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THE NONLOCAL BOUNDARY VALUE PROBLEM FOR ONE-DIMENSIONAL BACKWARD KOLMOGOROV EQUATION AND ASSOCIATED SEMIGROUP

This paper is devoted to a partial differential equation approach to the problem of construction of Feller semigroups associated with one-dimensional diffusion processes with boundary conditions in theory of stochastic processes. In this paper we investigate the boundary-value problem for a one-dimensional linear parabolic equation of the second order (backward Kolmogorov equation) in curvilinear bounded domain with one of the variants of nonlocal Feller-Wentzell boundary condition. We restrict our attention to the case when the boundary condition has only one term and it is of the integral type. The classical solution of the last problem is obtained by the boundary integral equation method with the use of the fundamental solution of backward Kolmogorov equation and the associated parabolic potentials. This solution is used to construct the Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle leaves the boundary of the domain by jumps.

Key words and phrases: parabolic potential, boundary integral equation method, Feller semigroup, nonlocal boundary condition.

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INTRODUCTION

Let $\Pi[0,T] = \{(s,x):\ 0 \le s \le T,\ x \in \mathbb{R}\}$ and let $S_t \subset \Pi[0,T]$ be the curvilinear domain

$$S_t = \{(s,x):\ 0 \le s < t \le T,\ r_1(s) < x < r_2(s)\},$$

where T is a fixed positive number, and r_1, r_2 are given functions defined on $[0,T]$. Denote by D_s the interval $(r_1(s), r_2(s))$ and by $\overline{S}_t$ and $\overline{D}_s$ the closure of S_t and D_s respectively. Denote also by $\mathcal{C}_i$ the curves $\{(s, r_i(s)):\ s \in [0,T]\}$ $(i = 1,2)$ and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

In $\Pi[0,T]$ we consider the parabolic operator of the second order with bounded continuous coefficients

$$\frac{\partial}{\partial s} + L_s \equiv \frac{\partial}{\partial s} + \frac{1}{2} b(s,x) \frac{\partial^2}{\partial x^2} + a(s,x) \frac{\partial}{\partial x}.$$

The main problem is to find a classical solution $u(s,x,t)$ of equation

$$\frac{\partial u}{\partial s} + L_s u = 0, \quad (s,x) \in S_t, \tag{1}$$

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which satisfies the "initial" condition

$$\lim_{s\uparrow t} u(s,x,t) = \varphi(x), \quad x \in \overline{D}_t, \tag{2}$$

and two boundary conditions

$$\int\limits_{D_s} [u(s,r_i(s),t) - u(s,y,t)]\mu_i(s,dy) = 0, \quad 0 \le s \le t \le T, \quad i = 1,2, \tag{3}$$

where φ is the given function and $\mu_i(s,\cdot)\,(s \in [0,T],\ i = 1,2)$ are given finite nonnegative measures on D_s, $s \in [0,T]$.

The problem (1)-(3) appears, in particular, in the theory of stochastic processes while studying the diffusion processes with boundary conditions. Recall that the most general form of boundary conditions for one-dimensional diffusion processes was established in works of W. Feller [2] and A. D. Wentzell [12] (see also [13], where the multidimensional case is considered). From the assertions proved there, it follows that if the ordinary differential operator of the second order is a generator of the Feller semigroup in $C[r_1,r_2]$ (r_1, r_2 are fixed, $-\infty < r_1 < r_2 < \infty$), then its domain of definition consists of functions satisfying nonlocal boundary conditions. In the general case, these boundary conditions contain the values of the function and its first-order derivatives with respect to the time variable and with respect to the spatial variable at points r_i, $i = 1,2$, and the nonlocal component of the integral type that correspond, respectively, to such properties of process after it reaches the boundary point r_i as its termination, delay, reflection and jump out of r_i.

In the present paper we shall establish the classical solvability of problem (1)-(3) by the boundary integral equation method with the use of the fundamental solution of the equation (1) and the associated parabolic potentials, and prove that its solution $u(s,x,t) \equiv T_{st}\varphi(x)$ can be treated as the two parameter semigroup of operators describing an inhomogeneous Feller process in $\mathbb{R}$ which trajectories are located in curvilinear domain $\overline{S}_T$. It is easy to understand that the trajectories of this process in $\overline{S}_T \setminus \mathcal{C}$ can be treated as the trajectories of the diffusion process generated by the operator L_s and at the points of curves $\mathcal{C}_i$ $(i = 1,2)$ their behavior is determined by Feller-Wentzell boundary conditions in (3). The conditions in (3) correspond to jump discontinuity of trajectories of process which is caused by inward jump of a Markovian particle from the boundary.

It is necessary to note that the scheme we shall use to solve the problem (1)-(3) is partially presented in work [6], where the similar problem was investigated in the case when the backward Kolmogotov equation is given in $\cup_{i=1}^{2} S_t^{(i)} = \cup_{i=1}^{2}\{(s,x) :\ 0 \le s < t \le T,\ (-1)^i(x - r(s)) > 0\}$ and, at the common boundary $x = r(s)$ of domains $S_t^{(1)}$ and $S_t^{(2)}$, the Feller-Wentzell conjugation condition, which, in addition to the integral term, contains also the local term corresponding to the termination of process, is imposed. We should also mention works [8], [11], which give the results concerning the construction of diffusion processes with nonlocal boundary conditions of the integral type by the methods of stochastics [8] and functional analysis [11].

We need the following conditions:

I. The operator $\partial/\partial s + L_s$ is uniformly parabolic in $\Pi[0,T]$, i.e., there exist constants b and B such that $0 < b \le b(s,x) \le B < \infty$ for all $(s,x) \in \Pi[0,T]$.

II. The coefficients of L_s are bounded and continuous functions in $\Pi[0,T]$ which belong to Hölder class $H^{\frac{\alpha}{2},\alpha}(\Pi[0,T])$, $0 < \alpha < 1$ (to recall the definitions of Hölder classes see [7, p.16]).

III. The function φ in (2) is assumed to be defined on $\mathbb{R}$ and belongs to the space of bounded continuous functions on $\mathbb{R}$, which we will denote by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\| = \sup_{s\in\mathbb{R}} |\varphi(x)|$. Furthermore, two fitting conditions

$$\int\limits_{D_t} [\varphi(r_i(t)) - \varphi(y)]\mu_i(t,dy) = 0, \quad i = 1,2, \quad \text{hold.}$$

IV. The nonnegative measures μ_i in (3) are such that $\mu_i(s, D_s) = 1$, $s \in [0,T]$ and for all $f \in C_b(\mathbb{R})$ the integrals

$$\int\limits_{D_s} f(y)\mu_i(s,dy), \quad i = 1,2,$$

belong to $H^{\frac{1+\alpha}{2}}([0,T])$ as functions of s.

V. The functions $r_i(s)$, $i = 1,2$, are continuous and belong to $H^{\frac{1+\alpha}{2}}([0,T])$.

Conditions I, II ensure the existence of the fundamental solution of the parabolic operator $\partial/\partial s + L_s$ in $\Pi[0,T]$ (see [7, Ch.IV, §15], [9, Ch.II, §3]), i.e., a function $G(s,x,t,y)$ defined for all (s,x) and (t,y) in $\Pi[0,T]$, $s < t$, satisfying the following condition:

for any $\varphi \in C_b(\mathbb{R})$, the function

$$u_0(s,x,t) = \int\limits_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy \tag{4}$$

satisfies the equation (1) if $0 \le s < t \le T$, $x \in \mathbb{R}$ and the condition (2) if $t \in (0,T]$, $x \in \mathbb{R}$.

Note that the function G admits the representation

$$G(s,x,t,y) = Z_0(s,x,t,y) + Z_1(s,x,t,y), \quad i = 1,2,$$

where

$$Z_0(s,x,t,y) = [2\pi b(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b(t,y)(t-s)}\right\},$$

$$Z_1(s,x,t,y) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} Z_0(s,x,\tau,z)Q(\tau,z,t,y)dz,$$

and the function $Q(s,x,t,y)$ is the solution of some singular Volterra integral equation of the second kind. Note also that

$$\left|D_s^r D_x^p Z_0(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{5}$$

$$\left|D_s^r D_x^p Z_1(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\} \tag{6}$$

($0 \le s < t \le T,\ x, y \in \mathbb{R}$), and that for the function u_0 defined by (4) ($\varphi \in C_b(\mathbb{R})$) which is called the Poisson potential in the theory of parabolic equations, the inequality

$$\left|D_s^r D_x^p u_0(s,x,t)\right| \le C\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \quad 0 \le s < t \le T,\ x \in \mathbb{R}, \tag{7}$$

holds. Here C and c are positive constants (we shall subsequently denote various positive constants by symbols C or c without specifying their values), r and p are the nonnegative integers for which $2r + p \le 2$, D_s^r is the partial derivative with respect to s of order r, D_x^p is the partial derivative with respect to x of order p.

In addition to the integral $u_0(s,x,t)$ we need to consider two more integrals

$$u_{i1}(s,x,t) = \int_s^t G(s,x,\tau,r_i(\tau))V_i(\tau,t)d\tau, \quad i = 1,2,$$

where $0 \le s < t \le T$, $x \in \mathbb{R}$ and V_1, V_2 are some functions. The function u_{i1} is called the parabolic simple-layer potential. If we assume that the density $V_i(\tau,t)$ is continuous for $\tau \in [s,t)$ and admits a weak singularity with an exponent of not less than $-\frac{1}{2}$ when $\tau = t$, then the function $u_{i1}(s,x,t)$, $i = 1,2$, is bounded continuous in $0 \le s \le t \le T$, $x \in \mathbb{R}$ and satisfies the equation (1) in $(s,x) \in [0,t) \times (\mathbb{R} \setminus r_i(s))$ with the initial condition: $u_{i1}(s,x,t) \to 0$ if $s \uparrow t$ ($x \in \mathbb{R}$, $i = 1,2$).

The important property of the function u_{i1} is reflected in the so-called theorem on the jump of conormal derivative of parabolic simple-layer potential (see, e.g. [3, Ch.V, §2], [7, Ch.IV, §15]). In the present paper this assertion is not used, and therefore we do not formulate it.

1 SOLVING THE PARABOLIC BOUNDARY VALUE PROBLEM

We shall find a solution u of problem (1)-(3) as a sum of Poisson potential u_0 and two simple-layer potentials u_{11} and u_{21}, namely:

$$u(s,x,t) = \int_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy + \sum_{j=1}^{2}\int_s^t G(s,x,\tau,r_j(\tau))V_j(\tau,t)d\tau, \quad (s,x) \in \overline{S}_t. \tag{8}$$

Here φ is the function in (2) and V_i, $i = 1,2$, are the unknown densities to be determined.

Note that since $\mu_i(s, D_s) = 1$ for every $s \in [0,T]$ (see the condition IV), the conditions (3) and the fitting conditions in III can be reduced to

$$u(s,r_i(s),t) - \int_{D_s} u(s,y,t)\mu_i(s,dy) = 0, \quad 0 \le s \le t \le T, i = 1,2, \tag{9}$$

and

$$\varphi(r_i(t)) - \int_{D_t} \varphi(y)\mu_i(t,dy) = 0, \quad i = 1,2, \tag{10}$$

respectively.

Substituting (8) into (9), we get the system of two Volterra integral equations of the first kind for the unknowns V_i, $i = 1, 2$, namely

$$\sum_{j=1}^{2} \int_s^t K_{ij}(s,\tau) V_j(\tau,t) d\tau = \Phi_i(s,t), \quad 0 \le s < t \le T,\ i = 1, 2, \tag{11}$$

where

$$K_{ij}(s,\tau) = G(s, r_i(s), \tau, r_j(\tau)) - \int_{D_s} G(s, y, \tau, r_j(\tau)) \mu_i(s, dy),$$

$$\Phi_i(s,t) = \int_{D_s} u_0(s,y,t)\mu_i(s,dy) - u_0(s, r_i(s), t).$$

Using Holmgren's method [4] (see also [5]) we shall reduce (11) to an equivalent system of Volterra integral equations of the second kind. For this purpose we consider the integro-differential operator

$$\mathcal{E}(s,t) f = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t (\rho - s)^{-\frac{1}{2}} f(\rho, t) d\rho, \quad 0 \le s < t \le T$$

and apply it to the both sides of each equation in (11).

The application of the operator $\mathcal{E}$ to the left-hand side of (11) gives the expression which after interchanging the order of integration takes on the form

$$I_i(s,t) \equiv \sum_{j=1}^{2} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t V_j(\tau, t) d\tau \int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}(\rho, \tau) d\rho.$$

Write K_{ij} as $K_{ij}(\rho,\tau) = K_{ij}^{(1)}(\rho,\tau) + K_{ij}^{(2)}(\rho,\tau) - K_{ij}^{(3)}(\rho,\tau)$, where

$$K_{ij}^{(1)}(\rho,\tau) = Z_0(\rho, r_i(\tau), \tau, r_j(\tau)),$$
$$K_{ij}^{(2)}(\rho,\tau) = Z_1(\rho, r_i(\tau), \tau, r_j(\tau)) + [G(\rho, r_i(\rho), \tau, r_j(\tau)) - G(\rho, r_i(\tau), \tau, r_j(\tau))],$$
$$K_{ij}^{(3)}(\rho,\tau) = \int_{D_\rho} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(\rho, dy) + \int_{D_\rho} Z_1(\rho, y, \tau, r_j(\tau)) \mu_i(\rho, dy),$$

and denote by $J_{ij}(s,\tau)$ the integral $\int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}(\rho,\tau) d\rho$, and by $J_{ij}^{(k)}(s,\tau)$ the integral $\int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}^{(k)}(\rho,\tau) d\rho$, $k = 1, 2, 3$.

Note that $J_{ij}^{(1)}(s,\tau)$ is equal to

$$\frac{1}{\sqrt{2\pi b(\tau, r_i(\tau))}} \int_s^\tau (\tau - \rho)^{-\frac{1}{2}} (\rho - s)^{-\frac{1}{2}} d\rho = \sqrt{\frac{\pi}{2b(\tau, r_i(\tau))}},$$

when $i = j$, and tends to zero as $s \uparrow \tau$ when $i \neq j$. Note also that application of the mean value theorem to difference $G(\rho, r_i(\rho), \tau, r_j(\tau)) - G(\rho, r_i(\tau), \tau, r_j(\tau))$ together with the condition V and the estimates (5), (6) lead to the estimate

$$|K_{ij}^{(2)}(\rho,\tau)| \le |Z_1(\rho, r_i(\tau), \tau, r_j(\tau))| + |D_x^1 G(\rho, x_0, \tau, r_j(\tau))| \cdot |r_i(\tau) - r_i(\rho)| \le C(\tau - \rho)^{-\frac{1}{2} + \frac{\alpha}{2}}$$

(x_0 is a point in the open interval with endpoints $r_i(\tau)$ and $r_i(\rho)$) from which it follows that $J_{ij}^{(2)}(s,\tau) \to 0$ as $s \uparrow \tau$.

Hence,

$$I_{ij}^{(k)}(s,t) \equiv \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t V_j(\tau,t) J_{ij}^{(k)}(s,\tau) d\tau = \sqrt{\frac{2}{\pi}} \int_s^t V_j(\tau,t) \frac{\partial}{\partial s} J_{ij}^{(k)}(s,\tau) d\tau \tag{12}$$

if $k=1$, $i \neq j$ or if $k=2$. If $k=1$ and $i=j$, then $I_{ij}^{(k)}(s,t) = -\frac{V_i(s,t)}{\sqrt{b(s,r_i(s))}}$.

Let us show that the relation (12) is true also for $k=3$. For this it suffices to prove that

$$\lim_{s \uparrow \tau} J_{ij}^{(3)}(s,\tau) = 0. \tag{13}$$

Let us denote by $K_{ij}^{(31)}$ the first term in the expression for $K_{ij}^{(3)}$ and by $J_{ij}^{(31)}$ the integral $J_{ij}^{(3)}$ with $K_{ij}^{(3)}$ replaced by $K_{ij}^{(31)}$. In view of (5) and (6), we may verify (13) only for $J_{ij}^{(31)}$.

Write $J_{ij}^{(31)}$ in the form $J_{ij}^{(31)}(s,\tau) = L_{ij}^{(1)}(s,\tau) + L_{ij}^{(2)}(s,\tau) + L_{ij}^{(3)}(s,\tau)$, $i=1,2$, $j=1,2$, where

$$L_{ij}^{(1)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} d\rho \Bigg[\int_{D_\rho} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-\rho)}\right\} \mu_i(\rho,dy)$$
$$- \int_{D_s} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-\rho)}\right\} \mu_i(s,dy) \Bigg],$$
$$L_{ij}^{(2)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_{D_s} \Bigg[\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\}$$
$$- \exp\left\{-\frac{(y-r_j(s))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\} \Bigg] R_j(s,\tau,y) \mu_i(s,dy),$$

$$L_{ij}^{(3)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_{D_s} \exp\left\{-\frac{(y-r_j(s))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\} R_j(s,\tau,y) \mu_i(s,dy),$$

and $R_j(s,\tau,y)$ denotes the integral

$$R_j(s,\tau,y) = \int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)} \cdot \frac{\rho-s}{\tau-\rho}\right\} d\rho,$$

which after the change of variables $z = \frac{\rho-s}{\tau-\rho}$ takes on the form

$$R_j(s,\tau,y) = \int_0^\infty z^{-\frac{1}{2}}(z+1)^{-1} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)} \cdot z\right\} dz,$$

and so

$$|R_j(s,\tau,y)| \leq C. \tag{14}$$

From this and IV it follows immediately that

$$|L_{ij}^{(1)}(s,\tau)| \le C(\tau - s)^{\frac{1+\alpha}{2}}, \tag{15}$$

$$|L_{ij}^{(3)}(s,\tau)| \le C\left(\mu_i(s, U_\delta(r_j(s))) + \exp\left\{-\frac{\delta^2}{2B(\tau - s)}\right\}\right), \tag{16}$$

where $U_\delta(r_j(s)) = \{y \in D_s : |y - r_j(s)| < \delta\}$, δ is any positive constant, B is the constant from I. Applying the mean value theorem to the difference of exponents within the braces in the expression for $L_{ij}^{(2)}$, we get, after using the condition V as well as the estimate (14) and the inequality $\sigma^\nu \exp\{-c\sigma\} \le C$ $(0 \le \sigma < \infty, 0 \le \nu < \infty)$,

$$|L_{ij}^{(2)}(s,\tau)| \le C(\tau - s)^{\frac{\alpha}{2}}. \tag{17}$$

The estimates (15)–(17) imply that $J_{ij}^{(31)}(s,\tau) \to 0$ as $s \uparrow \tau$. This completes the proof of (13). Thus, the relation (12) holds also for $k = 3$.

Let us apply the operator $\mathcal{E}$ to the right-hand side of (11). In order to simplify the expression for $\Upsilon_i(s,t) \equiv \mathcal{E}(s,t)\Phi_i(s,t)$ we need to prove the following two relations:

$$\Phi_i(s,t) \to 0 \text{ as } s \uparrow t, \tag{18}$$

$$|\Phi_i(s,t) - \Phi_i(\tilde{s},t)| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\tilde{s})^{\frac{1+\alpha}{2}}, \quad 0 \le \tilde{s} < s < t \le T. \tag{19}$$

Passing to the limit as $s \uparrow t$ in the expression for Φ_i $(i = 1,2)$, and recalling that the Poisson potential u_0 satisfies the condition (2), we get the expression which equals the left side of (10) taken with the opposite sign and which therefore vanishes. Thus (18) holds.

We proceed to prove (19). Write the difference $\Phi_i(s,t) - \Phi_i(\tilde{s},t)$ in the form

$$\begin{aligned} \Phi_i(s,t) - \Phi_i(\tilde{s},t) = &\int_{D_s} [u_0(s,y,t) - u_0(\tilde{s},y,t)]\mu_i(s,dy) \\ &+ \left[\int_{D_s} u_0(\tilde{s},y,t)\mu_i(s,dy) - \int_{D_{\tilde{s}}} u_0(\tilde{s},y,t)\mu_i(\tilde{s},dy)\right] \\ &+ [u_0(\tilde{s},r_i(\tilde{s}),t) - u_0(s,r_i(\tilde{s}),t)] + [u_0(s,r_i(\tilde{s}),t) - u_0(s,r_i(s),t)] \end{aligned} \tag{20}$$

and note that for $\tilde{s} < s$

$$\begin{aligned} |u_0(s,y,t) - u_0(\tilde{s},y,t)| &= |u_0(s,y,t) - u_0(\tilde{s},y,t)|^{\frac{1+\alpha}{2}} |u_0(s,y,t) - u_0(\tilde{s},y,t)|^{\frac{1-\alpha}{2}} \\ &\le \left|\left.\frac{\partial u_0(\hat{s},y,t)}{\partial \hat{s}}\right|_{\hat{s}=\tilde{s}+\theta(s-\tilde{s})} \cdot (s-\tilde{s})\right|^{\frac{1+\alpha}{2}} (|u_0(s,y,t)| + |u_0(\tilde{s},y,t)|)^{\frac{1-\alpha}{2}} \\ &\le C\|\varphi\|\left[(t - \tilde{s} - \theta(s-\tilde{s}))^{-1}(s-\tilde{s})\right]^{\frac{1+\alpha}{2}} \\ &\le C\|\varphi\|\left[((t-s) + (s-\tilde{s})(1-\theta))^{-1}(s-\tilde{s})\right]^{\frac{1+\alpha}{2}} \\ &\le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\tilde{s})^{\frac{1+\alpha}{2}}, \quad 0 < \theta < 1. \end{aligned}$$

Using this inequality for differences $u_0(s,y,t) - u_0(\tilde{s},y,t)$, $u_0(\tilde{s},r_i(\tilde{s}),t) - u_0(s,r_i(\tilde{s}),t)$ and the condition IV to estimate the difference of integrals in the second line of the expression (20) as well as the Lagrange formula together with the condition V and the inequality (7) (with $r = 0,\ p = 1$) to estimate the last term $u_0(s, r_i(\widetilde{s}), t) - u_0(s, r_i(s), t)$ in (20), we arrive at (19).

Taking into account (18) and (19) we see thus that the application of the operator $\mathcal{E}$ to the function Φ_i gives

$$\Upsilon_i(s,t) = \frac{1}{\sqrt{2\pi}} \int_s^t (\rho - s)^{-\frac{3}{2}} [\Phi_i(\rho, t) - \Phi_i(s,t)] d\rho - \sqrt{\frac{2}{\pi}} (t-s)^{-\frac{1}{2}} \Phi_i(s,t). \tag{21}$$

Having considered the action of the operator $\mathcal{E}$ on both sides of (11), we can now write the system of Volterra integral equations of the second kind for the unknowns V_i, $i = 1, 2$, which is equivalent to (11) and has the form

$$V_i(s,t) = \sum_{j=1}^{2} N_{ij}(s,\tau) V_j(\tau, t) d\tau + \Psi_i(s,t), \quad 0 \le s < t \le T,\ i = 1,2, \tag{22}$$

where

$$\begin{aligned}
\Psi_i(s,t) &= -\sqrt{b(s, r_i(s))}\,\Upsilon_i(s,t), \\
N_{ii}(s,\tau) &= \sqrt{\frac{2b(s, r_i(s))}{\pi}} \frac{\partial}{\partial s} \big(J_{ii}^{(2)}(s,\tau) - J_{ii}^{(3)}(s,\tau)\big), \quad i = j, \\
N_{ij}(s,\tau) &= \sqrt{\frac{2b(s, r_i(s))}{\pi}} \frac{\partial}{\partial s} J_{ij}(s,\tau), \quad i \ne j.
\end{aligned}$$

Note that from (21), (19) and (7) (with $r = p = 0$), it follows that

$$|\Psi_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}.$$

Unfortunately, the kernels N_{ij} do not have a weak singularity. We can not find the estimate for $N_{ij}(s,\tau)$ better than $C(\tau - s)^{-1}$. However this difficulty arises due to only one term

$$\int\limits_{U_\delta(r_j(s))} \frac{\partial}{\partial y} Z_0(s, y, \tau, r_j(\tau)) \mu_i(s, dy)$$

which appears after writing $\frac{\partial}{\partial s} J_{ij}^{(31)}(s,\tau)$ in the form

$$\begin{aligned}
\frac{\partial}{\partial s} J_{ij}^{(31)}(s,\tau) &= \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} \Big(\int\limits_{D_\rho} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(\rho, dy) \\
&- \int\limits_{D_{s_0}} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(s_0, dy) \Big)\Big|_{s_0 = s} + \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int\limits_{D_{s_0}} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(s_0, dy) \Big|_{s_0 = s}
\end{aligned}$$

and then taking the derivative of the last term in this expression. Namely,

$$
\begin{aligned}
&\frac{\partial}{\partial s}\int_s^{\tau}(\rho-s)^{-\frac{1}{2}}d\rho\int_{D_{s_0}} Z_0(\rho,y,\tau,r_j(\tau))\mu_i(s_0,dy)\Big|_{s_0=s} = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}}\\
&\times\frac{\partial}{\partial s}\int_{D_{s_0}}\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\}R_j(s,\tau,y)\mu_i(s_0,dy)\Big|_{s_0=s} = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}}\\
&\times\frac{\partial}{\partial s}\int_{D_{s_0}}\mu_i(s_0,dy)\int_0^{\infty}z^{-\frac{1}{2}}(z+1)^{-1}\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\cdot(z+1)\right\}dz\Big|_{s_0=s}\\
&=\sqrt{\frac{\pi b(\tau,r_j(\tau))}{2}}\int_{D_s}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy) = \sqrt{\frac{\pi b(\tau,r_j(\tau))}{2}}\\
&\times\Bigg(\int_{U_\delta(r_j(s))}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy)+\int_{D_s\setminus U_\delta(r_j(s))}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy)\Bigg).
\end{aligned}
$$

All other terms in the expression for N_{ij} can be estimated by $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$, where $C(\delta)$ is the positive constant depending on δ.

Despite the strong singularity of kernels N_{ij}, the system of equations (22) has a solution and this solution can be found by the method of successive approximations:

$$
V_i(s,t)=\sum_{n=0}^{\infty}V_i^{(n)}(s,t),\quad 0\le s<t\le T,\ i=1,2, \tag{23}
$$

where

$$
V_i^{(0)}(s,t)=\Psi_i(s,t),\qquad V_i^{(n)}(s,t)=\sum_{j=1}^{2}\int_s^t N_{ij}(s,\tau)V_j^{(n-1)}(\tau,t)d\tau,\quad n=1,2,\ldots.
$$

The convergence of series (23) and so the existence of the function V_i follows from the next inequality

$$
|V_i^{(n)}(s,t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{n}C_n^k a^{(n-k)}m^k,\quad 0\le s<t\le T,\ i=1,2, \tag{24}
$$

where

$$
\begin{aligned}
a^{(k)}&=\frac{\left(2C(\delta_0)T^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\right)^k\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1+k\alpha}{2}\right)},\quad k=0,1,\ldots,n,\\
m&=\max_{s\in[0,T]}\left\{\sum_{j=1}^{2}\mu_i(s,U_{\delta_0}(r_j(s))),\ i=1,2\right\}
\end{aligned}
$$

and the constant $\delta=\delta_0$ is chosen to be sufficiently small so that $m<1$. One can prove the estimate (24) by induction and by using the scheme analogous to those used in the proofs of (15), (16) and (17). Note also that the similar scheme was used in [10] in the study of the system of Volterra integral equations of the second kind with strong singularity in the kernels.

From (24) it also follows that the function $V_i(s,t)$, $i=1,2$, satisfies the inequality

$$|V_i(s,t)| \leq C\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{25}$$

Thus, the formula (23) represents the unique solution of (22), which is continuous in the domain $0 \leq s < t \leq T$ and satisfies the inequality (25).

From estimates (5) (with $r = p = 0$) and (25) it follows that there exist the simple-layer potentials $u_{i1}(s,x,t)$, $i=1,2$, in (8), and for them the condition $u_{i1}(s,x,t) \to 0$ if $s \uparrow t$ and the inequality

$$|u_{i1}(s,x,t)| \leq C\|\varphi\|, \quad (s,x) \in \overline{S}_t, \tag{26}$$

hold. It is obvious (see (7)) that the same inequality is also true for the Poisson potential $u_0(s,x,t)$ in (8) and thus for the function $u(s,x,t)$ as well. Recalling that $u_0(s,x,t) \to \varphi(x)$ if $s \uparrow t$ and that the functions $u_0(s,x,t)$ and $u_{i1}(s,x,t)$ satisfy equation (1) in the domain $(s,x) \in S_t$ we conclude that $u(s,x,t)$ is the desired classical solution of problem (1)-(3).

Let us prove the uniqueness of the solution of the problem (1)-(3). Suppose that the problem (1)-(3) has two solutions $u_1(s,x,t)$ and $u_2(s,x,t)$ which are continuous in $\overline{S}_t$. Then the function $\overline{u} \equiv u_1 - u_2$ satisfies equation (1), the initial condition (2) with $\varphi \equiv 0$ and two boundary conditions

$$u(s,r_i(s),t) = g_i(s,t), \quad 0 \leq s < t \leq T,\ i=1,2,$$

where

$$g_i(s,t) = \int\limits_{D_s} \overline{u}(s,y,t)\mu_i(s,dy).$$

The above problem is the first boundary value problem and since the function g_i is continuous in s, it has a unique classical solution, continuous in $\overline{S}_t$, which can be represented in the form (8) with $\varphi \equiv 0$. Thus, the function $\overline{u}$ can be expressed in the form (8) where there are no Poisson potential and V_i are the unknown functions, continuous in $s \in [0,t)$, which are determined by $g_i(s,t)$. Further, if we repeat the considerations of this section concerning the construction of solution of the problem (1)-(3), we obtain the system (22) with $\Psi_i \equiv 0$ for the unknowns V_i. Then $V_i \equiv 0$ and hence $\overline{u} \equiv 0$. This completes the proof of the uniqueness.

Thus we have proved the following theorem:

Theorem 1. *Let conditions I-V hold. Then problem* (1)-(3) *has a unique classical solution, continuous in $\overline{S}_t$ for all $t \in (0,T]$. Furthermore, this solution has the form* (8) *and satisfies the inequality* (26).

2 FELLER SEMIGROUP

Suppose that the conditions I-V hold and consider the two-parameter family of linear operators T_{st}, $0 \leq s < t \leq T$, acting on the function $\varphi \in C_b(\mathbb{R})$ by the rule:

$$T_{st}\varphi(x) = \int\limits_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy + \sum_{j=1}^{2}\int\limits_{s}^{t} G(s,x,\tau,r_j(\tau))V_j(\tau,t)d\tau, \tag{27}$$

where the pair of functions (V_1,V_2) is the solution of (22). Recall that the function V_i $(i=1,2)$ has the form (23) and satisfy the inequality (25).

We introduce the subspace $C_0(\mathbb{R})$ of $C_b(\mathbb{R})$ which consists of all functions $\varphi \in C_b(\mathbb{R})$ for which the fitting conditions in III holds. Since the subspace $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$, it is a Banach space. Furthermore, it is invariant under the operators T_{st}, i.e.,

$$\varphi \in C_0(\mathbb{R}) \Longrightarrow T_{st}\varphi \in C_0(\mathbb{R}).$$

Let us study properties of the family of operators T_{st} in $C_0(\mathbb{R})$.

First we note that if the sequence $\varphi_n \in C_b(\mathbb{R})$ is such that $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$ and, in addition, $\sup\limits_n \|\varphi_n\| < \infty$, then $\lim\limits_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \le s < t \le T$, $x \in \overline{D}_s$. The proof of this property is based on well known assertions of calculus on passage of the limit under the summation and integral signs (here this concerns series (23) and integrals on the right-hand side of (8)). This property allows us to prove the following properties of the operator family T_{st} without loss of generality, under the assumption that the function φ has a compact support.

Now we prove that the operators T_{st}, $0 \le s < t \le T$, remain the cone of nonnegative functions invariant.

Lemma 1. *If $\varphi \in C_0(\mathbb{R})$ and $\varphi(x) \ge 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \ge 0$ for all $x \in \overline{D}_s$, $0 \le s < t \le T$.*

Proof. Let φ be any nonnegative function in $C_0(\mathbb{R})$ with a compact support. Denote by γ the minimum of $T_{st}\varphi(x)$ in $\overline{S}_t$ and assume that $\gamma < 0$. From the minimum principle [3, Ch.II] it follows that the value γ may be attained only when $s \in (0, t)$ and $x = r_i(s)$, $i = 1, 2$. Fix $s_0 \in (0, t)$ and $i_0 \in \{1, 2\}$ for which $T_{s_0t}\varphi(r_{i_0}(s_0)) = \gamma$. But then

$$\int\limits_{D_{s_0}} [T_{s_0t}\varphi(r_{i_0}(s_0)) - T_{s_0t}\varphi(y)]\mu_{i_0}(s_0, dy) < 0$$

which contradicts (3). Therefore $\gamma \ge 0$ and the assertion of the lemma follows. □

Note also that $T_{st}\varphi_0(x) = 1$ for all $0 \le s < t \le T$, $x \in \overline{D}_s$ if $\varphi_0 \equiv 1$. This property together with the assertion of lemma 1 allow us to assert that operators T_{st} are contractive, i.e.,

$$\|T_{st}\varphi\| \le \|\varphi\|$$

for all $0 \le s < t \le T$.

Finally, we show that the operator family T_{st} has the semigroup property

$$T_{st} = T_{s\tau}T_{\tau t}, \quad 0 \le s < \tau < t \le T.$$

This property is a consequence of the assertion of uniqueness of the solution of the problem (1)-(3). Indeed, to find $u(s, x, t) = T_{st}\varphi(x)$, when it is given that $u(s, x, t) \to \varphi(x)$ as $s \uparrow t$, one can solve the problem first in time interval $[\tau, t]$ and then solve it in the time interval $[s, \tau]$ with that initial function $u(\tau, x, t) = T_{\tau t}\varphi(x)$ which was obtained; in other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x)$, $\varphi \in C_0(\mathbb{R})$ or $T_{st} = T_{s\tau}T_{\tau t}$.

The above properties of operators T_{st} imply the following assertion (see [1, Ch.II, §1]).

Theorem 2. *Let conditions I-V hold. Then the two-parameter family of operators T_{st}, $0 \le s < t \le T$, defined by formula (27) describes the inhomogeneous Feller process in $\mathbb{R}$ which trajectories are located in curvilinear domain $\overline{S}_T$. In $\overline{S}_T \setminus \mathcal{C}$, the trajectories of this process can be treated as the trajectories of the diffusion process generated by the operator L_s and at the points of curves $\mathcal{C}_i$ $(i = 1, 2)$ they behave according to boundary conditions in (3).*

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Шевчук Р.В., Савка І.Я., Нитребич З.М. *Нелокальна крайова задача для одновимірного оберненого рівняння Колмогорова і пов'язана з нею напівгрупа операторів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 463–474.

Стаття присвячена вивченню методами теорії диференціальних рівнянь в частинних похідних проблеми побудови напівгруп Феллера, які описують одновимірні дифузійні процеси в областях із заданими крайовими умовами. У цій статті ми досліджуємо крайову задачу для одновимірного лінійного параболічного рівняння другого порядку (оберненого рівняння Колмогорова) у криволінійній обмеженій області з одним із варіантів нелокальної крайової умови типу Феллера-Вентцеля. Ми зосереджуємо увагу на випадку, коли крайова умова Феллера-Вентцеля містить лише компоненту інтегрального типу. Класичну розв'язність останньої задачі одержано нами методом граничних інтегральних рівнянь з використанням фундаментального розв'язку оберненого рівняння Колмогорова і породжених ним параболічних потенціалів. Цей розв'язок використано для побудови напівгрупи Феллера, яка описує явище дифузії в обмеженій області з властивістю повернення дифундуючої частинки в середину області стрибками.

Ключові слова і фрази: параболічний потенціал, метод граничних інтегральних рівнянь, напівгрупа Феллера, нелокальна крайова умова.