ISSN 2075-9827 **e-ISSN 2313-0210** **https://journals.pnu.edu.ua/index.php/cmp**
Carpathian Math. Publ. 2021, **13** (3), 790–804 Карпатські матем. публ. 2021, Т.13, №3, С.790–804
doi:10.15330/cmp.13.3.790-804

An efficient hybrid technique for the solution of fractional-order partial differential equations

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In this paper, a hybrid technique called the homotopy analysis Sumudu transform method has been implemented solve fractional-order partial differential equations. This technique is the amalgamation of Sumudu transform method and the homotopy analysis method. Three examples are considered to validate and demonstrate the efficacy and accuracy of the present technique. It is also demonstrated that the results obtained from the suggested technique are in excellent agreement with the exact solution which shows that the proposed method is efficient, reliable and easy to implement for various related problems of science and engineering.

Key words and phrases: fractional differential equation, Sumudu transform, homotopy analysis method.

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Introduction

Newly, the fractional calculus (FC) and its various applications in mathematics, physics and engineering have received considerable attention. FC applications are found in many areas, such as dynamic device control theory, chemical mechanics, probability and statistics, electrical networks, corrosion electrochemistry, and optics and signal processing. Linear/nonlinear fractional-order differential equations may be successfully modeled. A fractional PDE is obtained from the classical diffusion equation of mathematical physics by replacing the nth order time derivative with a fractional-order derivative α, which is now the area of increasing interest apparent in the literature study [10–12, 19].

In recent decades, many of the numerical and analytical techniques have been implemented to solve fractional-order PDEs, such as the fractional variational iteration method [23,34,42,44, 45], fractional differential transform method [25, 36, 46], fractional series expansion method [9, 29], fractional Sumudu variational iteration method [20, 31], fractional natural decomposition method [32, 38], fractional Sumudu decomposition method [17, 30, 33], fractional Sumudu homotopy perturbation method [28], fractional reduce differential transform method [24, 26, 41], fractional Adomian decomposition method [16, 21, 47], fractional Laplace decomposition method [27], fractional Laplace homotopy perturbation method [14], fractional Laplace variational iteration method [13, 15, 18, 35, 37], variational iteration method [4–8] and local mesh less

УДК 517.95
2020 *Mathematics Subject Classification:* 34K37, 45J99, 34A08.

method [1–3, 22, 40]. As the main aim of this work the homotopy analysis Sumudu transform method is implemented to solve FPDEs and nonlinear system of FPDEs.

1 Fractional calculus

In this section, we demonstrate some notations and definitions that will be used further in the study. FC theory is almost more than two decades old in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper definition is due to Liouville as follows.

Definition 1 ([30,39]). *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$*, of a function* $\Psi(\tau) \in C_\varepsilon$, $\varepsilon \geq -1$*, is defined as*

$$I_\tau^\alpha \Psi(\tau) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\tau (\tau - s)^{\alpha-1} \Psi(s)\, ds, & \alpha > 0,\ \tau > 0, \\ \Psi(\tau), & \alpha = 0, \end{cases}$$

where $\Gamma(\cdot)$ *is the well-known Gamma function.*

Definition 2 ([30,39]). *The Caputo fractional derivative (CFD) with order* $\alpha > 0$ *of* $\Psi(\tau)$ *is defined as follows*

$$D_\tau^\alpha \Psi(\mu) = \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau - s)^{m-\alpha-1} \Psi^{(m)}(s)\, ds$$

for $m - 1 < \alpha < m$, $m \in \mathbb{N}$, $\tau > 0$*, and* $\varphi \in C_{-1}^m$.

The fundamental basic properties of the Caputo fractional derivative are given as:

(i)
$$D^\alpha I^\alpha \Psi(x, \tau) = \Psi(x, \tau);$$

(ii)
$$I^\alpha D^\alpha \Psi(x, \tau) = \Psi(x, \tau) - \sum_{k=0}^{m-1} \frac{\tau^k}{k!} \Psi^{(k)}(x, 0);$$

(iii)
$$D^\alpha \tau^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\beta+1)} \tau^{\beta-\alpha}, \quad \alpha > 0.$$

Definition 3 ([39]). *The Mittag–Leffler function* $E_\alpha(z)$ *with* $\alpha > 0$ *is defined as*

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha+1)}.$$

Definition 4 ([30,43]). *The Sumudu transform (ST) is defined by*

$$S[\Psi(\tau)] = \int_0^\infty e^{-\tau} \Psi(\omega\tau)\, d\tau, \quad \omega \in (-\omega_1, \omega_2).$$

Some properties of ST:

(i) $S[k] = k$ for any constant k;

(ii) $S[\tau^{n\alpha}/\Gamma(n\alpha+1)] = \omega^{n\alpha}$.

Definition 5 ([43]). *The ST of the CFD is defined as*

$$S[D_\tau^\alpha \Psi(x,\tau)] = \omega^{-\alpha} S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{(-\alpha+k)} \Psi^{(k)}(x,0), \quad m-1<\alpha<m.$$

2 Analysis of FHASTM

Let us consider a general fractional nonlinear PDE of the form

$$D_\tau^\alpha \Psi(x,\tau) + R\Psi(x,\tau) + N\Psi(x,\tau) = G(x,\tau), \quad m-1<\alpha\le m, \quad x\in R, \quad \tau>0, \tag{1}$$

subject to the initial condition $\Psi(x,0) = \Psi^{(k)}(x,0)$, $k=1,2,\dots,m-1$, where $D_\tau^\alpha \Psi(x,\tau)$ is the CFD of the function $\Psi(x,\tau)$ defined as

$$D_\tau^\alpha \Psi(x,\tau) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^\tau (\tau-s)^{m-\alpha-1} \frac{\partial^m \Psi(x,s)}{\partial \tau^m}\, ds, & m-1<\alpha<m, \\ \dfrac{\partial^m \Psi(x,\tau)}{\partial \tau^m}, & \alpha = m \in \mathbb{N}, \end{cases}$$

and R is the linear differential operator, N represents the general nonlinear differential operator, and $G(x,\tau)$ is the source term. Now taking the ST of both sides of equation (1) we have

$$S[D_t^\alpha \Psi(x,\tau)] + S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] = S[G(x,\tau)].$$

Using the differentiation properties of the ST and above initial condition, we have

$$\omega^{-\alpha} S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{k-\alpha} \Psi^{(k)}(x,0) + S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] = S[G(x,\tau)],$$

or

$$S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{k} \Psi^{(k)}(x,0) + \omega^\alpha (S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] - S[G(x,\tau)]) = 0.$$

We define the nonlinear operator

$$\begin{aligned} N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] &- \sum_{k=0}^{m-1} \omega^k \varnothing^{(k)}(x,0) \\ &+ \omega^\alpha (S[R\varnothing(x,\tau;q)] + S[N\varnothing(x,\tau;q)] - S[G(x,\tau)]), \end{aligned} \tag{2}$$

where $q\in[0,1]$ and $\varnothing(x,\tau;q)$ is a real function of x,τ and q, the so-called zero order deformation equation of the equation (2) has the form

$$(1-q)S[\varnothing(x,\tau;q) - \Psi_0(x,\tau)] = qhH(x,\tau)N[\varphi(x,\tau;q)], \tag{3}$$

where $q\in[0,1]$ is the embedding parameter, $H(x,\tau)$ denotes a nonzero auxiliary function, $h\neq 0$ is an auxiliary parameter $\Psi_0(x,\tau)$ is an initial guess of $\Psi(x,\tau)$ and $\varnothing(x,\tau;q)$ is an unknown function. Obviously, when the parameter $q=0$ and $q=1$, it holds

$$\varnothing(x,\tau;0) = \Psi_0(x,\tau), \quad \varnothing(x,\tau;1) = \Psi(x,\tau),$$

respectively. Thus as q increases from 0 to 1, the solution $\varnothing(x,\tau;q)$ varies from the initial guess $\Psi_0(x,\tau)$ to the solution $\Psi(x,\tau)$. Expanding $\varnothing(x,\tau;q)$ in Taylor's series with respect to q, we have

$$\varnothing(x,\tau;q) = \Psi_0(x,\tau) + \sum_{m=1}^{\infty} \Psi_m(x,\tau) q^m, \tag{4}$$

where

$$\Psi_m(x,\tau) = \frac{1}{m!} \frac{\partial^m \varnothing(x,\tau;q)}{\partial q^m}\Big|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen. The series (4) converges at $q = 1$, then we has

$$\Psi(x,\tau) = \Psi_0(x,\tau) + \sum_{m=1}^{\infty} \Psi_m(x,\tau), \tag{5}$$

which must be one of the solution of the original nonlinear equation (1). According to the definition of equation (5), the governing equation can be deduced from the zero-order deformation equation (3).

Define the vectors $\vec{\Psi}_m(x,\tau) = \{\Psi_0(x,\tau), \Psi_1(x,\tau), \ldots, \Psi_m(x,\tau)\}$. Differentiating the zero-order deformation equation (12) m-times with respect to q and then dividing by $m!$ and finally setting $q = 0$ we get the following mth order deformation equation

$$S[\Psi_m(x,\tau) - x_m \Psi_{m-1}(x,\tau)] = hH(x,\tau) R_m(\vec{\Psi}_{m-1}(x,\tau)).$$

Applying the inverse ST, we have

$$\Psi_m(x,\tau) = x_m \Psi_{m-1}(x,\tau) + S^{-1}[hH(x,\tau) R_m(\vec{\Psi}_{m-1}(x,\tau))],$$

where

$$R_m(\vec{\Psi}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varnothing(x,\tau;q)]}{\partial q^{m-1}}\Big|_{q=0}, \quad X_m = \begin{cases} 0, & x \leq 1, \\ 1, & x > 1. \end{cases}$$

In this way, it is easily to obtain $\Psi_m(x,\tau)$ for $m \geq 1$, at mth order, $h = -1$. We have

$$\Psi(x,\tau) = \sum_{m=0}^{\infty} \Psi_m(x,\tau).$$

3 Applications

Example 1. *Consider the following nonlinear FPDE*

$$D_\tau^\alpha \Psi + \Psi \Psi_x - \Psi_{xx} = 0, \quad 0 < \alpha \leq 1, \tag{6}$$

with $\Psi(x,0) = x$. *Applying ST to equation* (6), *we have*

$$\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} + S[\Psi\Psi_x - \Psi_{xx}] = 0 \quad \text{or} \quad S[\Psi] - x + \omega^\alpha S[\Psi\Psi_x - \Psi_{xx}] = 0.$$

We now define a nonlinear operator as

$$N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] + \omega^\alpha S\left[\varnothing(x,\tau;q)\frac{\partial \varnothing(x,\tau;q)}{\partial x} - \frac{\partial^2 \varnothing(x,\tau;q)}{\partial x^2}\right],$$

and thus

$$R_m(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1 - x_m)(x) + \omega^a S\left[\left(\sum_{i=0}^{m-1} \Psi_i (\Psi_{m-1-i})_x\right) - (\Psi_{m-1})_{xx}\right].$$

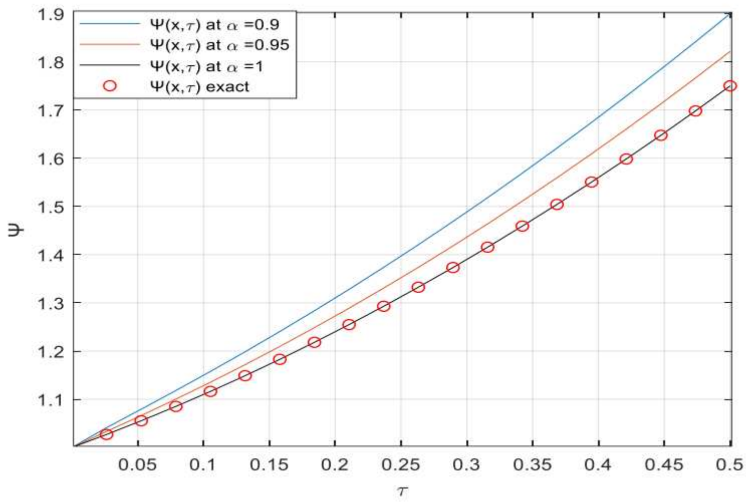

Figure 1. Plots of the exact and approximate solutions $\Psi(x, \tau)$ of (6) for values of α with the fixed value x.

The m*th order deformation equation is*

$$S[\Psi_m - x_w \Psi_{m-1}] = hH(x, \tau) R_m(\vec{\Psi}_{m-1}).$$

Applying the inverse ST we have

$$\Psi_m = x_m \Psi_{m-1} + hS^{-1}[H(x, \tau) R_m(\vec{\Psi}_{m-1})]. \tag{7}$$

Solve above the equation (7) *for* $m = 1, 2, \ldots$ *choosing* $H(x, \tau) = 1$. *Let us take the initial conditions* $\Psi_0 = x$,

$$\begin{aligned}
\Psi_1 &= x_1 \Psi_0 + hS^{-1}[R_1(\vec{\Psi}_0)] = (0)(x) + hS^{-1}[S(\Psi_0) - x + \omega^\alpha S(\Psi_0 \Psi_{0_x} - \Psi_{0_{xx}})] \\
&= hS^{-1}[\omega^\alpha x] = \frac{hx\tau^\alpha}{\Gamma(\alpha+1)}, \\
\Psi_2 &= x_2 \Psi_1 + hS^{-1}[R_2(\vec{\Psi}_1)] \\
&= (1)\left(\frac{hx\tau^\alpha}{\Gamma(\alpha+1)}\right) + hS^{-1}[S(\Psi_1) + \omega^\alpha S(\Psi_0 \Psi_{1_x} + \Psi_1 \Psi_{0_x} - \Psi_{1_{xx}})] \\
&= \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}\left[hx\omega^\alpha + \omega^\alpha S\left(\frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} - 0\right)\right] \\
&= \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}[hx\omega^\alpha + 2hx\omega^{2\alpha}] = \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + \frac{h^2 x\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2h^2 x\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \\
&\cdots\cdots\cdots
\end{aligned}$$

Setting then $h = -1$, *the series solutions of equation* (6) *are given by*

$$\Psi(x, \tau) = x + \frac{x\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2x\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.$$

The exact result of Example 1 when $\alpha = 1$ *is* $\Psi(x, \tau) = x/(1-\tau)$.

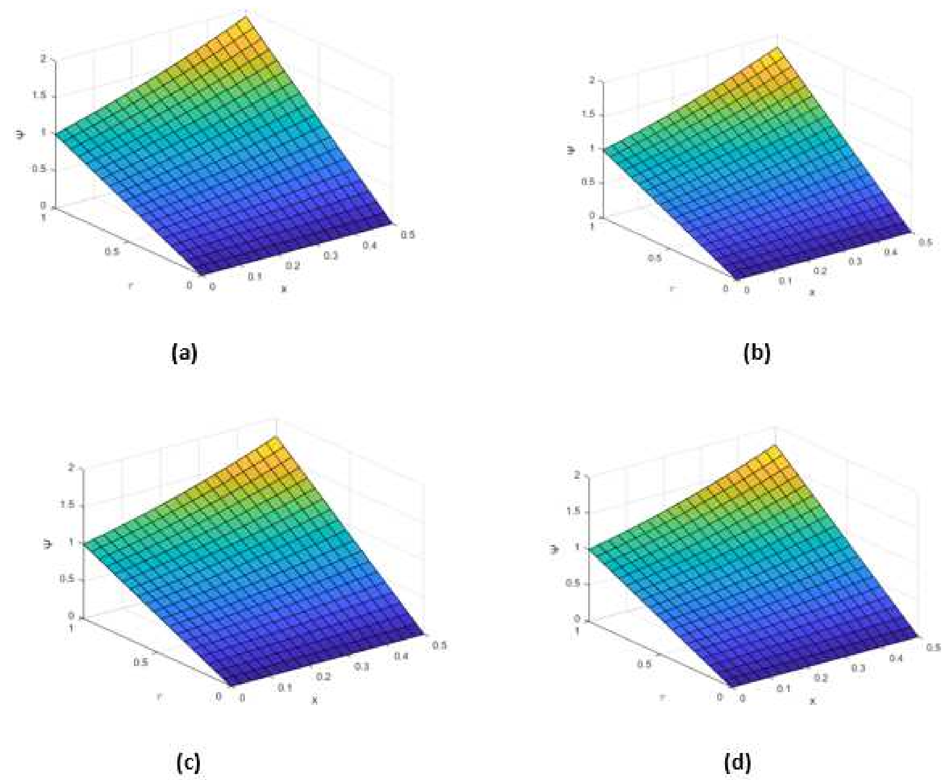

Figure 2. The surface graph of the approximate solutions $\Psi(x,\tau)$ of (6):
(a) $\Psi(x,\tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x,\tau)$ when $\alpha = 0.95$;
(c) $\Psi(x,\tau)$ when $\alpha = 1$, **(d)** $\varphi(\mu,\tau)$ exact solution.

In Figure 1, we plot the graph of the exact and approximate solutions for (6) when $\alpha = 0.9, 0.95, 1$. In Figure 2, we plot 3D surface solution for (6) when $\alpha = 0.9, 0.95, 1$.

Example 2. *Consider the non-linear FPDE*

$$D_\tau^\alpha \Psi - \Psi_x^2 - \Psi\Psi_x = 0, \quad 0 < \alpha \leq 1, \tag{8}$$

with the initial condition $\Psi(x,0) = x^2$. *Applying ST to the equation* (8) *we obtain*

$$\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} - S[\Psi_x^2 + \Psi\Psi_{xx}] = 0. \tag{9}$$

On simplifying and using the equation (9) *we have*

$$S[\Psi] - x^2 - \omega^\alpha S[\Psi_x^2 + \Psi\Psi_{xx}] = 0.$$

We now define a nonlinear operator as

$$N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] - x^2 - \omega^\alpha S\Big[\Big(\frac{\partial\varnothing(x,\tau;q)}{\partial x}\Big)^2 + \varnothing(x,\tau;q)\frac{\partial^2\varnothing(x,\tau;q)}{\partial x^2}\Big],$$

and thus

$$R_m(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1-x_m)x^2 - \omega^\alpha S\Big[\sum_{i=0}^{m-1}(\Psi_i)_x(\Psi_{m-1-i})_x + \sum_{i=0}^{m-1}\Psi_i(\Psi_{m-1-i})_{xx}\Big].$$

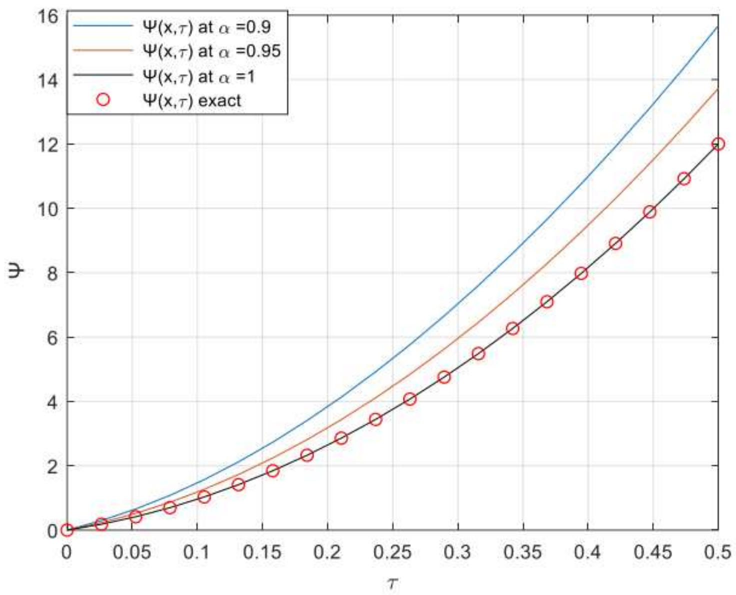

Figure 3. Plots of the exact and approximate solutions $\Psi(x,\tau)$ of (8) for different values of α with the fixed value x.

The m*th order deformation equation is*

$$S[\Psi_m - x_m\Psi_{m-1}] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}).$$

Applying the inverse ST we have

$$\Psi_m = x_m\Psi_{m-1} + hS^{-1}[H(x,\tau)R_m(\vec{\Psi}_{m-1})]. \tag{10}$$

Solve above the equation (10) *for* $m = 1,2,\ldots$ *choosing* $H(x,\tau) = 1$. *Let us take the initial conditions* $\Psi_0 = x^2$,

$$\begin{aligned}
\Psi_1 &= x_1\Psi_0 + hS^{-1}[R_1(\vec{\Psi}_0)] = (0)(x^2) + hS^{-1}[S(\Psi_0) - x^2 - \omega^\alpha S(\Psi_{0x}\Psi_{0_x} - \Psi_0\Psi_{0_{xx}})] \\
&= hS^{-1}[x^2 - x^2 - \omega^\alpha S(4x^2 + 2x^2)] = hS^{-1}[-\omega^\alpha(6x^2)] = \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)}, \\
\Psi_2 &= x_2\Psi_1 + hS^{-1}[R_2(\vec{\Psi}_1)] \\
&= (1)\Big(\frac{-6hx^2\tau^\alpha}{\Gamma(a+1)}\Big) + hS^{-1}[S(\Psi_1) - \omega^\alpha S(\Psi_{0_x}\Psi_{1_x} + \Psi_{1_x}\Psi_{0_x} + \Psi_0\Psi_{1_{xy}} + \Psi_1\Psi_{0_{xx}})] \\
&= \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}\Big[-6hx^2\omega^\alpha - \omega^\alpha S\Big(\frac{-72hx^2\tau^\alpha}{\Gamma(\alpha+1)}\Big)\Big] \\
&= \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}[-6hx^2\omega^\alpha + 2(6^2hx^2\omega^{2\alpha})] = \frac{-6hx^2\tau^\alpha}{\Gamma(a+1)} - \frac{6h^2x^2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2(6^2h^2x^2\tau^{2\alpha})}{\Gamma(2\alpha+1)}, \\
&\ldots\ldots
\end{aligned}$$

Setting the $h = -1$, *the series solutions of equation* (8) *are given by*

$$\Psi(x,\tau) = x^2 + \frac{6x^2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2(6^2x^2\tau^{2\pi})}{\Gamma(2\alpha+1)} + \cdots.$$

The exact result of Example 2 when $\alpha = 1$ *is* $\Psi_{(x,\tau)} = x^2/(1-6\tau)$.

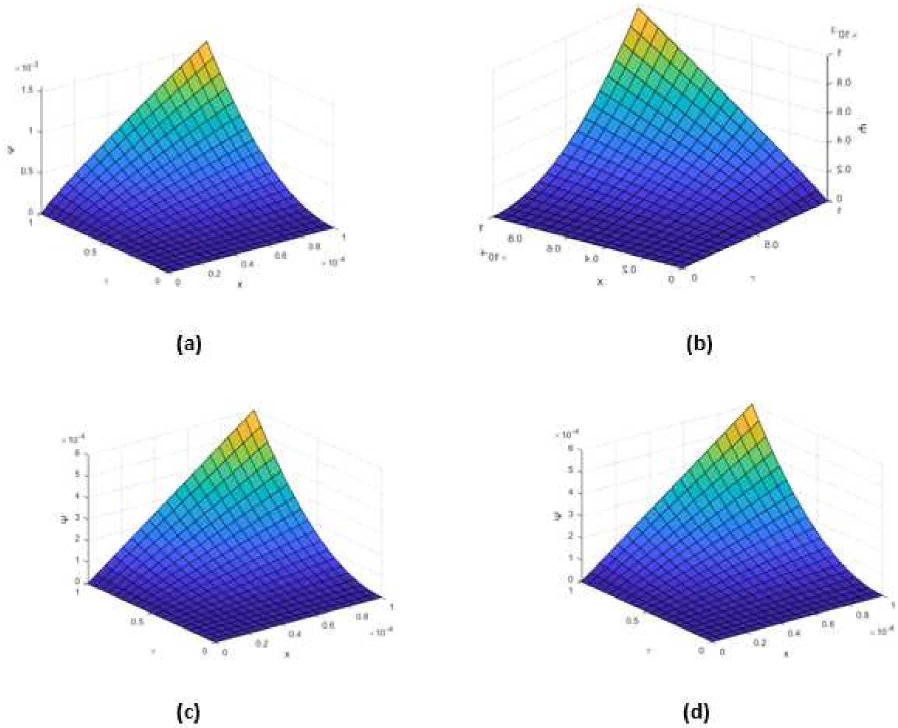

Figure 4. The surface graph of the approximate solutions $\Psi(x,\tau)$ of (8):
(a) $\Psi(x,\tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x,\tau)$ when $\alpha = 0.95$;
(c) $\Psi(x,\tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu,\tau)$ exact solution.

In Figure 3, we plot the graph of the exact and approximate solutions for (8) when $\alpha = 0.9, 0.95, 1$. In Figure 4, we plot 3D surface solution for (8) when $\alpha = 0.9, 0.95, 1$.

Example 3. *We consider the following non-linear system of equations*

$$\begin{aligned} &D_\tau^\alpha \Psi + u\Psi_x + \Psi - 1 = 0, \quad 0 < \alpha \leq 1, \\ &D_\tau^\beta u - \Psi u_x - u - 1 = 0, \quad 0 < \beta \leq 1, \end{aligned} \tag{11}$$

with the initial conditions $\Psi(x,0) = e^x$, $u(x,0) = e^{-x}$. *Applying ST on both sides in equations* (11), *we get*

$$\begin{aligned} &\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} + S[u\Psi_x + \Psi - 1] = 0, \\ &\frac{S[u]}{\omega^\alpha} - \frac{u(x,0)}{\omega^\alpha} + S[\Psi u_x - u - 1] = 0. \end{aligned} \tag{12}$$

On simplifying and using the equations (12) *we have*

$$\begin{aligned} &S[\Psi] - e^x + \omega^\alpha S[u\Psi_x + \Psi - 1] = 0, \\ &S[u] - e^{-x} - \omega^\beta S[\Psi u_x + u + 1] = 0. \end{aligned}$$

We now define a nonlinear operator as

$$\begin{aligned} N[\varnothing_1(x,\tau;q)] &= S[\varnothing_1] - e^x - \omega^\alpha + \omega^\alpha S\Big[\varnothing_2 \frac{\partial \varnothing_1}{\partial x} + \varnothing_1\Big], \\ N[\varnothing_2(x,\tau;q)] &= S[\varnothing_2] - e^{-x} - \omega^\beta - \omega^\beta S\Big[\varnothing_1 \frac{\partial \varnothing_2}{\partial x} + \varnothing_2\Big]. \end{aligned}$$

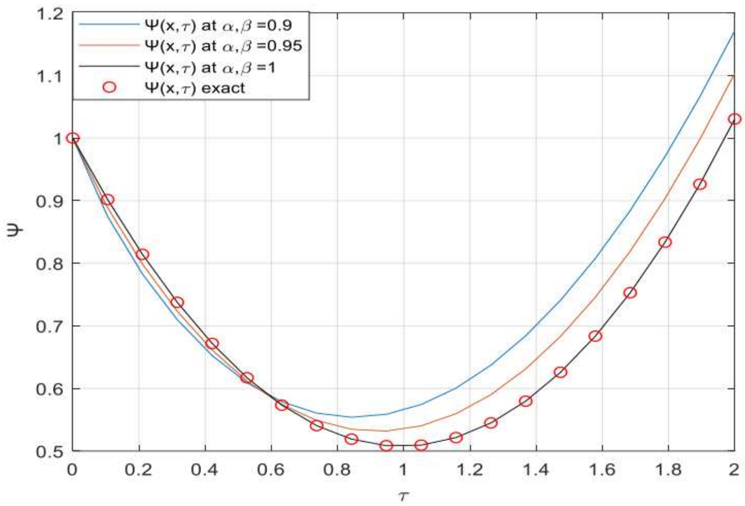

Figure 5. Plots of the exact and approximate solutions $\Psi(x,\tau)$ of (11) for different values of α with the fixed value x

Thus,

$$R_{1m}(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1-x_m)(e^x + \omega^\alpha) + \omega^\alpha S\left[\left(\sum_{i=0}^{m-1} u_i(\Psi_{m-1-i})_x\right) + \Psi_{m-1}\right]$$

$$R_{2m}(\vec{u}_{m-1}) = S(u_{m-1}) - (1-x_m)(e^{-x} + \omega^\beta) - \omega^\beta S\left[\left(\sum_{i=0}^{m-1} \Psi_i(u_{m-1-i})_x\right) + u_{m-1}\right].$$

Then mth order deformation equations are

$$S[\Psi_m - x_m\Psi_{m-1}] = hH(x,\tau)R_{1m}(\vec{\Psi}_{m-1}),$$
$$S[u_m - x_m u_{m-1}] = hH(x,\tau)R_{2m}(\vec{u}_{m-1}).$$

Applying the inverse ST we have

$$\Psi_m = x_m\Psi_{m-1} + hS^{-1}[H(x,\tau)R_{1m}(\vec{\Psi}_{m-1})],$$
$$u_m = x_m u_{m-1} + hS^{-1}[H(x,\tau)R_{2m}(u_{m-1})]. \tag{13}$$

Solve above the equations (13) *for $m = 1,2,\ldots$ choosing $H(x,\tau) = 1$. Let us take the initial conditions $\Psi_0(x,\tau) = e^x$, $u_0(x,\tau) = e^{-x}$,*

$$\begin{aligned}
\Psi_1 &= x_1\Psi_0 + hS^{-1}[R_{11}(\vec{\Psi}_0)] \\
&= (0)(e^x) + hS^{-1}[S(\Psi_0) - (1-0)(e^x + \omega^\alpha) + \omega^\alpha S(u_0\Psi_{0_x} + \Psi_0)] \\
&= hS^{-1}[e^x - e^x - \omega^\alpha + \omega^\alpha S(1+e^x)] = hS^{-1}[-\omega^\alpha + \omega^\alpha + \omega^\alpha e^x] = \frac{he^x\tau^\alpha}{\Gamma(\alpha+1)}, \\
u_1 &= x_1 u_0 + hS^{-1}[R_{21}(\vec{u}_0)] \\
&= (0)(e^{-x}) + hS^{-1}[S(u_0) - (1-0)(e^{-x} + \omega^\beta) - \omega^\beta S(\Psi_0 u_{0_x} + u_0)] \\
&= hS^{-1}[e^{-x} - e^{-x} - \omega^\beta - \omega^\beta S(-1 + e^{-x})] = hS^{-1}[-\omega^\beta + \omega^\beta - \omega^\beta e^{-x}] = \frac{-he^{-x}\tau^\beta}{\Gamma(\beta+1)},
\end{aligned}$$

$$\begin{aligned}
\Psi_2 &= x_2\Psi_1 + hS^{-1}[R_{12}(\vec{\Psi}_1)] \\
&= (1)\Big(\frac{-he^{-x}\tau^{\alpha}}{\Gamma(\alpha+1)}\Big) + hS^{-1}[S(\Psi_1) - \omega^{\alpha}S(u_0\Psi_{1_x} + u_1\Psi_{0_x} + \Psi_1)] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + hS^{-1}\Big[he^{x}\omega^{\alpha} + \omega^{\alpha}S\Big(\frac{h\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{h\tau^{\beta}}{\Gamma(\beta+1)} + \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + hS^{-1}[he^{x}\omega^{\alpha} + h\omega^{2\alpha} - h\omega^{\alpha+\beta} + he^{x}\omega^{2\alpha}] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2e^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{h^2\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{h^2e^{x}\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \\
u_2 &= x_2u_1 + hS^{-1}[R_{22}(\vec{u}_1)] \\
&= (1)\Big(\frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)}\Big) + hS^{-1}[S(u_1) - \omega^{\beta}S(\Psi_0u_{1_x} + \Psi_1u_{0_x} + u_1)] \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} + hS^{-1}\Big[-he^{-x}\omega^{\beta} - \omega^{\beta}S\Big(\frac{h\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{he^{-x}\tau^{\beta}}{\Gamma(\beta+1)}\Big)\Big] \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} + hS^{-1}[-he^{-x}\omega^{\beta} - h\omega^{2\beta} + h\omega^{\alpha+\beta} + he^{-x}\omega^{2\beta}], \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h^2e^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h^2\tau^{2\beta}}{\Gamma(2\beta+1)} + \frac{h^2\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{h^2e^{-x}\tau^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

$\ldots\ldots$

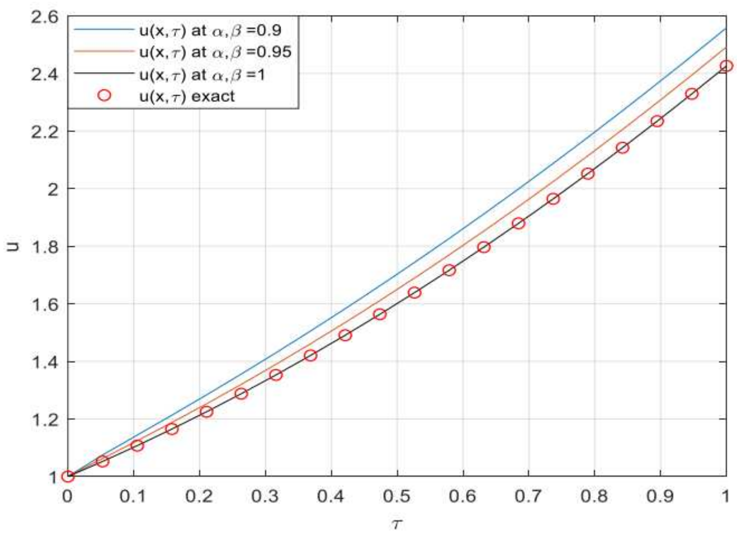

Figure 6. Plots of the exact and approximate solutions $u(x,\tau)$ of (11) for different values of α with the fixed value x

Setting the $h = -1$*, the series solutions of equations* (11) *are given by*

$$\Psi(x,\tau) = e^{x} - \frac{e^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{e^{x}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

$$u(x,\tau) = e^{-x} + \frac{e^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{e^{-x}\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots.$$

The exact result of Example 3 when $\alpha = \beta = 1$ *is* $\Psi(x,\tau) = e^{x-\tau}$, $u(x,\tau) = e^{-x+\tau}$.

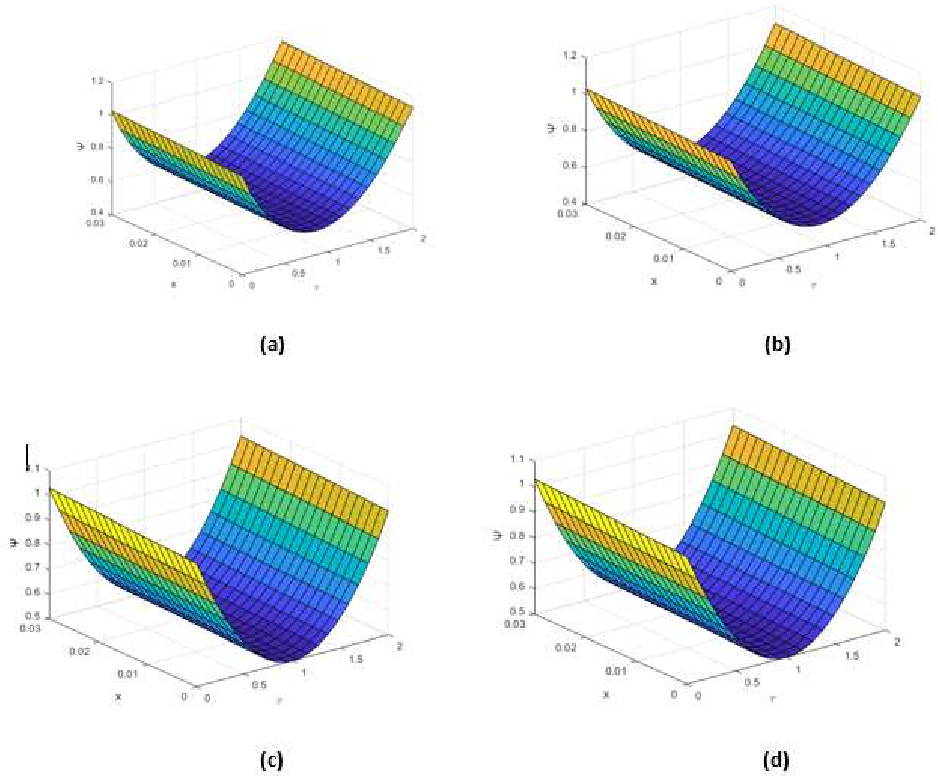

Figure 7. The surface graph of the approximate solutions $\Psi(x,\tau)$ of (11):
(a) $\Psi(x,\tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x,\tau)$ when $\alpha = 0.95$;
(c) $\Psi(x,\tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu,\tau)$ exact solution.

In Figures 5 and 6, we plot the graph of the exact and approximate solution for the equations (11) when $\alpha = 0.9, 0.95, 1$. In Figures 7 and 8, we plot 3D surface solutions for the equations (11) when $\alpha = 0.9, 0.95, 1$.

4 Conclusion

In this work, we utilized the HASTM to solve fractional-order PDEs and their approximate solutions were obtained. The HASTM was proved to be an effective approach for solving PDEs with CFD due to the excellent agreement between the obtained approximate solution and the exact solution. And it's rapid convergence shows that the procedure is reliable and introduces a significant improvement in solving linear and non-linear fractional-order PDEs.

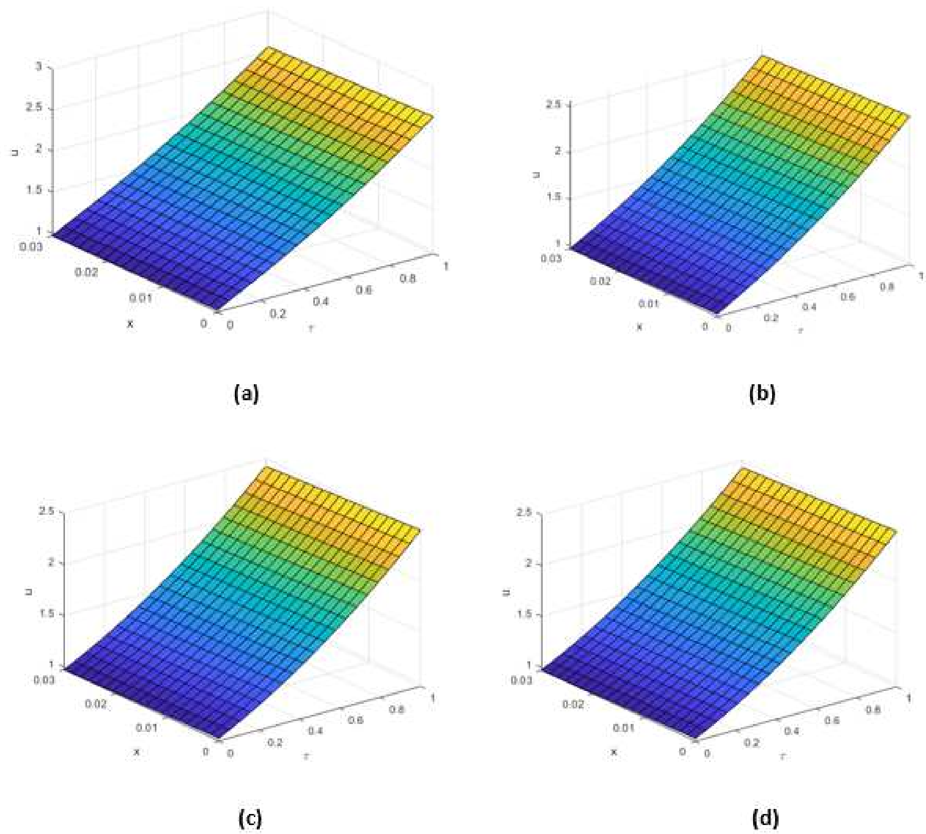

Figure 8. The surface graph of the approximate solutions $u(x, \tau)$ of (11):
(a) $u(x, \tau)$ when $\alpha = 0.9$; **(b)** $u(x, \tau)$ when $\alpha = 0.95$;
(c) $u(x, \tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu, \tau)$ exact solution.

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Received 17.05.2021
Revised 25.11.2021

Джассім Х.К., Ахмад Г., Шамаун А., Чезарано К. *Ефективний гібридний метод розв'язування диференціальних рівнянь з частинними похідними дробового порядку* // Карпатські матем. публ. — 2021. — Т.13, №3. — С. 790–804.

У цій роботі реалізовано гібридний метод, який називається гомотопічним аналізом за допомогою методу перетворення Сумуду, що дозволяє розв'язувати диференціальні рівнянння з частинними похідними дробового порядку. Цей метод є об'єднанням методу перетворення Сумуду та методу гомотопічного аналізу. Розглянуто три приклади для підтвердження і демонстрації ефективності та точності цієї методики. Також показано, що результати, отримані за допомогою запропонованої методики, чудово узгоджуються з точним розв'язком, що свідчить про ефективність, надійність та простоту реалізації запропонованого методу для різних суміжних проблем науки та техніки.

Ключові слова і фрази: диференціальне рівнянння з частинними похідними дробового порядку, перетворення Сумуду, метод гомотопічного аналізу.