((D^n\circ)(f^{(n)})^*\big) = \big\{G \in (L^2)_{-q}^{-\beta} : \|(D^n G)(f^{(n)})^*\|_{-q,-\beta}^2 = \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-q(m+n)} |G^{(m)} \diamond f^{(n)}|_{ext}^2 < \infty\big\},$$

this set is a dense one in $(L^2)_{-q}^{-\beta}$, hence $(D^n\circ)(f^{(n)})^{**} : (L^2)_q^\beta \to (L^2)_q^\beta$ is well defined. Now it remains to show that

$$\operatorname{dom}\big((D^n\circ)(f^{(n)})^{**}\big) = \operatorname{dom}\big((D^n\circ)(f^{(n)})\big). \tag{42}$$

By analogy with the consideration above, $F \in \operatorname{dom}\big((D^n\circ)(f^{(n)})^{**}\big)$ if and only if

$$(L^2)_{-q}^{-\beta} \supset \operatorname{dom}\big((D^n\circ)(f^{(n)})^*\big) \ni G \mapsto \langle\langle F, (D^n G)(f^{(n)})^*\rangle\rangle_{(L^2)}$$

is a linear continuous functional. The last is possible if and only if there exists $H \in (L^2)_q^\beta$ such that $\langle\langle F, (D^n G)(f^{(n)})^*\rangle\rangle_{(L^2)} = \langle\langle H, G\rangle\rangle_{(L^2)}$. It is clear that H has form (30), therefore equality (42) follows from (41). $\square$

Remark 7. *Let*

$$A_n := \big\{F \in (L^2)_q^\beta : \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F^{(m+n)}|_{ext}^2 < \infty\big\}, \quad n \in \mathbb{N},$$

here $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) *for F. For each $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ we define an operator $(\widetilde{D}^n\circ)(f^{(n)}) : (L^2)_q^\beta \to (L^2)_q^\beta$ with the domain A_n by formula* (30). *It follows from Proposition 4 that this operator is closable (its closure is equal to $(D^n\circ)(f^{(n)})$). Moreover, for each $F \in A_n$ the operator $(\widetilde{D}^n F)(\circ) : \mathcal{H}_{ext}^{(n)} \to (L^2)_q^\beta$ is a linear bounded (and, therefore, continuous) one: by* (30), (8) *and* (28)

$$\begin{aligned}\|(\widetilde{D}^n F)(f^{(n)})\|_{q,\beta}^2 &= \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |(F^{(m+n)}, f^{(n)})_{ext}|_{ext}^2 \\ &\le |f^{(n)}|_{ext}^2 \sum_{m=0}^{\infty} (m!)^{1+\beta} 2^{qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F^{(m+n)}|_{ext}^2.\end{aligned}$$

It is clear that the results of Theorems 1, 2 hold true (up to obvious modifications) for operators (40).

REFERENCES

[1] Benth F.E. *The Gross derivative of generalized random variables*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1999, **2** (3), 381–396.

[2] Benth F.E., Di Nunno G., Lokka A., Oksendal B., Proske F. *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*. Math. Finance 2003, **13** (1), 55–72.

[3] Berezansky Yu.M., Sheftel Z.G., Us G.F. Functional Analysis, Vol. 2. Birkhäuser Verlag, Basel–Boston–Berlin, 1996. (Russian edition: Vyshcha shkola, Kyiv, 1990.)

[4] Bertoin J. Lévy Processes. Cambridge University Press, Cambridge, 1996.

[5] Di Nunno G., Oksendal B., Proske F. Malliavin Calculus for Lévy Processes with Applications to Finance. In: Universitext. Springer-Verlag, Berlin, 2009.

[6] Di Nunno G., Oksendal B., Proske F. *White noise analysis for Lévy processes*. J. Funct. Anal. 2004, **206** (1), 109–148. doi:10.1016/S0022-1236(03)00184-8

[7] Dyriv M.M., Kachanovsky N.A. *Operators of stochastic differentiation on spaces of regular test and generalized functions in the Lévy white noise analysis*. Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2014, **4**, in print.

[8] Dyriv M.M., Kachanovsky N.A. *Stochastic integrals with respect to a Levy process and stochastic derivatives on spaces of regular test and generalized functions*. Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2013, **4**, 27–30.

[9] Gelfand I.M., Vilenkin N.Ya. Generalized Functions. In: Applications of Harmonic Analysis, 6. Academic Press, New York, London, 1964.

[10] Gihman I.I., Skorohod A.V. Theory of Random Processes, Vol. 2. Nauka, Moscow, 1973.

[11] Holden H., Oksendal B., Uboe J., Zhang T.-S. Stochastic Partial Differential Equations. A Modeling, White Noise Functional Approach. In: Probability and Its Applications. Birkhäuser, Boston, 1996.

[12] Itô K. *Spectral type of the shift transformation of differential processes with stationary increments*. Trans. Am. Math. Soc. 1956, **81**, 253–263.

[13] Kabanov Yu.M. *Extended stochastic integrals*. Teorija Verojatnostej i ee Pril. 1975, **20** (4), 725–737.

[14] Kabanov Yu.M., Skorohod A.V. *Extended stochastic integrals*. In: Proc. School-Symposium "Theory Stoch. Proc.", Druskininkai, Lietuvos Respublika, November 25–30, 1974, Inst. Phys. Math., Vilnus, 1975, 123–167.

[15] Kachanovsky N.A. *A generalized Malliavin derivative connected with the Poisson- and Gamma-measures*. Methods Funct. Anal. Topol. 2003, **9** (3), 213–240.

[16] Kachanovsky N.A. *A generalized stochastic derivative on the Kondratiev-type space of regular generalized functions of Gamma white noise*. Methods Funct. Anal. Topol. 2006, **12** (4), 363–383.

[17] Kachanovsky N.A. *Generalized stochastic derivatives on a space of regular generalized functions of Meixner white noise*. Methods Funct. Anal. Topol. 2008, **14** (1), 32–53.

[18] Kachanovsky N.A. *Generalized stochastic derivatives on parametrized spaces of regular generalized functions of Meixner white noise*. Methods Funct. Anal. Topol. 2008, **14** (4), 334–350.

[19] Kachanovsky N.A. *Extended stochastic integrals with respect to a Lévy process on spaces of generalized functions*. Math. Bull. Shevchenko Sci. Soc. 2013, **10**, 169–188.

[20] Kachanovsky N.A. *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*. Methods Funct. Anal. Topol. 2007, **13** (4), 338–379.

[21] Kachanovsky N.A. *On extended stochastic integrals with respect to Lévy processes*. Carpathian Math. Publ. 2013, **5** (2), 256–278. doi:10.15330/cmp.5.2.256-278

[22] Lytvynov E.W. *Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2003, **6** (1), 73–102.

[23] Meyer P.A. Quantum Probability for Probabilists. In: Lect. Notes in Math., 1538. Springer–Verlag, Berlin, 1993.

[24] Nualart D., Schoutens W. *Chaotic and predictable representations for Lévy processes*. Stochastic Process. Appl. 2000, **90** (1), 109–122.

[25] Protter P. Stochastic Integration and Differential Equations. Springer, Berlin, 1990.

[26] Sato K. Lévy Processes and Infinitely Divisible Distributions. In: Cambridge University Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.

[27] Schoutens W. Stochastic Processes and Orthogonal Polynomials. In: Lect. Notes in Statist., 146. Springer–Verlag, 2000.

[28] Skorohod A.V. Integration in Hilbert Space. Springer, New York–Heidelberg, 1974.

[29] Skorohod A.V. *On a generalization of a stochastic integral*. Teorija Verojatnostej i ee Pril. 1975, **20** (2), 223–238.

[30] Solé J.L., Utzet F., Vives J. *Chaos expansions and Malliavin calculus for Lévy processes*. In: Stoch. Anal. and Appl., Abel Symposium 2, Springer, Berlin, 2007, 595–612.

[31] Surgailis D. *On L^2 and non-L^2 multiple stochastic integration*. In: Lect. Notes in Control and Information Sciences, 36, Springer-Verlag, 1981, 212–226.

[32] Ustunel A.S. An Introduction to Analysis on Wiener Space. In: Lect. Notes in Math., Vol. 1610, Springer–Verlag, Berlin, 1995.

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Оператори стохастичного диференціювання, які є тісно пов'язаними із розширеним стохастичним інтегралом Скорохода та зі стохастичною похідною Хіди, грають важливу роль у класичному (гауссівському) аналізі білого шуму. Зокрема, ці оператори можна використовувати для вивчення властивостей розширеного стохастичного інтеграла та розв'язків стохастичних рівнянь з нелінійностями віківського типу.

У цій статті ми вводимо та вивчаємо обмежені і необмежені оператори стохастичного диференціювання у аналізі білого шуму Леві. Точніше, ми розглядаємо ці оператори на просторах параметризованого регулярного оснащення простору квадратично інтегровних за мірою білого шуму Леві функцій, використовуючи литвинівське узагальнення властивості хаотичного розкладу. Це дає можливість розширити на аналіз білого шуму Леві та поглибити відповідні результати класичного аналізу білого шуму.

Ключові слова і фрази: оператор стохастичного диференціювання, стохастична похідна, розширений стохастичний інтеграл, процес Леві.

Дырив М.Н., Качановский Н.А. *Об операторах стохастического дифференцирования на пространствах регулярных основных и обобщенных функций анализа белого шума Леви* // Карпатские матем. публ. — 2014. — Т.6, №2. — C. 212–229.

Операторы стохастического дифференцирования, тесно связанные с расширенным стохастическим интегралом Скорохода и со стохастической производной Хиды, играют важную роль в классическом (гауссовском) анализе белого шума. В частности, эти операторы можно использовать для изучения свойств расширенного стохастического интеграла и решений стохастических уравнений с нелинейностями виковского типа.

В этой статье мы вводим и изучаем ограниченные и неограниченные операторы стохастического дифференцирования в анализе белого шума Леви. Точнее, мы рассматриваем эти операторы на пространствах параметризованного регулярного оснащения пространства квадратично интегрируемых по мере белого шума Леви функций, используя литвиновское обобщение свойства хаотического разложения. Это дает возможность расширить на анализ белого шума Леви и углубить соответствующие результаты классического анализа белого шума.

Ключевые слова и фразы: оператор стохастического дифференцирования, стохастическая производная, расширенный стохастический интеграл, процесс Леви.