Carpathian mathematical publications

SCIENTIFIC JOURNAL
V.11, №2
2019

CONTENTS

Карпатські математичні публікації

Науковий журнал
Т.11, №2
2019

Зміст

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 213–227
doi:10.15330/cmp.11.2.213-227

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.213–227

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ANALYTIC VECTOR-FUNCTIONS IN THE UNIT BALL HAVING BOUNDED L-INDEX IN JOINT VARIABLES

In this paper, we consider a class of vector-functions, which are analytic in the unit ball. For this class of functions there is introduced a concept of boundedness of **L**-index in joint variables, where $\mathbf{L} = (l_1, l_2) : \mathbb{B}^2 \to \mathbb{R}^2_+$ is a positive continuous vector-function, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| = \sqrt{|z_1|^2 + |z_2|^2} \leq 1\}$. We present necessary and sufficient conditions of boundedness of **L**-index in joint variables. They describe the local behavior of the maximum modulus of every component of the vector-function or its partial derivatives.

Key words and phrases: bounded index, bounded **L**-index in joint variables, analytic function, unit ball, local behavior, maximum modulus, sup-norm, vector-valued function.

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1 INTRODUCTION

A concept of bounded index for entire function [19] draws attention of many mathematician (see a full bibliography in [5, 24, 26]) to investigations of these function class and its possible applications. It is interesting with its connections with value distribution theory, because every entire function has a bounded value distribution if and only if its derivative has a bounded index [15]. Also, there are many papers devoted to index boundedness of analytic solutions of differential equations [12, 13, 18]. It is important because any function of bounded index have its growth estimates, local behavior of derivatives and some uniform distribution of zeros. Moreover, some authors [26–33] study connection between p-valence and l-index boundedness of analytic functions, the existence of solutions of the second order linear differential equations with polynomial coefficients which are starlike, convex, close-to-convex and of bounded l-index ($l : \mathbb{C} \to \mathbb{R}_+$ is a continuous function). In other words, they combine analytic and geometric properties of functions of complex variable. Let us give a main definition introduced by B. Lepson [19]. An entire function f is said to be of bounded l-index if there exists an integer m, independent of z, such that for all p and all $z \in \mathbb{C}$ $\frac{|f^{(p)}(z)|}{p!} \leq \max\{\frac{|f^{(s)}(z)|}{s!} : 0 \leq s \leq m\}$. If we replace $p!$ by $p! l^p(z)$ and $s!$ by $s! l^s(z)$ in the definition, respectively, then we obtain the definition of entire function of bounded l-index [17]. The generalization was proposed by A.D. Kuzyk and M.M. Sheremeta to go beyond class of entire functions of exponential type because every entire function of bounded index is of exponential type [15].

Of course, there are papers on analytic curves of bounded l-index. This function class naturally appears if we consider systems of differential equations and investigate properties

УДК 517.55
2010 *Mathematics Subject Classification:* 32A10, 32A17, 32A37.

of their analytic solutions. A concept of bounded index for entire curves was introduced with the sup-norm [16] and with the Euclidean norm [23]. In these papers the authors replaced the modulus of function by the appropriate norm in the definition. Later there was proposed a definition of bounded ν-index [22] for entire curves with these norms. In this definition, R. Roy and S.M. Shah replaced $p!$ by $p!|z|^p$ and so on. Also M.T. Bordulyak and M.M. Sheremeta [14, 25] studied curves of bounded l-index which are analytic in arbitrary bounded domain on the complex plane. These mathematicians found sufficient conditions providing l-index boundedness of every analytic solutions for some system of differential equations.

Recently, there was published paper [21] about entire vector-valued bivariate functions having bounded index. The authors considered a concept of bounded index with the sup-norm. We will develop their approach and will investigate vector-valued functions which are analytic in the unit ball.

Our present investigation has used methods of A.I. Bandura and O.B. Skaskiv developed them for analytic functions in the unit ball [2–4, 10]. It is known that analytic function with unbounded multiplicities of zeros is of unbounded l-index for any positive continuous function l. The similar statement is valid for functions analytic in the unit ball [1]. In other words, functions with unbounded multiplicities of zero points are not still objects of investigations in theory of bounded index. But we can replace studying of properties of the function f with unbounded multiplicities of zero points by studying of properties of the map $(f, 1)$. Such approach allows to investigate any analytic functions in theory of bounded index.

2 NOTATIONS AND DEFINITIONS

Here we use some standard notations (see [3–5]). Let $\mathbb{R}_+ = [0; +\infty)$, $\mathbf{0} = (0, 0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1, 1) \in \mathbb{R}^2_+$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $|(z, w)| = \sqrt{|z|^2 + |w|^2}$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$, we will use formal notations without assumption of the existence of these expressions: $AB = (a_1b_1, a_2b_2)$, $A/B = (a_1/b_1, a_2/b_2)$, $A^B = (a_1^{b_1}, a_2^{b_2})$, and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, 2\}$; the relation $A \leq B$ is defined in the similar way. For $K = (k_1, k_2) \in \mathbb{Z}^2_+$ let us denote $K! = k_1! \cdot k_2!$. Addition, multiplication by scalar and conjugation in $\mathbb{C}^2$ is defined componentwise. For $a = (a_1, a_2) \in \mathbb{C}^2$, $b = (b_1, b_2) \in \mathbb{C}^2$ we define $\langle a, b\rangle = a_1\overline{b}_1 + a_2\overline{b}_2$, where $\overline{b}_1, \overline{b}_2$ is the complex conjugate of b_1, b_2.

The polydisc $\{(z, w) \in \mathbb{C}^2 : |z - z_0| < r_1, |w - w_0| < r_2\}$ is denoted by $\mathbb{D}^2((z_0, w_0), R)$, its skeleton $\{(z, w) \in \mathbb{C}^2 : |z - z_0| = r_1, |w - w_0| = r_2\}$ is denoted by $\mathbb{T}^2((z_0, w_0), R)$, the closed polydisc $\{(z, w) \in \mathbb{C}^2 : |z - z_0| \leq r_1, |w - w_0| \leq r_2\}$ is denoted by $\mathbb{D}^2[(z_0, w_0), R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0}; \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The open ball $\{(z, w) \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |w - w_0|^2} < r\}$ is denoted by $\mathbb{B}^2((z_0, w_0), r)$, the sphere $\{(z, w) \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |w - w_0|^2} = r\}$ is denoted by $\mathbb{S}^2((z_0, w_0), r)$, and the closed ball $\{(z, w) \in \mathbb{C}^2 : \sqrt{|z - z_0|^2 + |w - w_0|^2} \leq r\}$ is denoted by $\mathbb{B}^2[(z_0, w_0), r]$, $\mathbb{B}^2 = \mathbb{B}^2(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$.

Let $F(z, w) = (f_1(z, w), f_2(z, w))$ be an analytic vector-function in $\mathbb{B}^2$. Then at a point $(a, b) \in \mathbb{B}^2$ the function $F(z, w)$ has a bivariate vector-valued Taylor expansion:

$$F(z, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} C_{kl}(z - a)^k (w - b)^m,$$

where

$$C_{km} = \frac{1}{k!m!} \left(\frac{\partial^{k+m} f_1(z, w)}{\partial z^k \partial w^m}, \frac{\partial^{k+m} f_2(z, w)}{\partial z^k \partial w^m} \right) \Bigg|_{z=a, w=b} = \frac{1}{k!m!} F^{(k,m)}(a, b).$$

Let $\mathbf{L}(z,w) = (l_1(z,w), l_2(z,w))$, where $l_j(z,w) : \mathbb{B}^2 \to \mathbb{R}^2_+$ are positive continuous functions such that

$$\forall (z,w) \in \mathbb{B}^2 : \ \ l_j(z,w) > \frac{\beta}{1-\sqrt{|z|^2+|w|^2}}, \quad j \in \{1,2\}, \tag{1}$$

where $\beta > \sqrt{2}$ is a some constant.

Remark 1. *Note that from $R \in \mathbb{R}^2_+$, $|R| = \sqrt{r_1^2 + r_2^2} < \beta$, $(z_0, w_0) \in \mathbb{B}^2$ and $(z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)]$ it follows, that $(z,w) \in \mathbb{B}^2$.*

Indeed,

$$\begin{aligned}|(z,w)| &\le |(z,w)-(z_0,w_0)| + |(z_0,w_0)| \le \sqrt{\frac{r_1^2}{l_1^2(z_0,w_0)} + \frac{r_2^2}{l_2^2(z_0,w_0)}} + |(z_0,w_0)| \\ &< \frac{1-|(z_0,w_0)|}{\beta}\sqrt{r_1^2+r_2^2} + |(z_0,w_0)| \le \frac{1-|(z_0,w_0)|}{\beta}\beta + |z_0,w_0| = 1.\end{aligned}$$

The norm for the vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is defined as the sup-norm

$$\|F(z,w)\| = \max\{|f_1(z,w), |f_2(z,w)|\}.$$

We write

$$F^{(i,j)}(z,w) = \frac{\partial^{i+j}F(z,w)}{\partial z^i \partial w^j} = \left(\frac{\partial^{i+j}f_1(z,w)}{\partial z^i \partial w^j}, \frac{\partial^{i+j}f_2(z,w)}{\partial z^i \partial w^j}\right).$$

An analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ is said to be of bounded $\mathbf{L}$-index (in joint variables), if there exists $n_0 \in \mathbb{Z}_+$ such that $\forall (z,w) \in \mathbb{B}^2 \ \forall (i,j) \in \mathbb{Z}^2_+$:

$$\frac{\|F^{(i,j)}(z,w)\|}{i!j!l_1^i(z,w)l_2^j(z,w)} \le \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k,m \in \mathbb{Z}_+, k+m \le n_0\right\}. \tag{2}$$

The least such integer n_0 is called the $\mathbf{L}$-index in joint variables of the vector-function F and is denoted by $N(F, \mathbf{L}, \mathbb{B}^2)$. The concept of boundedness of $\mathbf{L}$-index in joint variables was considered for other classes of analytic functions. They have differed domains of analyticity: the unit ball [1,3,4,10], the polydisc [7,9], the Cartesian product of the unit disc and complex plane [8], n-dimensional complex space [1,6,11,12].

Example 1. *The function $f(z,w) = \exp\left\{\frac{1}{(1/\sqrt{2}-z)(1/\sqrt{2}-w)}\right\}$ has a bounded $\mathbf{L}$-index in joint variables $N(F, \mathbf{L}, \mathbb{D}^2((0,0),R)) = 0$ in the bidisk $\mathbb{D}^2((0,0),R)$ with $R = (1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{L}(z,w) = \left(\frac{1}{(1/\sqrt{2}-|z|)^2(1/\sqrt{2}-|w|)}, \frac{1}{(1/\sqrt{2}-|z|)(1/\sqrt{2}-|w|)^2}\right)$ (see details in [9]). But $|R| = 1$, therefore, it is easy to see, that the vector-function $F(z,w) = (f(z,w), 1)$ has the same bounded $\mathbf{L}$-index in joint variables $N(F, \mathbf{L}, \mathbb{B}^2) = 0$ in the unit ball $\mathbb{B}^2$.*

$Q(\mathbb{B}^2)$ stands for the function class of $\mathbf{L} : \mathbb{B}^2 \to \mathbb{R}^2_+$, which obey inequality (1) and for any $j \in \{1,2\}$ and some $R = (r_1, r_2)$, $|R| \le \beta$:

$$\sup_{(z_1,w_1),(z_2,w_2)\in\mathbb{B}^2}\left\{\frac{l_j(z_1,w_1)}{l_j(z_2,w_2)} : |z_1-z_2| \le \frac{r_1}{\min\{l_1(z_1,w_1), l_1(z_2,w_2)\}}, \right.$$
$$\left. |w_1-w_2| \le \frac{r_2}{\min\{l_2(z_1,w_1), l_2(z_2,w_2)\}}\right\} < \infty.$$

The function class $Q(\mathbb{B}^2)$ also can be defined as follows: for all $R \in \mathbb{R}^2_+$, $|R| \le \beta$, and for $j \in \{1,2\}$ the inequality $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty$ holds, where

$$\lambda_{1,j}(R) = \inf_{(z_0,w_0)\in\mathbb{B}^2} \inf\left\{ \frac{l_j(z,w)}{l_j(z_0,w_0)} : (z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)] \right\}, \tag{3}$$

$$\lambda_{2,j}(R) = \sup_{(z_0,w_0)\in\mathbb{B}^2} \sup\left\{ \frac{l_j(z,w)}{l_j(z_0,w_0)} : (z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)] \right\}. \tag{4}$$

3 LOCAL BEHAVIOR OF PARTIAL DERIVATIVES OF VECTOR-VALUED BIVARIATE ANALYTIC FUNCTIONS HAVING BOUNDED L-INDEX IN JOINT VARIABLES

The following theorem is basic in the theory of functions of bounded index. Our proof is similar to proof of the corresponding theorem [2] for analytic functions from B^n onto $\mathbb{C}$. For other classes of analytic functions it is proved in [5,8,9,20,24].

Theorem 1. *Let $\mathbf{L} \in Q(\mathbb{B}^2)$. An analytic vector-function $F : \mathbb{B}^2 \to \mathbb{C}^2$ has a bounded $\mathbf{L}$-index in joint variables if and only if for every $R \in \mathbb{R}^2$, $|R| \le \beta$ there exist $n_0 \in \mathbb{Z}_+$, $p > 0$ such that for all $(z_0,w_0) \in \mathbb{B}^2$ there exists 2-tuple $(k_0,m_0) \in \mathbb{Z}^2_+$, $k_0 + m_0 \le n_0$, satisfying inequality*

$$\max\left\{ \frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m \le n_0, (z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)] \right\} \le p_0 \frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{k_0!m_0!l_1^{k^0}(z_0,w_0)l_2^{m^0}(z_0,w_0)}. \tag{5}$$

Proof. Below we repeat considerations from [2], replacing modulus of function by the norm of vector-function.

Let F be an analytic vector-function of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{B}^2) < \infty$. For any $R \in \mathbb{R}^2_+$, $|R| < \beta$, we define

$$q = q(R) = [2(N+1)(r_1+r_2)\prod_{j=1}^{2}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N+1}] + 1,$$

where $[x]$ stands for the entire part of the real number x. For $p \in \{0,\dots,q\}$ and $(z_0,w_0) \in \mathbb{B}^2$ we denote:

$$S_p((z_0,w_0),R) = \max\Big\{ \frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m \le N, (z,w) \in \mathbb{D}^2\Big[(z_0,w_0), \frac{pR}{q\mathbf{L}(z_0,w_0)}\Big]\Big\},$$

$$S_p^*((z_0,w_0),R) = \max\Big\{ \frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z_0,w_0)l_2^m(z_0,w_0)} : k+m \le N, (z,w) \in \mathbb{D}^2\Big[(z_0,w_0), \frac{pR}{q\mathbf{L}(z_0,w_0)}\Big]\Big\}.$$

Using equality (3) and $\mathbb{D}^2[(z_0,w_0), \frac{pR}{q\mathbf{L}(z_0,w_0)}] \subset \mathbb{D}^2[(z_0,w_0), \frac{R}{\mathbf{L}(z_0,w_0)}]$, we have

$$\begin{aligned}
S_p((z_0,w_0),R) &= \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m\le N, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),\frac{pR}{q\mathbf{L}(z_0,w_0)}\right]\right\}\\
&= \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z_0,w_0)l_2^m(z_0,w_0)}\cdot\frac{l_1^k(z_0,w_0)l_2^m(z_0,w_0)}{l_1^k(z,w)l_2^m(z,w)} : k+m\le N,\right.\\
&\qquad\left.(z,w)\in\mathbb{D}^2[(z_0,w_0),\frac{pR}{q\mathbf{L}(z_0,w_0)}]\right\}\\
&\le S_p^*((z_0,w_0),R)\max\left\{\frac{l_1^k(z_0,w_0),l_2^m(z_0,w_0,)}{l_1^k(z,w),l_2^m(z,w)} : k+m\le N,\right.\\
&\qquad\left.(z,w)\in\mathbb{D}^2[(z_0,w_0),\frac{pR}{q\mathbf{L}(z_0,w_0)}]\right\}\\
&\le S_p^*((z_0,w_0),R)\max\{(\lambda_{1,1}(R))^{-k}(\lambda_{1,2}(R))^{-m} : k+m\le N\}\\
&\le S_p^*((z_0,w_0),R)(\lambda_{1,1}(R))^{-N}(\lambda_{1,2}(R))^{-N}\le S_p^*((z_0,w_0),R)\prod_{j=1}^{2}(\lambda_{1,j}(R))^{-N}.
\end{aligned}\tag{6}$$

Taking into account (4), we obtain

$$\begin{aligned}
S_p^*((z_0,w_0),R) &= \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)}\cdot\frac{l_1^k(z,w)l_2^m(z,w)}{l_1^k(z,w)l_2^m(z_0,w_0)} : k+m\le N,\right.\\
&\qquad\left.(z,w)\in\mathbb{D}^2[(z_0,w_0),\frac{(pr_1,pr_2)}{q\mathbf{L}(z_0,w_0)}]\right\}\\
&\le \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)}(\lambda_{2,1}(R))^k(\lambda_{2,2}(R))^m : k+m\le N,\right.\\
&\qquad\left.(z,w)\in\mathbb{D}^2[(z_0,w_0),\frac{(pr_1,pr_2)}{q\mathbf{L}(z_0,w_0)}]\right\}\\
&\le S_p((z_0,w_0),R)(\lambda_{2,1}(R))^N(\lambda_{2,2}(R))^N\le S_p((z_0,w_0),R)\prod_{j=1}^{2}(\lambda_{1,j}(R))^{N}.
\end{aligned}\tag{7}$$

Let $(k_p,m_p)\in\mathbb{Z}_+^2$, $k_p+m_p\le N$ and $(z_p,w_p)\in\mathbb{D}^2\left[(z_0,w_0),\frac{pR}{q\mathbf{L}(z_0,w_0)}\right]$ be such that

$$S_p^*((z_0,w_0),R)=\frac{\|F^{(k_p,m_p)}(z_p,w_p)\|}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)}\,. \tag{8}$$

Since by the maximum modulus principle we have $(z_p,w_p)\in\mathbb{T}^2\left((z_0,w_0),\frac{pR}{q\mathbf{L}(z_0,w_0)}\right)$, therefore $(z_p,w_p)\ne(z_0,w_0)$. We choose

$$\widetilde{z}_p=z_0+\frac{p-1}{p}\left(z_p-z_0\right),\ \ \widetilde{w}_p=w_0+\frac{p-1}{p}\left(w_p-w_0\right).$$

Then we have

$$|\widetilde{z}_p-\widetilde{z}_0|=\frac{p-1}{p}|z_p-z_0|=\frac{p-1}{p}\frac{pr_1}{ql_1(z_0,w_0)},\quad |\widetilde{w}_p-\widetilde{w}_0|=\frac{p-1}{p}\frac{pr_2}{ql_2(z_0,w_0)},$$

$$|\widetilde{z}_p-z_p|=|z_0+\frac{p-1}{p}(z_p-z_0)-z_p|=\frac{1}{p}|z_0-z_p|=\frac{r_1}{ql_1(z_0,w_0)}; \tag{9}$$

$$|\widetilde{w}_p-w_p|=|w_0+\frac{p-1}{p}(w_p-w_0)-w_p|=\frac{1}{p}|w_0-w_p|=\frac{r_2}{ql_2(z_0,w_0)}. \tag{10}$$

We obtain $(\widetilde{z}_p, \widetilde{w}_p) \in \mathbb{D}^2\left[(z_0,w_0), \frac{(p-1)R}{q(R)\mathbf{L}(z_0,w_0)}\right]$ and $S^*_{p-1}((z_0,w_0),R) \geq \frac{\|F^{(k_p,m_p)}(\widetilde{z}_p,\widetilde{w}_p)\|}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)}$.

From (8) by mean value theorem we have

$$\begin{aligned}
0 &\leq S^*_p((z_0,w_0),R) - S^*_{p-1}((z_0,w_0),R) \leq \frac{\|F^{(k_p,m_p)}(z_p,w_p)\| - \|F^{(k_p,m_p)}(\widetilde{z}_p,\widetilde{w}_p)\|}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)} \\
&= \frac{1}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)} \int_0^1 \frac{d}{dt}\|F^{(k_p,m_p)}(\widetilde{z}_p + t(z_p - \widetilde{z}_p), \widetilde{w}_p + t(w_p - \widetilde{w}_p))\|dt \\
&\leq \frac{1}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)} \int_0^1 |z^{(p)} - \widetilde{z}_p|\|F^{(k_p+1,m_p)}(\widetilde{z}_p + t(z_p - \widetilde{z}_p), \widetilde{w}_p + t(w_p - \widetilde{w}_p))\| \\
&\qquad + |w^{(p)} - \widetilde{w}_p|\|F^{(k_p,m_p+1)}(\widetilde{z}_p + t(z_p - \widetilde{z}_p), \widetilde{w}_p + t(w_p - \widetilde{w}_p))\|dt \\
&= \frac{1}{k_p!m_p!l_1^{k_p}(z_0,w_0)l_2^{m_p}(z_0,w_0)} \Big[|z^{(p)} - \widetilde{z}_p|\|F^{(k_p+1,m_p)}(\widetilde{z}_p + t^*(z_p - \widetilde{z}_p), \widetilde{w}_p + t^*(w_p - \widetilde{w}_p))\| \\
&\qquad + |w^{(p)} - \widetilde{w}_p|\|F^{(k_p,m_p+1)}(\widetilde{z}_p + t^*(z_p - \widetilde{z}_p), \widetilde{w}_p + t^*(w_p - \widetilde{w}_p))\|\Big],
\end{aligned} \tag{11}$$

where $0 \leq t^* \leq 1$, and $(\widetilde{z}_p + t^*(z_p - \widetilde{z}_p), \widetilde{w}_p + t^*(w_p - \widetilde{w}_p)) \in \mathbb{D}^2[(z_0,w_0), \frac{pR}{q\mathbf{L}(z_0,w_0)}]$. For $(z,w) \in \mathbb{D}^2[(z_0,w_0), \frac{pR}{q\mathbf{L}(z_0,w_0)}]$ and $(j_1,j_2) \in \mathbb{Z}^2_+$: $j_1 + j_2 \leq N+1$, we have

$$\begin{aligned}
&\frac{\|F^{(j_1,j_2)}(z,w)\|}{j_1!j_2!l_1^{j_1}(z_0,w_0)l_2^{j_2}(z_0,w_0)} \cdot \frac{l_1^{j_1}(z,w)l_2^{j_2}(z,w)}{l_1^{j_1}(z,w)l_2^{j_2}(z,w)} \\
&\qquad \leq \frac{\|F^{(j_1,j_2)}(z,w)\|}{j_1!j_2!l_1^{j_1}(z,w)l_2^{j_2}(z,w)} \max\left\{\frac{l_1^{j_1}(z,w)}{l_1^{j_1}(z_0,w_0)} \cdot \frac{l_2^{j_2}(z,w)}{l_2^{j_2}(z_0,w_0)} : j_1 + j_2 \leq N+1\right\} \\
&\qquad \leq \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m \leq N\right\} \cdot \big(\lambda_{2,1}(\frac{pR}{q})\big)^{N+1} \cdot \big(\lambda_{2,2}(\frac{pR}{q})\big)^{N+1} \\
&\qquad \leq (\lambda_{2,1}(R), \lambda_{2,2}(R))^{N+1} \cdot \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m \leq N\right\} \\
&\qquad = (\lambda_{2,1}(R)\lambda_{2,2}(R))^{N+1} \cdot S_p((z_0,w_0),R) \\
&\qquad \leq (\lambda_{2,1}(R)\lambda_{2,2}(R))^{N+1} \cdot S^*_p((z_0,w_0),R) \cdot (\lambda_{1,1}(R), \lambda_{1,2}(R))^{-N}.
\end{aligned}$$

Then from (11), (9) and (10) we obtain

$$\begin{aligned}
&0 \leq S^*_p((z_0,w_0),R) - S^*_{p-1}((z_0,w_0),R) \\
&\qquad \leq \prod_{j=1}^{2}(\lambda_{2,j}(R))^{N+1}\lambda_{1,j}(R))^{-N} S^*_p((z_0,w_0),R) \\
&\qquad\qquad \times \Big((k^{(p)}+1)(l_1(z_0,w_0))|z_j^{(p)} - \widetilde{z}_j^{(p)}| + (m^{(p)}+1)(l_2(z_0,w_0))|w_j^{(p)} - \widetilde{w}_j^{(p)}|\Big) \\
&\qquad = \prod_{j=1}^{2}(\lambda_{2,j}(R))^{N+1}(\lambda_{1,j}(R))^{-N}\frac{S^*_p((z_0,w_0),R)}{q(R)}((k_p+1)r_1 + (m_p+1)r_2) \\
&\qquad \leq \prod_{j=1}^{2}(\lambda_{2,j}(R))^{N+1}\lambda_{1,j}(R))^{-N}\frac{S^*_p((z_0,w_0),R)}{q(R)}(N+1)(r_1+r_2) \leq \frac{1}{2}S^*_p((z_0,w_0),R).
\end{aligned}$$

It follows that $S_p^*((z_0,w_0),R) \le 2S_{p-1}^*((z_0,w_0),R)$ and in view of (6) and (7) one has

$$
\begin{aligned}
S_p((z_0,w_0),R) &\le 2\prod_{j=1}^{2}(\lambda_{1,j}(R))^{-N}S_{p-1}^*((z_0,w_0),R)\\
&\le 2\prod_{j=1}^{2}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}S_{p-1}((z_0,w_0),R).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m\le N, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),\frac{qR}{q\mathbf{L}(z_0,w_0)}\right]\right\}\\
&= S_q((z_0,w_0),R) \le 2\prod_{j=1}^{2}(\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N}S_{q-1}((z_0,w_0),R)\\
&\le \ldots \le 2\prod_{j=1}^{2}((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}S_0((z_0,w_0),R)\\
&= 2\prod_{j=1}^{2}((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}\max\left\{\frac{\|F^{(k,m)}(z_0,w_0)\|}{k!m!l_1^k(z_0,w_0)l_2^m(z_0,w_0)} : k+m\le N\right\}.
\end{aligned}
\tag{12}
$$

This inequality implies (5) with $p_0 = 2\prod_{j=1}^{2}((\lambda_{1,j}(R))^{-N}(\lambda_{2,j}(R))^{N})^{q}$ and some k_0, m_0, such that $k_0+m_0\le N$. The necessity of condition (5) is proved.

Now we prove the sufficiency. Assume that for every $R\in\mathbb{R}_+^2$, $|R|\le\beta$, there exist $n_0\in\mathbb{Z}_+$, $p_0>1$, such that for every $(z_0,w_0)\in\mathbb{B}_+^2$ and for some $(k_0,m_0)\in\mathbb{Z}_+^2$, $(k_0+m_0\le n_0)$, inequality (5) holds. By Cauchy's integral formula we have $(\forall(z_0,w_0)\in\mathbb{B}^2)$, $(\forall(k,m)\in\mathbb{Z}_+^2)$, $(\forall(s,y)\in\mathbb{Z}_+^2)$:

$$
\frac{F^{(k+s,m+y)}(z_0,w_0)}{s!y!} = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2\left((z_0,w_0),\frac{R}{\mathbf{L}(z_0,w_0)}\right)}\frac{F^{(k,m)}(z,w)}{(z-z_0)^{s+1}(w-w_0)^{y+1}}dzdw.
$$

Hence, in view of (5), we obtain that

$$
\begin{aligned}
\frac{\|F^{(k+s,m+y)}(z_0,w_0)\|}{s!y!} &\le \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2\left((z_0,w_0),\frac{R}{\mathbf{L}(z_0,w_0)}\right)}\frac{\|F^{(k,m)}(z,w)\|}{|z-z_0|^{s+1}|w-w_0|^{y+1}}|dz||dw|\\
&\le \int_{\mathbb{T}^2\left((z_0,w_0),\frac{R}{\mathbf{L}(z_0,w_0)}\right)}\|F^{(k,m)}(z,w)\|\frac{l_1^{s+1}(z_0,w_0)l_2^{y+1}(z_0,w_0)}{(2\pi)^2r_1^{s+1}r_2^{y+1}}|dz||dw|\\
&\le \int_{\mathbb{T}^2\left((z_0,w_0),\frac{R}{\mathbf{L}(z_0,w_0)}\right)}\|F^{(k,m)}(z_0,w_0)\|\frac{k!m!p_0\lambda_{2,1}^k(R)\lambda_{2,2}^m(R)}{(2\pi)^2k_0!m_0!r_1^{s+1}r_2^{y+1}}\\
&\times\frac{l_1^{s+k+1}(z_0,w_0)l_2^{y+m+1}(z_0,w_0)}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}|dz||dw|\\
&= \|F^{(k,m)}(z_0,w_0)\|\frac{k!m!p_0\lambda_{2,1}^k(R)\lambda_{2,2}^m(R)l_1^{s+k}(z_0,w_0)l_2^{y+m}(z_0,w_0)}{k_0!m_0!r_1^sr_2^yl_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}\\
&= \|F^{(k,m)}(z_0,w_0)\|\frac{k!m!p_0\prod_{j=1}^{2}\lambda_{2,j}^{n_0}(R)l_1^{s+k}(z_0,w_0)l_2^{y+m}(z_0,w_0)}{k_0!m_0!r_1^sr_2^yl_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}.
\end{aligned}
$$

It follows that

$$\frac{\|F^{(k+s,m+y)}(z_0,w_0)\|}{(k+s)!(m+y)!l_1^{k+s}(z_0,w_0)l_2^{m+y}(z_0,w_0)} \leq \frac{\prod_{j=1}^{2}\lambda_{2,j}^{n_0}(R)k!m!p_0\|F^{(k_0,m_0)}(z_0,w_0)\|s!y!}{r_1^s r_2^y(k+s)!(m+y)!k_0!m_0 l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}. \tag{13}$$

It is obvious that $\frac{k!s!}{(k+s)!} = \frac{s!}{(k+1)\cdot\ldots\cdot(k+s)} \leq 1$, $\frac{m!y!}{(m+y)!} = \frac{y!}{(m+1)\cdot\ldots\cdot(m+y)} \leq 1$. We choose $r_j \in (1,\beta/\sqrt{2}]$, $j \in \{1,2\}$. Then $|R| = \sqrt{\sum_{j=1}^{2} r_j^2} \leq \beta$. Thus, $\frac{p_0\lambda_{2,1}^{k}(R)\lambda_{2,2}^{m}(R)}{r_1^s r_2^y} \to 0$ as $s+y \to \infty$, $k+m \leq n_0$.

Therefore, there exists s_0 such that for every $(s,y) \in \mathbb{Z}_+^2$ with $s+y \geq s_0$ the inequality holds

$$\frac{p_0 k!m!s!y!\lambda_{2,1}^{k}(R)\lambda_{2,2}^{m}(R)}{(k+s)!(m+y)!r_1^s r_2^y} = \frac{p_0 k!m!s!y!\prod_{j=1}^{2}\lambda_{2,j}^{n_0}(R)}{(k+s)!(m+y)!r_1^s r_2^y} \leq 1.$$

Then, in view of (13), one has

$$\frac{\|F^{(k+s,m+y)}(z_0,w_0)\|}{(k+s)!(m+y)!l_1^{k+s}(z_0,w_0)l_2^{m+y}(z_0,w_0)} \leq \frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{k_0!m_0!l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}.$$

It implies that for all $(j_1,j_2) \in \mathbb{Z}_+^2$

$$\frac{\|F^{(j_1,j_2)}(z_0,w_0)\|}{j_1!j_2!l_1^{j_1}(z_0,w_0)l_2^{j_2}(z_0,w_0)} \leq \max\left\{\frac{\|F^{(k,m)}(z_0,w_0)\|}{k!m!l_1^{k}(z_0,w_0)l_2^{m}(z_0,w_0)} : k+m \leq s_0+n_0\right\},$$

where s_0 and n_0 do not depend on (z_0,w_0). Then the analytic vector-function F in $\mathbb{B}^2$ has bounded $\mathbf{L}$-index in joint variables $N(F,\mathbf{L},\mathbb{B}^2) \leq s_0+n_0$. □

Note that instead of sup-norm $\|F(z,w)\| = \max_{1\leq j\leq 2}\{|f_j(z,w)|\}$ one can consider the Euclidean norm $\|F(z,w)\|_E = \sqrt{|f_1(z,w)|^2+|f_2(z,w)|^2}$.

Theorem 1 implies the following corollary.

Corollary 1. *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$. *An analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *has a bounded* $\mathbf{L}$*-index in joint variables in* sup*-norm if and only if it has a bounded* $\mathbf{L}$*-index in joint variables in the Euclidean norm.*

Proof. Obviously, that for all $(k,s) \in \mathbb{Z}_+^2$ and for all $(z,w) \in \mathbb{B}^2$ we obtain

$$\|F^{(k,s)}(z,w)\| \leq \|F^{(k,s)}(z,w)\|_E \leq \sqrt{2}\|F^{(k,s)}(z,w)\|.$$

Using the given double inequality and repeating arguments from Theorem 1 for the case of the Euclidean norm we can verify the equivalence of these norms for vector-functions having bounded $\mathbf{L}$-index in joint variables. □

Further, we will use only the sup-norm.

The following proposition was obtained for entire curves in [14]. Here we deduce it for vector-functions which are analytic in the unit ball.

Proposition 1. *Let* $\mathbf{L}$ *be a positive continuous function in* $\mathbb{B}^2$ *satisfying condition* (1) *and each component* f_j *of an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *is of bounded* $\mathbf{L}$*-index in joint variables. Then* F *is of bounded* $\mathbf{L}$*-index in joint variables by the sup-norm with* $N(\mathbf{L}; F) \leq \max\{N(l_s, f_s) : 1 \leq s \leq 2\}$ *and* F *is of bounded* $\mathbf{L}_*$*-index by the Euclidean norm with* $\mathbf{L}_*(z,w) \geq \sqrt{2}\mathbf{L}(z,w)$ *and*

$$N(\mathbf{L}_*, F) \leq \max\{N(l_s, f_s) : 1 \leq s \leq 2\}.$$

Proof. For all $i + j \geq N = \max\{N(\mathbf{L}, f_s) : 1 \leq s \leq 2\}$ we have

$$\begin{aligned}\frac{\|F^{(i,j)}(z,w)\|}{i!j!l_1^i(z,w)l_2^j(z,w)} &= \frac{\max\{|f_1^{(i,j)}(z,w)|, |f_2^{(i,j)}(z,w)|\}}{i!j!l_1^i(z,w)l_2^j(z,w)} \\ &\leq \max\left\{\frac{|f_s^{(k,m)}(z,w)|}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N, 1 \leq s \leq 2\right\} \\ &\leq \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N\right\},\end{aligned}$$

that is, $N(\mathbf{L}; F) \leq N = \max\{N(\mathbf{L}; f_s) : 1 \leq s \leq 2\}$. Also

$$\begin{aligned}\frac{\|F^{(i,j)}(z,w)\|_E}{i!j!l_1^i(z,w)l_2^j(z,w)} &= \frac{\sqrt{\sum_{s=1}^2 |f_s^{(i,j)}(z,w)|^2}}{i!j!l_1^i(z,w)l_2^j(z,w)} \\ &\leq \sqrt{\sum_{s=1}^{2}\left(\max\left\{\frac{|f_s^{(k,m)}(z,w)|}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N\right\}\right)^2} \\ &\leq \sqrt{2}\max\left\{\frac{|f_s^{(k,m)}(z,w)|}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N, 0 \leq s \leq 2\right\} \\ &\leq \sqrt{2}\max\left\{\frac{\|F^{(k,m)}(z,w)\|_E}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N\right\}\end{aligned}$$

and, thus, for $i + j \geq N + 1$

$$\begin{aligned}\frac{\|F^{(i,j)}(z,w)\|_E}{i!j!l_{*1}^i(z,w)l_{*2}^j(z,w)} &\leq \frac{1}{\sqrt{2}^{N+1}}\frac{\|F^{(i,j)}(z,w)\|_E}{i!j!l_1^i(z,w)l_2^j(z,w)} \\ &\leq \frac{1}{\sqrt{2}^N}\max\left\{\frac{\|F^{(k,m)}(z,w)\|_E}{k!m!l_1^k(z,w)l_2^m(z,w)} : 0 \leq k+m \leq N\right\} \\ &\leq \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_{*1}^k(z,w)l_{*2}^m(z,w)} : 0 \leq k+m \leq N\right\},\end{aligned}$$

that is, $N(\mathbf{L}_*, F) \leq \max\{N(\mathbf{L}, f_j) : 1 \leq j \leq 2\}$. Proposition is proved. □

Theorem 2. *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$*. In order that an analytic vector-function* $F : \mathbb{B}^2 \to \mathbb{C}^2$ *be of bounded* $\mathbf{L}$*-index in joint variables it is necessary that for all* $R \in \mathbb{R}^2$*,* $|R| \leq \beta$ *there exist* $n_0 \in \mathbb{Z}_+$*,* $p \geq 1$ *such that for all* $(z_0, w_0) \in \mathbb{B}^2$ *there exists* $(k_0, m_0) \in \mathbb{Z}_+^2$*,* $k_0 + m_0 \leq n_0$*, satisfying inequality*

$$\max\{\|F^{(k_0,m_0)}(z,w)\| : (z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)]\} \leq p\|F^{(k_0,m_0)}(z_0,w_0)\| \tag{14}$$

and it is sufficiently that for all $R \in \mathbb{R}^2$, $|R| \leq \beta$ there exist $n_0 \in \mathbb{Z}_+$, $p \geq 1$ $\forall (z_0, w_0) \in \mathbb{B}^2$ $\exists k_1^0 = (k_1^0, 0)$, $\exists m_2^0 = (0, m_2^0)$: $k_1^0 \leq n_0$, $m_2^0 \leq n_0$, and

$$\max\{\|F^{(k_1^0,0)}(z_0, w_0)\| : (z,w) \in \mathbb{D}^2[(z_0, w_0), R/\mathbf{L}(z_0, w_0)]\} \leq p\|F^{(k_1^0,0)}(z_0, w_0)\| \tag{15}$$

$$\max\{\|F^{(0,m_2^0)}(z, w)\| : (z,w) \in \mathbb{D}^2[(z_0, w_0), R/\mathbf{L}(z_0, w_0)]\} \leq p\|F^{(0,m_2^0)}(z_0, w_0)\|. \tag{16}$$

Proof. Then by Theorem 1 inequality (5) is obeyed for some tuple (k_0, m_0). We obtain

$$\frac{p_0}{k_0!m_0!}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}$$

$$\geq \max\left\{\frac{\|F^{(k_0,m_0)}(z,w)\|}{k_0!m_0!l_1^{k_0}(z,w)l_2^{m_0}(z,w)} : (z,w) \in \mathbb{D}^2\left[(z_0,w_0), R/\mathbf{L}(z_0,w_0)\right]\right\}$$

$$= \max\left\{\frac{\|F^{(k_0,m_0)}(z,w)\|}{k_0!m_0!}\frac{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)l_1^{k_0}(z,w)l_2^{m_0}(z,w)} :\right.$$

$$\left.(z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)]\right\}$$

$$= \max\left\{\frac{\|F^{(k_0,m_0)}(z,w)\|}{k_0!m_0!}\frac{\prod_{j=1}^2(\lambda_{2,j}(R))^{-n_0}}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)} : (z,w) \in \mathbb{D}^2\left[(z_0,w_0), R/\mathbf{L}(z_0,w_0)\right]\right\}.$$

From this inequality it follows

$$\frac{p_0(\lambda_{2,1}(R))^{n_0}(\lambda_{2,2}(R))^{n_0}}{k_0!m_0!}\cdot\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}$$

$$\geq \max\left\{\frac{\|F^{(k_0,m_0)}(z,w)\|}{k_0!m_0!l_1^{k_0}(z_0,w_0)l_2^{m_0}(z0,w_0)} : (z,w) \in \mathbb{D}^2\left[(z_0,w_0), R/\mathbf{L}(z_0,w_0)\right]\right\}.$$

From inequality (14) it follows (5) with $p = p_0(\lambda_{2,1}(R))^{n_0}(\lambda_{2,2}(R))^{n_0}$. The necessity of condition (14) is proved.

Now we prove the sufficiency of (15) and (16). Suppose that for each $R \in \mathbb{R}^2$, $|R| \leq \beta$ there exist $n_0 \in \mathbb{Z}_+$, $p \geq 1$ such that for every $(z_0, w_0) \in \mathbb{B}^2$ and some $k_1^0 \in \mathbb{Z}_+$, $m_2^0 \in \mathbb{Z}_+$ with $k_1^0 \leq n_0$, $m_2^0 \leq n_0$ inequalities (15) and (16) hold.

Let us write the Cauchy formula in the form $\forall (z_0, w_0) \in \mathbb{B}^2$ $\forall (s, y) \in \mathbb{Z}_+^2$

$$\frac{F^{(k_1^0+s,y)}(z_0,w_0)}{s!y!} = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}\frac{F^{(k_1^0,0)}(z,w)dzdw}{(z-z_0)^{s+1}(w-w_0)^{y+1}},$$

$$\frac{F^{(s,m_2^0+y)}(z_0,w_0)}{s!y!} = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}\frac{F^{(0,m_2^0)}(z,w)dzdw}{(z-z_0)^{s+1}(w-w_0)^{y+1}}.$$

We obtain that

$$\frac{\|F^{(k_1^0+s,y)}(z_0,w_0)\|}{s!y!} \leq \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}\frac{\|F^{(k_1^0,0)}(z,w)\|}{|z-z_0|^{s+1}|w-w_0|^{y+1}}|dz||dw|$$

$$\leq \frac{1}{(2\pi)^2}\max\left\{\|F^{(k_1^0,0)}(z,w)\| : (z,w) \in \mathbb{D}^2\left[(z_0,w_0), R/\mathbf{L}(z_0,w_0)\right]\right\}$$

$$\times\frac{l_1^{s+1}(z_0,w_0)l_2^{y+1}(z_0,w_0)}{r_1^{s+1}r_2^{y+1}}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}|dz||dw|$$

$$= \max\left\{\|F^{(k_1^0,0)}(z,w)\| : (z,w) \in \mathbb{D}^2[(z_0,w_0), R/\mathbf{L}(z_0,w_0)]\right\}\frac{l_1^s(z_0,w_0)l_2^y(z_0,w_0)}{r_1^s r_2^y},$$

$$\frac{\|F^{(s,m_2^0+y)}(z_0,w_0)\|}{s!y!} \le \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}\frac{\|F^{(0,m_2^0)}(z,w)\|}{|z-z_0|^{s+1}|w-w_0|^{y+1}}|dz||dw|$$
$$\le \frac{1}{(2\pi)^2}\max\left\{\|F^{(0,m_2^0)}(z,w)\| : (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/\mathbf{L}(z_0,w_0)\right]\right\}$$
$$\times\frac{l_1^{s+1}(z_0,w_0)l_2^{y+1}(z_0,w_0)}{r_1^{s+1}r_2^{y+1}}\int_{\mathbb{T}^2((z_0,w_0),R/\mathbf{L}(z_0,w_0))}|dz||dw|$$
$$=\max\left\{\|F^{(0,m_2^2)}(z,w)\| : (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/\mathbf{L}(z_0,w_0)\right]\right\}\frac{l_1^s(z_0,w_0)l_2^y(z_0,w_0)}{r_1^s r_2^y}.$$

Put $R=\left(\frac{\beta}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$. In view of (15) and (16) we have

$$\frac{\|F^{(k_1^0+s,y)}(z_0,w_0)\|}{s!y!}$$
$$\le\frac{l_1^s(z_0,w_0)l_2^y(z_0,w_0)}{(\beta/\sqrt{2})^{s+y}}\max\left\{\|F^{(k_1^0,0)}(z,w)\| : (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/L(z_0,w_0)\right]\right\}$$
$$\le\frac{pl_1^s(z_0,w_0)l_2^y(z_0,w_0)}{(\beta/\sqrt{2})^{s+y}}\|F^{(k_1^0,0)}(z_0,w_0)\|, \tag{17}$$

$$\frac{\|F^{(s,m_2^0+y)}(z_0,w_0)\|}{s!y!}$$
$$\le\frac{l_1^s(z_0,w_0)l_2^y(z_0,w_0)}{(\beta/\sqrt{2})^{s+y}}\max\left\{\|F^{(0,m_2^0)}(z,w)\| : (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/\mathbf{L}(z_0,w_0)\right]\right\}$$
$$\le\frac{pl_1^s(z_0,w_0)l_2^y(z_0,w_0)}{(\beta/\sqrt{2})^{s+y}}\|F^{(0,m_2^0)}(z_0,w_0)\|. \tag{18}$$

We choose $s,y\in\mathbb{Z}_+^2$ such that $s+y\ge s_0$, where $\frac{p}{(\beta/\sqrt{2})^{s_0}}\le 1$.

Then from (17) and (18) we obtain as $k_1^0\le n_0$, $m_2^0\le n_0$

$$\frac{\|F^{(k_1^0+s,y)}(z_0,w_0)\|}{l_1^{k_1^0+s}(z_0,w_0)l_2^y(z_0,w_0)(k_1^0+s)!y!}\le\frac{p}{(\beta/\sqrt{2})^{s+y}}\cdot\frac{s!y!k_1^0!}{(s+k_1^0)!y!}\cdot\frac{\|F^{(k_1^0,0)}(z_0,w_0)\|}{l_1^{k_1^0}(z_0,w_0)k_1^0!}$$
$$\le\frac{\|F^{(k_1^0,0)}(z_0,w_0)\|}{l_1^{k_1^0}(z_0,w_0)k_1^0!},$$
$$\frac{\|F^{(s,m_2^0+y)}(z_0,w_0)\|}{l_1^s(z_0,w_0)l_2^{m_2^0+y}(z_0,w_0)s!(m_2^0+y)!}\le\frac{p}{(\beta/\sqrt{2})^{s+y}}\cdot\frac{s!y!m_2^0!}{s!(m_2^0+y)!}\cdot\frac{\|F^{(0,m_2^0)}(z_0,w_0)\|}{l_2^{m_2^0}(z_0,w_0)m_2^0!}$$
$$\le\frac{\|F^{(0,m_2^0)}(z_0,w_0)\|}{l_2^{m_2^0}(z_0,w_0)m_2^0!}.$$

Therefore, $N(F,\mathbf{L},\mathbb{B}^2)\le n_0+s_0$. □

Lemma 1. *Let $\mathbf{L}_1$, $\mathbf{L}_2\in Q(\mathbb{B}^2)$ and for every point $(z,w)\in\mathbb{B}^2$ one has $\mathbf{L}_1(z,w)\le\mathbf{L}_2(z,w)$. If an analytic vector-function F in $\mathbb{B}^2$ has a bounded $\mathbf{L}_1$-index in joint variables, then the vector-function F has a bounded $\mathbf{L}_2$-index in joint variables and $N(F,\mathbf{L}_2,\mathbb{B}^2)\le 2N(F,\mathbf{L}_1,\mathbb{B}^2)$.*

Proof. Let $N(F, \mathbf{L}_1, \mathbb{B}^2) = n_0$. In view of (2) we obtain that

$$\begin{aligned}
\frac{\|F^{(i,j)}(z,w)\|}{i!j!\mathbf{L}_2^{i,j}(z,w)} &= \frac{\|F^{(i,j)}(z,w)\|}{i!j!l_{2,1}^i(z,w)l_{2,2}^j(z,w)} \\
&= \frac{l_{1,1}^i(z,w)l_{1,2}^j(z,w)}{l_{2,1}^i(z,w)l_{2,2}^j(z,w)} \cdot \frac{\|F^{(i,j)}(z,w)\|}{i!j!l_{1,1}^i(z,w)l_{1,2}^j(z,w)} \\
&\le \frac{l_{1,1}^i(z,w)l_{1,2}^j(z,w)}{l_{2,1}^i(z,w)l_{2,2}^j(z,w)} \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_{1,1}^k(z,w)l_{1,2}^m(z,w)} : (k,m) \in \mathbb{Z}_+^2, k+m \le n_0\right\} \\
&\le \frac{l_{1,1}^i(z,w)l_{1,2}^j(z,w)}{l_{2,1}^i(z,w)l_{2,2}^j(z,w)} \\
&\quad \times \max\left\{\frac{l_{2,1}^k(z,w)l_{2,2}^m(z,w)}{l_{1,1}^k(z,w)l_{1,2}^m(z,w)} \frac{\|F^{(k,m)}(z,w)\|}{k!m!l_{2,1}^k(z,w)l_{2,2}^m(z,w)} : (k,m) \in \mathbb{Z}_+^2, k+m \le n_0\right\} \\
&\le \max_{k+m\le n_0}\left\{\left(\frac{l_{1,1}(z,w)}{l_{2,1}(z,w)}\right)^{i-k} \cdot \left(\frac{l_{1,2}(z,w)}{l_{2,2}(z,w)}\right)^{j-m}\right\} \\
&\quad \times \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_{2,1}^k(z,w)l_{2,2}^m(z,w)} : (k,m) \in \mathbb{Z}_+^2, k+m \le n_0\right\}.
\end{aligned}$$

Since $\mathbf{L}_1(z,w) \le \mathbf{L}_2(z,w)$, for all $i + j \ge 2n_0$ we have

$$\frac{\|F^{(i,j)}(z,w)\|}{i!j!l_{2,1}^i(z,w)l_{2,2}^j(z,w)} \le \max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_{2,1}^k(z,w)l_{2,2}^m(z,w)} : (k,m) \in \mathbb{Z}_+^2, k+m \le n_0\right\}.$$

Therefore, the vector-function F has a bounded $\mathbf{L}_2$-index in joint variables and

$$N(F, \mathbf{L}_2, \mathbb{B}^2) \le 2N(F, \mathbf{L}_1, \mathbb{B}^2).$$

□

The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist $\theta_1 \in \mathbb{R}_+$, $\theta_2 \in \mathbb{R}_+$ such that for all $z \in \mathbb{B}^2$ and for each $j \in \{1,2\}$ we have

$$\theta_1 \widetilde{l}_j(z) \le l_j(z) \le \theta_2 \widetilde{l}_j(z).$$

Lemma 2. *Let* $\mathbf{L} \in Q(\mathbb{B}^2)$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta(\Theta_1) > 1$. *An analytic vector-function F in $\mathbb{B}^2$ has a bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if it has a bounded $\mathbf{L}$-index in joint variables.*

Proof. It is easy to prove that with $\mathbf{L} \in Q(\mathbb{B}^2)$ and $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ corresponding function $\widetilde{\mathbf{L}} \in Q(\mathbb{B}^2)$.

Let $N(F, \widetilde{\mathbf{L}}, \mathbb{B}^2) = \widetilde{n}_0 < +\infty$. Then by Theorem 1 for each $\widetilde{R} = (\widetilde{r}_1, \widetilde{r}_2) \in \mathbb{R}_+^2$, $|R| \le \beta$ there exists $\widetilde{p} \ge 1$ such that for all $(z_0, w_0) \in \mathbb{B}^2$ and some (k_0, m_0) with $k_0 + m_0 \le \widetilde{n}_0$ inequality (5) is true with $\widetilde{\mathbf{L}}$ and $\widetilde{R}$ instead of $\mathbf{L}$ and R, respectively. Hence, we have

$$\frac{\widetilde{p}}{k_0!m_0!}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)} = \frac{\widetilde{p}}{k_0!m_0!}\frac{\theta_2^{k_0+m_0}}{\theta_2^{k_0+m_0}}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)}$$
$$\geq \frac{\widetilde{p}}{k_0!m_0!}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{\theta_2^{k_0+m_0}\widetilde{l}_1^{k_0}(z_0,w_0)\widetilde{l}_2^{m_0}(z_0,w_0)}$$
$$\geq \frac{1}{\theta_2^{k_0+m_0}}\max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!\widetilde{l}_1^{k_0}(z,w)\widetilde{l}_2^{m_0}(z,w)} : k+m\leq \widetilde{n}_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/\mathbf{L}(z,w)\right]\right\}$$
$$\geq \frac{1}{\theta_2^{k_0+m_0}}\max\left\{\theta_1^{k+m}\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2{}^m(z,w)} : k+m\leq \widetilde{n}_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),\theta_1\widetilde{R}/\mathbf{L}(z,w)\right]\right\}$$
$$\geq \frac{\min\{1,\theta_1^{n_0}\}}{\max\{1,\theta_2^{n_0}\}}\max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2{}^m(z,w)} : k+m\leq \widetilde{n}_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),\theta_1\widetilde{R}/\widetilde{\mathbf{L}}(z,w)\right]\right\}.$$

By Theorem 1 we conclude that the vector-function F has a bounded $\mathbf{L}$-index in joint variables. □

Theorem 3. *Let* $\mathbf{L}\in Q(\mathbb{B}^2)$, $\beta>2$. *An analytic vector-function* $F:\mathbb{B}^2\to\mathbb{C}^2$ *has a bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $R\in\mathbb{R}^2_+$, $|R|\leq\beta$, $n_0\in\mathbb{Z}^2_+$ *and* $p_0>0$ *such that for all* $(z_0,w_0)\in\mathbb{B}^2$ *and for some* $(k_0,m_0)\in\mathbb{Z}^2_+$, $k_0+m_0\leq n_0$ *inequality* (5) *is valid.*

Proof. The necessity of this theorem follows from the necessity of Theorem 1.

Now we prove the sufficiency. From the proof of Theorem 1 with $R=\left(\frac{\beta}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$ we have that $N(F,\mathbf{L},\mathbb{B}^2)<+\infty$.

Let $\mathbf{L}^*(z,w) = \frac{R^0\mathbf{L}(z,w)}{R}$, that is $\left(l_1^*(z,w)=\frac{r_1^0 l_1(z,w)}{r_1}, l_2^*(z,w)=\frac{r_2^0 l_2(z,w)}{r_2}\right)$, where $R^0 = = (r_1^0,r_2^0) = \left(\frac{\beta}{\sqrt{2}},\frac{\beta}{\sqrt{2}}\right)$. In the general case, with validity of (5) for F, $\mathbf{L}$ and $R=(r_1,r_2)$ such that $|R|\leq\beta$, $R\neq R^0$, we get

$$\max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!(l_1^*(z,w))^k(l_2^*(z,w))^m} : k+m\leq n_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R_0/\mathbf{L}^*(z,w)\right]\right\}$$
$$=\max\left\{\frac{\|F^{(k,m)}(z,w)\|}{k!m!(r_1^0 l_1(z,w)/r_1)^k(r_2^0 l_2(z,w)/r_2)^m} : k+m\leq n_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),\frac{R_0}{R_0\mathbf{L}(z,w)/R}\right]\right\}$$
$$\leq\max\left\{\frac{2^{\frac{k+m}{2}}\|F^{(k,m)}(z,w)\|}{k!m!l_1^k(z,w)l_2^m(z,w)} : k+m\leq n_0, (z,w)\in\mathbb{D}^2\left[(z_0,w_0),R/\mathbf{L}(z_0,w_0)\right]\right\}$$
$$\leq\frac{p_0}{k_0!m_0!}\frac{2^{n_0/2}\|F^{(k_0,m_0)}(z_0,w_0)\|}{l_1^{k_0}(z_0,w_0)l_2^{m_0}(z_0,w_0)} = \frac{2^{n_0/2}(\beta/\sqrt{2})^{k_0+m_0}}{r_1^{k_0}r_2^{m_0}k_0!m_0!}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{(r_1^0 l_1(z,w)/r_1)^{k_0}(r_2^0 l_2(z,w)/r_2)^{m_0}}$$
$$\leq 2^{\frac{n_0}{2}}p_0\max\left\{\frac{(\beta/\sqrt{2})^{k_0+m_0}}{r_1^{k_0}r_2^{m_0}} : k_0+m_0\leq n_0\right\}\frac{\|F^{(k_0,m_0)}(z_0,w_0)\|}{k_0!m_0!(l_1^*(z,w))^{k_0}(l_2^*(z,w))^{m_0}},$$

i.e. (5) is true for F, $\mathbf{L}_*$ and $R_0=(\beta/\sqrt{2},\beta/\sqrt{2})$. Hence, by Theorem 1 the vector-function F is of bounded $\mathbf{L}_*$-index in joint variables. By Lemma 2 the vector-function F has a bounded $\mathbf{L}$-index in joint variables. □

REFERENCES

[1] Bandura A., Skaskiv O. *Analytic functions in the unit ball of bounded* **L***-index in joint variables and of bounded L-index in direction: a connection between these classes*. Demonstr. Math. 2019, **52** (1), 82–87. doi:10.1515/dema-2019-0008

[2] Bandura A., Skaskiv O. *Functions analytic in a unit ball of bounded* **L***-index in joint variables*. J. Math. Sci. 2017, **227** (1),1–12. doi:10.1007/s10958-017-3570-6

[3] Bandura A., Skaskiv O. *Sufficient conditions of boundedness of* **L***-index and analog of Hayman's Theorem for analytic functions in a ball*. Stud. Univ. Babeş-Bolyai Math. 2018, **63** (4), 483–501. doi:10.24193/subbmath.2018.4.06

[4] Bandura A. I., Skaskiv O. B. *Analytic functions in the unit ball of bounded* **L***-index: asymptotic and local properties*. Mat. Stud. 2017, **48** (1), 37–73. doi:10.15330/ms.48.1.37-73

[5] Bandura A., Skaskiv O. *Entire functions of several variables of bounded index*. Lviv: Publisher I.E. Chyzhykov, 2016, 128 p.

[6] Bandura A., Skaskiv O. *Asymptotic estimates of entire functions of bounded* **L***-index in joint variables*. Novi Sad J. Math. 2018, **48** (1), 103–116. doi:10.30755/NSJOM.06997

[7] Bandura A., Petrechko N., Skaskiv O. *Maximum modulus in a bidisc of analytic functions of bounded* **L***-index and an analogue of Hayman's theorem*. Mat. Bohemica. 2018, **143** (4), 339–354. doi:10.21136/MB.2017.0110-16

[8] Bandura A.I., Skaskiv O.B., Tsvigun V.L. *Some characteristic properties of analytic functions in* $\mathbb{D} \times \mathbb{C}$ *of bounded* **L***-index in joint variables*. Bukovyn. Mat. Zh. 2018, **6** (1-2), 21–31.

[9] Bandura A.I., Petrechko N.V., Skaskiv O.B. *Analytic in a polydisc functions of bounded* **L***-index in joint variables*. Mat. Stud. 2016, **46** (1), 72–80. doi:10.15330/ms.46.1.72-80

[10] Bandura A.I., Skaskiv O.B. *Partial logarithmic derivatives and distribution of zeros of analytic functions in the unit ball of bounded* **L***-index in joint variables*. J. Math. Sci. 2019, **239** (1), 17–29. doi: 10.1007/s10958-019-04284-z (translation of Ukr. Matem. Visn. 2018, 15 (2), 177–193. (in Ukrainian))

[11] Bandura A.I., Skaskiv O.B. *Exhaustion by balls and entire functions of bounded* **L***-index in joint variables*. Ufa Math. J. 2019, **11** (1), 100–113. doi:10.13108/2019-11-1-100

[12] Bandura A., Skaskiv O. *Analog of Hayman's Theorem and its Application to Some System of Linear Partial Differential Equations*. Zh. Mat. Fiz. Anal. Geom. 2019, 15 (2), 170–191. doi:10.15407/mag15.02.170

[13] Bordulyak M.T. *On the growth of entire solutions of linear differential equations*. Mat. Stud. 2000, **13** (2), 219–223.

[14] Bordulyak M.T., Sheremeta M.M. *Boundedness of l-index of analytic curves*. Mat. Stud. 2011, **36** (2), 152–161.

[15] Hayman W.K. *Differential inequalities and local valency*. Pacific J. Math. 1973, **44** (1), 117–137.

[16] Heath L.F. *Vector-valued entire functions of bounded index satisfying a differential equation*. J. Res. Bur. Stand. 1978, **83** (1), 75–79.

[17] Kuzyk A. D., Sheremeta M. N. *Entire functions of bounded l-distribution of values.* Math. Notes. 1986, **39** (1), 3–8. doi:10.1007/BF01647624 (translation of Mat. Zametki 1986, 39 (1), 3–13. (in Russian))

[18] Kuzyk A. D., Sheremeta, M. N. *On entire functions, satisfying linear differential equations*. Diff. equations. 1990, **26** (10), 1716–1722. (in Russian)

[19] Lepson B. *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*. Proc. Sympos. Pure Math. 1968, **2**, 298–307.

[20] Patterson R.F., Nura, F. *A characterization of holomorphic bivariate functions of bounded index*. Math. Slovaca 2017, **67** (3), 731–736. doi:10.1515/ms-2017-0005

[21] Nuray F., Patterson R.F. *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations*. Mat. Stud. 2018, **49** (1), 67–74. doi:10.15330/ms.49.1.67-74

[22] Roy R., Shah S.M. *Growth properties of vector entire functions satisfying differential equations*. Indian J. Math. 1986, **28** (1), 25–35.

[23] Roy R., Shah S.M. *Vector-valued entire functions satisfying a differential equation*. J. Math. Anal. Appl. 1986, **116** (2), 349–362.

[24] Sheremeta M. *Analytic functions of bounded index*. VNTL Publishers, Lviv, 1999, 141 p.

[25] Sheremeta M. *Boundedness of $l-M$-index of analytic curves*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2011, **75**, 226–231. (in Ukrainian)

[26] Sheremeta M. *Geometric properties of analytic solutions of differential equations*. Publisher I.E. Chyzhykov, Lviv, 2019, 164 p.

[27] Sheremeta Z., Sheremeta M. *Boundedness l-index of analytic functions represented by power series*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2006, **66**, 208–213. (in Ukrainian)

[28] Sheremeta Z. M. *Index boundedness of an entire solution of a differential equation*. Mat. Stud. 2003, **19** (2), 208–212.

[29] Sheremeta Z.M. *On the boundedness l-index of entire solution of some differential equation*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2004, **63**, 148–151.

[30] Sheremeta Z.M. *Properties of derivatives for entire solution of some differential equation*. Mat. Metodi Fiz.-Mekh.Polya. 2006, **49** (2), 80–85.

[31] Sheremeta Z. M., Sheremeta M. M. *On the boundedness l-index of entire solutions of some differential equation*. Dopov. Nats. Akad. Nauk Ukr. 2007, **2**, 31–36.

[32] Sheremeta Z.M., Sheremeta M.M. *Properties of entire solutions of differential equations*. Ukr. Math. J. 2006, **58** (12), 1924–1934. doi:10.1007/s11253-006-0177-3

[33] Zabolotskyi M. V., Sheremeta, Z. M. *On the boundedness of entire solution of some differential equation*. Mat. Metodi Fiz.-Mekh. Polya. 2004, **47** (2), 181–185. (in Ukrainian)

Received 18.07.2019

Бакса В.П. *Аналітичні в одичній кулі вектор-функції обмеженого* **L**-*індексу за сукупністю змінних* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 213–227.

У цій статті ми розглядаємо клас вектор-функцій, аналітичних в одиничній кулі. Для цього класу функцій введено поняття обмеженості **L**-індексу за сукупністю змінних, де $\mathbf{L} = (l_1, l_2) : \mathbb{B}^2 \to \mathbb{R}^2_+$ — додатна неперервна вектор-функція, $\mathbb{B}^2 = \{z \in \mathbb{C}^2 : |z| = \sqrt{|z_1|^2 + |z_2|^2} \leq 1\}$. Нами отримано необхідні й достатні умови обмеженості **L**-індексу за сукупністю змінних. Вони описують локальне поводження максимуму модуля кожного компонента вектор-функції чи її частинних похідних.

Ключові слова і фрази: обмежений індекс, обмежений **L**-індекс за сукупністю змінних, аналітична функція, одинична куля, локальне поводження, максимум модуля, sup-норма, векторнозначна функція.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 228–239
doi:10.15330/cmp.11.2.228-239

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.228–239

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THE NONLOCAL BOUNDARY VALUE PROBLEM WITH PERTURBATIONS OF MIXED BOUNDARY CONDITIONS FOR AN ELLIPTIC EQUATION WITH CONSTANT COEFFICIENTS. I

In this article we investigate a problem with nonlocal boundary conditions which are multipoint perturbations of mixed boundary conditions in the unit square G using the Fourier method.

The properties of a generalized transformation operator $R : L_2(G) \to L_2(G)$ that reflects normalized eigenfunctions of the operator L_0 of the problem with mixed boundary conditions in the eigenfunctions of the operator L for nonlocal problem with perturbations, are studied. We construct a system $V(L)$ of eigenfunctions of operator L. Also, we define conditions under which the system $V(L)$ is total and minimal in the space $L_2(G)$, and conditions under which it is a Riesz basis in the space $L_2(G)$. In the case if $V(L)$ is a Riesz basis in $L_2(G)$, we obtain sufficient conditions under which nonlocal problem has a unique solution in form of Fourier series by system $V(L)$.

Key words and phrases: differential equation with partial derivatives, eigenfunctions, Riesz basis.

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1 INTRODUCTION

The fundamentals of the theory of linear differential equations in partial derivatives with constant coefficients were established by L. Ehrenpreis, L. Hermander, V. Malgrange, I. Petrovsky.

Boundary value problems in bounded domains for certain classes of differential equations with constant coefficients have been studied in [1–13]. This paper is a continuation of the investigations that were begun in [3–6].

For our investigation we will use the following notations. Let $G := \{x := (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1, \ x_2 < 1\}$, $D_1, \ D_2$ are the operators of differentiation by the variables $x_1, \ x_2$ respectively; $H_0 := L_2(0,1)$, $H_1 := L_2(G)$; $H_2 := W_2^{2n}(G)$ be a Sobolev space with a scalar product and norm respectively

$$(u,v;H_2) := (u,v;H_1) + (D_1^{2n}u, D_1^{2n}v;H_1) + (D_2^{2n}u, D_2^{2n}v;H_1), \quad \|u;H_2\| := \sqrt{(u,u;H_2)};$$

$$W := \{v \in C[0,1] : \ v^{(s)} \in C[0,1], \ s = 1,\dots,2n-1, \ v^{(2n)} \in H_0\};$$

$$H_{0,s} := \{u(t) \in H_0 : \ u(t) \equiv (-1)^s u(1-t)\}, \quad s \in \{0,1\};$$

УДК 517.927.5
2010 *Mathematics Subject Classification:* 34G10, 34K10, 34K30, 34L10.

$W_r := W \cap H_{0,r}$, $r = 0, 1$; and $[H_0]$ be a set of linear continuous operators on the space H_0. Let us consider the boundary value problem

$$L(-D_1^2, -D_2^2)u := \sum_{j=0}^{n} a_j D_1^{2j} D_2^{2n-2j} u = f(x), \; x \in G, \tag{1}$$

$$\ell_{s,1}u := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} + \ell_{s,1}^0 u = 0, \tag{2}$$

$$\ell_{n+s,1}u := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0, \tag{3}$$

$$\ell_{s,2}u := D_2^{2s-2}u|_{x_1=0} + D_2^{2s-2}u|_{x_1=1} = 0, \tag{4}$$

$$\ell_{n+s,2}u := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} = 0, \; s = 1, \ldots, n, \tag{5}$$

where

$$\ell_{s,1}^0 u := \sum_{q=0}^{k_{s,1}} \sum_{r=0}^{n_1} b_{s,q,r} D_1^q u|_{x_1=x_{1,r}}, \; s = 1, \ldots, n, \tag{6}$$

$$0 = x_{1,1} < x_{1,2} < \cdots < x_{1,n_1} \le 1, \; a_j, \; b_{s,q,r} \in \mathbb{R},$$

$$q = 0, 1, \ldots, k_{s,1}, \; k_{s,1} < 2n, \; r = 0, 1, \ldots, n_1, \; s = 1, \ldots, n, \; j = 0, 1, \ldots, n.$$

Let $L : H_1 \to H_1$ be the operator of the problem (1)–(6) and

$$Lu := L(-D_1^2, -D_2^2)u, \; u \in D(L),$$

$$D(L) := \{u \in H_2 : \ell_{s,j}u = 0, \; s = 1, \ldots, 2n, \; j = 1, 2\}.$$

Definition. *The function* $y \in D(L)$, *that satisfies equality* $\|L(-D_1^2, -D_2^2)y - f; H_1\| = 0$, *is called a solution of problem* (1)–(6).

Let us consider the following assumptions and theorems, that are necessary for further investigation.

1. Assumption P_1: $b_{s,q,r} = -(-1)^q b_{s,q,n_1-r}$, $x_{1,r} = 1 - x_{1,n_1-r}$, $r = 0, 1, \ldots, n_1$, $s = 1, \ldots, n$.
2. Assumption P_2: $k_{s,1} \le 2s - 2$, $s = 1, \ldots, n$.
3. Assumption P_3: for any real numbers μ_1, μ_2 the positive number $C_1(L)$ exists, that the inequality $C_1(L)|\mu|^{2n} \le |L(\mu_1, \mu_2)|$, $\mu := (\mu_1, \mu_2)$, $|\mu|^2 := |\mu_1|^2 + |\mu_2|^2$, holds.

Theorem 1. *Let Assumption P_1 holds. Then, for an arbitrary* $a_q \in \mathbb{R}$, $q = 0, 1, \ldots, n$, $b_{s,q,r} \in \mathbb{R}$, *the operator L has a set of eigenvalues*

$$\sigma := \{\lambda_{k,m} := L(\mu_{1,k}, \mu_{2,m}), \; \mu_{1,k} = \pi^2 k^2, \; \mu_{2,m} = \pi^2 (2m-1)^2, \; k \in \mathbb{N}, \; m \in \mathbb{N}\}, \tag{7}$$

and the system $V(L)$ of eigenfunctions, which is complete and minimal in the space H_1.

Theorem 2. *Let Assumptions P_1–P_3 hold. Then, the operator L has the system $V(L)$ of eigenfunctions, which is the Riesz basis of the space H_1.*

Theorem 3. *Let Assumptions P_1–P_3 hold. Then, for arbitrary function $f \in H_1$ the unique solution of problem* (1)–(6) *exists.*

Let A_0 be the operator of boundary problem in the space H_0 :

$$-z^{(2)}(t) = g(t),\ t \in (0,1),\ z(0) = z(1) = 0;$$

$$A_0 z := -z^{(2)}(t),\ z(t) \in D(A_0),\ D(A_0) := \{z \in W_2^2(0,1) : z(0) = z(1) = 0\};$$

$$T_1 := \{\tau_{1,s,k}(t) \in H_0 :\ \tau_{1,s,k}(t) := \sqrt{2}\sin\rho_{s,k}t,\ \rho_{s,k} = \pi(2k+s-1),\ k \in \mathbb{N},\ s = 0,1\};$$

$$T_{1,s} := \{\tau_{1,s,k}(t) \in H_{0,s},\ k \in \mathbb{N}\},\ \ s = 0,1;$$

$$\sigma(A_0) := \{\mu_{1,k} = \pi^2 k^2,\ k \in \mathbb{N}\}.$$

Lemma 1. *The operator A_0 has the point spectrum $\sigma(A_0)$ and system of eigenfunctions T_1.*

Proof. A direct substitution proves that the elements of system T_1 are the eigenfunctions of operator A_0, which correspond to the eigenvalues $\sigma(A_0)$.

Taking into account that the subsystem of eigenfunctions $T_{1,s}$ of the operator A_0 is an orthonormal basis of spaces $H_{0,s},\ s = 0,1$, we obtain the statement of the lemma. □

Let $\Theta = \{\theta_k\}_{k=1}^{\infty}$ be any sequence of real numbers. We consider the operator $A_\Theta : H_0 \to H_0$, which has a set of eigenvalues $\sigma(A_0)$, and the system of eigenfunctions

$$V(A_\Theta) := \{v_{s,k}(t, A_\Theta) \in H_0 : v_{0,k}(t, A_\Theta) := \tau_{1,0,k}(t),$$
$$v_{1,k}(t, A_\Theta) := \tau_{1,1,k}(t) + \theta_k\sqrt{2}\cos 2k\pi t,\ k \in \mathbb{N}\}.$$

Lemma 2. *For an arbitrary sequence Θ the system of functions $V(A_\Theta)$ is complete and minimal in the space H_0. The system of functions $V(A_\Theta)$ is the Riesz basis of this space if and only if the sequence Θ is bounded.*

Proof. Suppose that the system $V(A_\Theta)$ is not complete in the space H_0.

Let us suppose that there exist functions $f = f_0 + f_1 \in H_0$, and $f_s \in H_{0,s},\ s = 0,1$, for which the conditions of orthogonality hold:

$$(f, v_{s,k}(t, A_\Theta); H_0) = 0,\ s = 0,1,\ k \in \mathbb{N}.$$

Taking into account, that the system of functions $\tau_{1,0,q}(t) = v_{0,q}(t, A_\Theta),\ q \in \mathbb{N}$, is an orthonormal basis of the space $H_{0,0}$ with respect to the condition of orthogonality, we obtain $f_0 = 0$. Thus $f = f_1 \in H_{0,1}$.

According to the condition of orthogonality we have the relation

$$(f, v_{1,k}(t, A_\Theta); H_0) = (f, \tau_{1,1,k}(t); H_0) = 0,\ k \in \mathbb{N}.$$

Taking into account the totality of the system of functions $V_1(L_0) = T_{1,1}$ in the space $H_{0,1}$, we have $f = f_1 \equiv 0$. Thus the system $V(A_\Theta)$ is total (complete) in the space H_0. Therefore, the operator A_Θ is defined on a dense set of the space H_0.

In the space H_0 let us define the operators

$$R(A_\Theta) := E + S(A_\Theta),\ \ S(A_\Theta)\tau_{1,0,q}(t) := 0,\ S(A_\Theta)\tau_{1,1,q}(t) := \theta_q\sqrt{2}\cos 2q\pi t \in H_{0,0},\ q \in \mathbb{N}.$$

According to equality $S^2(A_\Theta) = 0$ we get the relation $R^{-1}(A_\Theta) = E - S(A_\Theta)$. Therefore, the system of functions $V(A_\Theta)$ is minimal in the space H_0. Let us prove the second part of the lemma.

Necessity. We choose any bounded sequence Θ and show that $S(A_\Theta) : H_0 \to H_0$ is a bounded operator.

Let us expand an arbitrary function $h \in H_0$ into Fourier series

$$h = \sum_{k=1}^{\infty} \sum_{j=0}^{1} h_{j,k} \tau_{1,j,k}(t).$$

Consider $S(A_\Theta)h = \sum\limits_{k=1}^{\infty} \theta_m h_{1,k} \sqrt{2} \cos 2k\pi t.$

Taking into account that the system of functions $\{1, \cos 2k\pi t,\ k \in \mathbb{N}\}$ is an orthonormal basis of $H_{0,0}$ and using Cauchy's inequality, we obtain

$$\|S(A_\Theta)h; H_0\|^2 \le C_1 \|h; H_0\|^2,\ C_1 = \max |\theta_k|^2.$$

Thus $S(A_\Theta) \in [H_0]$.

Taking into account the relation $R^{-1}(A_\Theta) = E - S(A_\Theta),$ we obtain an estimate

$$\|R^{-1}(A_\Theta); [H_0]\|^2 \le C_2,\ C_2 = 2 + 2C_1.$$

Thus the system $V(A_\Theta)$ is the Riesz basis by definition.

Sufficiency. Let $V(A_\Theta)$ be the Riesz basis in the space H_0. Therefore, this system is almost normalized. Thus, for any positive numbers $C_3 \le C_4$ the next inequality holds:

$$C_3 \le \|v_{s,m}(t, A_\Theta); H_0\| \le C_4 < \infty,\ m \in \mathbb{N}.$$

Taking into account the equalities

$$\|v_{0,k}(t, A_\Theta); H_0\| = 1,\ \|v_{1,m}(t, A_\Theta); H_0\| = 1 + |\theta_m|,\ k = 0, 1, \ldots,\ m \in \mathbb{N},$$

we obtain the proof of sufficiency. □

Let B_0 be the operator of spectral problem

$$-z^{(2)}(t) = \mu z(t),\ \mu \in \mathbb{C},$$
$$\ell_1 z := z(0) + z(1) = 0,$$
$$\ell_2 z := z^{(1)}(0) + z^{(1)}(1) = 0,$$
$$B_0 z := -z^{(2)}(t),\ z(t) \in D(B_0),\ D(B_0) := \{z \in W_2^2(0,1) : \ell_s z = 0,\ s = 1, 2\},$$
$$T_2 := \{\tau_{2,r,m}(t) \in H_0 : \tau_{2,0,m}(t) := \sqrt{2} \sin \pi(2m-1)t,\ \tau_{2,1,m}(t) := \sqrt{2} \cos \pi(2m-1)t,\ m \in \mathbb{N}\},$$
$$\sigma(B_0) := \{\mu_{2,m} = \pi^2 (2m-1)^2,\ m \in \mathbb{N}\}.$$

Lemma 3. *The operator B_0 has the point spectrum $\sigma(B_0)$ and system of eigenfunctions T_2.*

Proof. After performing a direct substitution we obtain that

$$\tau_{2,r,m}(t) \in D(B_0),\ -\tau_{2,r,m}^{(2)}(t) = \mu_{2,m} \tau_{2,r,m}(t),\ r = 0, 1,\ m \in \mathbb{N}.$$

Thus operator L_0 has the system of eigenfunctions $V(L_0),$ which corresponds to the set of eigenvalues σ. □

For the equation (1) we consider the boundary conditions $\ell_{0,s,j}u = 0$, $s = 1,\dots,2n$, $j = 1,2$, which are the partial case of boundary conditions (2)–(6) for $\ell^1_{s,1}u = 0$, $s = 1,\dots,n$.

Let $L_0 : H_1 \to H_1$ be the operator of the obtained problem

$$L_0u := L\left(-D_1^2, -D_2^2\right)u,\ u \in D(L_0),\ D\left(L_0\right) := \{u \in H_2 : \ell_{0,s,j}u = 0,\ s = 1,\dots,2n,\ j = 1,2\},$$

and

$$V(L_0) := \{v_{r,s,k,m}(x, L_0) \in H_1 : v_{r,s,k,m}(x, L_0) := \tau_{1,s,k}(x_1)\tau_{2,r,m}(x_2),\ r,\ s \in \{0,1\},\ \ m,\ k \in \mathbb{N}\}$$

be the orthonormal basis of the space H_1.

Considering the ratio $L_0 = (-1)^n \sum\limits_{s=0}^{n} A_0^s B_0^{n-s}$, we obtain the following statement.

Lemma 4. *The operator L_0 has eigenvalues* (7) *and the system of eigenfunctions $V(L_0)$.*

2 The non self-ajoint problem for a differential equation of even order

For any fixed $p \in \{1,\dots,n\}$ we consider the problem

$$L(-D_1^2, -D_2^2)u := \sum_{s=0}^{n} a_s D_1^{2s} D_2^{2n-2s} u(x) = \lambda u(x),\ x \in G,\ \lambda \in \mathbb{C}, \tag{8}$$

$$\ell_{1,s,1}u := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} = 0,\ s \neq p,\ s = 1,\dots,n, \tag{9}$$

$$\ell_{1,p,1}u := D_1^{2p-2}u|_{x_1=0} + D_1^{2p-2}u|_{x_1=1} + \ell^0_{p,1}u = 0, \tag{10}$$

$$\ell_{1,n+s,1}u := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0,\ s \neq p,\ s = 1,\dots,n, \tag{11}$$

$$\ell_{1,s,2}u := D_2^{2s-2}u|_{x_2=0} + D_2^{2s-2}u|_{x_2=1} = 0,\ s = 1,\dots,n, \tag{12}$$

$$\ell_{1,n+s,2}u := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} = 0,\ s = 1,\dots,n. \tag{13}$$

Let $L_{1,p}$ be the operator of the problem (8)–(13):

$$L_{1,p}u := L(-D_1^2, -D_2^2)u,\ u \in D\left(L_{1,p}\right),$$
$$D\left(L_{1,p}\right) := \left\{u \in H_2 : \ell_{1,r,j}u = 0,\ r = 1,\dots,2n,\ j = 1,2\right\},$$

and $V\left(L_{1,p}\right)$ be the system of eigenfunctions of the operator $L_{1,p}$.

For any fixed $m \in \mathbb{N}$ let's consider the solutions of problem (8)–(13) in the form of product

$$u(x) := z\left(x_1\right)\tau_{2,s,m}\left(x_2\right),\ s \in \{0,1\}.$$

To determine the unknown function $z(x_1)$, we obtain the problem for eigenvalues

$$\sum_{q=0}^{n} a_q(-1)^{n-s}\mu_{2,m}^{n-q} z^{(2q)}(x_1) = \lambda z(x_1),\ \ x_1 \in (0,1),\ \ \lambda \in \mathbb{C}, \tag{14}$$

$$l^1_{s,1}z := z^{(2s-2)}(0) + z^{(2s-2)}(1) = 0,\ s \neq p,\ s = 1,\dots,n, \tag{15}$$

$$l^1_{p,1}z := z^{(2p-2)}(0) + z^{(2p-2)}(1) + l^0_{p,1}z = 0, \tag{16}$$

$$l^1_{n+s,1}z := z^{(2s-2)}(0) - z^{(2s-2)}(1) = 0,\ s = 1, \ldots, n, \tag{17}$$

where

$$l^0_{p,1}z := \sum_{q=0}^{k_{p,1}} \sum_{r=0}^{n_1} b_{p,q,r} z^{(q)}(x_{1,r}),\ p = 1, \ldots, n. \tag{18}$$

Let $L_{1,p,m}$ be the operator of problem (14)–(18):

$$L_{1,p,m}z := \sum_{s=0}^{n} a_s(-1)^{m-s}\mu_{2,m}^{n-s} z^{(2s)}, \quad z \in D\left(L_{1,p,m}\right),$$
$$D\left(L_{1,p,m}\right) := \left\{ z \in W : l^1_{j,1}z = 0,\ j = 1, \ldots, 2n \right\}.$$

Lemma 5. *Let Assumption P_1 holds. Therefore, for any $a_q \in \mathbb{R}$, $b_{p,q,r} \in \mathbb{R}$, $q = 0, 1, \ldots, k_{p,1}$, $r = 0, 1, \ldots, n_1$, m, $p \in \mathbb{N}$, the operator $L_{1,p,m}$ has the set of eigenvalues $\sigma_m := \{\lambda_{k,m} \in \sigma$, $k \in \mathbb{N}\}$, and the system of eigenfunctions $V\left(L_{1,p,m}\right)$, which is complete and minimal in the space H_0.*

Proof. The solutions $\omega_{r,m}(\lambda)$, $r = 1, \ldots, n$, of equation $\sum_{s=0}^{n} a_s(-1)^{n-s}\mu_{2,m}^{n-s}\omega^{2s} = \lambda$, which is characteristic for equations (14), we choose to fulfill the conditions

$$\operatorname{Re}\omega_{n,m}(\lambda) \le \operatorname{Re}\omega_{n-1,m}(\lambda) \le \cdots \le \operatorname{Re}\omega_{1,m}(\lambda) \le 0.$$

Let us determine the functions

$$z_{q,m}(x_1, \lambda) := \frac{1}{2}(\exp\omega_{q,m}(\lambda)\, x_1 + \exp\omega_{q,m}(\lambda)\,(1 - x_1)) \in H_{0,0},\ q = 1, \ldots, n,$$

$$z_{n+q,m}(x_1, \lambda) := \frac{1}{2}(\exp\omega_{q,m}(\lambda)\, x_1 - \exp\omega_{q,m}(\lambda)\,(1 - x_1)) \in H_{0,1},\ q = 1, \ldots, n,$$

$$z_m(x_1) = \sum_{j=1}^{2n} c_j z_{j,m}(x_1, \lambda),\ c_j \in \mathbb{R}. \tag{19}$$

Substituting expression (19) into boundary conditions (15)–(17), we obtain an equation for determining of eigenvalues for operator $L_{1,p,m}$:

$$\Delta_m(\lambda) = \det(l^1_{q,1} z_{j,m}(x, \lambda))_{j,q=1}^{2n} = 0.$$

According to the relations $z_{rn+q,m}(x_1, \lambda) \in H_{0,r}$, $l^1_{q+sn} \in W_s^*$, s, $r \in \{0, 1\}$, $l^0_{p,1} \in W_1^*$, we obtain

$$l^1_{n+q,1} z_{j,m}(x_1, \lambda) = 0,\ l^1_{q,1} z_{n+j,m}(x_1, \lambda) = 0,\ j,\ q = 1, \ldots, n,$$
$$\Delta_m(\lambda) = \Delta_{0,m}(\lambda)\Delta_{1,m}(\lambda),$$
$$\Delta_m(\lambda) = \prod_{q=1}^{n}(1 - e^{2\omega_{q,m}(\lambda)}) \prod_{1\le j<q\le n}(\omega_{j,m}(\lambda) - \omega_{q,m}(\lambda))^2 = 0. \tag{20}$$

Let $\omega_{r,k,m}$ be roots of the equation (20) for $\lambda = \lambda_{k,m}$, which are selected so that $\omega_{1,k,m} = \imath\pi k$, $\operatorname{Re}\omega_{n,k,m} \le \operatorname{Re}\omega_{n-1,k,m} \le \cdots \le \operatorname{Re}\omega_{1,k,m} \le 0$, $k \in \mathbb{N}$. Substituting expression (19) in boundary conditions (15)–(17), we can find the eigenfunctions of the operator $L_{1,p,m}$:

$$v_{0,k}\left(x_1, L_{1,p,m}\right) = \sqrt{2}\sin\rho_{0,k}x_1,\ \rho_{0,k} = \pi(2k-1),\ k \in \mathbb{N}. \tag{21}$$

Let us define the system of functions

$$z_{1,1,k,m}(x_1) = \sqrt{2}\cos\rho_{1,k}x_1,\ \rho_{1,k} = 2k\pi,\ k \in \mathbb{N}, \tag{22}$$

$$z_{1,q,k,m}(x_1) := \frac{1}{2}(1+\exp\omega_{q,k,m})^{-1}(\exp\omega_{q,k,m}x_1 + \exp\omega_{q,k,m}(1-x_1)),\ k \in \mathbb{N}, \tag{23}$$

and a square matrix of order n, elements of which we define by the following rule: pth row is defined by functions (22), (23), and elements of other rows is defined by numbers

$$\vartheta_{q,r,k,m} := \rho_{1,k}^{2-2r} l_{1,r,1} z_{1,q,k,m},\ v_{q,r,k,m} = \rho_{1,k}^{2-2r}\omega_{q,k,m}^{2r-2},\ q = 2,3,\ldots,n,\ r \neq p,\ r = 1,\ldots,n.$$

$$\vartheta_{1,r,k,m} = 2\sqrt{2},\ r \neq p,\ = 2,3,\ldots,n,\ k \in \mathbb{N}.$$

Determinant of the given matrix is denoted by $z_{2,p,k,m}(x_1),\ k \in \mathbb{N}$.

Remark 1. *For any fixed $m \in \mathbb{N}$ and $k \to \infty$, we obtain the relation*

$$\delta_{1,k,m} := \omega_{1,k,m}\rho_{1,k}^{-1} = \imath,$$

$$\delta_{q,k,m} := \rho_{1,k}^{-1}\omega_{q,k,m} = \varepsilon_q\left(1 + O\left(k^{-1}\right)\right),$$

where ε_q are the solutions of equation $(-1)^n(\varepsilon)^{2n} = 1,\ \varepsilon_1 = \imath,\ \operatorname{Im}\varepsilon_q < 0,\ q = 2,3,\ldots,n.$

Substituting function $z_{2,p,k,m}(x_1)$ in boundary conditions (14)–(17), we obtain the equalities

$$\ell_{1,s,1}z_{2,p,k,m} = 0,\ s \neq p,\ l_{1,p,1}z_{2,p,k,m} := c_{p,k,m},\ s = 1,\ldots,2n,\ k \in \mathbb{N},$$

where $c_{p,k,m} = \sqrt{2}2\rho_{1,k}^{2p-2}W_{k,m},\ W_{k,m} = W\left(\delta_{1,k,m}^2,\ldots,\delta_{n,k,m}^2\right)$ is Vandermonde determinant of order n, which is constructed by numbers $\delta_{q,k,m}^2,\ q = 1,\ldots,n.$

Remark 2. *For arbitrary $m \in \mathbb{N}$ and $k \to \infty$ the number sequence $\{W_{k,m}\}_{k=1}^{\infty}$ converges to Vandermonde determinant $W\left(\varepsilon_1^2,\ \varepsilon_2^2,\ldots,\varepsilon_n^2\right)$, which is constructed by numbers $\varepsilon_1^2,\ldots,\varepsilon_n^2$.*
Therefore, $\vartheta_{q,r,k,m} = \varepsilon_q^{2r-2}(1+O(\frac{1}{k})),\ k \to \infty,\ q = 1,\ldots,n.$

Thus, the positive numbers $C_5,\ C_6$ exist such that the following inequality holds:

$$0 < C_5 \leq \left|c_{p,k,m}\right|\rho_{1,k}^{2-2p} \leq C_6 < \infty,\ k \in \mathbb{N}.$$

Let us choose the functions

$$z_{3,p,k,m}(x_1) := W_{k,m}^{-1}z_{2,p,k,m}(x_1),\ k \in \mathbb{N}. \tag{24}$$

Taking into account equalities (24), we obtain the relations

$$\ell_{1,s}^1 z_{3,p,k,m} = 0,\ s \neq p,\ \ell_{1,p}^1 z_{3,p,k,m}(x_1) = 2\sqrt{2}\rho_{1,k}^{2p-2},\ s = 1,\ldots,n. \tag{25}$$

Let $\Delta_{j,s,k,m} := \det(\vartheta_{q,r,k,m})_{q,r=\overline{1,n}}^{q\neq j, r\neq s}$. Consider the functions $y_{p,k,m}(x_1) := \Delta_{1,1,k,m}^{-1}z_{3,p,k,m}(x_1)$,

$$y_{p,k,m}(x_1) = z_{1,1,k,m}(x_1) + \sum_{j=2}^{n}\gamma_{j,p,k,m}z_{1,j,k,m}(x_1),\ k \in \mathbb{N}, \tag{26}$$

where $\gamma_{j,p,k,m} = \Delta_{1,p,k,m}^{-1}\Delta_{j,p,k,m},\ j = 2,3,\ldots,n.$

From formulas (24)–(26) we obtain

$$y_{p,k,m}(x_1) = c_{1,p,k,m} z_{2,p,k,m}(x_1),$$

where

$$c_{1,p,k,m} = W_{k,m}^{-1} \Delta_{1,p,k,m},\ C_7 < c_{1,p,k,m} < C_8 < \infty.$$

Therefore,

$$l_{1,p}^{1} y_{p,k,m}(x_1) = c_{1,p,k,m} 2\sqrt{2} \rho_{1,k}^{2p-2},\ l_{1,s}^{1} y_{p,k,m}(x_1) = 0,\ s \neq p,\ s = 1, \ldots, n.$$

The eigenfunctions $v_{1,k}\left(x_1, L_{1,p,m}\right)$ of the operator $L_{1,p,m}$ we define by the equality

$$v_{1,k}(x_1, L_{1,p,m}) := \tau_{1,1,k}(x_1) + \eta_{p,k,m} y_{p,k,m}(x_1),\ k \in \mathbb{N}. \tag{27}$$

To determine the unknown parameters $\eta_{p,k,m}$, we substitute the expression (27) in the boundary conditions (16), (17).

Taking into account (24), we obtain

$$\eta_{p,k,m} = (-1)^p \sqrt{8^{-1}} c_{1,p,k,m}^{-1} \rho_{1,k}^{2-2p} l_{1,p}^{1} \tau_{1,1,k},\ k \in \mathbb{N}. \tag{28}$$

Thus, the operator $L_{1,p,m}$ has the system $V\left(L_{1,p,m}\right)$ of eigenfunctions (21), (24), (28).

The completeness of the system of functions $V\left(L_{1,p,m}\right)$ in the space H_0 is proved from the opposite, like in the proof of the Lemma 2.

Let us consider the operators

$$R(L_{1,p,m}),\ S(L_{1,p,m}):\ H_0 \to H_0,\ \ R(L_{1,p,m}) = E + S(L_{1,p,m}),$$

$$R(L_{1,p,m})\tau_{1,0,k}(x_1) := \tau_{1,0,k}(x_1),\ \ R(L_{1,p,m})\tau_{1,1,k}(x_1) := v_{1,k}(x_1, L_{1,p,m}),\ k \in \mathbb{N}.$$

From the definition of operator $S(L_{1,p,m})$ we obtain $S(L_{1,p,m}) : H_{0,0} \to 0,\ S(L_{1,p,m}) : H_{0,1} \to H_{0,0},\ S^2(L_{1,p,m}) = 0,\ R^{-1}(L_{1,p,m}) = E - S(L_{1,p,m})$. Therefore, the system of functions $V\left(L_{1,p,m}\right)$ is minimal in the space H_0. Lemma 5 is proved. □

Let $\theta_k = \eta_{p,k,m}$, then $A_{p,m} := A_\Theta,\ k,\ m \in \mathbb{N},\ p \in \{1, \ldots, n\}$.

Lemma 6. *If $\{\eta_{p,k,m}\}_{k=1}^{\infty}$ is a bounded sequence, then the system of functions $V(L_{1,p,m})$ is the Riesz basis in the space H_0.*

Proof. Taking into account the definition of the function $y_{p,k,m}(x_1)$ and the choice of numbers $\omega_{q,k,m},\ q = 1, \ldots, n$, we can conclude: if $\theta_k = \eta_{p,k,m},\ k \in \mathbb{N},\ p \in \{1, \ldots, n\}$, is a bounded sequence, then the systems of functions $V(L_{1,p,m}),\ V\left(A_{p,m}\right)$ are quadratically approximate for every $m \in \mathbb{N},\ p \in \{1, \ldots, n\}$.

Therefore, taking into account the Lemma 5 and the theorem N.K. Bari [10], we obtain the statement of Lemma 6. □

Let us choose an arbitrary sequence of real numbers $\Theta = \{\theta_k\}_{k=1}^{\infty}$, and define the operator $A_{\Theta,p,m} : H_0 \to H_0$, which has the set of eigenvalues $\sigma_{1,m} = \{\lambda_{k,m} \in \sigma,\ k \in \mathbb{N}\}$ and the system $V(A_{\Theta,p,m}) := \{v_{s,k,m}(x_1, A_{\Theta,p,m}) \in H_0 :\ \ s = 0, 1,\ k \in \mathbb{N}\}$ of eigenfunctions

$$v_{0,k,m}(x_1, A_{\Theta,p,m}) := \tau_{1,0,k}(x_1),\ v_{1,k,m}(x_1, A_{\Theta,p,m}) := \tau_{1,1,k}(x_1) + \theta_k y_{p,k,m}(x_1),\ k \in \mathbb{N}. \tag{29}$$

Consider the operators

$$R(A_{\Theta,p,m}) := E + S(A_{\Theta,p,m}),$$
$$S(A_{\Theta,p,m})\tau_{1,0,k}(x_1) := 0,$$
$$S(A_{\Theta,p,m})\tau_{1,1,k}(x_1) := \theta_k y_{p,k,m}(x_1),\ k \in \mathbb{N}.$$

Let $\Gamma_{1,p}(L_{0,m})$ be the set of operators, which have purely point spectrum $\sigma_{1,m}$ and the system of eigenfunctions (29).

We define on $\Gamma_{1,p}(L_{0,m})$ the commutative multiplication operation

$$R(A_{\Theta_1,p,m})R(A_{\Theta_2,p,m}) = E + S(A_{\Theta_1,p,m}) + S(A_{\Theta_2,p,m}) = R(A_{\Theta_2,p,m})R(A_{\Theta_1,p,m}),$$

$$A_{\Theta_2,p,m},\ A_{\Theta_1,p,m} \in \Gamma_{1,p}(L_0),$$

and inverse operator $R^{-1}(A_{\Theta,p,m}) = E - S(A_{\Theta,p,m}),\ A_{\Theta,p,m} \in \Gamma_1(L_{0,m})$.

Lemma 7. *For any real numbers $\theta_q \in \mathbb{R},\ q \in \mathbb{N},$ the system of functions $V(A_{\Theta,p,m})$ is complete and minimal in the space H_0. The system of functions $V(A_{\Theta,p,m})$ is the Riesz basis in H_0 if and only if the sequence Θ is bounded.*

Proof. The lemma can be proved by the schema of proof the Lemma 2. □

We define by the formulas

$$v_{s,r,k,m}\left(x, L_{1,p}\right) := v_{s,k}\left(x_1, L_{1,p,m}\right)\tau_{2,r,m}\left(x_2\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}, \tag{30}$$

the eigenfunctions of operator $L_{1,p}$.

Lemma 8. *Suppose that the Assumption P_1 holds. Then, for arbitrary $a_s \in \mathbb{R}$, $b_{p,q,r} \in \mathbb{R}$, the operator $L_{1,p}$ has the point spectrum σ, and the system of eigenfunctions $V\left(L_{1,p}\right) := \{v_{s,r,k,m}\left(x, L_{1,p}\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}\}$, which is complete and minimal in H_1.*

If the Assumptions P_1–P_3 hold, then the system of functions $V\left(L_{1,p}\right)$ is the Riesz basis in the space H_1.

Proof. Substituting functions (30) into the equations (8)–(13) makes sure that the numbers $\lambda_{k,m} \in \sigma$ are eigenvalues, if $k,\ m \in \mathbb{N}$.

In the space H_1 we define the operator $R\left(L_{1,p}\right) := E + S\left(L_{1,p}\right)$, which maps the system of functions $V\left(L_0\right)$ into $V\left(L_{1,p}\right)$.

The operator $R\left(L_{1,p}\right)$ has the form

$$R\left(L_{1,p}\right) := \sum_{r,m} R(L_{1,p,m}) \times \pi_{2,r,m},$$

where $\pi_{2,r,m}$ is the orthoprojector into the one-dimensional proper subspace in H_0, which corresponds to eigenfunction $\tau_{2,r,m}(x_2)$ of operator B_0.

We consider the operator $A_p : H_1 \to H_1,$ which has purely point spectrum $\sigma(A_p) := \{\lambda_{k,m} \in \mathbb{R} : \lambda_{k,m} = \mu_{1,k} + \mu_{2,m},\ k,\ m \in \mathbb{N}\}$ and the system of eigenfunctions

$$V(A_p) := \{v_{s,r,k,m}\left(x_1, x_2, A_p\right) := v_{s,k}\left(x_1, A_{p,m}\right)\tau_{2,r,m}\left(x_2\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}\}.$$

Let $R(A_p) := \sum\limits_{r,m} R(A_{p,m}) \times \pi_{2,r,m}$.

According to the Lemma 5, for an arbitrary $m \in \mathbb{N}$ the system of functions $W\left(L_{1,p,m}\right)$ exists, and it is biorthogonal to the system $V\left(L_{1,p,m}\right)$.

Therefore, we can define the elements of system $W\left(L_{1,p}\right)$, which is biorthogonal to system $V\left(L_{1,p}\right)$ in the space H_1 :

$$w_{s,r,k,m}(x_1, x_2, L_{1,p}) = w_{s,k}(x_1, L_{1,p,m})\tau_{2,r,m}(x_2),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}.$$

Thus, the system $V\left(L_{1,p}\right)$ is complete and minimal in H_1.

Therefore, when the Assumptions P_2 and P_3 hold, then we obtain the inequality $|\eta_{p,k,m}| \leq C_9 < \infty$, for arbitrary $m,\ k \in \mathbb{N}$. Taking into account the estimates $\|R(A_p); [H_1]\|^2 \leq C_{10}$, we obtain the statement: eigenfunctions (30) of operator A_p are almost normalized, and system $V\left(A_p\right)$ is the Riesz basis of the space H_1.

We consider the operator $R(L_{1,p}) = E + S(L_{1,p}) = (E + Q)(E + S(A_p))$. Then the operator $Q_p := S(L_{1,p}) - S(A_p)$ is completely continuous, because the systems of functions $V(L_{1,p,m})$, $V(A_{p,m})$ are quadratically approximate and the operator $Q_{p,m} := S(L_{1,p,m}) - S(A_{p,m})$ is idempotent: $Q_{p,m}^2 = 0$.

According to the definition of function $v_{s,r,k,m}(x, L_0)$, we obtain

$$\|Q_p v_{s,r,k,m}(x, L_0); H_1\| = O(m+k)^{-3},\ m,\ k \to \infty.$$

Then, for an arbitrary $h = \sum\limits_{s,r,m,k} h_{s,r,k,m} v_{s,r,k,m}(x, L_0) \in H_1$, from Cauchy's inequality we can get the inequality

$$\|Q_p h; H_1\|^2 = \| \sum_{s,r,k,m} h_{s,r,k,m} Q_p v_{s,r,k,m}(x, L_0); H_1\|^2 \leq C_{11} \|h; H_1\|^2.$$

Thus $\|Q_p; [H_1]\|^2 < \infty$, $(L_{1,p}) = Q_p + R(A_{1,p}) \in [H_1]$, $R(L_{1,p})^{-1} = (E - S(A_p)(E - Q) \in [H_1]$. □

Proof. Proof of the Theorem 1. Let $R(L) := \prod\limits_{p=1}^{n} R(L_{1,p})$. The eigenfunctions of operator L we can define in the form

$$v_{s,r,k,m}(x, L) := R(L) v_{s,r,k,m}(x, L_0),\ r,\ s \in \{0,1\},\ k,\ m \in \mathbb{N}.$$

Taking into account, that operators $R(L_{1,p})$ are elements of the group $\Gamma_{1,p}(L_0)$, we obtain

$$R(L) = E + S(L),\ R^{-1}(L) = E - S(L),\ S(L) := \sum_{p=1}^{n} S(L_{1,p}).$$

Therefore, the system of eigenfunctions $V\left(L\right)$ is complete and minimal in H_1. □

Proof. Proof of the Theorem 2. Let the Assumptions P_1–P_3 hold, then the system of eigenfunctions $V\left(L_{1,p}\right)$ is the Riesz basis in the space H_1, and $R(L) = \prod\limits_{p=1}^{n} R(L_{1,p}) \in [H_1]$. Therefore, taking into account the theorem N.K. Bari [10], we obtain the statement of the theorem. □

Let us define the elements of system $W(L)$, which is biorthogonal to system $V\left(L\right)$ in the space H_1:

$$w_{s,r,k,m}(x, L) := R(L)\tau_{1,s,k}(x_1)\tau_{2,r,m}(x_2),\ \ s, r \in \{0,1\},\ \ k,\ m \in \mathbb{N}.$$

Remark 3. *The positive numbers $C_1(L)$, $C_2(L)$ exist, such that for function*

$$f(x) = \sum_{s,r,k,m} f_{s,r,k,m} v_{s,r,k,m}(x_1, x_2, L),\ f_{s,r,k,m} := (f, w_{s,r,k,m}(x_1, x_2, L);\ H_1)$$

the following inequality holds

$$C_2(L)\|f; H_1\|^2 \leq \sum_{s,r,k,m} |f_{s,r,k,m}|^2,\ C_3(L)\|f; H_1\|^2. \tag{31}$$

Proof. Proof of the Theorem 3. We will use a solution of the problem (1)–(6) in the form of series

$$u(x) = \sum_{s,r,k,m} u_{s,r,k,m} v_{s,r,k,m}(x_1, x_2, L). \tag{32}$$

If we substitute series (31), (32) into equation (1), we obtain

$$u_{s,r,k,m} = \lambda_{k,m}^{-1} f_{s,r,k,m}.$$

Taking into account the Assumption P_3 and inequality $\lambda_{k,m}^{-1} \leq 1$, we can get

$$\|u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2,\ C_5(L) = C_2(L)^{-1}(L)C_3(L)C_1^{-2}(L),$$

$$\|D_1^{2n} u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2,$$

$$\|D_2^{2n} u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2.$$

Therefore, $\|u; H_2\|^2 \leq 3C_5(L)\|f; H_1\|^2$. □

REFERENCES

[1] Amanov D., Ashyralyev A. *Well-posedness of boundary value problems for partial differential equations of even order*. Electron. J. Diff. Equ. 2014, **108**, 1–18.

[2] Amanov D.J., Yuldasheva A.V. *Solvability and spectral properties of boundary value problems for equations of even order*. Malays. J. Math. Sci. 2009, **3** (2), 227–248.

[3] Baranetskij Ya.O., Ivasiuk I.Ya., Kalenyuk P.I., Solomko A.V. *The nonlocal boundary problem with perturbations of antiperiodicity conditions for the elliptic equation with constant coefficients*. Carpathian Math. Publ. 2018, **10** (2), 215–234. doi:10.15330/cmp.10.2.215-234

[4] Baranetskij Ya.O., Kalenyuk P.I., Kopach M.I. *The nonlocal boundary problem with multipoint perturbations for the differential equations with constant coefficients of even order*. Mat. Metodi Fiz.-Mekh. Polya 2018, **61** (1), 16–39. (in Ukrainian)

[5] Baranetskij Ya.O., Kalenyuk P.I., Kolyasa L.I., Kopach M.I. *The nonlocal problem for the differential-operator equation of the even order with the involution*. Carpathian Math. Publ. 2017, **9** (2), 109–119. doi:10.15330/cmp.9.2.109-119

[6] Baranetskij Ya.O., Demkiv I.I., Ivasiuk I.Ya., Kopach M.I. *The nonlocal problem for the differential equations the order 2n with an unbounded operator coefficients with the involution*. Carpathian Math. Publ. 2018, **10** (1), 14–30. doi:10.15330/cmp.10.1.14-30

[7] Demkiv I.I., Ivasiuk I.Ya., Kopach M.I. *Interpolation integral continued fraction with twofold node*. MMC 2019, **6** (1), 1–13. doi:10.23939/mmc2019.01.001

[8] Irgashev B.Yu. *Boundary problem for equations of high-even order*. Vestnik VGU. Seriya 1. Mathematica. Physica. 2016, **334** (3), 6–18. (in Russian) doi:10.15688/jvolsu1.2016.3.1

[9] Irgashev B.Yu. *On spectral problem for one equation of high even order*. Russian Math. (Iz. VUZ). 2016, **60** (7), 37–46.

[10] Gokhberg I.Ts., Krein M.G. Introduction to the Theory of Linear Non Self-Adjoint Operators. Nauka, Moscow, 1965. (in Russian)

[11] Koshanov B.D., Soldatov A.P. *Boundary value problem with normal derivatives for a higher-order elliptic equation on the plane*. Diff. Equations 2016, **52**, (12), 1594–1609.

[12] Koshanov B., Soldatov A. *About the generalized Dirichlet-Neumann problem for an elliptic equation of high order*. AIP Conference Proceedings 1997, 020013 (2018). doi:10.1063/1.5049007

[13] Yuldasheva A.V. *On solvability of the boundary value problem for one even-order equation*. Vestnik KRAUNC. Fiz.-Mat. Nauki 2016, **12** (1), 41–47.

Received 03.04.2019

Баранецький Я.О., Каленюк П.І., Копач М.І., Соломко А.В. *Нелокальна крайова задача зі збуреннями мішаних крайових умов для еліптичного рівняння зі сталими коефіцієнтами. I* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 228–239.

У роботі в одиничному квадраті G методом Фур'є досліджується задача з нелокальними умовами, які є багатоточковими збуреннями мішаних крайових умов. Вивчено властивості узагальненого оператора перетворення $R : L_2(G) \to L_2(G)$, який відображає нормовані власні функції оператора L_0 задачі із мішаними крайовими умовами у власні функції оператора L збуреної нелокальної задачі. Побудовано систему $V(L)$ власних функцій оператора L. Визначено умови, при яких система $V(L)$ повна та мінімальна в просторі $L_2(G)$, та умови, при яких вона є базисом Рісса у просторі $L_2(G)$. У випадку, якщо система $V(L)$ є базисом Рісса в просторі $L_2(G)$, встановлено достатні мови, при яких нелокальна задача має єдиний розв'язок у вигляді ряду Фур'є за системою $V(L)$.

Ключові слова і фрази: диференціальне рівняння з частинними похідними, кореневі функції, базис Рісса.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 240–249
doi:10.15330/cmp.11.2.240-249

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.240–249

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ON COMPARISON OF THE PRINCIPLES OF EQUIVALENT UTILITY AND ITS APPLICATIONS

An insurance premium principle is a way of assigning to every risk, represented by a non-negative bounded random variable on a given probability space, a non-negative real number. Such a number is interpreted as a premium for the insuring risk. In this paper the implicitly defined principle of equivalent utility is investigated. Using the properties of the quasideviation means, we characterize a comparison in the class of principles of equivalent utility under Rank-Dependent Utility, one of the important behavioral models of decision making under risk. Then we apply this result to establish characterizations of equality and positive homogeneity of the principle. Some further applications are discussed as well.

Key words and phrases: insurance premium, quasideviation mean, comparison, equality, positive homogeneity, risk loading.

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1 INTRODUCTION

Assume that $\mathcal{X}_+$ is a family of risks, represented by non-negative bounded random variables on a non-atomic probability space $(\Omega, \mathcal{F}, P)$. An insurance premium principle is a way of assigning to every $X \in \mathcal{X}_+$ a non-negative real number $H(X)$. The number $H(X)$ is interpreted as a premium for insuring X. There are many methods of defining the principles. In what follows we deal with the principle of equivalent utility. The principle, postulating a fairness in terms of utility, has been introduced in [2]. Under the Expected Utility model the premium for a risk $X \in \mathcal{X}_+$ is defined through the equation

$$E[u(w + H_{(w,u)}(X) - X)] = u(w), \tag{1}$$

where $w \in [0, \infty)$ is an initial wealth level and $u : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function such that $u(0)$=0. In general, (1) has no explicit solution. However, in some cases the premium can be expressed in an explicit way. In particular, if u is linear, then

$$H_{(w,u)}(X) = E[X] \quad \text{for} \quad X \in \mathcal{X}_+,$$

i.e. the principle of equivalent utility becomes the net premium principle. If $u(x) = a(1 - e^{-cx})$ for $x \in \mathbb{R}$, with some $a, c > 0$, then from (1) we deduce that the principle of equivalent utility reduces to the exponential principle

$$H_{(w,u)}(X) = \frac{1}{c} \ln E[e^{cX}] \quad \text{for} \quad X \in \mathcal{X}_+.$$

УДК 519.812
2010 *Mathematics Subject Classification:* 91B16, 91B30, 39B22.

Note that in both cases the premium for a given risk does not depend on an initial wealth level. Some properties of the principle of equivalent utility defined by (1) can be found e.g in [1,2,6,13].

In this paper we deal with the principle of equivalent utility under Rank-Dependent Utility, one of the behavioral models of decision making under risk. In this setting the principle has been introduced and investigated in [7]. In order to define it, recall that if $g : [0,1] \to [0,1]$ is a probability distortion function, that is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ then, for any bounded random variable X on $(\Omega, \mathcal{F}, P)$, the Choquet integral with respect to g is given by

$$E_g[X] = \int_{-\infty}^{0} (g(P(X > t)) - 1)\, dt + \int_{0}^{\infty} g(P(X > t))\, dt. \tag{2}$$

The premium for a risk $X \in \mathcal{X}_+$ under the Rank-Dependent Utility model is defined as a solution of the equation

$$E_g[u(w + H_{(w,u,g)}(X) - X)] = u(w). \tag{3}$$

It is known (cf. [4, Remark 1]) that if g is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function with $u(0) = 0$ then, for every $X \in \mathcal{X}_+$, the number $H_{(w,u,g)}(X)$ is uniquely determined by (3). Some properties of the premium defined by (3) have been investigated in [7] under the assumption that g is convex and u is concave and differentiable.

The main result of this paper provides a characterization of a comparison in the class of the principles of equivalent utility. Applying this result we establish characterizations of further natural properties of the principle, namely equality and positive homogeneity. Some results concerning the risk loading property of the principle of equivalent utility are presented as well.

It turns out that an effective tool for dealing with this issue is a notion of a quasideviation mean. Therefore, in the next section we present a definition of the mean and a result concerning a comparison of quasideviation means.

2 QUASIDEVIATION MEANS

The notion of the quasideviation mean has been introduced in [10]. In order to recall the notion, assume that $I \subseteq \mathbb{R}$ is an open interval. A function $D : I^2 \to \mathbb{R}$ is called a quasideviation if it satisfies the following three conditions:

(i) $D(x,x) = 0$ for $x \in I$ and $(x-y)D(x,y) > 0$ for $x, y \in I$ with $x \neq y$;

(ii) for every $x \in I$, the function $I \ni t \to D(x,t)$ is continuous;

(iii) for every $x, y \in I$, with $x < y$, the function $(x,y) \ni t \to \frac{D(y,t)}{D(x,t)}$ is strictly increasing.

Let

$$\Delta_n := [0,\infty)^n \setminus \{\overline{0}\}.$$

In [10] it has been proved that, if $D : I^2 \to \mathbb{R}$ is a quasideviation, then for every $n \in \mathbb{N}$, $\overline{x} = (x_1, ..., x_n) \in I^n$ and $\overline{\lambda} = (\lambda_1, ..., \lambda_n) \in \Delta_n$, equation

$$\sum_{i=1}^{n} \lambda_i D(x_i, t) = 0 \tag{4}$$

has a unique solution t_0. Moreover

$$\min\{x_i : i \in \{1,..,n\}\} \le t_0 \le \max\{x_i : i \in \{1,..,n\}\}.$$

Thus, equation (4) defines a mean, called a D-quasideviation mean of $\overline{x}$ weighted by $\overline{\lambda}$. Following [10], we denote the mean by $\tilde{\mathfrak{M}}_D(\overline{x};\overline{\lambda})$. Several properties of quasideviation means have been proved in [11]. In our considerations we will need the following result, which is a particular case of [11, Theorem 7].

Theorem 1. *Assume that $I \subseteq \mathbb{R}$ is an open interval and $D_1, D_2 : I^2 \to \mathbb{R}$ are quasideviations. Then the following statements are equivalent:*

- (i) $\tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)) \le \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda))$ *for* $x_1, x_2 \in I$, $\lambda \in [0,1]$;
- (ii) *there exists a function* $A : I \to (0,\infty)$ *such that*

$$D_1(x,y) \le A(y)D_2(x,y) \quad \text{for} \quad x,y \in I.$$

3 Preliminary remarks

Remark 1. *Let g be a probability distortion function. It is known (cf. [5, Proposition 5.1]) that the Choquet integral is monotone and positively homogeneous. Furthermore, for every bounded random variable X on (Ω,Σ,P), we get*

$$E_g[X+c] = E_g[X] + c \quad \text{for} \quad c \in \mathbb{R} \tag{5}$$

and

$$E_g[-X] = -E_{\bar{g}}[X], \tag{6}$$

where $\bar{g} : [0,1] \to [0,1]$, given by

$$\bar{g}(p) = 1 - g(1-p) \quad \text{for} \quad p \in [0,1], \tag{7}$$

is the probability distortion function conjugated to g.

Remark 2. *Note that if g is the identity on $[0,1]$ then $E_g[X] = E[X]$ for every bounded random variable X on (Ω,Σ,P). Therefore, applying [5, Proposition 5.2 (iii)], we conclude that:*

- *if $g(p) \le p$ for $p \in [0,1]$ then $E_g[X] \le E[X]$ for every bounded random variable X on (Ω,Σ,P);*
- *if $g(p) \ge p$ for $p \in [0,1]$ then $E_g[X] \ge E[X]$ for every bounded random variable X on (Ω,Σ,P).*

Remark 3. *Let g be a continuous probability distortion function. Since the Choquet integral is monotone, for every $X \in \mathcal{X}_+$, the function*

$$\mathbb{R} \ni t \to E_g[u(w+t-X)] - u(w)$$

is nondecreasing. Furthermore, $H_{(w,u,g)}(X)$ is its unique zero.

Remark 4. *In view of* (3) *the premium for a given risk depends only on a probability distribution of the risk. Thus, we identify the risks with their probability distributions. Note also (cf. e.g. [12, Lemma 2.7.1]) that, as the probability space* (Ω, Σ, P) *is non-atomic, for every* $x, y \in \mathbb{R}$, *with* $x < y$, *and every* $p \in (0,1)$, *there exists a random variable* X *on* (Ω, Σ, P) *such that* $P(X = x) = p$ *and* $P(X = y) = 1 - p$. *Denote any such a random variable by* $\langle x, y; 1-p, p\rangle$. *Furthermore, let* $\mathcal{X}^{(2)}$ *be a family of all such random variables and*

$$\mathcal{X}_+^{(2)} := \{\langle x, y; 1-p, p\rangle \in \mathcal{X}^{(2)} : x \geq 0\}.$$

Remark 5. *If* $X = \langle x_1, x_2; 1-p, p\rangle \in \mathcal{X}^{(2)}$ *then, in view of* (2), *we get (cf. [8])*

$$E_g[X] = (1 - g(p))x_1 + g(p)x_2.$$

Remark 6. *Assume that* $w \in [0, \infty)$, g *is a continuous probability distortion function and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function such that* $u(0) = 0$. *Then, taking* $X = \langle x, y; p, 1-p\rangle \in \mathcal{X}_+^{(2)}$, *we obtain*

$$u(w + H_{(w,u,g)}(X) - X) = \langle (u(w + H_{(w,u,g)}(X) - y), u(w + H_{(w,u,g)}(X) - x)); 1-p, p\rangle.$$

Therefore, applying Remark 5, from (3) *we derive that* $H_{(w,u,g)}(X)$ *is a unique solution of the equation*

$$(1 - g(p))(u(w + H_{(w,u,g)}(X) - y) + g(p)u(w + H_{(w,u,g)}(X) - x) = u(w). \tag{8}$$

4 RESULTS

The following theorem is the main result of this paper.

Theorem 2. *Let* $w_1, w_2 \in [0, \infty)$. *Assume that* g *is a continuous probability distortion function and* $u, v : \mathbb{R} \to \mathbb{R}$ *are strictly increasing continuous functions such that* $u(0) = v(0) = 0$. *Then the following statements are pairwise equivalent:*

(i)

$$H_{(w_1,v,g)}(X) \leq H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+^{(2)}; \tag{9}$$

(ii)

$$H_{(w_1,v,g)}(X) \leq H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+; \tag{10}$$

(iii) *there exists* $c \in (0, \infty)$ *such that*

$$u(x) \leq cv(x + w_1 - w_2) + u(w_2) - cv(w_1) \quad \textit{for} \quad x \in \mathbb{R}. \tag{11}$$

Proof. Let $D_1, D_2 : (0, \infty)^2 \to \mathbb{R}$ be given by

$$D_1(x, y) = v(w_1) - v(w_1 + y - x) \quad \text{for} \quad x, y \in (0, \infty), \tag{12}$$

and

$$D_2(x, y) = u(w_2) - u(w_2 + y - x) \quad \text{for} \quad x, y \in (0, \infty), \tag{13}$$

respectively. Then, as one can easily check, D_1 and D_2 are quasideviations. Furthermore, since g is continuous with $g(0) = 0$ and $g(1) = 1$, for every $\lambda \in (0,1)$ there exists (not necessarily unique) $p_\lambda \in (0,1)$ such that

$$g(p_\lambda) = \lambda. \tag{14}$$

First we show that *(i)* $\Longrightarrow$ *(iii)*. Assume that (9) holds. Let $x_1, x_2 \in (0,\infty)$ and $\lambda \in [0,1]$. We claim that

$$\tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)) \leq \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda)). \tag{15}$$

If $x_1 = x_2$ or $\lambda = 1$, then both sides of (15) are equal to x_1. Moreover, if $\lambda = 0$, then both sides of (15) are equal to x_2. So, assume that $\lambda \in (0,1)$ and $x_1 \neq x_2$, say $x_1 < x_2$. Let $X = \langle x_1, x_2; p_\lambda, 1-p_\lambda\rangle$, where $p_\lambda \in (0,1)$ satisfies (14). Then $X \in \mathcal{X}_+^{(2)}$ whence, taking into account (8) and (12), we get

$$\begin{aligned}
&\lambda D_1(x_1, H_{(w_1,v,g)}(X)) + (1-\lambda)D_1(x_2, H_{(w_1,v,g)}(X)) \\
&\quad = g(p_\lambda)(v(w_1) - v(w_1 + H_{(w_1,v,g)}(X) - x_1)) + (1-g(p_\lambda))(v(w_1) - v(w_1 + H_{(w_1,v,g)}(X) - x_2)) \\
&\quad = v(w_1) - ((1-g(p_\lambda))v(w_1 + H_{(w_1,v,g)}(X) - x_2) + g(p_\lambda)v(w_1 + H_{(w_1,v,g)}(X) - x_1)) = 0.
\end{aligned}$$

Thus

$$H_{(w_1,v,g)}(X) = \tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)).$$

The similar arguments show that

$$H_{(w_2,u,g)}(X) = \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda)).$$

Hence, in view of (9), we get (15). In this way we have proved that (15) holds for every $x_1, x_2 \in (0,\infty)$ and $\lambda \in [0,1]$. Therefore, applying Theorem 1 and making use of (12)-(13), we obtain that there exists a function $A : (0,\infty) \to (0,\infty)$ such that

$$v(w_1) - v(w_1 + y - x) \leq A(y)(u(w_2) - u(w_2 + y - x)) \quad \text{for} \quad x, y \in (0,\infty).$$

Since u and v are strictly increasing with $u(0) = v(0) = 0$, replacing in the last inequality x by $y - x$, we get

$$v(w_1) - v(w_1 + x) \leq A(y)(u(w_2) - u(w_2 + x)) \quad \text{for} \quad x \in \mathbb{R},\ y \in (\max\{0,x\},\infty).$$

Thus

$$\frac{u(w_2) - u(w_2+x)}{v(w_1) - v(w_1+x)} \leq \frac{1}{A(y)} \quad \text{for} \quad x \in (0,\infty),\ y > x \tag{16}$$

and

$$\frac{u(w_2) - u(w_2+x)}{v(w_1) - v(w_1+x)} \geq \frac{1}{A(y)} \quad \text{for} \quad x \in (-\infty,0],\ y \in (0,\infty). \tag{17}$$

Hence, taking

$$c := \sup\left\{\frac{1}{A(y)} : y \in (0,\infty)\right\},$$

we conclude that $0 < c < \infty$. Moreover, it follows from (16) that the inequality

$$u(w_2 + x) \leq cv(w_1 + x) + u(w_2) - cv(w_1) \tag{18}$$

holds for all $x \in (0, \infty)$. Furthermore, taking in (17) the supremum over all $y \in (0, \infty)$, we obtain

$$c = \sup \left\{ \frac{1}{A(y)} : y \in (0, \infty) \right\} \leq \frac{u(w_2) - u(w_2 + x)}{v(w_1) - v(w_1 + x)} \text{ for } x \in (-\infty, 0],$$

which implies (18) for $x \in (-\infty, 0]$. Therefore, (18) holds for all $x \in \mathbb{R}$. Replacing in (18) x by $x - w_2$, we obtain (11). So, $(i) \Rightarrow (iii)$.

Now, assume that (11) is satisfied. Then, as the Choquet integral is monotone and positively homogeneous, in view of (3) and (5), for every $X \in \mathcal{X}_+$, we have

$$E_g[u(w_2 + H_{(w_1,v,g)}(X) - X)] - u(w_2) \leq c(E[v(w_1 + H_{(w_1,v,g)}(X) - X)] - v(w_1)) = 0.$$

Moreover, according to Remark 3, for every $X \in \mathcal{X}_+$, the function

$$\mathbb{R} \ni t \to E_g[u(w_2 + t - X)] - u(w_2)$$

is nondecreasing and $H_{(w_2,u,g)}(X)$ is its unique zero. Hence, (10) is valid. In this way we have proved that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

From Theorem 2 we derive the following characterization of equality in the class of principles of equivalent utility under the Rank-Dependent Utility model.

Corollary 1. *Let $w_1, w_2 \in [0, \infty)$. Assume that g is a continuous probability distortion function and $u, v : \mathbb{R} \to \mathbb{R}$ are strictly increasing continuous functions such that $u(0) = v(0) = 0$. Then the following statements are pairwise equivalent:*

(i)

$$H_{(w_1,v,g)}(X) = H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+^{(2)}; \tag{19}$$

(ii)

$$H_{(w_1,v,g)}(X) = H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+;$$

(iii) *there exists $c \in (0, \infty)$ such that*

$$u(x) = cv(x + w_1 - w_2) + u(w_2) - cv(w_1) \quad \textit{for} \quad x \in \mathbb{R}. \tag{20}$$

Proof. Assume that (19) holds. Then, according to Theorem 2, there exist $c, \tilde{c} \in (0, \infty)$ such that (11) is valid and

$$v(x) \leq \tilde{c}u(x + w_2 - w_1) + v(w_1) - \tilde{c}u(w_2) \quad \text{for} \quad x \in \mathbb{R}.$$

Hence

$$u(x) - u(w_2) \leq c(v(x + w_1 - w_2) - v(w_1)) \leq \tilde{c}c(u(x) - u(w_2)) \quad \text{for} \quad x \in \mathbb{R}.$$

Therefore, since v is strictly increasing, we get $c\tilde{c} = 1$ and so (20) is valid. This proves that $(i) \Rightarrow (iii)$.

If (20) holds then, replacing x by $x + w_2 - w_1$, we get

$$v(x) = \frac{1}{c}u(x + w_2 - w_1) + v(w_1) - \frac{1}{c}u(w_2) \quad \text{for} \quad x \in \mathbb{R}. \tag{21}$$

Taking into account (20) and (21), from Theorem 2 we derive (19). Thus $(iii) \Rightarrow (ii)$. Obviously, we have also $(ii) \Rightarrow (i)$. □

Applying Corollary 1 we are going to characterize the positive homogeneity of the principle of equivalent utility. Recall that the principle $H_{(w,u,g)}$ is positively homogeneous provided, for every $X \in \mathcal{X}_+$ and $\lambda \in (0,\infty)$, it holds

$$H_{(w,u,g)}(\lambda X) = \lambda H_{(w,u,g)}(X). \tag{22}$$

If (22) holds for every $X \in \mathcal{X}_+^{(2)}$ and $\lambda \in (0,\infty)$, then the principle $H_{(w,u,g)}$ is said to be positively homogeneous on $\mathcal{X}_+^{(2)}$. The positive homogeneity of $H_{(w,u,g)}$ in the case $w = 0$ has been characterized in [7]. It is proved there that if g is convex and u is concave and differentiable then $H_{(0,u,g)}$ is positively homogeneous if and only if u is linear.

Theorem 3. *Assume that $w \in [0,\infty)$, g is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. Then the following statements are pairwise equivalent:*

(i) *$H_{(w,u,g)}$ is positively homogeneous on $\mathcal{X}_+^{(2)}$;*

(ii) *$H_{(w,u,g)}$ is positively homogeneous;*

(iii) *there exist $a, b, r \in (0,\infty)$ and $\gamma \in \mathbb{R}$ such that*

$$u(x) = \begin{cases} -a(w-x)^r + \gamma & \text{for } x \in (-\infty, w], \\ b(x-w)^r + \gamma & \text{for } x \in (w,\infty). \end{cases} \tag{23}$$

Proof. Assume that (i) holds. For every $t \in (0,\infty)$, define $u_t : \mathbb{R} \to \mathbb{R}$ as follows

$$u_t(x) = u(w + tx) - u(w) \quad \text{for} \quad x \in \mathbb{R}. \tag{24}$$

Then, taking into account (3) and (5), for every $X \in \mathcal{X}_+^{(2)}$ and $t \in (0,\infty)$, we get

$$\begin{aligned} E_g[u_t(H_{(w,u,g)}(X) - X)] &= E_g[u(w + tH_{(w,u,g)}(X) - tX)] - u(w) \\ &= E_g[u(w + H_{(w,u,g)}(tX) - tX)] - u(w) = 0 = u_t(0) = E_g[u_t(H_{(0,u_t,g)}(X) - X)]. \end{aligned}$$

Therefore,

$$H_{(0,u_t,g)}(X) = H_{(w,u,g)}(X) \quad \text{for} \quad X \in \mathcal{X}_+^{(2)},\ t \in (0,\infty)$$

and so, applying Corollary 1 with $w_1 = 0$, $w_2 = w$ and $v = u_t$, we conclude that for every $t \in (0,\infty)$ there exists $c(t) \in (0,\infty)$ such that

$$u(x) = c(t)u_t(x-w) + u(w) \quad \text{for} \quad x \in \mathbb{R}.$$

Hence, replacing x by $x + w$, in view of (24), we get

$$u_t(x) = \frac{1}{c(t)}u_1(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,\infty).$$

Moreover, it follows from (24) that

$$u_t(x) = u_1(tx) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,\infty).$$

Thus, we have

$$u_1(tx) = \frac{1}{c(t)} u_1(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,\infty). \tag{25}$$

Since $u_1(x) > 0$ for $x \in (0,\infty)$, applying (25) with $x = 1$, we obtain

$$c(t) = \frac{u_1(1)}{u_1(t)} \quad \text{for} \quad t \in (0,\infty).$$

Hence (25) becomes

$$\bar{u}(tx) = \bar{u}(t)\bar{u}(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0,\infty), \tag{26}$$

where $\bar{u} : \mathbb{R} \to \mathbb{R}$ is given by

$$\bar{u}(x) = \frac{u_1(x)}{u_1(1)} \quad \text{for} \quad x \in \mathbb{R}. \tag{27}$$

Note that as u is strictly increasing and continuous, so is $\bar{u}$. Moreover, it follows from (26) that

$$\bar{u}(tx) = \bar{u}(t)\bar{u}(x) \quad \text{for} \quad x, t \in (0,\infty).$$

Thus, according to [9, Theorem 13.3.8], there exist $\beta, r \in (0,\infty)$ such that

$$\bar{u}(x) = \beta x^r \quad \text{for} \quad x \in (0,\infty).$$

Furthermore, replacing in (26) x and t by -1 and $-x$, respectively, we get

$$\bar{u}(x) = \bar{u}(-1)\bar{u}(-x) \quad \text{for} \quad x \in (-\infty,0).$$

Therefore, as $u(0) = 0$ and, in view of (24),

$$u(x) = u_1(x-w) + u(w) \quad \text{for} \quad x \in \mathbb{R},$$

taking into account (27), we obtain (23) with $a := -\beta u_1(-1) > 0$, $b := \beta u_1(1) > 0$ and $\gamma := u(w)$. In this way we have proved that $(i) \Rightarrow (iii)$.

If u is of the form (23) with some $a, b, r \in (0,\infty)$ and $\gamma \in \mathbb{R}$ then, for every $x \in \mathbb{R}$ and $\lambda \in (0,\infty)$, we have

$$u(w+\lambda x) = \lambda^r u(w+x) + (1-\lambda^r)\gamma = \lambda^r u(w+x) + (1-\lambda^r)u(w).$$

Thus, as the Choquet integral is positively homogeneous, in view of (3) and (5), for every $X \in \mathcal{X}_+$ and $\lambda \in (0,\infty)$, we obtain

$$\begin{aligned} E_g[u(w+\lambda H_{(w,u,g)}(X) - \lambda X)] &= \lambda^r E_g[u(w+H_{(w,u,g)}(X) - X)] + (1-\lambda^r)u(w) \\ &= \lambda^r u(w) + (1-\lambda^r)u(w) = u(w) = E_g[u(w+H_{(w,u,g)}(\lambda X) - \lambda X)]. \end{aligned}$$

Hence

$$H_{(w,u,g)}(\lambda X) = \lambda H_{(w,u,g)}(X) \quad \text{for} \quad X \in \mathcal{X}_+,\ \lambda \in (0,\infty).$$

This means that $H_{(w,u,g)}$ is positively homogeneous and shows that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

Corollary 2. *Assume that $w \in [0,\infty)$, g is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. Then the following statements are pairwise equivalent:*

(i)
$$H_{(w,u,g)}(X) \geq E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+^{(2)};$$

(ii)
$$H_{(w,u,g)}(X) \geq E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+; \tag{28}$$

(iii) there exists $c \in (0, \infty)$ *such that*
$$u(x) \leq c(x - w) + u(w) \quad \text{for} \quad x \in \mathbb{R}. \tag{29}$$

Proof. Let v be the identity on $\mathbb{R}$. Then, taking into account (3) and (5)-(6), for every $X \in \mathcal{X}_+$, we get
$$\begin{aligned} w = v(w) =& E_g[v(w + H_{(w,v,g)}(X) - X)] \\ =& E_g[w + H_{(w,v,g)}(X) - X] = w + H_{(w,v,g)}(X) - E_{\bar{g}}[X] \end{aligned}$$
which implies that
$$H_{(w,v,g)}(X) = E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+.$$
Therefore, applying Theorem 2, we get the assertion. □

The next result concerns the risk loading property of the principle of equivalent utility under the Rank-Dependent Utility model. Let us recall that the principle $H_{(w,u,g)}$ has the risk loading property, provided
$$H_{(w,u,g)}(X) \geq E[X] \quad \text{for} \quad X \in \mathcal{X}_+. \tag{30}$$

Corollary 3. *Assume that* $w \in [0, \infty)$ *and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function with* $u(0) = 0$*. Let* g *be a continuous probability distortion function such that*
$$g(p) \geq p \quad \text{for} \quad p \in [0, 1]. \tag{31}$$
If the premium principle $H_{(w,u,g)}$ *has the risk loading property, then there exists* $c \in (0, \infty)$ *such that* (29) *holds.*

Proof. It follows from (7) and (31) that $\bar{g}(p) \leq p$ for $p \in [0, 1]$. Therefore, if $H_{(w,u,g)}$ has the risk loading property then, applying Remark 2, we get (28). Hence, according to Corollary 2, (29) is valid with some $c \in (0, \infty)$. □

Remark 7. *Suppose that* $g(p) \leq p$ *for* $p \in [0, 1]$*. Then* $\bar{g}(p) \geq p$ *for* $p \in [0, 1]$ *and so, according to Remark 2, we have*
$$E[X] \leq E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+.$$
Hence, if (29) *is valid, then using a monotonicity of the Choquet integral, in view of* (3) *and* (5)-(6)*, for every* $X \in \mathcal{X}_+$*, we get*
$$E_g[u(w + E[X] - X)] \leq E_g[u(w + E_{\bar{g}}[X] - X)] \leq c(E_g[E_{\bar{g}}[X] - X]) + u(w) = u(w).$$
Therefore, applying Remark 3, we conclude that (30) *holds, that is* $H_{(w,u,g)}$ *has the risk loading property.*

We complete the paper with a result which is a direct consequence of Corollary 3 and Remark 7. In fact, it is a slight generalization of [3, Theorem 7].

Corollary 4. *Assume that* $w \in [0, \infty)$*,* g *is the identity on* $[0, 1]$ *and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function with* $u(0) = 0$*. Then the premium principle* $H_{(w,u,g)}$ *has the risk loading property if and only if there exists* $c \in (0, \infty)$ *such that* (29) *holds.*

REFERENCES

[1] Bowers N.L., Gerber H.U., Hickman J.C., Jones D.A., Nesbitt C.J. Actuarial Mathematics. The Society of Actuaries, Itasca, Illinois, 1986.

[2] Bühlmann H. Mathematical Models in Risk Theory. Springer, Berlin, 1970.

[3] Chudziak J. *On applications of inequalities for quasideviation means in actuarial mathematics*. Math. Inequal. Appl. 2018, **21** (3), 601–610. doi:10.7153/mia-2018-21-44

[4] Chudziak J. *On existence and uniqueness of the principle of equivalent utility under Cumulative Prospect Theory*. Insurance: Mathematics and Economics 2018, **79**, 243–246. doi:10.1016/j.insmatheco.2018.02.001

[5] Denneberg D. Lectures on Non-Additive Measure and Integral. Kluwer Academic, Boston, 1994.

[6] Gerber H. U. An Introduction to Mathematical Risk Theory. S.S. Huebner Foundation, Homewood Illinois, 1979.

[7] Heilpern S. *A rank-dependent generalization of zero utility principle*. Insurance: Mathematics and Economics 2003, **33** (1), 67–73. doi:10.1016/S0167-6687(03)00144-6

[8] Kałuszka M., Krzeszowiec M. *Pricing insurance contracts under Cumulative Prospect Theory*. Insurance: Mathematics and Economics 2012, **50** (1), 159–166. doi:10.1016/j.insmatheco.2011.11.001

[9] Kuczma M. An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's equation and Jensen's inequality. Birkhäuser, Berlin, 2009.

[10] Páles Zs. *Characterization of quasideviation means*. Acta Math. Hungar. 1982, **40** (3-4), 243–260. doi:10.1007/BF01903583

[11] Páles Zs. *General inequalities for quasideviation means*. Aequationes Math. 1988, **36** (1), 32–56. doi:10.1007/BF01837970

[12] Rachev S. T., Klebanov L. B., Stoyanov S. V., Fabozzi F. J. The Methods of Distances in the Theory of Probability and Statistics. Springer, New York, 2013.

[13] Rolski T., Schmidli H., Schmidt V., Teugels J. Stochastic Processes for Insurance and Finance. John Wiley & Sons., New York, 1999.

Received 09.09.2019

Худзяк М. *Порівняння принципів еквівалентної корисності та його застосування* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 240–249.

Принцип страхової винагороди є способом поставити у відповідність кожному ризику, зображеному за допомогою невід'ємної обмеженої випадкової величини на заданому ймовірнісному просторі, деяке дійсне невід'ємне число. Таке число можна інтерпретувати як винагороду за страховий ризик. У цій статті досліджено неявно заданий принцип еквівалентної корисності. Використовуючи властивості середнього квазівідхилення ми характеризуємо порівняння в класі принципів еквівалентної корисності за ранг-залежною корисністю, однією з важливих поведінкових моделей прийняття рішення в умовах ризику. Ми використовуємо цей результат для встановлення рівності і додатної однорідності цього принципу. Також висвітлено деякі інші застосування.

Ключові слова і фрази: страхова винагорода, середнє квазівідхилення, порівняння, рівність, додатна однорідність, ризик.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 250–257
doi:10.15330/cmp.11.2.250-257

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.250–257

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LEIBNIZ ALGEBRAS: A BRIEF REVIEW OF CURRENT RESULTS

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a,b,c \in L$. This paper is a brief review of some current results, which related to finite- dimensional and infinite-dimensional Leibniz algebras

Key words and phrases: Leibniz algebra, cyclic Leibniz algebra, ideal, subideal, contraideal, center, lower (upper) central series, finite-dimensional Leibniz algebra, nilpotent Leibniz algebra, Leibniz T-algebra, anticenter, antinilpotent Leibniz algebra.

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To Professor L.A. Kurdachenko on the occasion of his 70th birthday

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a,b,c \in L$.

Leibniz algebras appeared first in the papers of A.M. Bloh [5–7], in which he called them the *D-algebras*. However, a real interest to Leibniz algebras rose after the paper of J.-L. Loday [25] (see also [26, Section 10.6]), who rediscovered these algebras and used the term *Leibniz algebras* since it was G.W. Leibniz who discovered and proved the Leibniz rule for differentiation of functions.

Note that the Leibniz algebras have many connections with some areas of mathematics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, non- commutative geometry, and physics (see, for example, [8,12,13]).

The theory of Leibniz algebras has been developing intensively but very sporadic. On the one hand, many analogues of important results from the theory of Lie algebras were proven (see, for example, a survey [18]). On the other hand, many natural questions about the structure of Leibniz algebras are not considered. For example, we can note the situation about the structure of finite-dimensional Leibniz algebras. The first natural step in the study of all types of algebras is the description of algebras having small dimensions. Unlike the simpler cases of 1- and 2-dimensional Leibniz algebras, the structure of 3- dimensional Leibniz algebras is more complex, as well as it is more complex than the structure of 3- dimensional Lie algebras. The study of Leibniz algebras, having dimension 3, has been conducted in the papers [1,2,9,11]

УДК 512.554
2010 *Mathematics Subject Classification:* 17A32, 17A60.

for the fields of characteristic 0, moreover for the field $\mathbb{C}$ of complex numbers or algebraically closed fields of characteristic 0. In [33], Yashchuk V.S. considered the opposite situation. She described the structure of Leibniz algebras of dimension 3 over finite fields. In some cases, the structure of such algebras essentially depends on the characteristic of the field. In other words, it depends on the solvability of specific equations in the fields, and so on. It is also worth mentioning here that the description of Leibniz algebras of dimension 3 is very different from the description of Lie algebras of dimension 3, which indicates a significant difference between these types of algebras.

Note that if L is a Lie algebra, then $[[a,b],c]+[[b,c],a]+[[c,a],b]=0$. It follows that

$$\begin{aligned}[[a,b],c] &= -[[b,c],a]-[[c,a],b]\\ &= [a,[b,c]]+[b,[c,a]]\\ &= [a,[b,c]]-[b,[a,c]],\end{aligned}$$

which shows that every Lie algebra is a Leibniz algebra.

Conversely, suppose that $[a,a]=0$ for all elements $a\in L$. Then for arbitrary elements $a,b\in L$ we have $0=[a+b,a+b]=[a,a]+[a,b]+[b,a]+[b,b]=[a,b]+[b,a]$. It follows that $[a,b]=-[b,a]$. Then

$$\begin{aligned}0 &= [[a,b],c]-[a,[b,c]]+[b,[a,c]]\\ &= [[a,b],c]+[[b,c],a]-[[a,c],b]\\ &= [[a,b],c]+[[b,c],a]+[[c,a],b]\end{aligned}$$

for all $a,b,c\in L$. In other words, Lie algebras can be characterized as Leibniz algebras in which $[a,a]=0$ for every element $a\in L$.

Recall some basic definitions.

A Leibniz algebra L is called *abelian* if $[a,b]=0$ for all elements $a,b\in L$. Thus, an abelian Leibniz algebra is a Lie algebra.

Let L be a Leibniz algebra over a field F. If A,B are subspaces of L, then $[A,B]$ will denote a subspace generated by all elements $[a,b]$, where $a\in A$, $b\in B$. A subspace A of L is called a *subalgebra* of L, if $[x,y]\in A$ for every $x,y\in A$. It follows that $[A,A]\leqslant A$.

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L, then $\langle M\rangle$ denote the subalgebra of L generated by M.

A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[y,x]\in A$ (respectively $[x,y]\in A$) for every $x\in A$, $y\in L$. In other words, if A is a left (respectively right) ideal, then $[L,A]\leqslant A$ (respectively $[A,L]\leqslant A$).

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, that is $[y,x],[x,y]\in A$ for every $x\in A$, $y\in L$.

If A is an ideal of L, we can consider a *factor-algebra* L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

Denote by $\mathbf{Leib}(L)$ the subspace, generated by the elements $[a, a]$, $a \in L$. Note that $\mathbf{Leib}(L)$ is an ideal of L, which is called the *Leibniz kernel* of algebra L.

The *left* (respectively *right*) *center* $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{x \in L |\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{right}(L) = \{x \in L |\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. The right center is a subalgebra of L, and in general, the left and right centers are different. Moreover, they even may have different dimensions as shows an example 2.1 from [19].

The *center* $\zeta(L)$ of L is the intersection of the left and right centers, that is

$$\zeta(L) = \{x \in L |\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Clearly, the center $\zeta(L)$ is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

Now we define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \dots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \dots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup\limits_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

Let L be a Leibniz algebra. Define the *lower central series*

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \dots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \dots \gamma_\delta(L)$$

of L by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap\limits_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. For the last term $\gamma_\delta(L)$ we have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

The introduced here concepts of the upper and lower central series for Leibniz algebras are an analogous of others similar concepts, which became standard in several algebraic structures. They play an important role, for example, in Lie algebras and groups. Following this analogy, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$.

We note that in [22] Kurdachenko L.A., Subbotin I.Ya. and Semko N.N. proved a series of results, which connected with (locally) nilpotent and hypercentral Leibniz algebras. In particular, these results are analogues of well-known group-theoretical results.

It is a well-known that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length. The same result is also true for Leibniz algebras (see, for example, [19]).

Let L be a Leibniz algebra. Let us define the *lower derived series*

$$L = \delta_0(L) \geqslant \delta_1(L) \geqslant \ldots \delta_\alpha(L) \geqslant \delta_{\alpha+1}(L) \geqslant \ldots \delta_\nu(L)$$

of L by the following rule: $\delta_0(L) = L$, $\delta_1(L) = [L, L]$, and recursively $\delta_{\alpha+1}(L) = [\delta_\alpha(L), \delta_\alpha(L)]$ for all ordinals α and $\delta_\lambda(L) = \bigcap_{\mu<\lambda} \delta_\mu(L)$ for the limit ordinals λ. For the last term $\delta_\nu(L)$ we have $\delta_\nu(L) = [\delta_\nu(L), \delta_\nu(L)]$. If $\delta_n(L) = \langle 0 \rangle$ for some positive integer n, then we say that L is a *soluble* Leibniz algebra.

One of the first questions that naturally arises in the study of any algebraic structure is the question of the structure of its cyclic substructures. Unlike Lie algebras, associative algebras, groups, etc., cyclic Leibniz algebras are no necessarily abelian. In [10, Theorem 1.1] Chupordia V.A., Kurdachenko L.A. and Subbotin I.Ya. described the structure of such Leibniz algebras. This description made it possible to obtain a structure of the Leibniz algebras, whose proper subalgebras are Lie algebras. Such algebras are either Lie algebras, or nilpotent cyclic algebras, or they can be represented as a direct sum of an abelian ideal (from the left center of algebra) and Lie subalgebra of dimension 1 with some additional properties [10, Theorem 1.2]. As a corollary it was described Leibniz algebras whose proper subalgebras are abelian [10, Corollary 1.1]. This result implies that a description of Leibniz algebras, whose proper subalgebras are abelian, can be deduced to the case of Lie algebras, whose proper subalgebras are abelian. Such Lie algebras are either simple, or soluble. Soluble minimal non-abelian Lie algebras (even soluble minimal non-nilpotent Lie algebras) were described in [16, 30, 31]. Simple minimal non-abelian Lie algebras were studied in [14, 15], but their complete description remains an open problem.

Another natural question concerns the relationship of the subalgebras and ideals. In particular, what is a structure of Leibniz algebras, all of whose subalgebras are ideals? It is not hard to prove that a Lie algebra, all of whose subalgebras are ideals, is abelian. For groups the situation is different: there exists non-abelian groups, all of whose subgroups are normal. Such groups have been described in [3]. For Leibniz algebras the situation is quite diverse. Recall that a Leibniz algebra L is called an *extraspecial* algebra if it satisfies the following condition: $\zeta(L)$ is non-trivial and has dimension 1, and $L/\zeta(L)$ is abelian. It is important to observe that there are extraspecial Leibniz algebras in which not every subalgebra is an ideal. In [20] Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. proved that if L is a Leibniz algebra over a field F, all of whose subalgebras are ideals and L is non-abelian, then $L = E \oplus Z$ where $Z \leqslant \zeta(L)$, and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.

Consider now some other natural questions of the general theory of Leibniz algebras. Note that the relation "to be a subalgebra of a Leibniz algebra" is transitive. However, the relation "to be an ideal" is not transitive even for Lie algebras. Therefore it is natural to ask the question about the structure of Leibniz algebras, in which the relation "to be an ideal" is transitive. In this context, the following important type of subalgebras naturally arises. A subalgebra A of a Leibniz algebra L is called a *left* (respectively *right*) *subideal* of L, if there is a finite series of subalgebras $A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$ such that A_{j-1} is a left (respectively, right) ideal of A_j, $1 \leqslant j \leqslant n$.

Similarly, a subalgebra A of a Leibniz algebra L is called a *subideal* of L, if there is a finite series of subalgebras $A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$ such that A_{j-1} is an ideal of A_j, $1 \leqslant j \leqslant n$.

We note the following property of nilpotent Leibniz algebras (see, for example [18]): if L is a nilpotent Leibniz algebra over a field F, then every subalgebra of L is a subideal of L.

A Leibniz algebra L is called a *T-algebra*, if a relation "to be an ideal" is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that in a Leibniz T-algebra every subideal is an ideal. Lie algebras, in which a relation "to be an ideal" is transitive have been studied by I. Stewart [28] and A.G. Gejn and Yu.N. Muhin [17]. In particular, soluble T-algebras and finite dimensional T-algebras over a field of characteristic 0 has been described. As in the mentioned above cases, the situation in Leibniz algebras is much more complex and diverse than it was in Lie algebras (see, for examples [18]). The description of Leibniz T-algebras has been obtained by Kurdachenko L.A., Subbotin I.Ya. and Yashchuk V.S. in the paper [24].

Consider now some new approach in Leibniz algebra theory. Two ideals are naturally associated with each subalgebra A of a Leibniz algebra L: the ideal A^L which is the intersection of all ideals including A (that is an ideal, generated by A); and the ideal $\mathbf{Core}_L(A)$ which is the sum of all ideals that are contained in A. A subalgebra A of L is called an *contraideal* of L, if $A^L = L$. From the definition it follows that the contraideals are natural antipodes to the concepts of ideals. Therefore, the study of Leibniz algebras whose subalgebras are either ideals or contraideals is very natural. The description of such Leibniz algebras was obtained by Kurdachenko L.A., Subbotin I.Ya. and Yashchuk V.S. in the paper [23]. As a corollary, the authors obtained the structure of Lie algebras, whose subalgebras are either ideals or contraideals [23].

As we noted above, the fact that $\gamma_{c+1}(L) = \langle 0 \rangle$ is equivalent to the fact that $\zeta_c(L) = L$, i.e. the lower and the upper central series in nilpotent Leibniz algebras have the same length. The next natural step is the consideration of the case, when the upper (respectively lower) central series has finite length. For this case the question about the relationships between $L/\zeta_k(L)$ and $\gamma_{k+1}(L)$ naturally appears.

If L is a Lie algebra such that $L/\zeta_k(L)$ is finite-dimensional, then $\gamma_{k+1}(L)$ is also finite-dimensional [29]. A corresponding result for groups has been obtained early by R. Baer [4]. Kurdachenko L.A., Otal J. and Pypka A.A. in the paper [19] obtained the following analog of these theorems: if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta_k(L)) = d$ is finite for some positive integer k, then $\gamma_{k+1}(L)$ has finite dimension; moreover $\mathbf{dim}_F(\gamma_{k+1}(L)) \leqslant 2^{k-1}d^{k+1}$.

An important specific case here is the case when the center of a Leibniz algebra L has finite codimension. For Lie algebras the following result is well known (see, for example [32]). A corresponding result for groups was proved much earlier: if G is a group and C is a subgroup of the center $\zeta(G)$ such that G/C is finite, then the derived subgroup $[G, G]$ is finite. In this formulation, for the first time it appears in the paper of B.H. Neumann [27]. This theorem was obtained also by R. Baer [4].

For Leibniz algebras the following analog of these results was proved by Kurdachenko L.A., Otal J. and Pypka A.A. in [19]: *if L is a Leibniz algebra over a field F, $\mathbf{codim}_F(\zeta^{left}(L)) = d$ and $\mathbf{codim}_F(\zeta^{right}(L)) = r$ are finite, then $[L, L]$ has finite dimension; moreover, $\mathbf{dim}_F([L, L]) \leqslant d(d + r)$.*

In this connection, the following question appears: suppose that only $\mathbf{codim}_F(\zeta^{left}(L))$ is finite. Is $\mathbf{dim}_F([L, L])$ finite? The Example 3.1 from [19] gives a negative answer on this question. However, if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta(L)) = d$ is finite, then $[L, L]$ has finite dimension; in particular, $\mathbf{dim}_F([L, L]) \leqslant d^2$ [19]. Moreover, if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta(L)) = d$ is finite, then the Leibniz kernel of L has finite dimension at most $\frac{1}{2}d(d-1)$ [19].

Finally, we note that in [21] Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. introduced the concepts of anticenter of Leibniz algebras and antinilpotent Leibniz algebras. Let L be a Leibniz algebra. Put

$$\alpha(L) = \{z \in L | [a, z] = -[z, a] \text{ for each element } a \in L\}.$$

This subset is called the *anticenter* of a Leibniz algebra L. Note that the anticenter is an ideal of L. Note also that we must consider the case, when $char(F) \neq 2$, because in the case when $char(F) = 2$ anticenter in general is not ideal [21].

For this concept the above authors proved some analogs of result from Leibniz algebra theory. In particular, in [21] they proved that if L is a Leibniz algebra over a field F and the anticenter of L has finite codimension d, then the Leibniz kernel of L has finite dimension at most d^2.

Starting from the anticenter, we define the *upper anticentral series*

$$\langle 0 \rangle = \alpha_0(L) \leqslant \alpha_1(L) \leqslant \ldots \alpha_\lambda(L) \leqslant \alpha_{\lambda+1}(L) \leqslant \ldots \alpha_\gamma(L) = \alpha_\infty(L)$$

of a Leibniz algebra L by the following rule: $\alpha_1(L) = \alpha(L)$ is the anticenter of L, and recursively, $\alpha_{\lambda+1}(L)/\alpha_\lambda(L) = \alpha(L/\alpha_\lambda(L))$ for all ordinals λ,and $\alpha_\mu(L) = \bigcup\limits_{\nu<\mu} \alpha_\nu(L)$ for the limit ordinals μ. By definition, each term of this series is an ideal of L. The last term $\alpha_\infty(L)$ of this series is called the *upper hyperanticenter* of L. A Leibniz algebra L is said to be *hyperanticentral* if it coincides with the upper hyperanticenter. Denote by $al(L)$ the length of upper anticentral series of L. If L is hyperanticentral and $al(L)$ is finite, then L is said to be *antinilpotent*.

If U, V the ideals of L, then we denote by (U, V) a subspace, generated by all elements $[u, v] + [v, u]$, $u \in U, v \in V$. Note that $[u, v] + [v, u] \in \zeta^{left}(L)$ and (U, V) is an ideal of L [21]. Define the *lower anticentral series* of L

$$L = \kappa_1(L) \geqslant \kappa_2(L) \geqslant \ldots \kappa_\alpha(L) \geqslant \kappa_{\alpha+1}(L) \geqslant \ldots \kappa_\delta(L)$$

by the following rule: $\kappa_1(L) = L$, $\kappa_2(L) = (L, L)$, and recursively $\kappa_{\lambda+1}(L) = (L, \kappa_\lambda(L))$ for all ordinals λ and $\kappa_\mu(L) = \bigcap\limits_{\nu<\mu} \kappa_\nu(L)$ for the limit ordinals μ. For the last term $\kappa_\delta(L)$ we have $\kappa_\delta(L) = (L, \kappa_\delta(L))$.

As we noted above in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length. For antinilpotent Leibniz algebras Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. [21] proved the analog of this statement: if L is an antinilpotent Leibniz algebra, then the length of the lower anticentral series coincides with the length of the upper anticentral series; moreover, the length of these two series is the smallest among the lengths of all anticentral series of L.

REFERENCES

[1] Ayupov S.A., Omirov B.A. *On 3-dimensional Leibniz algebras*. Uzbek Math. J. 1999, **1**, 9–14.

[2] Albeverio S., Omirov B.A., Rakhimov I.S. *Varieties of nilpotent complex Leibniz algebras of dimension less than five*. Comm. Algebra 2005, **33** (5), 1575–1585. doi:10.1081/AGB-200061038

[3] Baer R. *Situation der Untergruppen und Struktur der Gruppe*. S.-B. Heidelberg Acad. Math.-Nat. Klasse 1933, **2**, 12–17.

[4] Baer R. *Endlichkeitskriterien für Kommutatorgruppen*. Math. Ann. 1952, **124** (1), 161–177. doi:10.1007/BF01343558

[5] Bloh A.M. *On a generalization of the concept of Lie algebra*. Dokl. Akad. Nauk SSSR 1965, **165** (3), 471–473. (in Russian)

[6] Bloh A.M. *Cartan-Eilenberg homology theory for a generalized class of Lie algebras*. Dokl. Akad. Nauk SSSR 1967, **175** (8), 824–826. (in Russian)

[7] Bloh A.M. *A certain generalization of the concept of Lie algebra*. Algebra and number theory. Moskov. Gos. Ped. Inst. Uchen. Zap. 1971, **375**, 9–20.

[8] Butterfield J., Pagonis C. From Physics to Philosophy. Cambridge Univ. Press, Cambridge, 1999.

[9] Casas J.M., Insua M.A., Ladra M., Ladra S. *An algorithm for the classification of 3-dimensional complex Leibniz algebras*. Linear Algebra Appl. 2012, **436** (9), 3747–3756. doi:10.1016/j.laa.2011.11.039

[10] Chupordia V.A., Kurdachenko L.A., Subbotin I. Ya. *On some "minimal" Leibniz algebras*. J. Algebra Appl. 2017, **16** (05), 1750082 (16 pages). doi:10.1142/S0219498817500827

[11] Demir I., Misra K.C., Stitzinger E. *On some structures of Leibniz algebras*. Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics 2014, **623**, 41–54. doi:10.1090/conm623/12456

[12] Dobrev V. Lie Theory and its applications in physic. IX International workshop. Springer, Tokyo, 2013.

[13] Duplij S., Wess J. Noncommutative Structures in Mathematics and Physics, Proceedings of the NATO advanced research workshop. Springer, Kiev, 2001.

[14] Farnsteiner R. *On the structure of simple-semiabelian Lie algebras*. Pacific J. Math. 1984, **111**, 287–299. doi:10.2140/pjm.1984.111.287

[15] Gejn A.G. *Minimal noncommutative and minimal nonabelian algebras*. Comm. Algebra 1985, **13**, 305–328. doi:10.1080/00927878508823161

[16] Gejn A.G., Kuznetsov S.V., Mukhin Yu.N. *On minimal non-nilpotent Lie algebras*. Ural. Gos. Univ. Mat. Zap. 1972, **8**, 18–27.

[17] Gejn A., Mukhin Yu.N. *Complements to subalgebras of Lie algebras*. Ural. Gos. Univ. Mat. Zap. 1980 **12** (2), 24–48.

[18] Kirichenko V.V., Kurdachenko L.A., Pypka A.A., Subbotin I.Ya. *Some aspects of Leibniz algebra theory*. Algebra Discrete Math. 2017 **24** (1), 1–33.

[19] Kurdachenko L.A., Otal J., Pypka A.A. *Relationships between factors of canonical central series of Leibniz algebras*. Eur. J. Math. 2016, **2** (2), 565–577. doi:10.1007/s40879-016-0093-5

[20] Kurdachenko L.A., Semko N.N., Subbotin I.Ya. *The Leibniz algebras whose subalgebras are ideals*. Open Math. 2017, **15** (1), 92–100. doi:10.1515/math-2017-0010

[21] Kurdachenko L.A., Semko N.N., Subbotin I.Ya. *On the anticommutativity in Leibniz algebras*. Algebra Discrete Math. 2018 **26** (1), 97–109.

[22] Kurdachenko L.A., Subbotin I.Y., Semko N.N. *From Groups to Leibniz Algebras: Common Approaches, Parallel Results*. Adv. Group Theory Appl. 2018 **5**, 1–31. doi:10.4399/97888255161421

[23] Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S. *On Some Antipodes of the Ideals in Leibniz Algebras*. J. Algebra Appl. 2019. doi:10.1142/S0219498820501133

[24] Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S. *Leibniz Algebras Whose Subideals are Ideals*. J. Algebra Appl. 2018, **17** (08) 1850151. doi:10.1142/S0219498818501517

[25] Loday J.-L. *Une version non commutative des algèbres de Lie: les algèbras de Leibniz*. Enseign. Math. 1993, **39**, 269–293.

[26] Loday J.-L. Cyclic homology. Grundlehren Math. Wiss. Vol. 301, 2nd ed., Springer, Verlag, Berlin, 1998.

[27] Neumann B.H. *Groups with finite classes of conjugate elements*. Proc. Lond. Math. Soc. 1951, **3** (1), 178–187. doi:10.1112/plms/s3-1.1.178

[28] Stewart I.N. Subideals of Lie algebras. Ph.D. Thesis, University of Warwick. 1969.

[29] Stewart I.N. *Verbal and marginal properties of non-associative algebras in the spirit of infinite group theory*. Proc. Lond. Math. Soc. 1974, **3** (28), 129–140. doi:10.1112/plms/s3-28.1.129

[30] Stitzinger E.L. *Minimal nonnilpotent solvable Lie algebras*. Proc. Amer. Math. Soc. 1971, **28**, 47–49. doi:10.1090/S0002-9939-1971-0271178-X

[31] Towers D.A. *Lie algebras all whose proper subalgebras are nilpotent*. Linear Algebra Appl. 1980, **32**, 61–73. doi:10.1016/0024-3795(80)90007-5

[32] Vaughan-Lee M.R. *Metabelian BFC p-groups*. J. Lond. Math. Soc. 1972, **5** (4), 673–680. doi:10.1112/jlms/s2-5.4.673

[33] Yashchuk V.S. *On some Leibniz algebras, having small dimension*. Algebra Discrete Math. 2019, **27** (2), 292–308.

Received 11.09.2019

Чупордя В.А., Пипка О.О., Семко М.М., Ящук В.С. *Алгебри Лейбніца: короткий огляд сучасних результатів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 250–257.

Нехай L – алгебра над полем F з двома бінарними операціями $+$ та $[\cdot,\cdot]$. Тоді L називатимемо лівою алгеброю Лейбніца, якщо вона задовольняє ліву тотожність Лейбніца $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ для всіх $a,b,c \in L$. Дана стаття є коротким оглядом деяких сучасних результатів, пов'язаних зі скінченновимірними та нескінченновимірними алгебрами Лейбніца.

Ключові слова і фрази: алгебра Лейбніца, циклічна алгебра Лейбніца, ідеал, субідеал, контраідеал, центр, верхній (нижній) центральний ряд, скінченновимірна алгебра Лейбніца, нільпотентна алгебра Лейбніца, T-алгебра Лейбніца, антицентр, антинільпотентна алгебра Лейбніца.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 258–267
doi:10.15330/cmp.11.2.258-267

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.258–267

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SOME DISTANCE BASED INDICES OF GRAPHS BASED ON FOUR NEW OPERATIONS RELATED TO THE LEXICOGRAPHIC PRODUCT

For a (molecular) graph, the Wiener index, hyper-Wiener index and degree distance index are defined as $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$, $WW(G) = W(G) + \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)^2$, and $DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)(d(u/G) + d(v/G))$, respectively, where $d(u/G)$ denotes the degree of a vertex u in G and $d_G(u,v)$ is distance between two vertices u and v of a graph G. In this paper, we study Wiener index, hyper-Wiener index and degree distance index of graphs based on four new operations related to the lexicographic product, subdivision and total graph.

Key words and phrases: Wiener index, degree distance index, hyper-Wiener index, lexicographic product, subdivision, total graph.

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INTRODUCTION

In this paper G is a simple and connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *degree of a vertex* v in G is the number of edges incident to v and denoted by $d(v/G)$. The *distance* $d_G(u,v)$ between any two vertices u and v of a graph G is equal to the length of a shortest path connecting them. A *line graph*, $L(G)$, is the graph whose vertices correspond to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

In chemical graph theory, a graphical invariant is a number related to a graph which is structurally invariant. These invariant numbers are also known as the topological indices. The well-known Zagreb indices are one of the oldest graph invariants firstly introduced by Gutman and Trinajstić [18], where they examined the dependence of total π-electron energy on molecular structures, and this was elaborated on in [17]. For a (molecular) graph G, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$, are:

$$M_1(G) = \sum_{v\in V(G)} d(v/G)^2 = \sum_{uv\in E(G)} [d(u/G) + d(v/G)],$$

and

$$M_2(G) = \sum_{uv\in E(G)} d(u/G)d(v/G).$$

УДК 519.17
2010 *Mathematics Subject Classification:* 05C07, 05C12, 05C76.
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For properties of the two Zagreb indices see [4–6] and the papers cited therein. In recent years, some novel variants of Zagreb indices have been put forward, such as Zagreb coindices [2, 10, 15], reformulated Zagreb indices [20, 24], Zagreb hyper index [3, 25], multiplicative Zagreb indices [13, 30], multiplicative sum Zagreb index [11, 28], and multiplicative Zagreb coindices [29], etc. The Zagreb coindices are defined as:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d(u/G) + d(v/G)],$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} d(u/G)d(v/G).$$

The *Wiener index* of G is denoted by $W(G)$ and is defined by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v). \tag{1}$$

The name Wiener index or Wiener number for the quantity defined in Equation (1) is usual in chemical literature, since Harold Wiener [27] in 1947 seems to be the first who considered it. Wiener himself conceived W only for acyclic molecules and defined it in a slightly different-yet equivalent-manner; the definition of the Wiener index in terms of distances between vertices of a graph, such as in Equation (1), was first given by Hosoya [19]. Eliasi et. al [12], determined the Wiener index of some graph operations.

The *hyper-Wiener index* of G is denoted by $WW(G)$, and is defined as

$$WW(G) = W(G) + \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)^2.$$

Lukovits [23] derived formulas for the hyper-Wiener index of chains and trees which contain one trivalent or tetravalent branching vertex, and this index is studied by several authors in [1, 8, 16, 22]. Khalifeh et. al [21], determined the hyper-Wiener index of graph operations.

The *degree distance* of a graph G, $DD(G)$, was introduced by Dobrynin and Kochetova [9] and Gutman [14] as a weighted version of the Wiener index, and is defined as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)(d(u/G) + d(v/G)).$$

In this paper, we study of the Wiener, hyper-Wiener and degree distance indices of graphs based on operations related to the lexicographic, subdivision and total graph. For this purpose, we recall some operations on graphs in the following.
The *composition* or *lexicographic product* of two connected graphs G_1 and G_2, denoted by $G_1[G_2]$, is a graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $G_1[G_2]$ are adjacent if and only if either u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent with v_2. For a connected graph G, there are four related graphs as follows:

(i) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2;

(ii) $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and joining each new vertex to the end vertices of the corresponding edge;

(iii) $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and joining those pairs of new vertices on adjacent edges of G;

(iv) $T(G)$ is the graph with vertex set $V(G) \cup E(G)$ and adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graphs $S(G)$ and $T(G)$ are called the subdivision graph and the total graph of G, respectively.

Based on the lexicographic product of two connected graphs G_1 and G_2, Sarala et al. [26], introduced four new operations on these graphs.

Let $F \in \{S, R, Q, T\}$. The F-product of G_1 and G_2, denoted by $G_1[G_2]_F$, is defined by $F(G_1)[G_2] - E^*$, where $E^* = \{(u, v_1)(u, v_2) \in E(F(G_1)[G_2]) : u \in V(F(G_1)) - V(G_1)$ and $v_1v_2 \in E(G_2)\}$, i.e., $G_1[G_2]_F$ is a graph with the vertex set $V(G_1[G_2]_F) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ of $G_1[G_2]_F$ are adjacent if and only if either $[u_1 = u_2 \in V(G_1)$ and $v_1v_2 \in E(G_2)]$ or $[u_1u_2 \in E(F(G_1))$ and $v_1, v_2 \in V(G_2)]$.

Sarala et al. [26] determined the Zagreb indices of F-product of G_1 and G_2 where $F \in \{S, R, Q, T\}$, and Dehgardi et. al [7] computed the leap Zagreb indices of these graphs.

We will use the following results.

Theorem 1 ([7]). *Let G_1 and G_2 be two connected graphs, and let $G = G_1[G_2]_F$ be the F-product of G_1 and G_2. Then*

$$d_G((u,x),(v,y)) = \begin{cases} 1 & \text{if } \ u = v \in V(G_1), xy \in E(G_2) \\ 2 & \text{if } \ u = v \in V(G_1), xy \notin E(G_2) \\ 2 & \text{if } \ u = v \in V(F(G_1)) - V(G_1) \\ d_{F(G_1)}(u,v) & \text{if } \ u \neq v. \end{cases}$$

Theorem 2 ([15]). *Let G be a graph with n vertices and m edges. Then*

$$M_1(G) + \overline{M_1}(G) = 2m(n-1).$$

Theorem 3 ([15]). *Let G be a graph with n vertices and m edges. Then*

$$M_2(G) + \overline{M_2}(G) = 2m^2 - \frac{1}{2}M_1(G).$$

Theorem 4 ([31]). *Let G be a graph. Then for any $v, v' \in V(G)$,*

$$\frac{1}{2}d_{S(G)}(v,v') = d_{T(G)}(v,v') = d_{R(G)}(v,v') = d_{Q(G)}(v,v') - 1 = d_G(v,v').$$

Theorem 5 ([31]). *Let G be a graph. Then for any $e, e' \in E(G)$,*

$$\frac{1}{2}d_{S(G)}(e,e') = d_{T(G)}(e,e') = d_{R(G)}(e,e') - 1 = d_{Q(G)}(e,e') = d_{L(G)}(e,e').$$

1 WIENER, HYPER WIENER, AND DEGREE DISTANCE INDICES FOR F-PRODUCT OF GRAPHS

In this section, we consider $F \in \{S, Q, R, T\}$, and compute the Wiener, hyper Wiener, and degree distance indices for F-product of two connected graphs G_1 and G_2. Let $|V(G_i)| = n_i$, and $|E(G_i)| = \varepsilon_i$ for $i = 1, 2$. Throughout this section we assume that

$\sum_1 := \sum_{\{(u,x),(v,y)\} \subseteq V(G), u=v \in V(G_1), xy \in E(G_2)}$,
$\sum_2 := \sum_{\{(u,x),(v,y)\} \subseteq V(G), u=v \in V(G_1), xy \notin E(G_2)}$,
$\sum_3 := \sum_{\{(u,x),(v,y)\} \subseteq V(G), u=v \in V(F(G_1)) - V(G_1), x,y \in V(G_2)}$, and
$\sum_4 := \sum_{\{(u,x),(v,y)\} \subseteq V(G), u \neq v, x,y \in V(G_2)}$.

1.1 Wiener index and hyper Wiener index

Theorem 6. *Let G_1 and G_2 be two connected graphs, and let $G = G_1[G_2]_F$. Then*

$$W(G) = n_1n_2(n_2-1) - n_1\varepsilon_2 + \varepsilon_1 n_2(n_2-1) + n_2^2 W(F(G_1)).$$

Proof. By Theorem 1, we have

$$\begin{aligned} W(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)} d_G((u,x),(v,y)) \\ &= \textstyle\sum_1 1 + \sum_2 2 + \sum_3 2 + \sum_4 d_{F(G_1)}(u,v) \\ &= n_1\varepsilon_2 + 2n_1(\tfrac{n_2(n_2-1)}{2} - \varepsilon_2) + 2\varepsilon_1 \tfrac{n_2(n_2-1)}{2} + n_2^2 W(F(G_1)) \\ &= n_1n_2(n_2-1) - n_1\varepsilon_2 + \varepsilon_1 n_2(n_2-1) + n_2^2 W(F(G_1)). \end{aligned}$$

□

Theorem 7. *Let G_1 and G_2 be two connected graphs, and let $G = G_1[G_2]_F$. Then*

$$WW(G) = -4n_1\varepsilon_2 + 3n_2(n_2-1)(n_1+\varepsilon_1) + n_2^2 WW(F(G_1)).$$

Proof. By Theorem 1, we have

$$\begin{aligned} WW(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)} [d_G((u,x),(v,y)) + d_G^2((u,x),(v,y))] \\ &= \textstyle\sum_1 2 + \sum_2 6 + \sum_3 6 + \sum_4 [d_{F(G_1)}(u,v) + d^2_{F(G_1)}(u,v)] \\ &= 2n_1\varepsilon_2 + 6n_1(\tfrac{n_2(n_2-1)}{2} - \varepsilon_2) + 6\varepsilon_1 \tfrac{n_2(n_2-1)}{2} + n_2^2 WW(F(G_1)) \\ &= -4n_1\varepsilon_2 + 3n_2(n_2-1)(n_1+\varepsilon_1) + n_2^2 WW(F(G_1)). \end{aligned}$$

□

1.2 Degree distance index

1.2.1 The case F=S

Theorem 8 ([26]). *If G_1 and G_2 are two connected graphs of orders n_1 and n_2, respectively, and $G = G_1[G_2]_S$, then*

$$d((u,x)/G) = \begin{cases} n_2 d(u/G_1) + d(x/G_2) & \text{if} \quad u \in V(G_1), \\ 2n_2 & \text{if} \quad u \in V(S(G_1)) - V(G_1). \end{cases}$$

Theorem 9. *Let G_i be a connected graph of order n_i, and size ε_i for $i = 1,2$, and let $G = G_1[G_2]_S$. Then*

$$\begin{aligned} DD(G) &= 2(n_2-1)(4\varepsilon_1 n_2^2 + \varepsilon_2 n_1) - 4n_2\varepsilon_1\varepsilon_2 + n_1\overline{M}_1(G_2) + 2n_2^3 DD(G_1) \\ &+ 4n_2(\varepsilon_2 - n_2^2)W(G_1) + 4n_2^2(n_2^2 - \varepsilon_2)W(L(G_1)) + (2n_2\varepsilon_2 + 2n_2^3)W(S(G_1)) \\ &+ n_2^3 \textstyle\sum_{u\in V(G_1), v\in V(S(G_1))-V(G_1)} d(u/G_1) d_{S(G_1)}(u,v). \end{aligned}$$

Proof. Let e_u be the corresponding edge to the new vertex u. We deduce from Theorems 1, 2, 3, 4, 5 and 8, that

$$\begin{aligned} DD(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)} [d((u,x)/G) + d((v,y)/G)] d_G((u,x),(v,y)) \\ &= \textstyle\sum_1 [d((u,x)/G) + d((u,y)/G)] \\ &+ 2\textstyle\sum_2 [d((u,x)/G) + d((u,y)/G)] \\ &+ 2\textstyle\sum_3 [d((u,x)/G) + d((u,y)/G)] \\ &+ \textstyle\sum_4 [(d((u,x)/G) + d((v,y)/G)) d_{S(G_1)}(u,v)], \end{aligned}$$

and

$$\begin{aligned}\textstyle\sum_1[d((u,x)/G)+d((u,y)/G)] &= \textstyle\sum_1[2n_2d(u/G_1)+d(x/G_2)+d(y/G_2)]\\ &= 4n_2\varepsilon_2\varepsilon_1+n_1M_1(G_2).\end{aligned}$$

$$\begin{aligned}2\textstyle\sum_2[d((u,x)/G)+d((u,y)/G)] &= 2\textstyle\sum_2[2n_2d(u/G_1)+d(x/G_2)+d(y/G_2)]\\ &= 4n_2^2\varepsilon_1(n_2-1)-8n_2\varepsilon_1\varepsilon_2+2n_1\overline{M}_1(G_2).\end{aligned}$$

$$2\textstyle\sum_3[d((u,x)/G)+d((u,y)/G)] = 2\textstyle\sum_3 4n_2 = 4n_2^2\varepsilon_1(n_2-1).$$

$$\begin{aligned}&\sum_4[(d((u,x)/G)+d((v,y)/G))d_{S(G_1)}(u,v)]\\
&= \textstyle\sum_{u\neq v,u,v\in V(G_1),x,y\in V(G_2)}[n_2(d(u/G_1)+d(v/G_1))+d(x/G_2)+d(y/G_2)]d_{S(G_1)}(u,v)\\
&+\textstyle\sum_{u\neq v,u,v\in V(S(G_1))-V(G_1),x,y\in V(G_2)} 4n_2d_{S(G_1)}(u,v)\\
&+\textstyle\sum_{u\in V(G_1),v\in V(S(G_1))-V(G_1),x,y\in V(G_2)}[n_2d(u/G_1)+d(x/G_2)+2n_2]d_{S(G_1)}(u,v)\\
&= 2n_2^3DD(G_1)+2W(G_1)(2M_1(G_2)+2\overline{M}_1(G_2)+4\varepsilon_2)\\
&+4n_2^3\textstyle\sum_{e_u,e_v\in V(L(G_1))} 2d_{L(G_1)}(e_u,e_v)+n_2^3\textstyle\sum_{u\in V(G_1),v\in(V(S(G_1))-V(G_1))} d(u/G_1)d_{S(G_1)}(u,v)\\
&+(2n_2\varepsilon_2+2n_2^3)\textstyle\sum_{u\in V(G_1),v\in V(S(G_1))-V(G_1)} d_{S(G_1)}(u,v)\\
&= 2n_2^3DD(G_1)+8\varepsilon_2n_2W(G_1)+8n_2^3W(L(G_1))\\
&+n_2^3\textstyle\sum_{u\in V(G_1),v\in V(S(G_1))-V(G_1)} d(u/G_1)d_{S(G_1)}(u,v)\\
&+(2n_2\varepsilon_2+2n_2^3)[W(S(G_1))-2W(G_1)-2W(L(G_1))].\end{aligned}$$

Therefore

$$\begin{aligned}DD(G) &= 2(n_2-1)(4\varepsilon_1n_2^2+\varepsilon_2n_1)-4n_2\varepsilon_1\varepsilon_2+n_1\overline{M}_1(G_2)+2n_2^3DD(G_1)\\
&+ 4n_2(\varepsilon_2-n_2^2)W(G_1)+4n_2^2(n_2^2-\varepsilon_2)W(L(G_1))+(2n_2\varepsilon_2+2n_2^3)W(S(G_1))\\
&+ n_2^3\textstyle\sum_{u\in V(G_1),v\in V(S(G_1))-V(G_1)} d(u/G_1)d_{S(G_1)}(u,v).\end{aligned}$$

□

1.2.2 The case F=R

Theorem 10 ([26]). *If G_1 and G_2 are two connected graphs of orders n_1 and n_2, respectively, and let $G=G_1[G_2]_R$. Then*

$$d((u,x)/G)=\begin{cases}2n_2d(u/G_1)+d(x/G_2) & \text{if } u\in V(G_1)\\ 2n_2 & \text{if } u\in V(R(G_1))-V(G_1).\end{cases}$$

Theorem 11. *Let G_i be a connected graph of order n_i, and size ε_i for $i=1,2$, and let $G=G_1[G_2]_R$. Then*

$$\begin{aligned}DD(G) &= 2(n_2-1)(6\varepsilon_1n_2^2+\varepsilon_2n_1)-8n_2\varepsilon_1\varepsilon_2+n_1(\overline{M_1}(G_2)+2n_2^3DD(G_1)\\
&+ 4\varepsilon_2n_2W(G_1)+4\varepsilon_2)+4n_2^3[W(L(G_1))+\tfrac{\varepsilon_1(\varepsilon_1-1)}{2}]\\
&+ (2n_2\varepsilon_2+2n_2^3)[W(R(G_1))-W(G_1)-W(L(G_1))-\tfrac{\varepsilon_1(\varepsilon_1-1)}{2}]\\
&+ 2n_2^3\textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1)} d(u/G_1)d_{R(G_1)}(u,v).\end{aligned}$$

Proof. Let e_u be the corresponding edge to the new vertex u. By Theorems 1, 2, 3, 4, 5 and 10,

$$\begin{aligned} DD(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)}[d((u,x)/G)+d((v,y)/G)]d_G((u,x),(v,y)) \\ &= \textstyle\sum_1[d((u,x)/G)+d((u,y)/G)] \\ &+ 2\textstyle\sum_2[d((u,x)/G)+d((u,y)/G)] \\ &+ 2\textstyle\sum_3[d((u,x)/G)+d((u,y)/G)] \\ &+ \textstyle\sum_4[(d((u,x)/G)+d((v,y)/G))d_{R(G_1)}(u,v)], \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_1[d((u,x)/G)+d((u,y)/G)] &= \textstyle\sum_1[4n_2d(u/G_1)+d(x/G_2)+d(y/G_2)] \\ &= 8n_2\varepsilon_2\varepsilon_1+n_1M_1(G_2), \\ 2\textstyle\sum_2[d((u,x)/G)+d((u,y)/G)] &= 2\textstyle\sum_2[4n_2d(u/G_1)+d(x/G_2)+d(y/G_2)] \\ &= 8n_2^2\varepsilon_1(n_2-1)-16n_2\varepsilon_1\varepsilon_2+2n_1\overline{M}_1(G_2), \end{aligned}$$

$$2\textstyle\sum_3[d((u,x)/G)+d((u,y)/G)] = 2\textstyle\sum_3 4n_2 = 4n_2^2\varepsilon_1(n_2-1),$$

$$\begin{aligned} &\sum_4[(d((u,x)/G)+d((v,y)/G))d_{R(G_1)}(u,v)] \\ &= \textstyle\sum_{u\neq v,u,v\in V(G_1),x,y\in V(G_2)}[2n_2(d(u/G_1)+d(v/G_1))+d(x/G_2)+d(y/G_2)]d_{R(G_1)}(u,v) \\ &+ \textstyle\sum_{u\neq v,u,v\in V(R(G_1))-V(G_1),x,y\in V(G_2)} 4n_2d_{R(G_1)}(u,v) \\ &+ \textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1),x,y\in V(G_2)}[2n_2d(u/G_1)+d(x/G_2)+2n_2]d_{R(G_1)}(u,v) \\ &= 2n_2^3DD(G_1)+W(G_1)(2M_1(G_2)+2\overline{M}_1(G_2)+4\varepsilon_2) \\ &+4n_2^3\textstyle\sum_{e_u,e_v\in V(L(G_1))}(d_{L(G_1)}(e_u,e_v)+1)+2n_2^3\textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1)} d(u/G_1)d_{R(G_1)}(u,v) \\ &+(2n_2\varepsilon_2+2n_2^3)\textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1)} d_{R(G_1)}(u,v) \\ &= 2n_2^3DD(G_1)+4\varepsilon_2n_2W(G_1)+4n_2^3[W(L(G_1))+\frac{\varepsilon_1(\varepsilon_1-1)}{2}] \\ &+2n_2^3\textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1)} d(u/G_1)d_{R(G_1)}(u,v) \\ &+(2n_2\varepsilon_2+2n_2^3)[W(R(G_1))-W(G_1)-W(L(G_1))-\frac{\varepsilon_1(\varepsilon_1-1)}{2}]. \end{aligned}$$

Then

$$\begin{aligned} DD(G) &= 2(n_2-1)(6\varepsilon_1n_2^2+\varepsilon_2n_1)-8n_2\varepsilon_1\varepsilon_2+n_1\overline{M_1}(G_2)+2n_2^3DD(G_1) \\ &+ 4\varepsilon_2n_2W(G_1)+4n_2^3[W(L(G_1))+\tfrac{\varepsilon_1(\varepsilon_1-1)}{2}] \\ &+ (2n_2\varepsilon_2+2n_2^3)[W(R(G_1))-W(G_1)-W(L(G_1))-\tfrac{\varepsilon_1(\varepsilon_1-1)}{2}] \\ &+ 2n_2^3\textstyle\sum_{u\in V(G_1),v\in V(R(G_1))-V(G_1)} d(u/G_1)d_{R(G_1)}(u,v). \end{aligned}$$

□

1.2.3 The case F=T

Theorem 12 ([26]). *If G_1 and G_2 are two connected graphs of order n_1, and n_2, respectively, and let $T(G_1)$ be the defined graph of G_1 such that u is the new vertex corresponding to the edge $e_u = ww'$. Then in graph $G = G_1[G_2]_T$ we have*

$$d((u,x)/G) = \begin{cases} n_2d(u/G_1)+d(x/G_2) & \text{if } u\in V(G_1), \\ n_2d(e_u) & \text{if } u\in V(T(G_1))-V(G_1). \end{cases}$$

Theorem 13. *Let G_i be a connected graph of order n_i, and size ε_i for $i = 1, 2$, and let $G = G_1[G_2]_T$. Then*

$$\begin{aligned} DD(G) &= 2(n_2-1)(2\varepsilon_1 n_2^2 + \varepsilon_2 n_1) - 4n_2\varepsilon_1\varepsilon_2 + n_1\overline{M}_1(G_2) + 2n_2^2(n_2-1)M_1(G_1) \\ &+ n_2^3 DD(G_1) + 4[\varepsilon_2 n_2 W(G_1) + W(L(G_1))] + n_2^3 DD(L(G_1)) \\ &+ 2n_2\varepsilon_2[W(T(G_1)) - W(G_1) - W(L(G_1))] \\ &+ n_2^3 \textstyle\sum_{u\in V(G_1), v\in(V(T(G_1))-V(G_1))}[d(u/G_1) + d(e_u)]d_{T(G_1)}(u,v). \end{aligned}$$

Proof. Let e_u be the corresponding edge to the new vertex u. We deduce from Theorems 1, 2, 3, 4, 5 and 12, that

$$\begin{aligned} DD(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)}[d((u,x)/G) + d((v,y)/G)]d_G((u,x),(v,y)) \\ &= \textstyle\sum_1[d((u,x)/G) + d((u,y)/G)] \\ &+ 2\textstyle\sum_2[d((u,x)/G) + d((u,y)/G)] \\ &+ 2\textstyle\sum_3[d((u,x)/G) + d((u,y)/G)] \\ &+ \textstyle\sum_4[(d((u,x)/G) + d((v,y)/G))d_{T(G_1)}(u,v)]. \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_1[d((u,x)/G) + d((u,y)/G)] &= \textstyle\sum_1[2n_2 d(u/G_1) + d(x/G_2) + d(y/G_2)] \\ &= 4n_2\varepsilon_2\varepsilon_1 + n_1 M_1(G_2). \end{aligned}$$

$$\begin{aligned} 2\textstyle\sum_2[d((u,x)/G) + d((u,y)/G)] &= 2\textstyle\sum_2[2n_2 d(u/G_1) + d(x/G_2) + d(y/G_2)] \\ &= 4n_2^2\varepsilon_1(n_2-1) - 8n_2\varepsilon_1\varepsilon_2 + 2n_1\overline{M}_1(G_2). \end{aligned}$$

$$\begin{aligned} 2\textstyle\sum_3[d((u,x)/G) + d((u,y)/G)] &= 2\textstyle\sum_3 2n_2 d(e_u) \\ &= 2n_2^2(n_2-1)M_1(G_1). \end{aligned}$$

$$\begin{aligned} &\sum_4[(d((u,x)/G) + d((v,y)/G))d_{T(G_1)}(u,v)] \\ &= \sum\nolimits_{u\neq v, u,v\in V(G_1), x,y\in V(G_2)}[n_2(d(u/G_1) + d(v/G_1)) + d(x/G_2) + d(y/G_2)]d_{T(G_1)}(u,v) \\ &+ \sum\nolimits_{u\neq v, u,v\in V(T(G_1))-V(G_1), x,y\in V(G_2)}[n_2(d(e_u) + d(e_v))d_{T(G_1)}(u,v)] \\ &+ \sum\nolimits_{u\in V(G_1), v\in V(T(G_1))-V(G_1), x,y\in V(G_2)}[n_2 d(u/G_1) + d(x/G_2) + n_2 d(e_v)]d_{T(G_1)}(u,v) \\ &= n_2^3 DD(G_1) + W(G_1)(2M_1(G_2) + 2\overline{M}_1(G_2) + 4\varepsilon_2) \\ &+ n_2^3 \sum\nolimits_{u\neq v, u,v\in V(T(G_1))-V(G_1)}[d(e_u/L(G_1)) + d(e_v/L(G_1)) + 4]d_{L(G_1)}(e_u,e_v) \\ &+ n_2^3 \sum\nolimits_{u\in V(G_1), v\in V(T(G_1))-V(G_1)}[d(u/G_1) + d(e_u)]d_{T(G_1)}(u,v) \\ &+ 2n_2\varepsilon_2 \sum\nolimits_{u\in V(G_1), v\in V(T(G_1))-V(G_1)} d_{T(G_1)}(u,v) \\ &= n_2^3 DD(G_1) + n_2^3 DD(L(G_1)) + 4[\varepsilon_2 n_2 W(G_1) + W(L(G_1))] \\ &+ n_2^3 \sum\nolimits_{u\in V(G_1), v\in V(T(G_1))-V(G_1)}[d(u/G_1) + d(e_u)]d_{T(G_1)}(u,v) \\ &+ 2n_2\varepsilon_2[W(T(G_1)) - W(G_1) - W(L(G_1))]. \end{aligned}$$

Hence

$$\begin{aligned} DD(G) &= 2(n_2-1)(2\varepsilon_1 n_2^2 + \varepsilon_2 n_1) - 4n_2\varepsilon_1\varepsilon_2 + n_1\overline{M}_1(G_2) + 2n_2^2(n_2-1)M_1(G_1) \\ &+ n_2^3 DD(G_1) + 4[\varepsilon_2 n_2 W(G_1) + W(L(G_1))] + n_2^3 DD(L(G_1)) \\ &+ 2n_2\varepsilon_2[W(T(G_1)) - W(G_1) - W(L(G_1))] \\ &+ n_2^3 \textstyle\sum_{u\in V(G_1), v\in(V(T(G_1))-V(G_1))}[d(u/G_1) + d(e_u)]d_{T(G_1)}(u,v). \end{aligned}$$

□

1.2.4 The case F=Q

Theorem 14 ([26]). *If G_1 and G_2 are two connected graphs of order n_1 and n_2, respectively, and let $Q(G_1)$ be the graph obtained from G_1 by inserting a new vertex into each edge of G_1, then joining with edges those pairs of new vertices on adjacent edges of G_1. Suppose that u is the new vertex inserted at the edge $e_u = ww'$. Then in graph $G = G_1[G_2]_Q$ we have*

$$d((u,x)/G) = \begin{cases} n_2 d(u/G_1) + d(x/G_2) & \text{if } u \in V(G_1), \\ n_2 d(e_u) & \text{if } u \in V(Q(G_1)) - V(G_1). \end{cases}$$

Theorem 15. *Let G_i be a connected graph of order n_i, and size ε_i for $i = 1,2$, and let $G = G_1[G_2]_Q$. Then*

$$\begin{aligned} DD(G) &= 2(n_2-1)(2\varepsilon_1 n_2^2 + \varepsilon_2 n_1) - 4n_2\varepsilon_1\varepsilon_2 + n_1\overline{M}_1(G_2) + 2n_2^2(n_2-1)M_1(G_1) \\ &+ n_2^3(DD(G_1) + M_1(G_1)) + 2\varepsilon_2 n_2(2W(G_1) + n_2(n_2-1)) + n_2^3 DD(L(G_1)) \\ &+ 4W(L(G_1)) + 2n_2\varepsilon_2[W(Q(G_1)) - W(G_1) - W(L(G_1)) - \tfrac{n_1(n_1-1)}{2}] \\ &+ n_2^3 \textstyle\sum_{u\in V(G_1), v\in V(Q(G_1))-V(G_1)}[(d(u/G_1) + d(e_u))d_{Q(G_1)}(u,v)]. \end{aligned}$$

Proof. Let e_u be the corresponding edge to the new vertex u. By Theorems 1, 2, 3, 4, 5 and 14,

$$\begin{aligned} DD(G) &= \textstyle\sum_{\{(u,x),(v,y)\}\subseteq V(G)}[d((u,x)/G) + d((v,y)/G)]d_G((u,x),(v,y)) \\ &= \textstyle\sum_1[d((u,x)/G) + d((u,y)/G)] + 2\sum_2[d((u,x)/G) + d((u,y)/G)] \\ &+ 2\textstyle\sum_3[d((u,x)/G) + d((u,y)/G)] \\ &+ \textstyle\sum_4[(d((u,x)/G) + d((v,y)/G))d_{Q(G_1)}(u,v)] \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_1[d((u,x)/G) + d((u,y)/G)] &= \textstyle\sum_1[2n_2d(u/G_1) + d(x/G_2) + d(y/G_2)] \\ &= 4n_2\varepsilon_2\varepsilon_1 + n_1M_1(G_2). \end{aligned}$$

$$\begin{aligned} 2\textstyle\sum_2[d((u,x)/G) + d((u,y)/G)] &= 2\textstyle\sum_2[2n_2d(u/G_1) + d(x/G_2) + d(y/G_2)] \\ &= 4n_2^2\varepsilon_1(n_2-1) - 8n_2\varepsilon_1\varepsilon_2 + 2n_1\overline{M}_1(G_2). \end{aligned}$$

$$2\sum_3[d((u,x)/G) + d((u,y)/G)] = 2\sum_3 2n_2 d(e_u) = 2n_2^2(n_2-1)M_1(G_1).$$

$$\begin{aligned} &\sum_4[(d((u,x)/G) + d((v,y)/G))d_{Q(G_1)}(u,v)] \\ &= \textstyle\sum_{u\neq v, u,v\in V(G_1), x,y\in V(G_2)}[n_2(d(u/G_1) + d(v/G_1)) + d(x/G_2) + d(y/G_2)]d_{Q(G_1)}(u,v) \\ &+ \textstyle\sum_{u\neq v, u,v\in V(Q(G_1))-V(G_1), x,y\in V(G_2)}[n_2(d(e_u) + d(e_v))d_{Q(G_1)}(u,v)] \\ &+ \textstyle\sum_{u\in V(G_1), v\in V(Q(G_1))-V(G_1), x,y\in V(G_2)}[n_2d(u/G_1) + d(x/G_2) + n_2d(e_v)]d_{Q(G_1)}(u,v) \\ &= n_2^3(DD(G_1) + M_1(G_1)) + (W(G_1) + \frac{n_2(n_2-1)}{2})(2M_1(G_2) + 2\overline{M}_1(G_2) + 4\varepsilon_2) \\ &+ n_2^3\textstyle\sum_{u\neq v, u,v\in V(T(G_1))-V(G_1)}[d(e_u/L(G_1)) + d(e_v/L(G_1)) + 4]d_{L(G_1)}(e_u,e_v) \\ &+ n_2^3\textstyle\sum_{u\in V(G_1), v\in V(Q(G_1))-V(G_1)}[(d(u/G_1) + d(e_v))d_{Q(G_1)}(u,v)] \\ &+ 2n_2\varepsilon_2\textstyle\sum_{u\in V(G_1), v\in V(Q(G_1))-V(G_1)} d_{Q(G_1)}(u,v) \\ &= n_2^3(DD(G_1) + M_1(G_1)) + 2\varepsilon_2 n_2(2W(G_1) + n_2(n_2-1)) + n_2^3DD(L(G_1)) + 4W(L(G_1)) \\ &+ n_2^3\textstyle\sum_{u\in V(G_1), v\in V(Q(G_1))-V(G_1)}[(d(u/G_1) + d(e_u))d_{Q(G_1)}(u,v)] \\ &+ 2n_2\varepsilon_2[W(Q(G_1)) - W(G_1) - W(L(G_1)) - \frac{n_1(n_1-1)}{2}]. \end{aligned}$$

Hence,

$$\begin{aligned} DD(G) &= 2(n_2-1)(2\varepsilon_1 n_2^2+\varepsilon_2 n_1)-4n_2\varepsilon_1\varepsilon_2+n_1\overline{M}_1(G_2)+2n_2^2(n_2-1)M_1(G_1)\\ &+ n_2^3(DD(G_1)+M_1(G_1))+2\varepsilon_2 n_2(2W(G_1)+n_2(n_2-1))+n_2^3 DD(L(G_1))\\ &+ 4W(L(G_1))+2n_2\varepsilon_2[W(Q(G_1))-W(G_1)-W(L(G_1))-\tfrac{n_1(n_1-1)}{2}]\\ &+ n_2^3\textstyle\sum_{u\in V(G_1),v\in V(Q(G_1))-V(G_1)}[(d(u/G_1)+d(e_u))d_{Q(G_1)}(u,v)]. \end{aligned}$$

□

REFERENCES

[1] Aringhieri R., Hansen P., Malucelli F. *A linear algorithm for the hyper-Wiener index of chemical trees*. J. Chem. Inf. Comput. Sci. 2001, **41**, 958–963. doi:10.1021/ci0001536

[2] Ashrafi A., Došlić T., Hamzeh A. *The Zagreb coindices of graph operations*. Discrete Appl. Math. 2010, **158**, 1571–1578.

[3] Basavanagoud B., Patil S. *A note on hyper-zagreb index of graph operations*. Iran. J. Math. Chem. 2016, **7**, 89–92. doi:10.22052/IJMC.2016.12405

[4] Borovicanin B., Das K.C., Furtula B., Gutman I. *Bounds for Zagreb indices*. MATCH Commun. Math. Comput. Chem. 2017, **78**, 17–100.

[5] Das K.C., Gutman I. *Some properties of the second Zagreb index*. MATCH Commun. Math. Comput. Chem. 2004, **52**, 103–112.

[6] Das K.C., Gutman I. *The first Zagreb index 30 years after*. MATCH Commun. Math. Comput. Chem. 2004, **50**, 83–92.

[7] Dehgardi N., Sheikholeslami S.M., Soroudi M. *On the leap Zagreb indices of graphs*. (submitted).

[8] Diudea M. V., Parv B. *Molecular topology. 25. HypernïSWiener index of dendrimers*. J. Chem. Inf. Comput. Sci. 1995, **35**, 1015–1018.

[9] Dobrynin A. A., Kochetova A. A. *Degree Distance of a Graph: A Degree Analogue of the Wiener Index*. J. Chem. Inf. Comput. Sci. 1994, **34**, 1082–1086.

[10] Došlić T. *Vertex-weighted Wiener polynomials for composite graphs*. Ars Math. Contemp. 2008, **1**, 66–80.

[11] Eliasi M., Iranmanesh A., Gutman I. *Multiplicative versions of first zagreb index*. MATCH Commun. Math. Comput. Chem. 2012, **68**, 217–230.

[12] Eliasi M., Raeisi G., Taeri B. *Wiener index of some graph operations*. Discrete Appl. Math. 2012, **160**, 1333–1344.

[13] Gutman I. *Multiplicative zagreb indices of trees*. Bull. Soc. Math. Banja Luka 2011, **18**, 17–23.

[14] Gutman I. *Selected Properties of the Schultz Molecular Topological Index*. J. Chem. Inf. Comput. Sci. 1994, **34**, 1087–1089.

[15] Gutman I., Furtula B., Vukićević K., Popivoda G. *On Zagreb indices and coindices*. MATCH Commun. Math. Comput. Chem. 2015, **74**, 5–16.

[16] Gutman I., Linert W., Lukovits I., Dobrynin A. A. *Trees with extremal hyper-Wiener index: Mathematical basis and chemical applications*. J. Chem. Inf. Comput. Sci. 1997, **37** (2), 349–354. doi:10.1021/ci960139m

[17] Gutman I., Rucčič B., Trinajstić N., Wilcox C.F. *Graph theory and molecular orbitals. XII. acyclic polyenes*. J. Chem. Phys. 1975, **62**, 3399–3405. doi:10.1063/1.430994

[18] Gutman I., Trinajsti N. *Graph theory and molecular orbitals. total π-electron energy of alternant hydrocarbons*. Chem. Phys. Lett. **17** (1972), 535–538. doi:10.1016/0009-2614(72)85099-1

[19] Hosoya H. *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*. Bull. Chem. Soc. Jpn. 1971, **4**, 2332–2339.

[20] Ilić A., Zhou B. *On reformulated Zagreb indices*. Discrete Appl. Math. 2012, **160** (3), 204–209. doi:10.1016/j.dam.2011.09.021

[21] Khalifeh M.H., Yousefi-Azari H., Ashrafi A.R. *The hyper-Wiener index of graph operations*. Comput. Math. Appl. 2008, **56**, 1402–1407.

[22] Klavžar S., Žigert P., Gutman I. *An algorithm for the calculation of the hyper-Wiener index of benzenoid hydrocarbons*. MATCH Commun. Math. Comput. Chem. 2000, **24**, 229–233.

[23] Lukovits I. *Formulas for the hyper-Wiener index of trees*. J. Chem. Inf. Comput. Sci. 1994, **34**, 1079–1081.

[24] Miličević A., Nikolić S., Trinajstić N. *On reformulated zagreb indices*. Mol Divers. 2004, **8** (4), 393–399. doi:10.1023/b:modi.0000047504.14261.2a

[25] Pattabiraman K., Vijayaragavan M. *Hyper zagreb indices and its coindices of graphs*. Bull. Int. Math. Virtual Inst. 2017, **7**, 31–41.

[26] Sarala D., Deng H., Ayyaswamy S.K., Balachandran S. *The Zagreb indices of graphs based on four new operations related to the lexicographic product*. Appl. Math. Comput. 2017, **309**, 156–169.

[27] Wiener H. *Structural determination of paraffin boiling points*. J. Amer. Chem. Soc. 1947, **69**, 17–20.

[28] Xu K., Das K.C. *Trees, unicyclic, and bicyclic graphs extremal with respect to multiplicative sum zagreb index*. MATCH Commun. Math. Comput. Chem. 2012, **68**, 257–272.

[29] Xu K., Das K.C., Tang K. *On the multiplicative zagreb coindex of graphs*. Opuscula Math. 2013, **33**, 191–204.

[30] Xu K., Hua H. *A unified approach to extremal multiplicative zagreb indices for trees, unicyclic and bicyclic graphs*. MATCH Commun. Math. Comput. Chem. 2012, **68**, 241–256.

[31] Yan W., Yang B-Y., Yeh Y-N. *The behavior of Wiener indices and polynomials of graphs under five graph decorations*. MATCH Commun. Appl. Math. Lett. 2007, **20**, 290–295.

Received 18.05.2018

Дегарді Н, Шейхолесламі С.М., Сороуді М. *Деякі дистанційні індекси графів, що ґрунтуються на чотирьох нових операціях, які відносяться до лексикографічного добутку* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 258–267.

Для (молекулярного) графу індекс Вінера, гіпервінерівський індекс і індекс степеневої відстані визначаються як $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$, $WW(G) = W(G) + \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)^2$ і $DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)(d(u/G) + d(v/G))$ відповідно. $d(u/G)$ позначає степінь вершини u в G і $d_G(u,v)$ — відстань між двома вершинами u і v в графі G. У цій статті ми вивчаємо індекс Вінера, гіпервінерівський індекс і індекс степеневої відстані у графах, що ґрунтуються на чотирьох нових операціях, які відносяться до лексикографічного добутку, підроздільності та тотального графу.

Ключові слова і фрази: індекс Вінера, індекс степеневої відстані, гіпервінерівський індекс, лексикографічний добуток, підроздільність, тотальний граф.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 268–280
doi:10.15330/cmp.11.2.268-280

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.268–280

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PROPERTIES OF INTEGRALS WHICH HAVE THE TYPE OF DERIVATIVES OF VOLUME POTENTIALS FOR ONE KOLMOGOROV TYPE ULTRAPARABOLIC ARBITRARY ORDER EQUATION

In weighted Hölder spaces it is studied the smoothness of integrals, which have the structure and properties of derivatives of volume potentials which generated by fundamental solutions of the Cauchy problem for one ultraparabolic arbitrary order equation of the Kolmogorov type. The coefficients in this equation depend only on the time variable. Special distances and norms are used for constructing of the weighted Hölder spaces.

The results of the paper can be used for establishing of the correct solvability of the Cauchy problem and estimates of solutions of the given non-homogeneous equation in corresponding weighted Hölder spaces.

Key words and phrases: ultraparabolic Kolmogorov type arbitrary order equation, an integral which have the type of derivatives of the volume potential, weight Hölder norm, Hölder space of increasing functions.

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INTRODUCTION

Properties of the corresponding volume potentials are very important when the fundamental solution is being constructed and investigated, correct solvability of the Cauchy problem is being established and estimates of solutions for parabolic equations are being obtained. Such properties have been established for parabolic equations in the sense of Petrovsky and for $\overrightarrow{2b}$-parabolic equations in the sense of Eidelman without any degenerations in works [5, 6, 8] and for equations with degenerations on the initial hyperplane in works [6, 7, 10, 12, 13]. Volume potentials for the degenerated arbitrary order parabolic equations of the Kolmogorov type (ultraparabolic equations of the Kolmogorov type) were studied in [1–4, 6] and properties of volume potentials with density from Hölder spaces of bounded functions which are increasing as $|x| \to \infty$ were established only for the second order equations.

It is convenient to obtain such properties if the statements about properties of integrals which have the type of derivatives of volume potentials are proved first at all. These properties are described by belonging such integrals to corresponding functional spaces according to the type of spaces which density and kernel of the integral belong to. Statesments of such type are

УДК 517.956.4
2010 *Mathematics Subject Classification:* 35K70.

proved in works [6,8,9,11] for parabolic equations in the sense of Petrovsky and for parabolic equations in the sense of Eidelman. By the way they have their own value.

In this paper there is an attempt to prove the corresponding statements in case of the Kolmogorov type parabolic equations. The major part of these equations are parabolic in the sense of Petrovsky with respect to basic indepedent variables.

1 NOTATIONS AND ASSUMPTIONS

Let b, n_1, n_2, n_3 be given positive integer numbers such that $1 \le n_3 \le n_2 \le n_1$, $n := n_1 + n_2 + n_3$; $x := (x_1, x_2, x_3) \in \mathbb{R}^n$, $x_l := (x_{l1}, \ldots, x_{ln_j}) \in \mathbb{R}^{n_l}$, $l \in L := \{1,2,3\}$; T is a positive number; if $k_1 := (k_{11}, \ldots, k_{1n_1}) \in \mathbb{Z}_+^{n_1}$ is a n_1-dimensional index, then $|k_1| := k_{11} + \ldots + k_{1n_1}$, $\partial_{x_1}^{k_1} := \partial_{x_{11}}^{k_{11}} \cdot \ldots \cdot \partial_{x_{1n_1}}^{k_{1n_1}}$.

The paper is concerned with the study of properties of integrals of the type

$$u(t,x) := \int_0^t d\tau \int_{\mathbb{R}^n} M(t,x;\tau,\xi) f(\tau,\xi) d\xi, \quad (t,x) \in \Pi_{(0,T]} := (0,T] \times \mathbb{R}^n. \tag{1}$$

The kernel M is a complex-valued function which has properties of the derivatives of the fundamental solution G of the Cauchy problem for the equation

$$(\partial_t - \sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}} - \sum_{j=1}^{n_3} x_{2j}\partial_{x_{3j}} - \sum_{|k_1|\le 2b} a_{k_1}(t)\partial_{x_1}^{k_1}) u(t,x) = f(t,x), \quad (t,x) \in \Pi_{(0,T]}. \tag{2}$$

In the equation (2) $\partial_t - \sum_{|k_1|\le 2b} a_{k_1}(t)\partial_{x_1}^{k_1}$ is parabolic by Petrovsky differential expression, and coefficients a_{k_1} are continuous on $[0,T]$ functions.

The equation (2) belongs to a class of ultraparabolic equations arbitrary order $2b$ and it generalize known equation of A.N.Kolmogorov of diffusion with inertia. In [6] it was established a structure and properties of the function G and its derivatives.

Let us describe properties of the kernel M of integral (1). For the purpose we denote: $q := 2b/(2b-1)$, $N := (n_1 + (2b+1)n_2 + (4b+1)n_3)/(2b)$, $\Delta_x^{x'} f(t,x) := f(t,x) - f(t,x')$, $\rho(t,x,\xi) := t^{1-q}\sum_{j=1}^{n_1} |x_{1j} - \xi_{1j}|^q + t^{1-2q}\sum_{j=1}^{n_2} |x_{2j} + tx_{1j} - \xi_{2j}|^q + t^{1-3q}\sum_{j=1}^{n_3} |x_{3j} + tx_{2j} + 2^{-1}t^2 x_{1j} - \xi_{3j}|^q$, $d(x;x') := \sum_{l=1}^{3} |x_l - x'_l|^{1/(2b(l-1)+1)}$, $d_1(x;x';\lambda) := |x_1 - x'_1|^\lambda + \sum_{l=2}^{3} |x_l - x'_l|^{(\lambda+1)/(2b(l-1)+1)}$, $d_2(x;x';\lambda) := |x_1 - x'_1|^\lambda + |x_2 - x'_2|^{(\lambda+1)/(2b+1)} + |x_3 - x'_3|^{(\lambda+2b+1)/(4b+1)}$, if $t \in (0,T]$, $\{x,x',\xi\} \subset \mathbb{R}^n$, $\lambda \in (0,1]$.

Note, that if $d(x;x') < 1$, then

$$d_2(x;x';\lambda) \le d_1(x;x';\lambda) \le 4^{1-\lambda} d(x;x')^\lambda, \quad \{x,x'\} \subset \mathbb{R}^n, \lambda \in (0,1].$$

As the kernel of the integral (1), let us take the function M, which can be represented in the form

$$M(t,x;\tau,\xi) := (t-\tau)^{-\nu-N}\Omega(t,x;\tau,\xi), \quad 0 \le \tau < t \le T, \ \{x,\xi\} \subset \mathbb{R}^n, \tag{3}$$

where $\nu \in (0, 2b + 1/(2b)]$, and the function Ω, with the values in $\mathbb{C}$, is continuous and it satisfies the conditions below with some numbers $c > 0$ and $\gamma \in (0,1]$:

A_1. $\forall\{t,\tau\}\subset(0,T],\ \tau<t,\ \forall x\in\mathbb{R}^n:$

$$\int\limits_{\mathbb{R}^n}\Omega(t,x;\tau,\xi)d\xi=0\ \text{ for } \nu\in(1-1/(2b),1],$$

$$\int\limits_{\mathbb{R}^{n_2+n_3}}\Omega(t,x;\tau,\xi)d\xi_2 d\xi_3=0\ \text{ for } \nu\in(1,1+1/(2b)], \tag{4}$$

$$\int\limits_{\mathbb{R}^{n_3}}\Omega(t,x;\tau,\xi)d\xi_3=0\ \text{ for } \nu\in(1+1/(2b),2b+1/(2b)];$$

A_2. $\exists C>0\ \forall\{t,\tau\}\subset(0,T],\ \tau<t,\ \forall\{x,\xi\}\subset\mathbb{R}^n:$

$$|\Omega(t,x;\tau,\xi)|\le C\exp\{-c\rho(t-\tau,x,\xi\};\tag{5}$$

A_3. $\exists C>0\ \forall\{t,\tau\}\subset(0,T],\ \tau<t,\ \forall\{x,x',\xi\}\subset\mathbb{R}^n,\ d(x;x')<(t-\tau)^{1/(2b)}:$

$$|\Delta_x^{x'}\Omega(t,x;\tau,\xi)|\le C(d(x;x'))^{\gamma}(t-\tau)^{-\gamma/(2b)}\exp\{-c\rho(t-\tau,x,\xi)\}.\tag{6}$$

The definition of the function M contains the number ν, c, and γ, which assume are considered to be given. By $\mathcal{M}(\nu,c,\gamma)$ we denote a set of all functions M determined by formula (3), in which the function Ω satisfies conditions A_1 – A_3 with given $\gamma\in(0,1]$, $\nu\in(0,2b+1/(2b)]$, $c\in\mathbb{R}_+$.

It should be noted that for $\nu\in[1,2b+1/(2b)]$ integral (1) with the function $M\in\mathcal{M}(\nu,c,\gamma)$ is treated as the limit

$$\lim_{h\to0}\int\limits_0^{t-h}d\tau\int\limits_{\mathbb{R}^n}M(t,x;\tau,\xi)f(\tau,\xi)d\xi,$$

which exists for suitable f, because of condition A_1.

Let us define spaces to which the functions f and u belong. They are the spaces of functions which are continuous or satisfy Hölder condition and which have certain restrictions as $|x|\to\infty$. Their behavior as $|x|\to\infty$ will be described by the functions

$$\varphi(t,x):=\exp\sum_{l=1}^{3}k_l(t,a_l)|x_l|^q$$

or

$$\psi(t,x):=\exp\sum_{l=1}^{3}s_l(t)|x_l|^q,\ t\in[0,T],\ x\in\mathbb{R}^n.$$

Here for a fixed number c_0 from the interval $(0,c)$, where c is the constant from conditions A_2 and A_3, and for a set $a:=(a_1,a_2,a_3)$ of non-negative numbers a_l, $l\in L$, such that $T<\min\limits_{l\in L}(c_0/a_l)^{(2b-1)/(2b(l-1)+1)}$:

$$k_l(t,a_l):=c_0a_l(c_0^{2b-1}-a_l^{2b-1}t^{2b(l-1)+1})^{1-q},\ l\in L;$$

$$s_1(t):=k_1(t,a_1)+2^{q-1}t^qk_2(t,a_2)+2^{q-2}t^{2q}k_3(t,a_3),$$

$$s_2(t):=2^{q-1}k_2(t,a_2)+4^{q-1}t^qk_3(t,a_3),\ \ s_3(t):=4^{q-1}k_3(t,a_3),\ t\in[0,T].$$

The functions $k(t) := (k_1(t, a_1), k_2(t, a_2), k_3(t, a_3))$ and $s(t) := (s_1(t), s_2(t), s_3(t))$, $t \in [0, T]$, have the following properties [6]:

$$k(0) = a,\ a_l \le k_l(\tau, a_l) < k_l(t, a_l) < s_l(t),\ 0 \le \tau < t \le T,\ l \in L; \tag{7}$$

$$k_l(t - \tau, k_l(\tau, a_l)) \le k_l(t, a_l),\ 0 \le \tau < t \le T,\ l \in L; \tag{8}$$

$$-c_0\rho(t - \tau, x, \xi) + \sum_{l=1}^{3} a_l |\xi_l|^q \le \sum_{l=1}^{3} k_l(t, a_l)|\bar{x}_l(t)|^q \le \sum_{l=1}^{3} s_l(t)|x_l|^q,$$

$$0 \le \tau < t \le T, \{x, \xi\} \subset \mathbb{R}^n, \tag{9}$$

where $\bar{x}_l(t) := (\bar{x}_{l1}(t), \bar{x}_{l2}(t), \ldots, \bar{x}_{ln_l}(t))$, $l \in L$; $\bar{x}_{1j}(t) := x_{1j}$, $j \in \{1, \ldots, n_1\}$; $\bar{x}_{2j}(t) := x_{2j} + tx_{1j}$, $j \in \{1, \ldots, n_2\}$; $\bar{x}_{3j}(t) := x_{3j} + tx_{2j} + 2^{-1}t^2 x_{1j}$, $j \in \{1, \ldots, n_3\}$.

From these properties it is follows that

$$\varphi(\tau, X_1(t - \tau)) \le \varphi(t, X_1(t)) \le \psi(t, x),$$

$$\exp\{-c_0\rho(t - \tau, x, \xi)\}\varphi(\tau, \xi) \le \psi(t, x),\ \ 0 \le \tau < t \le T,\ \ \{x, \xi\} \subset \mathbb{R}^n, \tag{10}$$

where $X_1(t) := (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$.

For a given number $\lambda \in (0, 1]$ we denote by C^0, C^λ_φ, $C^\lambda_{1,\varphi}$ and $C^\lambda_{2,\varphi}$ spaces of continuous functions $u : \Pi_{[0,T]} \to \mathbb{C}$, for which the corresponding norms $||u||^0_\varphi$, $||u||^\lambda_\varphi := ||u||^0_\varphi + [u]^\lambda_\varphi$, $||u||^\lambda_{1,\varphi} := ||u||^0_\varphi + [u]^\lambda_{1,\varphi}$ and $||u||^\lambda_{2,\varphi} := ||u||^0_\varphi + [u]^\lambda_{2,\varphi}$, where

$$||u||^0_\varphi := \sup_{(t,x) \in \Pi_{[0,T]}} \frac{|u(t, x)|}{\varphi(t, x)},$$

$$[u]^\lambda_\varphi := \sup_{\substack{\{(t,x),(t,x')\} \subset \Pi_{[0,T]} \\ (t,x) \ne (t,x')}} \frac{|\Delta^{x'}_x u(t, x)|}{(d(x; x'))^\lambda(\varphi(t, x) + \varphi(t, x'))},$$

$$[u]^\lambda_{1,\varphi} := \sup_{\substack{\{(t,x),(t,x')\} \subset \Pi_{[0,T]} \\ (t,x) \ne (t,x')}} \frac{|\Delta^{x'}_x u(t, x)|}{d_1(x; x'; \lambda)(\varphi(t, x) + \varphi(t, x'))},$$

$$[u]^\lambda_{2,\varphi} := \sup_{\substack{\{(t,x),(t,x')\} \subset \Pi_{[0,T]} \\ (t,x) \ne (t,x')}} \frac{|\Delta^{x'}_x u(t, x)|}{d_2(x; x'; \lambda)(\varphi(t, x) + \varphi(t, x'))}$$

are finite.

Except these spaces we will use the space C^λ_ψ. The definition of this space is obtained if in the definition of the space C^λ_φ the function φ replace by the function ψ.

2 MAIN THEOREM

Let us formulate the main results of this paper.

Theorem. *Let $M \in \mathcal{M}(\nu, c, \gamma)$ and function u is determined by formula* (1)*. Then the following statements are valid:*

a) *if $\nu \leq 1 - 1/(2b)$ and $f \in C^0$, then $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^0_\varphi; \tag{11}$$

b) *if $\nu \in (1 - 1/(2b), 1]$ and $f \in C^\lambda_\varphi$, $\lambda \in (0, 1]$, then with $\nu + (\gamma - \lambda)/(2b) < 1$ we have $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^\lambda_\varphi, \tag{12}$$

and with $\nu + (\gamma - \lambda)/(2b) > 1$ we have $u \in C^\lambda_\psi$ and

$$||u||^\lambda_\psi \leq C||f||^\lambda_\varphi; \tag{13}$$

c) *if $\nu \in (1, 1 + 1/(2b)]$ and $f \in C^\lambda_{1,\varphi}$, $\lambda \in (0, 1]$, then with $\nu + (\gamma - 1 - \lambda)/(2b) < 1$ we have $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^\lambda_{1,\varphi}, \tag{14}$$

and with $\nu + (\gamma - 1 - \lambda)/(2b) > 1$ we have $u \in C^\lambda_\psi$ and

$$||u||^\lambda_\psi \leq C||f||^\lambda_{1,\varphi}; \tag{15}$$

d) *if $\nu \in (1 + 1/(2b), 2b + 1/(2b)]$ and $f \in C^\lambda_{2,\varphi}$, $\lambda \in (0, 1]$, then with $\nu + 1 - 2b + (\gamma - 1 - \lambda)/(2b) < 1$ we have $u \in C^\gamma_\psi$ and*

$$||u||^\gamma_\psi \leq C||f||^\lambda_{2,\varphi}, \tag{16}$$

and with $\nu + 1 - 2b + (\gamma - 1 - \lambda)/(2b) > 1$ we have $u \in C^\lambda_\psi$ and

$$||u||^\lambda_\psi \leq C||f||^\lambda_{2,\varphi}. \tag{17}$$

The constants C in inequalities (11)–(17) *depend only on the constant C from conditions A_2 and A_3, and also they depend on the numbers n_1, n_2, n_3, b, ν, c, γ and λ.*

Proof. Below various constants we will denote by same letters if we have no interest in constant's values.

a) Using the equality [6]

$$\int\limits_{\mathbb{R}^n} (t - \tau)^{-N} \exp\{-c'\rho(t - \tau, x, \xi)\} d\xi = C, \;\; 0 < \tau < t \leq T, \; x \in \mathbb{R}^n, \; c' > 0, \tag{18}$$

with the help of (3), (5), (10) and of the definition of the norm $||f||^0_\varphi$ we have

$$\begin{aligned} |u(t, x)| &\leq C \int\limits_0^t (t - \tau)^{-\nu - N} d\tau \int\limits_{\mathbb{R}^n} \exp\{-c\rho(t - \tau, x, \xi)\}|f(\tau, \xi)| d\xi = C \int\limits_0^t (t - \tau)^{-\nu - N} d\tau \\ &\int\limits_{\mathbb{R}^n} \exp\{-c_0\rho(t - \tau, x, \xi)\}\varphi(\tau, \xi)\frac{|f(\tau, \xi)|}{\varphi(\tau, \xi)} \exp\{-(c - c_0)\rho(t - \tau, x, \xi)\} d\xi \\ &\leq C\psi(t, x) \int\limits_0^t (t - \tau)^{-\nu} d\tau ||f||^0_\varphi = C\psi(t, x) t^{1-\nu} ||f||^0_\varphi, \; (t, x) \in \Pi_{(0,T]}. \end{aligned} \tag{19}$$

Let x and x' be arbitrary fixed points from $\mathbb{R}^n$ and $d := d(x; x')$. Let us estimate the difference $\Delta_x^{x'} u$.

When $d^{2b} > t$, with the help of estimate (19) we obtain

$$\begin{aligned} &|\Delta_x^{x'} u(t,x)| \le |u(t,x)| + |u(t,x')| \le C(\psi(t,x) + \psi(t,x'))t^{1-\nu}||f||_{\varphi}^0 \\ &\le C(\psi(t,x) + \psi(t,x'))(d(x;x'))^{\gamma} t^{1-\nu-\gamma/(2b)}||f||_{\varphi}^0, \ \ t \in (0,T], \ \{x,x'\} \subset \mathbb{R}^n, \ \gamma \in (0,1]. \end{aligned} \tag{20}$$

Let us consider the case $d^{2b} < t$. We have

$$|\Delta_x^{x'} u(t,x)| \le \int_0^t d\tau \int_{\mathbb{R}^n} |\Delta_x^{x'} M(t,x;\tau,\xi)|\, |f(\tau,\xi)| d\xi, \ \ t \in (0,T], \ \{x,x'\} \subset \mathbb{R}^n. \tag{21}$$

Let us prove for the difference $\Delta M := \Delta_x^{x'} M(t,x;\tau,\xi)$ the inequality

$$|\Delta M| \le C d^{\gamma} (t-\tau)^{-\gamma/(2b)-\nu-N} \exp\{-c\rho(t-\tau,x,\xi)\}. \tag{22}$$

We shall distinguish the following cases: 1) $d^{2b} \ge t-\tau$, 2) $d^{2b} < t-\tau$.

In the first case, we obtain estimate (22) immediately from (3), (5) and from the inequality $|\Delta M| \le |M(t,x;\tau,\xi)| + |M(t,x';\tau,\xi)|$. In case 2) note that

$$\Delta M = (t-\tau)^{-\nu-N} \Delta_x^{x'} \Omega(t,x;\tau,\xi).$$

Because of (6) we have estimate (22) in case 2).

With the help of (10), (18), (21) and (22) we get

$$\begin{aligned} &|\Delta_x^{x'} u(t,x)| \le C(\psi(t,x) + \psi(t,x'))d^{\gamma} t^{1-\nu-\gamma/(2b)}||f||_{\varphi}^0, \\ &t \in (0,T], \ \{x,x'\} \subset \mathbb{R}^n, \gamma \in (0, 1-1/(2b)]. \end{aligned} \tag{23}$$

From (20) and (23) the estimate

$$[u]_{\psi}^{\gamma} \le C||f||_{\varphi}^0$$

follows and by this result and (19) the estimate (11) holds.

b) Let $\nu \in (1-1/(2b), 1]$. Because of the first condition from (4) we represent integral (1) in the form

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^n} M(t,x;\tau,\xi) \Delta_{\xi}^{X_1(t-\tau)} f(\tau,\xi) d\xi, \ \ (t,x) \in \Pi_{(0,T]}, \tag{24}$$

where $X_1(t) := (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ as in (10).

With the help of (3), (5) and (7)–(10) we get

$$\begin{aligned} |u(t,x)| &\le C \int_0^t (t-\tau)^{-\nu-N} d\tau \int_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} \exp\{-c_0\rho(t-\tau,x,\xi)\} \\ &\times (\varphi(\tau,\xi) + \varphi(\tau, X_1(t-\tau))) \frac{|\Delta_{\xi}^{X_1(t-\tau)} f(\tau,\xi)|}{\varphi(\tau,\xi) + \varphi(\tau, X_1(t-\tau))} d\xi \le C \int_0^t (t-\tau)^{-\nu-N} d\tau \\ &\times \int_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} (d(\xi, X_1(t-\tau)))^{\lambda} d\xi \psi(t,x) [f]_{\varphi}^{\lambda}. \end{aligned}$$

Now let us use the inequality [6]

$$\begin{aligned}&(d(\xi, X_1(t-\tau)))^\lambda \exp\{-\bar{c}\rho(t-\tau,x,\xi)\} \le C(t-\tau)^{\lambda/(2b)}\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\},\\&0\le\tau<t\le T,\ \{x,\xi\}\subset\mathbb{R}^n,\ 0<\bar{c}_1<\bar{c},\ \lambda\in(0,1].\end{aligned} \tag{25}$$

For $\bar{c}=c-c_0$ with the help of (18) we have

$$\begin{aligned}|u(t,x)| &\le C\int\limits_0^t (t-\tau)^{-\nu-N+\lambda/(2b)}d\tau\int\limits_{\mathbb{R}^n}\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}d\xi\psi(t,x)[f]^\lambda_\varphi\\&= C\psi(t,x)[f]^\lambda_\varphi\int\limits_0^t(t-\tau)^{-\nu+\lambda/(2b)}d\tau = C\psi(t,x)[f]^\lambda_\varphi t^{1-\nu+\lambda/(2b)},\ \ (t,x)\in\Pi_{(0,T]}.\end{aligned} \tag{26}$$

Then

$$||u||^0_\psi \le C[f]^\lambda_\varphi. \tag{27}$$

Let us estimate the difference $\Delta^{x'}_x u$. If $d^{2b}\ge t$, where $d := d(x;x')$, then under condition (26) we have the estimate

$$|\Delta^{x'}_x u(t,x)| \le C(\psi(t,x)+\psi(t,x'))[f]^\lambda_\varphi t^{1-\nu+\lambda/(2b)},\ \ t\in(0,T],\ \{x,\xi\}\subset\mathbb{R}^n.$$

We obtain

$$\begin{aligned}|\Delta^{x'}_x u(t,x)| &\le C(\psi(t,x)+\psi(t,x'))[f]^\lambda_\varphi d^\lambda t^{1-\nu}\\&\le C(\psi(t,x)+\psi(t,x'))d^\lambda[f]^\lambda_\varphi,\ t\in(0,T],\ \{x,\xi\}\subset\mathbb{R}^n;\end{aligned} \tag{28}$$

and with $\nu+(\gamma-\lambda)/(2b)<1$ we receive

$$\begin{aligned}|\Delta^{x'}_x u(t,x)| &\le C(\psi(t,x)+\psi(t,x'))[f]^\lambda_\varphi t^{1-\nu-(\gamma-\lambda)/(2b)}t^{\gamma/(2b)}\\&\le C(\psi(t,x)+\psi(t,x'))[f]^\lambda_\varphi t^{1-\nu-(\gamma-\lambda)/(2b)}d^\gamma\\&\le C(\psi(t,x)+\psi(t,x'))d^\gamma[f]^\lambda_\varphi,\ t\in(0,T],\ \{x,\xi\}\subset\mathbb{R}^n.\end{aligned} \tag{29}$$

It is sufficient to consider the case, where $d^{2b}<t$. By the first condition from (4) like (24) we write

$$\begin{aligned}\Delta^{x'}_x u(t,x) = &\int\limits_0^{t-d^{2b}} d\tau\int\limits_{\mathbb{R}^n}\Delta^{x'}_x M(t,x;\tau,\xi)\Delta^{X_1(t-\tau)}_\xi f(\tau,\xi)d\xi\\&+\int\limits_{t-d^{2b}}^t d\tau\int\limits_{\mathbb{R}^n} M(t,x;\tau,\xi)\Delta^{X_1(t-\tau)}_\xi f(\tau,\xi)d\xi\\&-\int\limits_{t-d^{2b}}^t d\tau\int\limits_{\mathbb{R}^n} M(t,x';\tau,\xi)\Delta^{X'_1(t-\tau)}_\xi f(\tau,\xi)d\xi =: \sum_{l=1}^3 K_l,\end{aligned}$$

where $X'_1(t) := X_1(t)|_{x=x'}$.

Using (3), (6), the second inequality from (9), (10), we get

$$|K_1| \le C\int\limits_0^{t-d^{2b}}(t-\tau)^{-\nu-N}d\tau\int\limits_{\mathbb{R}^n}(d(x;x'))^\gamma(t-\tau)^{-\gamma/(2b)}\exp\{-c\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau)))\frac{|\Delta_\xi^{X_1(t-\tau)}f(\tau,\xi)|}{\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau))}d\xi \le C\int_0^{t-d^{2b}}(t-\tau)^{-\nu-N-\gamma/(2b)}d\tau$$

$$\times\int_{\mathbb{R}^n}\psi(t,x)\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}(d(\xi;X_1(t-\tau)))^\lambda d\xi d^\gamma[f]^\lambda_\varphi.$$

Now let us use the inequality (25) and equality (18). We get

$$|K_1| \le Cd^\gamma\int_0^{t-d^{2b}}(t-\tau)^{-\nu-(\gamma-\lambda)/(2b)}d\tau\psi(t,x)[f]^\lambda_\varphi. \tag{30}$$

If $\nu+(\gamma-\lambda)/(2b)<1$, then from (30) we obtain

$$|K_1| \le Cd^\gamma\psi(t,x)[f]^\lambda_\varphi(t-\tau)^{1-\nu-(\gamma-\lambda)/(2b)}|^0_{\tau=t-d^{2b}}$$

$$= Cd^\gamma\psi(t,x)[f]^\lambda_\varphi(t^{1-\nu-(\gamma-\lambda)/(2b)}-d^{2b(1-\nu)-\gamma+\lambda}) \le Cd^\gamma\psi(t,x)[f]^\lambda_\varphi.$$

If $\nu+(\gamma-\lambda)/(2b)>1$, then from (30) we obtain

$$|K_1| \le Cd^\gamma\psi(t,x)[f]^\lambda_\varphi(t-\tau)^{1-\nu-(\gamma-\lambda)/(2b)}|^{t-d^{2b}}_{\tau=0} = Cd^\gamma\psi(t,x)[f]^\lambda_\varphi(d^{2b(1-\nu)-\gamma+\lambda)}$$

$$-t^{1-\nu-(\gamma-\lambda)/(2b)}) \le Cd^{2b(1-\nu)+\lambda}\psi(t,x)[f]^\lambda_\varphi = Cd^\lambda\psi(t,x)[f]^\lambda_\varphi.$$

Let us estimate K_2. With the help of (3), (9), (10) and (25) we obtain

$$|K_2| \le C\int_{t-d^{2b}}^t(t-\tau)^{-\nu-N}d\tau\int_{\mathbb{R}^n}(d(\xi;X_1(t-\tau)))^\lambda\exp\{-c\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_1(t-\tau)))d\xi[f]^\lambda_\varphi \le C\int_{t-d^{2b}}^t(t-\tau)^{-\nu-N}d\tau$$

$$\times\int_{\mathbb{R}^n}(d(\xi;X_1(t-\tau)))^\lambda\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^\lambda_\varphi$$

$$\le C\int_{t-d^{2b}}^t(t-\tau)^{-\nu-N+\lambda/(2b)}d\tau\int_{\mathbb{R}^n}\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^\lambda_\varphi.$$

Using (18) with $c'=\bar{c}_1$, we have

$$|K_2| \le C\int_{t-d^{2b}}^t(t-\tau)^{-\nu+\lambda/(2b)}d\tau\psi(t,x)[f]^\lambda_\varphi.$$

Since $-\nu+\lambda/(2b)>-1$, we obtain

$$|K_2| \le C(t-\tau)^{1-\nu+\lambda/(2b)}|^{t-d^{2b}}_{\tau=t}\psi(t,x)[f]^\lambda_\varphi = Cd^{2b(1-\nu)+\lambda}\psi(t,x)[f]^\lambda_\varphi \tag{31}$$

and thus, we have

$$|K_2| \le Cd^\lambda d^{2b(1-\nu)}\psi(t,x)[f]^\lambda_\varphi \le Cd^\lambda\psi(t,x)[f]^\lambda_\varphi,$$

if $\nu + (\gamma - \lambda)/(2b) > 1$. In case, where $\nu + (\gamma - \lambda)/(2b) < 1$, we receive from (31) the following inequality

$$|K_2| \leq C d^\gamma d^{2b(1-\nu)+\lambda-\gamma} \psi(t,x)[f]^\lambda_\varphi \leq C d^\gamma \psi(t,x)[f]^\lambda_\varphi.$$

By the similar way we obtain

$$|K_3| \leq C d^\lambda \psi(t,x')[f]^\lambda_\varphi$$

in case, where $\nu \in (1 - 1/(2b), 1]$, and

$$|K_3| \leq C d^\gamma \psi(t,x')[f]^\lambda_\varphi$$

in case, where $\nu \in (1 - 1/(2b), 1]$ and $\nu - (\gamma - \lambda)/(2b) < 1$.

From (27), (28), (29) and from the estimates for K_l, $l \in L$, the estimates (12) and (13) follow with $\nu \in (1 - 1/(2b), 1]$.

c) Let $\nu \in (1, 1 + 1/(2b)]$. Because of the second condition from (4) we represent integral (1) in the form

$$u(t,x) = \int_0^t d\tau \int_{\mathbb{R}^{n_1}} \Big(\int_{\mathbb{R}^{n_2+n_3}} (t-\tau)^{-\nu-N} \Omega(t,x;\tau,\xi) \Delta_\xi^{X_2(t-\tau)} f(\tau,\xi) d\xi_2 d\xi_3 \Big) d\xi_1, \tag{32}$$
$$(t,x) \in \Pi_{(0,T]},$$

where $X_2(t) := (\xi_1, \bar{x}_2(t), \bar{x}_3(t))$, with $\bar{x}_l(t)$, $l \in \{2,3\}$, which were determined in (9).

With the help of (3), (5) and (7)–(10) we get

$$|u(t,x)| \leq C \int_0^t (t-\tau)^{-\nu-N} d\tau \int_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}$$

$$\times \exp\{-c_0\rho(t-\tau,x,\xi)\}(\varphi(\tau,\xi) + \varphi(\tau,X_2(t-\tau))) \frac{|\Delta_\xi^{X_2(t-\tau)} f(\tau,\xi)|}{\varphi(\tau,\xi) + \varphi(\tau,X_2(t-\tau))} d\xi$$

$$\leq C \int_0^t (t-\tau)^{-\nu-N} d\tau \int_{\mathbb{R}^n} \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} d_1(\xi; X_2(t-\tau); \lambda) d\xi \psi(t,x)[f]^\lambda_{1,\varphi}.$$

The inequality below follows from definitions of d, d_1 and X_2.

$$\begin{aligned} d_1(\xi; X_2(t-\tau); \lambda) &= \sum_{l=2}^3 |\xi_l - \bar{x}_l(t-\tau)|^{(\lambda+1)/(2b(l-1)+1)} \\ &\leq C \left(\sum_{l=2}^3 |\xi_l - \bar{x}_l(t-\tau)|^{1/(2b(l-1)+1}) \right)^{\lambda+1} = C\left(d(\xi; X_2(t-\tau))\right)^{\lambda+1}, \\ &\qquad 0 \leq \tau < t \leq T, \quad \{x,\xi\} \subset \mathbb{R}^n, \quad \lambda \in (0,1]. \end{aligned}$$

Here $C > 0$ is some constant. Then taking into account inequality (25) we have

$$\begin{aligned} d_1(\xi; X_2(t-\tau); \lambda) \exp\{-\bar{c}\rho(t-\tau,x,\xi)\} &\leq C(d(\xi; X_2(t-\tau)))^{1+\lambda} \exp\{-\bar{c}\rho(t-\tau,x,\xi)\} \\ &\leq C(t-\tau)^{(1+\lambda)/(2b)} \exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}, \\ &0 \leq \tau < t \leq T,\ \{x,\xi\} \subset \mathbb{R}^n,\ 0 < \bar{c}_1 < \bar{c},\ \lambda \in (0,1]. \end{aligned} \tag{33}$$

For $\bar{c} = c - c_0$ with the help of (18) we have

$$|u(t,x)| \le C \int\limits_0^t (t-\tau)^{-\nu-N+(1+\lambda)/(2b)} d\tau \int\limits_{\mathbb{R}^n} \exp\{-\bar{c}_1 \rho(t-\tau,x,\xi)\} d\xi \psi(t,x) [f]^{\lambda}_{1,\varphi}$$
$$= C\psi(t,x)[f]^{\lambda}_{1,\varphi} \int\limits_0^t (t-\tau)^{-\nu+(1+\lambda)/(2b)} d\tau \tag{34}$$
$$= C\psi(t,x)[f]^{\lambda}_{1,\varphi} t^{1-\nu+(1+\lambda)/(2b)}, \ \ (t,x) \in \Pi_{(0,T]}.$$

Then

$$||u||^0_{\psi} \le C[f]^{\lambda}_{1,\varphi}. \tag{35}$$

Let us estimate the difference $\Delta^{x'}_x u$. If $d^{2b} \ge t$, where $d := d(x;x')$, then under estimate (34) we have the inequality

$$|\Delta^{x'}_x u(t,x)| \le C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{1,\varphi} d^{\lambda} t^{1-\nu+1/(2b)}$$
$$\le C(\psi(t,x) + \psi(t,x')) d^{\lambda} [f]^{\lambda}_{1,\varphi}, \ t \in (0,T], \ \{x,\xi\} \subset \mathbb{R}^n,$$

and with $\nu + (\gamma - 1 - \lambda)/(2b) < 1$ we receive

$$|\Delta^{x'}_x u(t,x)| \le C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{1,\varphi} t^{1-\nu-(\gamma-1-\lambda)/(2b)} t^{\gamma/(2b)}$$
$$\le C(\psi(t,x) + \psi(t,x'))[f]^{\lambda}_{1,\varphi} t^{1-\nu-(\gamma-1-\lambda)/(2b)} d^{\gamma} \tag{36}$$
$$\le C(\psi(t,x) + \psi(t,x')) d^{\gamma} [f]^{\lambda}_{1,\varphi}, \ t \in (0,T], \ \{x,\xi\} \subset \mathbb{R}^n.$$

It is sufficient to consider the case, where $d^{2b} < t$. By the second condition from (4) like (32) we write

$$\Delta^{x'}_x u(t,x) = \int\limits_0^{t-d^{2b}} d\tau \int\limits_{\mathbb{R}^{n_1}} \Big(\int\limits_{\mathbb{R}^{n_2+n_3}} \Delta^{x'}_x M(t,x;\tau,\xi) \Delta^{X_2(t-\tau)}_{\xi} f(\tau,\xi) d\xi_2 d\xi_3 \Big) d\xi_1$$
$$+ \int\limits_{t-d^{2b}}^{t} d\tau \int\limits_{\mathbb{R}^{n_1}} \Big(\int\limits_{\mathbb{R}^{n_2+n_3}} M(t,x;\tau,\xi) \Delta^{X_2(t-\tau)}_{\xi} f(\tau,\xi) d\xi_2 d\xi_3 \Big) d\xi_1 \tag{37}$$
$$- \int\limits_{t-d^{2b}}^{t} d\tau \int\limits_{\mathbb{R}^{n_1}} \Big(\int\limits_{\mathbb{R}^{n_2+n_3}} M(t,x';\tau,\xi) \Delta^{X'_2(t-\tau)}_{\xi} f(\tau,\xi) d\xi_2 d\xi_3 \Big) d\xi_1 =: \sum_{l=1}^{3} K'_l,$$

where $X'_2(t) := X_2(t)|_{x=x'}$.

Using (3), (6), the second inequality from (9), (10), we get

$$|K'_1| \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} (d(x;x'))^{\gamma} (t-\tau)^{-\gamma/(2b)} \exp\{-c\rho(t-\tau,x,\xi)\}$$
$$\times (\varphi(\tau,\xi) + \varphi(\tau,X_2(t-\tau))) \frac{|\Delta^{X_2(t-\tau)}_{\xi} f(\tau,\xi)|}{\varphi(\tau,\xi) + \varphi(\tau,X_2(t-\tau))} d\xi \le C \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N-\gamma/(2b)} d\tau$$
$$\times \int\limits_{\mathbb{R}^n} \psi(\tau,x) \exp\{-(c-c_0)\rho(t-\tau,x,\xi)\} d_1(\xi;X_2(t-\tau);\lambda) d\xi d^{\gamma} [f]^{\lambda}_{1,\varphi}.$$

Now let us use the inequalities (33) and equality (18). We get

$$|K_1'| \le Cd^\gamma \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-N-\gamma/(2b)+(1+\lambda)/(2b)} d\tau \int\limits_{\mathbb{R}^n} \psi(\tau,x)\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}d\xi$$

$$\times d^\gamma [f]^\lambda_{1,\varphi} = Cd^\gamma \int\limits_0^{t-d^{2b}} (t-\tau)^{-\nu-(\gamma-1-\lambda)/(2b)} d\tau \psi(t,x)[f]^\lambda_{1,\varphi}.$$

If $\nu + (\gamma-1-\lambda)/(2b) > 1$, then

$$|K_1'| \le Cd^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}(t-\tau)^{1-\nu-(\gamma-1-\lambda)/(2b)}|^{t-d^{2b}}_{\tau=0} = Cd^\gamma\psi(t,x)[f]^\lambda_{1,\varphi}(d^{2b(1-\nu)-\gamma+1+\lambda}$$

$$-t^{1-\nu-(\gamma-1-\lambda)/(2b)}) \le Cd^{2b(1-\nu)+1+\lambda}\psi(t,x)[f]^\lambda_{1,\varphi} \le Cd^\lambda\psi(t,x)[f]^\lambda_{1,\varphi}.$$

If $\nu + (\gamma-1-\lambda)/(2b) < 1$, then

$$|K_1'| \le Cd^\gamma \psi(t,x)[f]^\lambda_{1,\varphi}(t-\tau)^{1-\nu-(\gamma-1-\lambda)/(2b)}|^{0}_{\tau=t-d^{2b}} = Cd^\gamma\psi(t,x)[f]^\lambda_{1,\varphi}$$

$$\times(t^{1-\nu-(\gamma-1-\lambda)/(2b)} - d^{2b(1-\nu)-\gamma+1+\lambda}) \le Cd^\gamma\psi(t,x)[f]^\lambda_{1,\varphi}.$$

Let us estimate K_2'. With the help of (3), (9), (10) and (33) we obtain

$$|K_2'| \le C \int\limits_{t-d^{2b}}^{t} (t-\tau)^{-\nu-N} d\tau \int\limits_{\mathbb{R}^n} d_1(\xi; X_2(t-\tau);\lambda)\exp\{-c\rho(t-\tau,x,\xi)\}$$

$$\times(\varphi(\tau,\xi)+\varphi(\tau,X_2(t-\tau)))d\xi[f]^\lambda_{1,\varphi} \le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu-N}d\tau$$

$$\times\int\limits_{\mathbb{R}^n} d_1(\xi;X_2(t-\tau);\lambda)\exp\{-(c-c_0)\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^\lambda_{1,\varphi}$$

$$\le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu-N+(1+\lambda)/(2b)}d\tau\int\limits_{\mathbb{R}^n}\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}\psi(t,x)d\xi[f]^\lambda_{1,\varphi}.$$

Using (18) with $c' = \bar{c}_1$, we have

$$|K_2'| \le C\int\limits_{t-d^{2b}}^{t}(t-\tau)^{-\nu+(1+\lambda)/(2b)}d\tau\psi(t,x)[f]^\lambda_{1,\varphi}.$$

Since $\nu - (1+\lambda)/(2b) < 1$, we obtain

$$|K_2'| \le C(t-\tau)^{1-\nu+(1+\lambda)/(2b)}|^{t-d^{2b}}_{\tau=t}\psi(t,x)[f]^\lambda_{1,\varphi} = Cd^{2b(1-\nu)+1+\lambda}\psi(t,x)[f]^\lambda_{1,\varphi}. \tag{38}$$

The estimate

$$|K_2'| \le Cd^\gamma d^{2b(1-\nu)+1+\lambda-\gamma}\psi(t,x)[f]^\lambda_{1,\varphi} \le Cd^\gamma\psi(t,x)[f]^\lambda_{1,\varphi}$$

follow from (38) if $\nu + (\gamma - 1 - \lambda)/(2b) < 1$, and the estimate

$$|K_2'| \leq Cd^\lambda d^{2b(1-\nu)+1}\psi(t,x)[f]^\lambda_{1,\varphi} \leq Cd^\lambda \psi(t,x)[f]^\lambda_{1,\varphi}$$

if $\nu + (\gamma - 1 - \lambda)/(2b) > 1$.

By the similar way we obtain

$$|K_3'| \leq Cd^\gamma \psi(t,x')[f]^\lambda_{1,\varphi}$$

in case, where $\nu + (\gamma - 1 - \lambda)/(2b) < 1$, and

$$|K_3'| \leq Cd^\lambda \psi(t,x')[f]^\lambda_{1,\varphi}$$

in the case, where $\nu + (\gamma - 1 - \lambda)/(2b) > 1$.

From (35), (36), (37) and from estimates for K_l', $l \in L$, the estimates (14) and (15) follow.

d) This case can be proved by the similar way as the case **c)**. We must use the third equality from (4); representation of the integral (1) in the form

$$u(t,x) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n_1+n_2}} \Big(\int\limits_{\mathbb{R}^{n_3}} (t-\tau)^{-\nu-N}\Omega(t,x;\tau,\xi)\Delta_\xi^{X_3(t-\tau)} f(\tau,\xi)d\xi_3\Big)d\xi_1 d\xi_2,\ (t,x) \in \Pi_{(0,T]},$$

where $X_3(t) := (\xi_1, \xi_2, \bar{x}_3(t))$, with $\bar{x}_3(t)$, which was determined in (9); and estimates

$$\begin{aligned} d_2(\xi; X_3(t-\tau);\lambda)\exp\{-\bar{c}\rho(t-\tau,x,\xi)\} &\leq C(d(\xi;X_3(t-\tau)))^{1+2b+\lambda}\exp\{-\bar{c}\rho(t-\tau,x,\xi)\} \\ &\leq C(t-\tau)^{(1+2b+\lambda)/(2b)}\exp\{-\bar{c}_1\rho(t-\tau,x,\xi)\}, \\ &0 \leq \tau < t \leq T, \qquad \{x,\xi\} \subset \mathbb{R}^n, \\ &0 < \bar{c}_1 < \bar{c}, \qquad \lambda \in (0,1]. \end{aligned}$$

These estimates are obtained in the same way as estimates (33). □

REFERENCES

[1] Dron' V.S. *On correct solvability in weighted Hölder spaces for the Cauchy problem for a class of degenerate parabolic equations of the Kolmogorov type*. Nauk. Visnyk Chernivtsi University 2000, **76**, 32–41. (in Ukrainian)

[2] Dron' V.S., Ivasyshen S.D. *On properties of the volume potential and correct solvability of the Cauchy problem for a model ultraparabolic equation*. Nauk. Visnyk Chernivtsi University 1999, **46**, 36–43. (in Ukrainian)

[3] Dron' V.S., Ivasyshen S.D. *Properties of the volume potential for one class ultraparabolic equation of arbitrary order*. Bukovinian Math. J. 2016, **4**(3–4), 47–56. (in Ukrainian)

[4] Dron' V.S., Ivasyshen S.D. *On smoothness of volume potential for degenerate $\overrightarrow{2b}$-parabolic equations of Kolmogorov type*. Nonclassical problems of the theory of differential equations: Call. of scientific works dedicated to 80-anniversary of B.Yo.Ptashnyk. IAPMM, Lviv, 2017, 38–53. (in Ukrainian)

[5] Eidelman S.D. Parabolic Systems. North-Holland, Amsterdam, 1969.

[6] Eidelman S.D., Ivasyshen S.D., Kochubei A.N. *Analytic methods in the theory of differential and pseudo-differential equations of parabolic type*. In: Operator Theory: Advances and Applications, 152. Birkhäuser Verlag, Basel-Boston-Berlin, 2004. doi:10.1007/978-3-0348-7844-9

[7] Ivasyshen S.D., Voznyak O.G. *The Cauchy problem for parabolic systems with degeneration on the initial hypeplane*. Dopov. Nats. Akad. Nauk Ukr. 1994, **6**, 7–11. (in Ukrainian)

[8] Ivasyshen S.D., Eidelman S.D. $\overrightarrow{2b}$*-parabolic systems*. Trudy Sem. Funkt. Anal. Kiev, Institute of Mathematics 1968, **1**, 3–175, 271–273. (in Russian)

[9] Ivasyshen S.D., Medynsky I.P. *Properties of integrals which have the type of derivatives of volume potentials for parabolic systems with degeneration on the initial hypeplane*. Mat. Studii 2000, **13** (1), 33–46.

[10] Ivasyshen S.D., Medynsky I.P. *On correct solvability of parabolic systems with degeneration on the initial hypeplane*. Nauk. Visnyk Chernivtsi University 2000, **76**, 71–76. (in Ukrainian)

[11] Ivasyshen S.D., Medynsky I.P. *Properties of integrals which have the type of derivatives of volume potentials for* $\overrightarrow{2b}$*-parabolic systems with degeneration on the initial hypeplane*. Mat. Metodi Fiz.-Mekh. Polya 2002, **45** (4), 76–86. (in Ukrainian)

[12] Ivasyshen S.D., Medynsky I.P. *Cauchy problem for* $\overrightarrow{2b}$*-parabolic systems with degeneration on the initial hypeplane*. Mat. Metody Fiz.-Mech. Polya 2003, **46** (3), 15–24. (in Ukrainian)

[13] Medynsky I.P. *On a priori estimates for solutions of parabolic systems with degeneration on the initial hypeplane*. Visnyk Nat. Univ. Lvivska Polytechnika 2000, **407**, 185–194. (in Ukrainian)

Received 10.10.2018

Дронь В.С., Івасишен С.Д., Мединський І.П. *Властивості інтегралів типу похідних від об'ємного потенціалу для одного ультрапараболічного рівняння типу Колмогорова довільного порядку* // Карпатські матем. публ. — 2019. — Т.11, №2. — C. 268–280.

Розглядаються інтеграли, які мають структуру та властивості, подібні до похідних від об'ємних потенціалів, породжених фундаментальним розв'язком задачі Коші для ультрапараболічного рівняння типу Колмогорова довільного порядку. Коефіцієнти цього рівняння залежать тільки від часової змінної. Встановлюється належність цих інтегралів до відповідних вагових просторів Гельдера, залежно від того, до яких просторів належить густина та ядро інтеграла.

Для побудови просторів Гельдера використовуються спеціальні відстані та вагові норми. Відстані враховують анізотропність за просторовими змінними рівняння, яке породжує інтеграли, що розглядаються. Ваговими функціями є експоненти, які необмежено зростають при $|x| \to \infty$ і тип їх зростання спеціальним способом залежить від змінної t.

Результати роботи можуть бути використані для встановлення коректної розв'язності задачі Коші та оцінок розв'язків даного неоднорідного рівняння у відповідних вагових просторах Гельдера.

Ключові слова і фрази: ультрапараболічне рівняння типу Колмогорова довільного порядку, інтеграл типу похідних від об'ємного потенціалу, вагова гельдерова норма, простір Гельдера зростаючих функцій.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 281–295
doi:10.15330/cmp.11.2.281-295

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.281–295

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ESTIMATES OF APPROXIMATIVE CHARACTERISTICS OF THE CLASSES $B^{\Omega}_{p,\theta}$ OF PERIODIC FUNCTIONS OF SEVERAL VARIABLES WITH GIVEN MAJORANT OF MIXED MODULI OF CONTINUITY IN THE SPACE L_q

In this paper, we continue the study of approximative characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables whose majorant of the mixed moduli of continuity contains both exponential and logarithmic multipliers. We obtain the exact-order estimates of the orthoprojective widths of the classes $B^{\Omega}_{p,\theta}$ in the space L_q, $1 \leq p < q < \infty$, and also establish the exact-order estimates of approximation for these classes of functions in the space L_q by using linear operators satisfying certain conditions.

Key words and phrases: orthoprojective width, mixed modulus of continuity, linear operator, Vallée-Poussin kernel, Fejér kernel.

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INTRODUCTION

Let $\mathbb{R}^d, d \geq 1$ denote d-dimensional space with elements

$$x = (x_1, \dots, x_d),\ (x, y) = x_1 y_1 + \dots + x_d y_d$$

and let $L_p(\pi_d)$, $1 \leq p < \infty$, be the space of functions $f(x) = f(x_1, \dots, x_d)$, which are 2π-periodic in each variable and summable in degree p on the cube $\pi_d = \prod\limits_{j=1}^{d} [0; 2\pi]$ for which the norm is defined as follows:

$$\|f\|_{L_p(\pi_d)} = \|f\|_p = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Respectively, $L_\infty(\pi_d)$ is the space of essentially bounded functions $f(x) = f(x_1, \dots, x_d)$, which are 2π- periodic in each variable, with the norm

$$\|f\|_{L_\infty(\pi_d)} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|.$$

Further, we assume that, for functions $f \in L_p(\pi_d)$, the following additional condition holds:

$$\int_0^{2\pi} f(x) dx_j = 0 \quad j = \overline{1, d}.$$

УДК 517.51
2010 *Mathematics Subject Classification:* 42B99.

For $f \in L_p(\pi_d), 1 \le p \le \infty$, and $t = (t_1, \dots, t_d)$, $t_j \ge 0$, $j = \overline{1,d}$, we consider the mixed modulus of continuity of the order l

$$\Omega_l(f,t)_p = \sup_{\substack{|h_j| \le t_j \\ j=\overline{1,d}}} \|\Delta_h^l f(\cdot)\|_p,$$

where $l \in \mathbb{N}$, $\Delta_h^l f(x) = \Delta_{h_1}^l \dots \Delta_{h_d}^l f(x) = \Delta_{h_d}^l(\dots(\Delta_{h_1}^l f(x)))$ is a mixed difference of the order l with a vector step $h = (h_1, \dots, h_d)$, and the difference of the l^{th} order with a step h_j in the variable x_j is defined as follows:

$$\Delta_{h_j}^l f(x) = \sum_{n=0}^{l} (-1)^{l-n} C_l^n f(x_1, \dots, x_{j-1}, x_j + nh_j, x_{j+1}, \dots, x_d).$$

Let $\Omega(t) = \Omega(t_1, \dots, t_d)$ be a given function of the type of a mixed modulus of continuity of the order l, which satisfies the following conditions:

1) $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1,d}$; $\Omega(t) = 0$, $\prod\limits_{j=1}^{d} t_j = 0$;

2) $\Omega(t)$ is nondecreasing in each variable;

3) $\Omega(m_1 t_1, \dots, m_d t_d) \le \left(\prod\limits_{j=1}^{d} m_j \right)^l \Omega(t)$, $m_j \in \mathbb{N}$, $j = \overline{1,d}$;

4) $\Omega(t)$ is continuous for $t_j \ge 0$, $j = \overline{1,d}$.

We assume that $\Omega(t)$ satisfies also the conditions (S) and (S_l), which are called the Bari-Stechkin conditions [1]. This means the following.

A function of one variable $\varphi(\tau) \ge 0$ satisfies the condition (S) if $\varphi(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$, i.e., there exists a constant $C_1 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \le C_1 \frac{\varphi(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

A function $\varphi(\tau) \ge 0$ satisfies the condition (S_l) if $\varphi(\tau)/\tau^\gamma$ almost decreases for some $0 < \gamma < l$, i.e., there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 and such that

$$\frac{\varphi(\tau_1)}{\tau_1^\gamma} \ge C_2 \frac{\varphi(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

We say that $\Omega(t)$ satisfies the conditions (S) and (S_l) if $\Omega(t)$ satisfies these conditions in each variable t_j for fixed t_i, $i \ne j$.

Thus, let $1 \le p \le \infty$, $1 \le \theta \le \infty$, and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l. Then the classes $B_{p,\theta}^\Omega$ are defined in the following way [21]:

$$B_{p,\theta}^\Omega = \{ f \in L_p(\pi_d) : \|f\|_{B_{p,\theta}^\Omega} \le 1 \},$$

where

$$\|f\|_{B_{p,\theta}^\Omega} = \left\{ \int\limits_{\pi_d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^\theta \prod_{j=1}^{d} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty,$$

$$\|f\|_{B^{\Omega}_{p,\infty}} = \sup_{t>0} \frac{\Omega_l(f,t)_p}{\Omega(t)},$$

(the expression $t > 0$ for $t = (t_1, \dots, t_d)$ is equivalent to $t_j > 0, j = \overline{1,d}$).

We note that, for $\theta = \infty$, the classes $B^{\Omega}_{p,\theta}$ coincide with the classes H^{Ω}_{p}, which were considered by N.N. Pustovoitov in [13].

In the subsequent, it will be convenient to use the equivalent (to within absolute constants) definition of the classes $B^{\Omega}_{p,\theta}$. For this purpose, we need the corresponding notations.

To every vector $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$, we put the set

$$\rho(s) = \left\{k = (k_1, \dots, k_d) : 2^{s_j-1} \le |k_j| < 2^{s_j}, k_j \in \mathbb{Z}, j = \overline{1,d}\right\}$$

in correspondence, and, for $f \in L_p(\pi_d), 1 < p < \infty$, we denote

$$\delta_s(f) := \delta_s(f,x) = \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where

$$\widehat{f}(k) = (2\pi)^{-d} \int_{\pi_d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f.

Let $1 < p < \infty$, $1 \le \theta \le \infty$ and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l that satisfies the conditions 1 – 4, (S) and (S_l). Then, to within absolute constants, the classes $B^{\Omega}_{p,\theta}$ can be defined as follows [21]:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\theta}} = \left(\sum_s \Omega^{-\theta}(2^{-s}) \|\delta_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}} \le 1 \right\} \tag{1}$$

for $1 \le \theta < \infty$ and

$$B^{\Omega}_{p,\infty} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\infty}} = \sup_s \frac{\|\delta_s(f)\|_p}{\Omega(2^{-s})} \le 1 \right\}. \tag{2}$$

Here and below, $\Omega(2^{-s}) = \Omega(2^{-s_1}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

The given definitions of the classes $B^{\Omega}_{p,\theta}$ can be extended also to the extreme values $p = 1$ and $p = \infty$, by modifying the "blocks" $\delta_s(f)$ in (1) and (2). Let $V_n(t)$ stand for a Vallée-Poussin kernel of the order $2n - 1$, i.e.,

$$V_n(t) = 1 + 2\sum_{k=1}^{n} \cos kt + 2\sum_{k=n+1}^{2n-1} \left(1 - \frac{k-n}{n}\right) \cos kt.$$

To every vector $s = (s_1, \dots, s_d), s_j \in \mathbb{N}, j = \overline{1,d}$, we put the polynomial

$$A_s(x) = \prod_{j=1}^{d} \left(V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j) \right)$$

in correspondence. For $f \in L_p(\pi_d), 1 \le p \le \infty$, by $A_s(f)$ we denote the convolution

$$A_s(f) := A_s(f,x) = (f * A_s)(x).$$

Then, to within absolute constants, the classes $B^{\Omega}_{p,\theta}$, $1 \le p \le \infty$, can be defined as follows:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\theta}} = \left(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}} \le 1 \right\} \tag{3}$$

for $1 \le \theta < \infty$ and

$$B^{\Omega}_{p,\infty} = \left\{ f \in L_p(\pi_d) : \|f\|_{B^{\Omega}_{p,\infty}} = \sup_s \frac{\|A_s(f)\|_p}{\Omega(2^{-s})} \le 1 \right\}. \tag{4}$$

We note that relations (3) and (4) were obtained in works [18] and [13], respectively.

We note also that, for $\Omega(t) = \prod\limits_{j=1}^{d} t_j^{r_j}$, $0 < r_j < l$, the classes $B^{\Omega}_{p,\theta}$ are analogs of the well-known Besov $B^{r}_{p,\theta}$, $1 \le \theta < \infty$, and Nikol'skii $B^{r}_{p,\infty} = H^{r}_{p}$ classes (see, e.g., [8]).

In what follows, we study the classes $B^{\Omega}_{p,\theta}$ that are defined by the function $\Omega(t)$:

$$\Omega(t) = \Omega(t_1, \dots, t_d) = \begin{cases} \prod\limits_{j=1}^{d} \dfrac{t_j^{r}}{\left(\log \frac{1}{t_j}\right)_+^{b_j}}, & \text{if } t_j > 0,\ j = \overline{1,d}; \\ 0, & \text{if } \prod\limits_{j=1}^{d} t_j = 0. \end{cases} \tag{5}$$

Here and below, we consider the logarithms with base 2, and

$$\left(\log \frac{1}{t_j} \right)_+ = \max \left\{ 1, \log \frac{1}{t_j} \right\}.$$

In addition, we assume that $b_j \in \mathbb{R}$, $j = \overline{1,d}$, and $0 < r < l$. Hence, properties 1–4 and the conditions (S) and (S_l) are satisfied for the function $\Omega(t)$ of the form (5).

In the present paper we obtain the exact-order estimates of orthoprojective widths of the classes $B^{\Omega}_{p,\theta}$ in the space L_q, $1 \le p < q < \infty$. We recall that the notion of orthoprojective width was introduced by V. N. Temlyakov [23].

Let $\{u_i\}_{i=1}^{M}$ be an orthonormalized system of functions $u_i \in L_\infty(\pi_d)$, $f \in L_q(\pi_d)$, $1 \le q \le \infty$. We set

$$(f, u_i) = (2\pi)^{-d} \int\limits_{\pi_d} f(x) \overline{u}_i(x) dx,$$

where $\overline{u}_i$ is the function complex conjugate to the function u_i.

To every function $f \in L_q(\pi_d)$, $1 \le q \le \infty$, we put an approximation of the form $\sum\limits_{i=1}^{M} (f, u_i) u_i$ in correspondence, i.e., the orthogonal projection of the function f onto the subspace generated by the system of functions $\{u_i\}_{i=1}^{M}$. Then, for the functional class $F \subset L_q(\pi_d)$, the quantity

$$d_M^{\perp}(F, L_q) = \inf_{\{u_i\}_{i=1}^{M}} \sup_{f \in F} \left\| f - \sum_{i=1}^{M} (f, u_i) u_i \right\|_q \tag{6}$$

is called the orthoprojective width (the Fourier-width) of this class in the space $L_q(\pi_d)$.

In addition to orthoprojective widths, we study the quantities $d_M^{B}(F, L_q)$ introduced by V.N. Temlyakov [22]). They are defined as follows:

$$d^{B}_{M}(F,L_q)=\inf_{G\in L_M(B)_q}\ \sup_{f\in F\cap D(G)}\|f-Gf\|_q. \tag{7}$$

Here, $L_M(B)_q$ stands for a set of linear operators satisfying the conditions:

a) the domain of definition $D(G)$ of these operators contains all trigonometric polynomials, and their domain of values is contained in a subspace with dimension M of the space $L_q(\pi_d)$;

b) there exists a number $B\geq 1$ such that, for all vectors $k=(k_1,\dots,k_d)$, $k_j\in\mathbb{Z}$, $j=\overline{1,d}$, the inequality $\left\|Ge^{i(k,\cdot)}\right\|_2\leq B$ holds.

We note that $L_M(1)_2$ contains the operators of orthogonal projection onto the spaces with dimension M and the operators that are set on an orthonormalized system of functions with the help of the multiplier defined by a sequence $\{\lambda_m\}$ such that $|\lambda_m|\leq 1$ for all m.

From (6) and (7), it is easy to see that the quantities $d^{\perp}_{M}(F,L_q)$ and $d^{B}_{M}(F,L_q)$ are connected with each other by the inequality

$$d^{B}_{M}(F,L_q)\leq d^{\perp}_{M}(F,L_q). \tag{8}$$

At present, a lot of works are known, in which the quantities $d^{\perp}_{M}(F,L_q)$ and $d^{B}_{M}(F,L_q)$ were studied for various classes of functions. We mention works [14,16,17,22,24], where the quantities (6) and (7) were considered for the classes of functions of many variables $W^{r}_{p,\alpha}$, H^{r}_{p}, $B^{r}_{p,\theta}$, and H^{Ω}_{p} (see also numerous references therein). The quantities $d^{\perp}_{M}(B^{\Omega}_{p,\theta},L_q)$ and $d^{B}_{M}(B^{\Omega}_{p,\theta},L_q)$ for the classes of functions of many variables with a given function $\Omega(t)$ of the form (5) under the condition $b_j<r$, $j=\overline{1,d}$, were considered in works [4–7].

1 AUXILIARY ASSERTIONS

We now give several known assertions, which are used in the subsequent considerations.

As was noted above, $\Omega(t)$ is a function of the form (5). For a natural N, we set

$$\chi(N)=\left\{s=(s_1,\dots,s_d):\ s_j\in\mathbb{N},\ j=\overline{1,d},\ \Omega(2^{-s})\geq\frac{1}{N}\right\},$$

$$Q(N)=\bigcup_{s\in\chi(N)}\rho(s).$$

We note that the approximation of certain classes of periodic functions of many variables with mixed generalized smoothness by trigonometric polynomials with "numbers" of harmonics from the sets that are analogs of $Q(N)$ was started in work [15]. Later, the approximations by trigonometric polynomials with "numbers" of harmonics from the sets $Q(N)$ were studied in works [4], [19], [20] and other ones.

The following proposition is true.

Lemma 1 ([14]). *For the number of elements of the set $Q(N)$, the following ordinal equalities hold:*

$$|Q(N)|\asymp N^{\frac{1}{r}}\left(\log N\right)^{-\frac{b_1}{r}-\dots-\frac{b_\nu}{r}+\nu-1},$$

if $b_1 \le \ldots \le b_\nu < r < b_{\nu+1} \le \ldots \le b_d$;

$$|Q(N)| \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}},$$

if $r \le b_1 \le \ldots \le b_d$, $b_2 > r$.

Here and below, the notation $\mu_1 \ll \mu_2$ for positive functions $\mu_1(N)$ and $\mu_2(N)$ means that there exists a constant $C > 0$ such that, $\forall N \in \mathbb{N}$, the inequality $\mu_1(N) \le C\mu_2(N)$ holds.

The relation $\mu_1 \asymp \mu_2$ holds if $\mu_1 \ll \mu_2$ and $\mu_1 \gg \mu_2$. We note also that all constants $C_i, i = 1, 2, \ldots$, which are used in what follows, can depend only on parameters that are contained in the definitions of a class and a dimension d of the space $\mathbb{R}^d$.

To formulate the following assertions, we note that, according to (5), the definition of a set $\chi(N)$ takes the form

$$\chi(N) = \Big\{s = (s_1, \ldots, s_d) : \ s_j \in \mathbb{N},\ j = \overline{1,d},\ \prod_{j=1}^{d} 2^{rs_j} s_j^{b_j} \le N\Big\}.$$

Therefore,

$$\chi^\perp(N) = \mathbb{N}^d \setminus \chi(N).$$

Let

$$\Theta(N) = \Big\{s = (s_1, \ldots, s_d) : \ s_j \in \mathbb{N}, j = \overline{1,d},\ \frac{1}{2^l N} \le \Omega(2^{-s}) < \frac{1}{N}\Big\}.$$

In work [11], it was established that the number of elements of the set $\Theta(N)$ satisfies the ordinal equality

$$|\Theta(N)| \asymp (\log N)^{d-1}.$$

Lemma 2 ([14]). *For the function* $\Omega(t)$ *defined by equality* (5) *for* $0 < \beta < r$, $0 < p < \infty$ *the relation*

$$\sum_{s \in \chi^\perp(N)} \big(\Omega(2^{-s}) 2^{\|s\|_1 \beta}\big)^p \ll \sum_{s \in \Theta(N)} \big(\Omega(2^{-s}) 2^{\|s\|_1 \beta}\big)^p$$

holds, where $\|s\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{N}$.

Lemma 3 ([14]). *If* $\gamma_1 \le \ldots \le \gamma_\nu < 1 < \gamma_{\nu+1} \le \ldots \le \gamma_d$, *then*

$$\sum_{s \in \Theta(N)} \prod_{j=1}^{d} s_j^{-\gamma_j} \asymp \big(\log N\big)^{-\gamma_1 - \ldots - \gamma_\nu + \nu - 1}.$$

If $1 \le \gamma_1 \le \ldots \le \gamma_d$, $\gamma_2 > 1$, *then*

$$\sum_{s \in \Theta(N)} \prod_{j=1}^{d} s_j^{-\gamma_j} \asymp \big(\log N\big)^{-\gamma_1}.$$

Lemma 4 ([22]). *Let* $1 \le p < q < \infty$ *and* $f \in L_p(\pi_d)$. *Then*

$$\|f\|_q^q \ll \sum_s \left(\|\delta_s(f)\|_p \, 2^{\|s\|_1 \left(\frac{1}{p} - \frac{1}{q}\right)} \right)^q.$$

Lemma 5 ([24]). *Let A be the linear operator given by the equality*

$$Ae^{i(k,x)} = \sum_{m=1}^{\overline{M}} a_m^k \psi_m(x),$$

where $\{\psi_m(x)\}_{m=1}^{\overline{M}}$ is the set of functions for which

$$\|\psi_m(\cdot)\|_2 \le 1, \ \ m = 1, \dots, \overline{M}.$$

Then, for any trigonometric polynomial t, the following inequality holds:

$$\min_{y=x} Re\, At(x-y) \le \left(\overline{M} \sum_{m=1}^{\overline{M}} \sum_{k} |a_m^k \widehat{t}(k)|^2 \right)^{\frac{1}{2}}.$$

Theorem 1 ([10]). *Let T_n be a trigonometric polynomial of the order $n = (n_1, \dots, n_d)$, i.e.,*

$$T_n(x) = \sum_{|k_1| \le n_1} \dots \sum_{|k_d| \le n_d} c_{k_1,\dots,k_d} e^{i(k,x)},$$

where n_j, $j = \overline{1,d}$ are natural numbers, and $c_{k_1,\dots,k_d}$ are any coefficients. Then, for $1 \le p < q \le \infty$ the inequality

$$\|T_n\|_q \le 2^d \left(\prod_{j=1}^{d} n_j \right)^{\frac{1}{p} - \frac{1}{q}} \|T_n\|_p \tag{9}$$

holds.

Inequality (9) was established by S. M. Nikol'skii and is called the "inequality of different metrics". In the one-dimensional case for $p = \infty$, the corresponding inequality was proved by D. Jackson [3].

Theorem 2 (Littlewood-Paley theorem; see, e.g., [9], p. 65). *Let $p \in (1, \infty)$. Then there exist positive numbers $C_3(p)$ and $C_4(p)$ such that, for every function $f \in L_p(\pi_d)$, the following relations are true:*

$$C_3(p)||f||_p \le \left\| \left(\sum_s |\delta_s(f)|^2 \right)^{\frac{1}{2}} \right\|_p \le C_4(p) \|f\|_p \,.$$

2 MAIN RESULTS

Passing to the statement of the propositions and their proof, we assume that $M = |Q(N)|$. First, we consider case $b_1 \le \ldots \le b_\nu < r < b_{\nu+1} \le \ldots \le b_d$. Then, according to Lemma 1, we have

$$M \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r} + \nu - 1},$$
$$\log M \asymp \log N, \ \ N \asymp M^r \big(\log M\big)^{b_1 + \ldots + b_\nu - (\nu - 1)r}.$$

The following theorem is true.

Theorem 3. *Let $1 \le p < q < \infty$, $q < \theta < \infty$, and let $\Omega(t)$ be a function of the form* (5). *Then, for $\frac{1}{p} - \frac{1}{q} < r < l$, $b_1 \le \ldots \le b_\nu < \frac{r}{\frac{q}{p}-1} < b_{\nu+1} \le \ldots \le b_d$, the relations*

$$d_M^{\perp}(B^{\Omega}_{p,\theta}, L_q) \asymp d_M^{B}(B^{\Omega}_{p,\theta}, L_q) \asymp M^{-r+\frac{1}{p}-\frac{1}{q}}\big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)} \tag{10}$$

hold.

Proof. First, we establish the upper bounds in (10). According to (8), it is sufficient to obtain the upper bound for the orthoprojective width $d_M^{\perp}(B^{\Omega}_{p,\theta}, L_q)$.

For this purpose, we consider an approximation of the functions $f \in B^{\Omega}_{p,\theta}$ by trigonometric polynomials $t_{Q(N)}$ of the form

$$t_{Q(N)}(x) = \sum_{s\in\chi(N)} \delta_s(f,x).$$

Let q_0 be any number that satisfies the condition $p < q_0 < q$.

Then, using Lemma 4, and the relation

$$\|\delta_s(f)\|_{q_0} \asymp \|A_s(f)\|_{q_0},\ 1 < q_0 < \infty,$$

for $f \in B^{\Omega}_{p,\theta}$ we have

$$\|f - t_{Q(N)}\|_q = \Big\| f - \sum_{s\in\chi(N)} \delta_s(f)\Big\|_q = \Big\| \sum_{s\in\chi^{\perp}(N)} \delta_s(f)\Big\|_q$$

$$\ll \left(\sum_{s\in\chi^{\perp}(N)} \|\delta_s(f)\|^q_{q_0} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \asymp \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|^q_{q_0} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} = I_1.$$

Then, applying to $A_s(f)$ the Nikol'skii inequality of different metrics, we continue the estimate as follows:

$$I_1 \ll \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|^q_{p} 2^{\|s\|_1(\frac{1}{p}-\frac{1}{q_0})q} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} = \left(\sum_{s\in\chi^{\perp}(N)} \|A_s(f)\|^q_{p} 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}}$$

$$= \left(\sum_{s\in\chi^{\perp}(N)} \Omega^{-q}(2^{-s})\|A_s(f)\|^q_{p} \Omega^{q}(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)q}\right)^{\frac{1}{q}} = I_2.$$

Using first the Hölder inequality with index $\frac{\theta}{q}$ and then Lemma 2, we get

$$I_2 \le \left(\sum_{s\in\chi^{\perp}(N)} \Omega^{-\theta}(2^{-s})\|A_s(f)\|^{\theta}_{p}\right)^{\frac{1}{\theta}} \cdot \left(\sum_{s\in\chi^{\perp}(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\right)^{\frac{\theta q}{\theta-q}}\right)^{\frac{\theta-q}{\theta q}}$$

$$\ll \|f\|_{B^{\Omega}_{p,\theta}} \left(\sum_{s\in\chi^{\perp}(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1\left(\frac{1}{p}-\frac{1}{q}\right)}\right)^{\frac{\theta q}{\theta-q}}\right)^{\frac{\theta-q}{\theta q}}$$

$$\ll \left(\sum_{s\in\Theta(N)} \left(\Omega(2^{-s}) 2^{\|s\|_1 \left(\frac{1}{p}-\frac{1}{q}\right)} \right)^{\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}} \leq N^{-1} \left(\sum_{s\in\Theta(N)} 2^{\|s\|_1 \left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}} = I_3.$$

Taking into account that, for $s \in \Theta(N)$,

$$2^{\|s\|_1} \asymp N^{\frac{1}{r}} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}},$$

and using Lemma 3, we have

$$I_3 \asymp N^{-1} \left(\sum_{s\in\Theta(N)} N^{\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$= N^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\sum_{s\in\Theta(N)} \prod_{j=1}^{d} s_j^{-\frac{b_j}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\frac{\theta q}{\theta-q}} \right)^{\frac{\theta-q}{\theta q}}$$

$$\asymp N^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\log N\right)^{\left(-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)+(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}$$

$$\asymp \left(M^r \left(\log M\right)^{b_1+\ldots+b_\nu-(\nu-1)r} \right)^{-1+\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\log M\right)^{\left(-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)+(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}$$

$$= M^{-r+\frac{1}{p}-\frac{1}{q}} \left(\log M\right)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}.$$

Thus, in view of the definition of orthoprojective width, the above reasoning gives the upper bound for $d^{\perp}_M(B^{\Omega}_{p,\theta}, L_q)$, and, respectively, for the quantity $d^{B}_M(B^{\Omega}_{p,\theta}, L_q)$.

Let us find the lower bounds in (10). Since inequality (8) holds, it is sufficient to obtain the lower bound for the quantity $d^{B}_M(B^{\Omega}_{p,\theta}, L_q)$.

With the help of the reasoning analogous to that in [12], we can prove the existence of a set $\Theta_1(N) \subset \Theta(N)$ such that, for $s = (s_1, \ldots, s_d) \in \Theta_1(N)$, the following relations are satisfied:

$$s_j \asymp \log N, \ \ j = \overline{1,d} \ \text{ and } \ |\Theta_1(N)| \asymp \left(\log N\right)^{d-1}.$$

Also we can assert that there exists a set

$$\Theta_1^{(\nu)}(N) = \{s \in \Theta(N) : s_j \asymp \log N, \ j = 1, \ldots, \nu, \ s_j = 1, \ j = \nu+1, \ldots, d\}$$

such that

$$|\Theta_1^{(\nu)}(N)| \asymp \left(\log N\right)^{\nu-1}.$$

Consider the set $\widetilde{Q}(N) = \bigcup\limits_{s\in\Theta_1^{(\nu)}(N)} \rho(s)$. By $T(\widetilde{Q}(N))$ we denote the set of trigonometric polynomials with the "numbers" of harmonics from $\widetilde{Q}(N)$.

Let K_n be the Fejér kernel of the order n, i.e.,

$$K_n(t) = \sum_{|k|\leq n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}.$$

We set

$$g_1(x) = \sum_{s \in \Theta_1^{(\nu)}(N)} \mathcal{K}_s^{(\nu)}(x) \prod_{j=\nu+1}^{d} e^{ix_j},$$

where

$$\mathcal{K}_s^{(\nu)}(x) = \prod_{j=1}^{\nu} e^{ik_j^{s_j} x_j} K_{2^{s_j-2}}(x_j),$$

$$k_j^{s_j} = \begin{cases} 2^{s_j-1} + 2^{s_j-2}, & s_j \geq 2; \\ 1, & s_j = 1, j = \overline{1, \nu}. \end{cases}$$

Suppose that the operator G belongs to $L_M(B)_q, 1 < q < \infty$. Consider the operator $A = S_{\widetilde{Q}(N)} G$, where $S_{\widetilde{Q}(N)}$ is the operator of taking partial Fourier sum corresponding to the set $\widetilde{Q}(N)$. Then $A \in L_M(B)_q$ and the domain of values of the operator A is a subspace A_M of the space $T(\widetilde{Q}(N))$, whose dimension $\dim A_M = \overline{M} \leq M$. It follows from Theorem 2 that for $f \in T(\widetilde{Q}(N))$, the following relation is satisfied:

$$\|f - Af\|_q \ll \|f - Gf\|_q.$$

Consider the quantity

$$I = \sup_{y} \|g_1(x-y) - Ag_1(x-y)\|_\infty.$$

Obviously,

$$I \geq g_1(0) - \min_{y=x} Re A g_1(x-y).$$

Using Lemma 5, we obtain

$$\min_{y=x} Re A g_1(x-y) \leq M^{\frac{1}{2}} B \left(\sum_k |\widehat{g}_1(k)|^2 \right)^{\frac{1}{2}} \ll M^{\frac{1}{2}} B |\widetilde{Q}(N)|^{\frac{1}{2}}. \tag{11}$$

Further, taking into account the relation

$$|\Theta_1^{(\nu)}(N)| \asymp \left(\log N\right)^{\nu-1},$$

as well as

$$|\rho(s)| = 2^{||s||_1} \asymp N^{\frac{1}{r}} \left(\log N\right)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r}}, s \in \Theta_1^{(\nu)}(N),$$

we can write

$$|\widetilde{Q}(N)| \asymp N^{\frac{1}{r}} \left(\log N\right)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r} + \nu - 1}. \tag{12}$$

On the other hand,

$$g_1(0) \asymp N^{\frac{1}{r}} \left(\log N\right)^{-\frac{b_1}{r} - \ldots - \frac{b_\nu}{r} + \nu - 1} \asymp |\widetilde{Q}(N)|. \tag{13}$$

Using (11) and (12), we can chose a number N so that $|\widetilde{Q}(N)| \asymp M$ and the right-hand side of (13) will be at least twice as large as the right-hand side of (11).

For some $y^* = (y_1^*, \ldots, y_d^*)$, for this N we have

$$\|g_1(x-y^*) - Ag_1(x-y^*)\|_\infty \gg M. \tag{14}$$

Consider the function

$$g_2(x) = C_5 N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}} g_1(x),\ C_5>0.$$

We now show that, at the corresponding choice of the constant C_5, this function belongs to the class $B^{\Omega}_{p,\theta}$. Indeed, since

$$||K_n||_p \asymp n^{1-\frac{1}{p}}\, 1\le p\le\infty,$$

for the Fejér kernel, we have

$$\left\|\mathcal{K}_s^{(\nu)}\right\|_p \asymp 2^{\|s\|_1\left(1-\frac{1}{p}\right)}\ 1\le p\le\infty.$$

Thus, we can write

$$\begin{aligned}
\|g_2\|_{B^{\Omega}_{p,\theta}} &= \left(\sum_s \Omega^{-\theta}(2^{-s})\|A_s(g_2)\|_p^\theta\right)^{\frac{1}{\theta}}\\
&\ll N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\cdot\left(\sum_{s\in\Theta_1^{(\nu)}(N)}\Omega^{-\theta}(2^{-s})\|A_s(g_1)\|_p^\theta\right)^{\frac{1}{\theta}} \qquad (15)\\
&\ll \Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\cdot\left(\sum_{s\in\Theta_1^{(\nu)}(N)} 2^{\|s\|_1\left(1-\frac{1}{p}\right)\theta}\right)^{\frac{1}{\theta}} = I_4.
\end{aligned}$$

Taking into account the fact that, for $s\in\Theta_1^{(\nu)}(N)\subset\Theta(N)$

$$2^{\|s\|_1}\asymp N^{\frac{1}{r}}\prod_{j=1}^d s_j^{-\frac{b_j}{r}},$$

and

$$s_j\asymp\log N,\ j=1,\ldots,\nu,\ s_j=1,\ j=\nu+1,\ldots d,\ |\Theta_1^{(\nu)}(N)|\asymp\big(\log N\big)^{\nu-1},$$

we get

$$\begin{aligned}
I_4 &\asymp \Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\\
&\times\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{1-\frac{1}{p}}|\Theta_1^{(\nu)}(N)|^{\frac{1}{\theta}}\asymp\big(\log N\big)^{-\frac{\nu-1}{\theta}}\big(\log N\big)^{\frac{\nu-1}{\theta}}=1.
\end{aligned}\qquad(16)$$

By comparing (15) and (16), we may conclude that $g_2\in B^{\Omega}_{p,\theta}$ with the corresponding constant $C_5>0$.

It was established in work [14] that for $t\in T(\widetilde{Q}(N))$, the following estimate is satisfied:

$$\|t\|_\infty\ll\|t\|_q\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{q}}\big(\log N\big)^{(\nu-1)\left(1-\frac{1}{q}\right)}.$$

Taking into account the last relation and using estimate (14), we get

$$\|g_2(x-y^*)-Gg_2(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\|g_1(x-y^*)-Gg_1(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}\|g_1(x-y^*)-Ag_1(x-y^*)\|_q$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{\frac{1}{p}-1}\big(\log N\big)^{-\frac{\nu-1}{\theta}}$$

$$\times\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_\nu}{r}}\Big)^{-\frac{1}{q}}\big(\log N\big)^{-(\nu-1)\left(1-\frac{1}{q}\right)}\|g_1(x-y^*)-Ag_1(x-y^*)\|_\infty$$

$$\gg N^{-1}\Big(N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}-\ldots-\frac{b_d}{r}+d-1}\Big)^{\frac{1}{p}-\frac{1}{q}-1}\big(\log N\big)^{(d-1)\left(-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}M$$

$$\asymp M^{-r}\big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)r}M^{\frac{1}{p}-\frac{1}{q}-1}\big(\log M\big)^{(\nu-1)\left(-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}M$$

$$=M^{-r+\frac{1}{p}-\frac{1}{q}}\big(\log M\big)^{-b_1-\ldots-b_\nu+(\nu-1)\left(r-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)}.$$

The lower bounds in (10) are established. Theorem 3 is proved. □

In the following proposition, we consider other relations for the numbers $r, b_1, \ldots, b_d$. Let $r \le b_1 \le \ldots \le b_d$, $b_2 > r$. In this case, by Lemma 1, we obtain

$$M \asymp N^{\frac{1}{r}}\big(\log N\big)^{-\frac{b_1}{r}},$$
$$\log M \asymp \log N,\ \ N \asymp M^r\big(\log M\big)^{b_1}.$$

Assume that

$$b_1=\ldots=b_\nu<b_{\nu+1}\le\ldots\le b_d.$$

Then, for $\nu=1$, the inequality $r \le b_1 < b_2$ holds. But $\nu \ge 2$, then $b_1 > r$.

Theorem 4. *Let* $1 \le p < q < \infty$, $q < \theta < \infty$, *and let* $\Omega(t)$ *be a function of the form* (5)*. Then, for* $\frac{1}{p}-\frac{1}{q} < r < l$, $b_2 > \frac{r}{\frac{q}{p}-1}$, *the order estimates*

$$d^{\perp}_M(B^{\Omega}_{p,\theta}, L_q) \asymp d^{B}_M(B^{\Omega}_{p,\theta}, L_q) \asymp M^{-r+\frac{1}{p}-\frac{1}{q}}\big(\log M\big)^{-b_1} \tag{17}$$

hold.

Proof. For $q<\theta<\infty$, the embedding $B^{\Omega}_{p,\theta} \subset H^{\Omega}_p$, is valid. Therefore, the upper bounds in (17) follow from the corresponding estimate $d^{\perp}_M(H^{\Omega}_p, L_q)$, proved in [14].

To get the lower bounds in (17), it is sufficient to get the corresponding lower bound for the quantity $d^B_M(B^{\Omega}_{p,\theta}, L_q)$.

We choose a vector $\tilde{s}=(\tilde{s}_1,\ldots,\tilde{s}_d) \in \Theta(N)$ so that

$$\tilde{s}_1 \asymp \log N,\quad \tilde{s}_2=\ldots=\tilde{s}_d=1,$$

and set

$$g_3(x)=\mathcal{K}_{\tilde{s}}(x)=e^{i(k^{\tilde{s}},x)}K_{2^{\tilde{s}_1-2}}(x_1),$$

where $k^{\tilde{s}} = (2^{\tilde{s}_1-1} + 2^{\tilde{s}_1-2}, 1, \dots, 1)$.

Suppose that the operator G belongs to $L_M(B)_q, 1 < q < \infty$. Consider the operator $A = S_{\rho(\tilde{s})}G$, where $S_{\rho(\tilde{s})}$ is the operator of taking partial Fourier sum corresponding to the set $\rho(\tilde{s})$.

Taking into account that

$$2^{\|\tilde{s}\|_1} \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}},$$

and using lemma 5, we get

$$\min_{y=x} ReAg_3(x-y) \le M^{\frac{1}{2}} B \bigg(\sum_k |\widehat{g}_3(k)|^2 \bigg)^{\frac{1}{2}} \ll M^{\frac{1}{2}} \big(2^{\|\tilde{s}\|_1}\big)^{\frac{1}{2}} \asymp M^{\frac{1}{2}} N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}}. \tag{18}$$

On the other hand,

$$g_3(0) \asymp 2^{\|\tilde{s}\|_1} \asymp N^{\frac{1}{r}} \big(\log N\big)^{-\frac{b_1}{r}}. \tag{19}$$

Therefore, we can chose a number N so that $|Q(N)| \asymp M$ and the right-hand side of (19) will be at least twice as large as the right-hand side of (18). For some $y^* = (y_1^*, \dots, y_d^*)$, for this N we have

$$\|g_3(x-y^*) - Ag_3(x-y^*)\|_\infty \gg M. \tag{20}$$

Consider the function

$$g_4(x) = C_6 N^{-1} 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} g_3(x),\ C_6 > 0.$$

We now show that, at the corresponding choice of the constant C_6, the function g_4 belongs to the class $B^{\Omega}_{p,\theta}$.

Indeed, in view of the properties of the Fejér kernel, we have

$$\|g_4\|_{B^{\Omega}_{p,\theta}} = \bigg(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(g_4)\|_p^\theta \bigg)^{\frac{1}{\theta}} \ll N^{-1} 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} \Big(\Omega^{-\theta}(2^{-\tilde{s}}) \|A_{\tilde{s}}(g_3)\|_p^\theta \Big)^{\frac{1}{\theta}}$$

$$\ll 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} \|A_{\tilde{s}}(g_3)\|_p \asymp 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} 2^{||\tilde{s}||_1 \left(1-\frac{1}{p}\right)} = 1.$$

Hence, $g_4 \in B^{\Omega}_{p,\theta}$ with the corresponding constant $C_6 > 0$.

It was established in work [14] that for a trigonometric polynomial t with "numbers" of harmonics from the set $\rho(\tilde{s})$, the following relation is satisfied:

$$\|t\|_\infty \ll \|t\|_q 2^{\frac{\|\tilde{s}\|_1}{q}}.$$

Taking into account the last relation and using estimate (20), we get

$$\|g_4(x-y^*) - Gg_4(x-y^*)\|_q \gg N^{-1} 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} \|g_3(x-y^*) - Gg_3(x-y^*)\|_q$$

$$\gg N^{-1} 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} \|g_3(x-y^*) - Ag_3(x-y^*)\|_q$$

$$\gg N^{-1} 2^{||\tilde{s}||_1 \left(\frac{1}{p}-1\right)} 2^{-\frac{\|\tilde{s}\|_1}{q}} \|g_3(x-y^*) - Ag_3(x-y^*)\|_\infty$$

$$\gg M^{-r} \big(\log M\big)^{-b_1} M^{\frac{1}{p}-\frac{1}{q}-1} M = M^{-r+\frac{1}{p}-\frac{1}{q}} \big(\log M\big)^{-b_1}.$$

The lower bounds in (17) are established. Theorem 4 is proved. □

Remark 1. *Results, corresponding to Theorems 3 and 4, but for the classes $B^{\Omega}_{p,\theta}$ in the space L_∞, are obtained in [2].*

Remark 2. *The analogues of Theorems 3 and 4 for the classes H^{Ω}_{p} are obtained by N.N. Pustovoitov in [14]. Moreover, if the conditions of Theorem 4 are satisfied, the ordinal relations*

$$d^{\perp}_{M}(B^{\Omega}_{p,\theta}, L_q) \asymp d^{B}_{M}(B^{\Omega}_{p,\theta}, L_q) \asymp d^{\perp}_{M}(H^{\Omega}_{p}, L_q) \asymp d^{B}_{M}(H^{\Omega}_{p}, L_q)$$

hold. In other words, the orders of the quantities $d^{\perp}_{M}(B^{\Omega}_{p,\theta}, L_q)$ and $d^{B}_{M}(B^{\Omega}_{p,\theta}, L_q)$ are independent on the parameter θ.

REFERENCES

[1] Bari N.K., Stechkin S.B. *The best approximations and differential properties of two conjugate functions*. Trans. Moscow Math. Soc. 1956, **5**, 483–522. (in Russian)

[2] Fedunyk-Yaremchuk O.V., Solich K.V. *Estimates of approximative characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables with given majorant of mixed continuity moduli in the space L_∞*. J. Math. Sci. (N.Y.) 2018, **231** (1), 28–40. doi:10.1007/s10958-018-3803-3 (translation of Ukr. Mat. Visn. 2017, **14** (3), 345–360. (in Ukrainian))

[3] Jackson D. *Certain problem of closest approximation*. Bull. Amer. Math. Soc. 1933, **39**, 889–906.

[4] Konograi A.F. *Estimates of the approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables with given majorant of mixed moduli of continuity*. Math. Notes 2014, **95** (5), 656–189. doi: 10.1134/S0001434614050095 (translation of Mat. Zametki 2014, **95** (5), 734–749. doi:10.4213/mzm10118 (in Russian))

[5] Konograi A. F., Fedunyk-Yaremchuk O. V. *Estimates of approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables with a given majorant of the mixed moduli of continuity* . Approx. Theory of Functions and Related Problems: Proc. Inst. Math. NAS Ukr. 2013, **10** (1), 148–160. (in Ukrainian)

[6] Konograi A. F., Fedunyk-Yaremchuk O. V. *Estimates of orthoprojection widths of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables with given majorant of mixed moduli of continuity*. Approx. Theory of Functions and Related Problems: Proc. Inst. Math. NAS Ukr. 2014, **11** (3), 143–165. (in Ukrainian)

[7] Konograi A.F., Fedunyk-Yaremchuk O.V. *Estimates of orthogonal widths of the classes $B^{\Omega}_{\infty,\theta}$ of periodic functions of many variables with given majorant of mixed moduli of continuity*. Approx. Theory of Functions and Related Problems: Proc. Inst. Math. NAS Ukr. 2015, **12** (4), 205–215. (in Ukrainian)

[8] Lizorkin P. I., Nikol'skii S. M. *Spaces of functions with mixed smoothness from the decomposition point of view*. Proc. Steklov Inst. Math. 1990, **187**, 163–184. (translation of Tr. Mat. Inst. Steklova 1989, **187**, 143–161. (in Russian))

[9] Nikol'skii S.M. Approximation of Functions of Several Variables and Embedding Theorems. Nauka, Moscow, 1969. (in Russian)

[10] Nikol'skii S.M. *Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of severaly variables*. Tr. Mat. Inst. Steklova 1951, **38**, 244–278. (in Russian)

[11] Pustovoitov N.N. *Approximation of multidimensional functions with a given majorant of mixed moduli of continuity*. Math. Notes 1999, **65** (1), 89–98. doi:10.1007/BF02675013 (translation of Mat. Zametki 1999, **65** (1), 107–117. doi:10.4213/mzm1032 (in Russian))

[12] Pustovoitov N. N. *On the approximation and characterization of periodic functions of many variables, whose majorant of mixed continuity moduli has a special form*. Anal. Math. 2003, **29**, 201–218. doi:10.1023/A:1025415204826 (in Russian)

[13] Pustovoitov N.N. *Representation and approximation of periodic functions of several variables with given mixed modulus of continuity*. Anal. Math. 1994, **20**, 35–48. doi:10.1007/BF01908917 (in Russian)

[14] Pustovoitov N.N. *The orthowidths of classes of multidimensional periodic functions, for which the majorant of mixed continuity moduli contains power and logarithmic multipliers*. Anal. Math. 2008, **34**, 187–224. doi:10.1007/s10476-008-0303-6 (in Russian)

[15] Romanyuk A.S. *Approximation of classes of periodic functions of several variables*. Ukrainian Math. J. 1992, **44** (5), 596–606. doi:10.1007/BF01056698 (translation of Ukrainian Math. J. 1992, **44** (5), 662–672. (in Russian))

[16] Romanyuk A.S. *Best approximations and widths of classes of periodic functions of several variables*. Sb. Math. 2008, **199** (2), 253–275. doi:10.1070/SM2008v199n02ABEH003918 (translation of Mat. Sb. 2008, **199** (2), 93–114. doi:10.4213/sm3685 (in Russian))

[17] Romanyuk A. S. *Diameters and best approximation of the classes $B^{r}_{p,\theta}$ of periodic functions of several variables*. Anal. Math. 2011, **37**, 181–213. doi:10.1007/s10476-011-0303-9 (in Russian)

[18] Stasyuk S. A., Fedunyk O. V. *Approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables*. Ukrainian Math. J. 2006, **58** (5), 779–793. doi:10.1007/s11253-006-0101-x (translation of Ukrainian Math. J. 2006, **58** (5), 692–704. (in Ukrainian))

[19] Stasyuk S. A. *Approximation by Fourier sums and the Kolmogorov widths for the classes $MB^{\Omega}_{p,\theta}$ of periodic functions of several variables*. Tr. Inst. Mat. Mekh. 2014, **20** (1), 247–257. (in Russian)

[20] Stasyuk S. A. *Best approximations of periodic functions of several variables from the classes $B^{\Omega}_{p,\theta}$*. Math. Notes 2010, **87** (1-2), 102–114. doi:10.1134/S000143461001013X (translation of Mat. Zametki 2010, **87** (1), 108–121. doi:10.4213/mzm4053 (in Russian))

[21] Yongsheng S., Heping W. *Representation and approximation of multivariate periodic functions with bounded mixed moduli of smoothness*. Tr. Mat. Inst. Steklova 1997, **219**, 356–377.

[22] Temlyakov V.N. *Approximation of functions with bounded mixed derivative*. Proc. Steklov Inst. Math. 1989, **178**, 1–121. (translation of Tr. Mat. Inst. Steklova 1986, **178**, 3–113. (in Russian))

[23] Temlyakov V. N., *Diameters of some classes of functions of several variables*. Dokl. Akad. Nauk 1982, **267** (2), 314–317. (in Russian)

[24] Temlyakov V. N. *Estimates of the asymptotic characteristics of classes of functions with bounded mixed derivative or difference*. Proc. Steklov Inst. Math. 1990, **189**, 161–197. (translation of Tr. Mat. Inst. Steklova 1989, **189**, 138–168. (in Russian))

Received 29.08.2019

Федуник-Яремчук О.В., Гембарська С.Б. *Оцінки апроксимативних характеристик класів $B^{\Omega}_{p,\theta}$ періодичних функцій багатьох змінних із заданою мажорантою мішаних модулів неперервності у просторі L_q* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 281–295.

В роботі продовжується вивчення апроксимативних характеристик класів $B^{\Omega}_{p,\theta}$ періодичних функцій багатьох змінних, мажоранта мішаних модулів неперервності яких містить як степеневі, так і логарифмічні множники. Одержано точні за порядком оцінки ортопроекційних поперечників класів $B^{\Omega}_{p,\theta}$ у просторі L_q, $1 \leq p < q < \infty$, а також встановлено точні за порядком оцінки наближення цих класів функцій у просторі L_q за допомогою лінійних операторів, які підпорядковані певним умовам.

Ключові слова і фрази: ортопроекційний поперечник, мішаний модуль неперервності, лінійний оператор, ядро Валле-Пуссена, ядро Фейєра.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 296–310
doi:10.15330/cmp.11.2.296-310

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.296–310

GUTIK O.V., SAVCHUK A.S.

ON INVERSE SUBMONOIDS OF THE MONOID OF ALMOST MONOTONE INJECTIVE CO-FINITE PARTIAL SELFMAPS OF POSITIVE INTEGERS

In this paper we study submonoids of the monoid $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ of almost monotone injective co-finite partial selfmaps of positive integers $\mathbb{N}$. Let $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ be a submonoid of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ which consists of cofinite monotone partial bijections of $\mathbb{N}$ and $\mathscr{C}_{\mathbb{N}}$ be a subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ which is generated by the partial shift $n \mapsto n+1$ and its inverse partial map. We show that every automorphism of a full inverse subsemigroup of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_{\mathbb{N}}$ is the identity map. We construct a submonoid $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ with the following property: if S is an inverse submonoid of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ as a submonoid, then every non-identity congruence $\mathfrak{C}$ on S is a group congruence. We show that if S is an inverse submonoid of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid then S is simple and the quotient semigroup $S/\mathfrak{C}_{\mathbf{mg}}$, where $\mathfrak{C}_{\mathbf{mg}}$ is the minimum group congruence on S, is isomorphic to the additive group of integers. Also, we study topologizations of inverse submonoids of $\mathscr{I}_{\infty}^{\Rsh\nearrow}(\mathbb{N})$ which contain $\mathscr{C}_{\mathbb{N}}$ and embeddings of such semigroups into compact-like topological semigroups.

Key words and phrases: inverse semigroup, isometry, partial bijection, congruence, bicyclic semigroup, semitopological semigroup, topological semigroup, discrete topology, embedding, Bohr compactification.

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1 INTRODUCTION AND PRELIMINARIES

In this paper all spaces will be assumed to be Hausdorff. Furthermore we shall follow the terminology of [14, 16, 20, 35, 39]. We shall denote the set of all positive integers by $\mathbb{N}$, the first infinite ordinal by ω and the cardinality of the set A by $|A|$. If A is a subset of a semigroup S, then by $\langle A\rangle$ we shall denote a subsemigroup of S generated by the elements of the set A.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruences on S, and a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. If $\mathfrak{C}$ is a congruence on a semigroup S then by $\mathfrak{C}^{\sharp}$ we denote the natural homomorphism from S onto the quotient semigroup $S/\mathfrak{C}$.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a as

УДК 512.534
2010 *Mathematics Subject Classification:* 20M18, 20M20, 20M30, 22A15, 54D40, 54D45, 54H10.

band (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

An inverse subsemigroup T of an inverse semigroup S is called *full* if $E(S) = E(T)$.

By $(\mathscr{P}_{<\omega}(\lambda), \cup)$ we shall denote the *free semilattice with identity* over a set of cardinality $\lambda \geqslant \omega$, i.e., $(\mathscr{P}_{<\omega}(\lambda), \cup)$ is the set of all finite subsets (with the empty set) of λ with the semilattice operation "union".

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [16]). A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

A (*semi*)*topological semigroup* is a topological space with a (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology τ on a semigroup S is called:
semigroup if (S, τ) is a topological semigroup;
semigroup inverse if S is an inverse semigroup and (S, τ) is a topological inverse semigroup;
shift-continuous if (S, τ) is a semitopological semigroup.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q, subject only to the condition $pq = 1$.

The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p, q)$ under h is a cyclic group (see [16, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example a well-known Andersen's result [1] states that a (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain an isomorphic copy of the bicyclic semigroup. The bicyclic monoid admits only the discrete semigroup Hausdorff topology. Bertman and West in [13] extended this result for the case of Hausdorff semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic monoid [3, 33]. The problem of embedding of the bicyclic monoid into compact-like topological semigroups was studied in [5, 6, 28]. Independently to Eberhart-Selden results on topolozabilty of the bicyclic semigroup, in [41] Taimanov constructed a commutative semigroup $\mathfrak{A}_\kappa$ of cardinality κ which admits only the discrete semigroup topology. Also, Taimanov [42] gave sufficient conditions for a commutative semigroup to have a non-discrete semigroup topology. In the paper [23] it was showed that for the Taimanov semigroup $\mathfrak{A}_\kappa$ from [41] the following conditions hold: every T_1-topology τ on the semigroup $\mathfrak{A}_\kappa$ such that $(\mathfrak{A}_\kappa, \tau)$ is a topological semigroup is discrete; $\mathfrak{A}_\kappa$ is closed in any T_1-topological semigroup containing $\mathfrak{A}_\kappa$ and every homomorphic non-isomorphic image of $\mathfrak{A}_\kappa$ is a zero-semigroup.

Non-discrete topologizations of some bicyclic-like semigroups were studied in [7, 8, 9, 10, 11, 12, 22, 25, 34, 36, 40]. In particular in [21] it is proved that the discrete topology is the unique shift-continuous Hausdorff topology on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$. We observe that for many (0-)bisimple semigroups S the following statement holds: *every shift-continuous Hausdorff Baire (in particular locally compact) topology on S is discrete* (see [15, 24, 26, 27, 29, 30]).

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of a set X of cardinality λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}, \qquad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [16]). The symmetric inverse semigroup was introduced by Wagner [43] and it plays a major role in the theory of semigroups.

Remark 1. *We observe that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}$, which is generated by partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows:*

$$\operatorname{dom}\alpha = \mathbb{N}, \quad \operatorname{ran}\alpha = \mathbb{N}\setminus\{1\}, \quad (n)\alpha = n+1$$

and

$$\operatorname{dom}\beta = \mathbb{N}\setminus\{1\}, \quad \operatorname{ran}\beta = \mathbb{N}, \quad (n)\beta = n-1$$

(see Exercise IV.1.11(ii) in [38]).

Let $\mathbb{N}$ be the set of all positive integers. We shall denote the semigroup of monotone, non-decreasing, injective partial transformations φ of $\mathbb{N}$ such that the sets $\mathbb{N}\setminus\operatorname{dom}\varphi$ and $\mathbb{N}\setminus\operatorname{rank}\varphi$ are finite by $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$. The semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is called *the semigroup of cofinite monotone partial bijections* of $\mathbb{N}$.

In [29] Gutik and Repovš studied the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. They showed that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also, they proved that every locally compact inverse semigroup topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is discrete and described the closure of $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ in a topological semigroup.

Doroshenko in [18, 19] studied the semigroups of endomorphisms of linearly ordered sets $\mathbb{N}$ and $\mathbb{Z}$ and their subsemigroups of cofinite endomorphisms $\mathcal{O}_{fin}(\mathbb{N})$ and $\mathcal{O}_{fin}(\mathbb{Z})$. In [19] he described the Green relations, groups of automorphisms, conjugacy, centralizers of elements, growth, and free subsemigroups in these subgroups. Especially in [19] it is proved that the group of automorphisms consists only of the identity mapping, whereas the groups of automorphisms of $\mathcal{O}_{fin}(\mathbb{Z})$ is isomorphic to the semigroup of integers with operation of addition and consist only of inner automorphisms. In [18] there was shown that both these semigroups do not admit an irreducible system of generators. In their subsemigroups of cofinite functions all irreducible systems of generators are described there. Also, here the last semigroups are presented in terms of generators and relations.

A partial map $\alpha\colon \mathbb{N} \rightharpoonup \mathbb{N}$ is called *almost monotone* if there exists a finite subset A of $\mathbb{N}$ such that the restriction $\alpha\,|_{\mathbb{N}\setminus A}\colon \mathbb{N}\setminus A \rightharpoonup \mathbb{N}$ is a monotone partial map.

By $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ we shall denote the semigroup of monotone, almost non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N}\setminus\operatorname{dom}\varphi$ and $\mathbb{N}\setminus\operatorname{rank}\varphi$ are finite for all $\varphi\in\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$ and the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ too. The semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is called *the semigroup of co-finite almost monotone partial bijections* of $\mathbb{N}$.

In the paper [15] the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is studied. It was shown that the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also it was proved that every Baire shift-continuous T_1-topology τ on $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ is discrete, described the closure of $(\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N}),\tau)$ in a topological semigroup and constructed non-discrete Hausdorff semigroup topologies on $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$.

A partial transformation $\alpha\colon (X,d) \rightharpoonup (X,d)$ of a metric space (X,d) is called *isometric* or a *partial isometry*, if $d(x\alpha, y\alpha) = d(x,y)$ for all $x, y \in \operatorname{dom}\alpha$. It is obvious that the composition of two partial isometries of a metric space (X,d) is a partial isometry, and the converse partial map to a partial isometry is a partial isometry. Hence the set of partial isometries of a metric space (X,d) with the operation of composition of partial isometries is an inverse submonoid of the symmetric inverse monoid over the set X.

Let $\mathbf{I}\mathbb{N}_\infty$ be the set of all partial cofinite isometries of the set of positive integers $\mathbb{N}$ with the usual metric $d(n,m) = |n-m|$, $n,m \in \mathbb{N}$. Then $\mathbf{I}\mathbb{N}_\infty$ with the operation of composition of partial isometries is an inverse submonoid of $\mathscr{I}_\omega$. The semigroup $\mathbf{I}\mathbb{N}_\infty$ of all partial co-finite isometries of positive integers is studied in [32]. There we describe the Green relations on the semigroup $\mathbf{I}\mathbb{N}_\infty$, its band and proved that $\mathbf{I}\mathbb{N}_\infty$ is a simple E-unitary F-inverse semigroup. Also in [32], the least group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathbf{I}\mathbb{N}_\infty$ is described and proved that the quotient-semigroup $\mathbf{I}\mathbb{N}_\infty/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. An example of a non-group congruence on the semigroup $mathbf{I}\mathbb{N}_\infty$ is presented. Also we proved that a congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is group if and only if its restriction onto an isomorphic copy of the bicyclic semigroup in $\mathbf{I}\mathbb{N}_\infty$ is a group congruence.

In this paper we show that every automorphism of a full inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_\mathbb{N}$ is the identity map. We construct a submonoid $\mathbf{I}\mathbb{N}_\infty^{[1]}$ of $\mathscr{I}_\infty^{\,\mathrel{\overset{\to}{\nearrow}}}(\mathbb{N})$ with the following property: if S be an inverse subsemigroup of $\mathscr{I}_\infty^{\,\mathrel{\overset{\to}{\nearrow}}}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_\infty^{[1]}$ as a submonoid, then every non-identity congruence $\mathfrak{C}$ on S is a group congruence. We show that if S is an inverse submonoid of $\mathscr{I}_\infty^{\,\mathrel{\overset{\to}{\nearrow}}}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a subsubmonoid then S is simple and the quotient semigroup $S/\mathfrak{C}_{\mathbf{mg}}$, where $\mathfrak{C}_{\mathbf{mg}}$ is the minimum group congruence on S, is isomorphic to the additive group of integers. Also, we study topologizations of inverse submonoids of $\mathscr{I}_\infty^{\,\mathrel{\overset{\to}{\nearrow}}}(\mathbb{N})$ which contain $\mathscr{C}_\mathbb{N}$ and embeddings of such semigroups into compact-like topological semigroups.

2 MAIN ALGEBRAIC RESULTS

We recall for a semigroup S a homomorphism $\Phi\colon S \to S$ is called an *endomorphism of S* and every bijective endomorphism (isomorphism) $\Phi\colon S \to S$ is called an *automorphism of S*. We observe that in the case when S is a monoid with the unit 1_S, then an endomorphism $\Phi\colon S \to S$ with $(1_S)\Phi = 1_S$ is called a *monoid endomorphism*. It is obvious that $(1_S)\Phi = 1_S$ for any automorphism $\Phi\colon S \to S$ of a monoid with the unit 1_S.

Recall [37] a semigroup S is combinatorial if it has no non-trivial subgroups. A regular (an inverse) semigroup S is combinatorial if all its $\mathscr{H}$-classes are singleton. It is obvious that any subsemigroup of a combinatorial semigroup is combinatorial.

Lemma 1. *Let $\Psi\colon S \to S$ be an automorphism of a combinatorial inverse semigroup S. If $(e)\Psi = e$ for all $e \in E(S)$, then Ψ is the identity map.*

Proof. Fix an arbitrary $s \in S \setminus E(S)$. Then $(ss^{-1})\Psi = ss^{-1}$ and $(s^{-1}s)\Psi = s^{-1}s$. Since in any inverse semigroup the following condition hold: $x\mathscr{H}y$ if and only if $xx^{-1} = yy^{-1}$ and $x^{-1}x = y^{-1}y$ (see [35, Section 3.2, p. 82]), we have that

$$(s)\Psi(s^{-1})\Psi = (ss^{-1})\Psi = ss^{-1} \qquad \text{and} \qquad (s^{-1})\Psi(s)\Psi = (s^{-1}s)\Psi = s^{-1}s,$$

and hence $(s)\Psi\mathscr{H}s$. Since S is a combinatorial inverse semigroup, $(s)\Psi = s$. $\square$

For any positive integer i by $\varepsilon(i)$ we denote the identity map of the set $\mathbb{N} \setminus \{i\}$. It is obvious that $\varepsilon(i) \in E(\mathbf{I}\mathbb{N}_\infty)$ for any positive integer i.

Lemma 2. *Let S be a full inverse submonoid of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\Phi\colon S \to S$ be an automorphism. Then $(\varepsilon(1))\Phi = \varepsilon(1)$.*

Proof. Since $\Phi\colon S \to S$ is an automorphism, $(\mathbb{I})\Phi = \mathbb{I}$. Suppose to the contrary that $(\varepsilon(1))\Phi \neq \varepsilon(1)$. Since the restriction $\Phi|_{E(S)\setminus\{\mathbb{I}\}}\colon E(S) \setminus \{\mathbb{I}\} \to E(S) \setminus \{\mathbb{I}\}$ of the automorphism Φ onto $E(S) \setminus \{\mathbb{I}\}$ is an automorphism, there exist (not necessary distinct) idempotents $\iota, \upsilon \in S \setminus \{\mathbb{I}, \varepsilon(1)\}$ such that $(\varepsilon(1))\Phi = \upsilon$, $(\iota)\Phi = \varepsilon(1)$ and $|\mathbb{N} \setminus \operatorname{dom} \upsilon| = |\mathbb{N} \setminus \operatorname{dom} \iota| = 1$.

We shall show that $1 \in \operatorname{dom} \varphi \cap \operatorname{ran} \varphi$ and moreover $(1)\varphi = 1$ for any $\varphi \in \langle(\alpha)\Phi, (\beta)\Phi\rangle$. Our assumption implies that $\varepsilon(1)) = \beta\alpha$ and hence

$$(1)(\beta\alpha)\Phi = 1 = (1)(\alpha\beta)\Psi = (1)(\mathbb{I})\Phi = (1)\mathbb{I} = 1.$$

This implies that $1 \in \operatorname{dom}(\alpha)\Phi$ and $1 \in \operatorname{dom}(\beta)\Phi$. If $(1)(\beta)\Phi \neq 1$, then the monotonicity of β implies that $1 \notin \operatorname{dom}(\alpha)\Phi$, and hence $1 \notin \operatorname{dom}(\alpha\beta)\Phi = \mathbb{N}$, a contradiction. Since α is inverse of β in S, the equality $(1)(\beta)\Phi = 1$ implies that $1 = (1)(\beta\alpha)\Phi = ((1)(\beta)\Phi)(\alpha)\Psi = (1)(\alpha)\Phi$. This implies that $(1)(\beta^i\alpha^j)\Phi = 1$ for all non-negative integers i and j.

By Remark 1, $\langle\alpha, \beta\rangle$ is a submonoid of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ which is isomorphic to the bicyclic monoid, and since $\Phi\colon S \to S$ is an automorphism, $\langle(\alpha)\Phi, (\beta)\Phi\rangle$ is isomorphic to the bicyclic monoid, too. By Lemma 2.6 of [29] for every idempotent $\varepsilon \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ there exists a positive integer n_ε such that $\varepsilon \cdot \beta^n\alpha^n = \beta^n\alpha^n$ for any positive integer $n \geqslant n_\varepsilon$. Then there exists a positive integer n_ι such that $\iota\beta^n\alpha^n = \beta^n\alpha^n$ and hence $(\iota\beta^n\alpha^n)\Phi = (\beta^n\alpha^n)\Phi$ for all $n \geqslant n_\iota$. Since $(\iota)\Phi = \beta\alpha$ we have that $(\iota\beta^n\alpha^n)\Phi = (\iota)\Phi(\beta^n\alpha^n)\Phi = \varepsilon(1)(\beta^n\alpha^n)\Phi$ and hence $1 \notin \operatorname{dom} \beta\alpha$ for all $n \geqslant n_\iota$. This contradicts the previous part of the proof. The obtained contradiction implies the statement of the lemma. □

Lemma 3. *Let S be a full inverse submonoid of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\Phi\colon S \to S$ be an automorphism. Then $(\beta^i\alpha^j)\Phi = \beta^i\alpha^j$ for all non-negative integers i and j.*

Proof. By Lemma 2, $(\beta\alpha)\Phi = (\varepsilon(1))\Phi = \varepsilon(1) = \beta\alpha$ and since $(\mathbb{I})\Phi = \mathbb{I}$, we have that

$$(\beta)\Phi(\alpha)\Phi = \beta\alpha \qquad \text{and} \qquad (\alpha)\Phi(\beta)\Phi = \mathbb{I}.$$

By Proposition 2.1(iii) from [29] the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is combinatorial and hence S is combinatorial, too. Then the arguments presented in the proof of Lemma 1 imply that $(\beta)\Phi = \beta$ and $(\alpha)\Phi = \alpha$. Therefore we get

$$(\beta^i\alpha^j)\Phi = (\beta^i)\Phi(\alpha^j)\Phi = ((\beta)\Phi)^i((\alpha)\Phi)^j = \beta^i\alpha^j$$

for all non-negative integers i and j. □

Lemma 4. *Let S be a full inverse submonoid of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\Phi\colon S \to S$ be an automorphism. Then $(\varepsilon)\Phi = \varepsilon$ for each idempotent $\varepsilon \in S$.*

Proof. Since the restriction $\Phi|_{E(S)\setminus\{\mathbb{I}\}}\colon E(S) \setminus \{\mathbb{I}\} \to E(S) \setminus \{\mathbb{I}\}$ of Φ onto $E(S) \setminus \{\mathbb{I}\}$ is an automorphism, the equality $(\iota)\Phi = \upsilon$ for $\iota, \upsilon \in E(S) \setminus \{\mathbb{I}, \varepsilon(1)\}$ implies that $|\mathbb{N} \setminus \operatorname{dom} \upsilon| = |\mathbb{N} \setminus \operatorname{dom} \iota|$. Fix so elements $\iota, \upsilon \in E(S) \setminus \{\mathbb{I}, \varepsilon(1)\}$ with $|\mathbb{N} \setminus \operatorname{dom} \upsilon| = |\mathbb{N} \setminus \operatorname{dom} \iota| = 1$. Then there exist positive integers k and l such that $\upsilon = \varepsilon(k)$ and $\iota = \varepsilon(l)$. Suppose to the contrary that $\iota \neq \upsilon$. If $k > l > 1$ then,

$$\beta^l\alpha^l = (\beta^l\alpha^l)\Phi = (\beta^l\alpha^l \cdot \varepsilon(l))\Phi = \beta^l\alpha^l \cdot (\varepsilon(l))\Phi = \beta^l\alpha^l \cdot (\varepsilon(l))\Phi = \beta^l\alpha^l \cdot \varepsilon(k) \neq \beta^l\alpha^l.$$

If $l > k > 1$, then

$$\begin{aligned}\beta^k\alpha^k = (\beta^k\alpha^k)\Phi^{-1} &= (\beta^k\alpha^k \cdot \varepsilon(k))\Phi^{-1} = \beta^k\alpha^k \cdot (\varepsilon(k))\Phi^{-1}\\ &= \beta^k\alpha^k \cdot (\varepsilon(k))\Phi^{-1} = \beta^k\alpha^k \cdot \varepsilon(l) \neq \beta^k\alpha^k.\end{aligned}$$

The obtained contradictions and Lemma 3 imply that $(\iota)\Phi = \iota$ for every $\iota \in E(S)$ with $|\mathbb{N} \setminus \operatorname{dom}\iota| = 1$.

By Proposition 2.1 of [29] for every idempotent $\varepsilon \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ there exists a finite subset $\{n_1, \ldots, n_k\}$ of positive integers such that ε is the identity map of $\mathbb{N} \setminus \{n_1, \ldots, n_k\}$. This implies that $\varepsilon = \varepsilon(n_1)\cdots\varepsilon(n_k)$. Hence we get that

$$(\varepsilon)\Phi = (\varepsilon(n_1)\cdots\varepsilon(n_k))\Phi = (\varepsilon(n_1))\Phi\cdots(\varepsilon(n_k))\Phi = \varepsilon(n_1)\cdots\varepsilon(n_k) = \varepsilon,$$

which completes the proof of the lemma. □

It is well known that every automorphism Φ of the bicyclic semigroup $\mathscr{C}(p,q)$ is trivial. i.e., Φ is the identity map of $\mathscr{C}(p,q)$. The following theorem shows that every full inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_{\mathbb{N}}$ has such property.

Theorem 1. *Let S be a full inverse submonoid of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_{\mathbb{N}}$. Then every automorphism of S is the identity map.*

Proof. By Lemma 4 for each automorphism $\Phi\colon S \to S$ the band $E(\mathscr{I}_\infty^{\nearrow}(\mathbb{N}))$ is the set of fixed points of Φ. By Proposition 2.1 of [29], $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is combinatorial inverse semigroup, and hence by Proposition 3.2.11 of [35] so is S. Next we apply Lemma 1. □

Theorem 1 implies the following two corollaries.

Corollary 1. *Every automorphism of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is trivial.*

Corollary 2. *Every automorphism of the semigroup $\mathbf{I}\mathbb{N}_\infty$ is trivial.*

Remark 2. *By Lemma 1.1 from [15] the band of the monoid $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is isomorphic to the free semilattice $(\mathscr{P}_{<\omega}(\omega), \cup)$. Next we identify $\mathbb{N}$ with ω. Then every bijective transformation of $\mathbb{N}$ extends to an automorphism of the free semilattice $(\mathscr{P}_{<\omega}(\omega), \cup)$. This implies that the monoid $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ contains a full inverse subsemigroup which has $\mathfrak{c}$ distinct automorphisms.*

An example of a non-group congruence on the semigroup $\mathbf{I}\mathbb{N}_\infty$ is presented in [32]. Later we shall establish what submonoids of $\mathscr{I}_\infty^{\,\Rsh\!\nearrow}(\mathbb{N})$ admit only a group non-identity congruence.

For an arbitrary positive integer n_0 we denote $[n_0) = \{n \in \mathbb{N} \colon n \geqslant n_0\}$. Since the set of all positive integers is well ordered, the definition of the semigroup $\mathscr{I}_\infty^{\,\Rsh\!\nearrow}(\mathbb{N})$ implies that for every $\alpha \in \mathscr{I}_\infty^{\,\Rsh\!\nearrow}(\mathbb{N})$ there exists the smallest positive integer $n_\alpha^{\mathbf{d}} \in \operatorname{dom}\alpha$ such that the restriction $\alpha|_{[n_\alpha^{\mathbf{d}})}$ of the partial map $\alpha\colon \mathbb{N} \rightharpoonup \mathbb{N}$ onto the set $[n_\alpha^{\mathbf{d}})$ is an element of the semigroup $\mathscr{C}_{\mathbb{N}}$, i.e., $\alpha|_{[n_\alpha^{\mathbf{d}})}$ is a some partial shift of $[n_\alpha^{\mathbf{d}})$. For every $\alpha \in \mathscr{I}_\infty^{\,\Rsh\!\nearrow}(\mathbb{N})$ we put $\overrightarrow{\alpha} = \alpha|_{[n_\alpha^{\mathbf{d}})}$, i.e.

$$\operatorname{dom}\overrightarrow{\alpha} = \left[n_\alpha^{\mathbf{d}}\right), \qquad (x)\overrightarrow{\alpha} = (x)\alpha \quad \text{for all} \quad x \in \operatorname{dom}\overrightarrow{\alpha} \qquad \text{and} \qquad \operatorname{ran}\overrightarrow{\alpha} = \left(\operatorname{dom}\overrightarrow{\alpha}\right)\alpha.$$

Also, we put

$$\underline{n}_{\alpha}^{\mathbf{d}} = \min\{j \in \mathbb{N} \colon j \in \operatorname{dom}\alpha\} \qquad \text{for} \quad \alpha \in \mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N}),$$

and

$$\overline{n}_{\alpha}^{\mathbf{d}} = \max\left\{j \in \operatorname{dom}\alpha \colon j < n_{\alpha}^{\mathbf{d}}\right\} \qquad \text{for} \quad \alpha \in \mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N}) \setminus \mathscr{C}_{\mathbb{N}}.$$

It is obvious that $\underline{n}_{\alpha}^{\mathbf{d}} \leqslant n_{\alpha}^{\mathbf{d}}$ when $\alpha \in \mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ and $\underline{n}_{\alpha}^{\mathbf{d}} \leqslant \overline{n}_{\alpha}^{\mathbf{d}} < n_{\alpha}^{\mathbf{d}}$ when $\alpha \in \mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N}) \setminus \mathscr{C}_{\mathbb{N}}$.

The following theorem is proved in [32].

Theorem 2 ([32, Theorem 9])**.** *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathbf{I}\mathbb{N}_{\infty}$. Then the following conditions are equivalent:*

(1) *$\mathfrak{C}$ is a group congruence;*

(2) *there exists a subsemigroup S of $\mathbf{I}\mathbb{N}_{\infty}$ which is isomorphic to the bicyclic semigroup and S contains two distinct $\mathfrak{C}$-equivalent elements;*

(3) *every subsemigroup of $\mathbf{I}\mathbb{N}_{\infty}$, which is isomorphic to the bicyclic semigroup, has two distinct $\mathfrak{C}$-equivalent elements.*

The following lemma completes the statements of Theorem 2.

Lemma 5. *Let $\mathfrak{C}$ be a congruence on the semigroup $\mathbf{I}\mathbb{N}_{\infty}$, $\varepsilon \in E(\mathscr{C}_{\mathbb{N}})$, $\iota \in E(\mathbf{I}\mathbb{N}_{\infty}) \setminus E(\mathscr{C}_{\mathbb{N}})$ and $\iota \leqslant \varepsilon$. Then $\varepsilon\mathfrak{C}\iota$ implies that $\mathfrak{C}$ is a group congruence on $\mathbf{I}\mathbb{N}_{\infty}$.*

Proof. The assumptions of the lemma imply that $n_{\iota}^{\mathbf{d}} < n_{\varepsilon}^{\mathbf{d}}$. Put $\varepsilon_{n_{\iota}^{\mathbf{d}}+1} \colon \mathbb{N} \rightharpoonup \mathbb{N}$ and $\varepsilon_{n_{\iota}^{\mathbf{d}}} \colon \mathbb{N} \rightharpoonup \mathbb{N}$ are identity maps of the sets $\left[n_{\iota}^{\mathbf{d}}+1\right)$ and $\left[n_{\iota}^{\mathbf{d}}\right)$, respectively. It is obvious that $\varepsilon_{n_{\iota}^{\mathbf{d}}+1}, \varepsilon_{n_{\iota}^{\mathbf{d}}} \in E(\mathscr{C}_{\mathbb{N}})$,

$$\varepsilon_{n_{\iota}^{\mathbf{d}}} = \varepsilon_{n_{\iota}^{\mathbf{d}}} \cdot \varepsilon_{n_{\iota}^{\mathbf{d}}+1} = \varepsilon_{n_{\iota}^{\mathbf{d}}} \cdot \iota = \varepsilon_{n_{\iota}^{\mathbf{d}}+1} \cdot \iota \qquad \text{and} \qquad \varepsilon_{n_{\iota}^{\mathbf{d}}+1} = \varepsilon_{n_{\iota}^{\mathbf{d}}+1} \cdot \varepsilon,$$

and hence $\varepsilon_{n_{\iota}^{\mathbf{d}}+1}\mathfrak{C}\varepsilon_{n_{\iota}^{\mathbf{d}}}$. Then Theorem 2 and Corollary 1.32 [16] imply that $\mathfrak{C}$ is a group congruence on $\mathbf{I}\mathbb{N}_{\infty}$. $\square$

Definition 1. *Put* $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}} = \left\{\alpha \in \mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N}) \colon \text{ the restriction } \alpha|_{\operatorname{dom}\alpha \setminus \{\underline{n}_{\alpha}^{\mathbf{d}}\}} \text{ is a partial isometry of } \mathbb{N}\right\}$.

It is obvious that $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ is an inverse submonoid of the inverse monoid $\mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N})$, $\mathbf{I}\mathbb{N}_{\infty}$ is an inverse submonoid of $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ and $E(\mathbf{I}\mathbb{N}_{\infty}) = E(\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}) = E(\mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N})) = E(\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N}))$.

Lemma 6. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ as a submonoid. Let $\mathfrak{C}$ be a congruence on S such that two distinct idempotents ε and ι of $\mathbf{I}\mathbb{N}_{\infty}^{\boxed{1}}$ are $\mathfrak{C}$-equivalent. Then $\mathfrak{C}$ is a group congruence on S.*

Proof. If ε and ι are idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}$ of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$, then the statement of our lemma follows from Theorem 2. Hence, we assume that at least one of idempotents ε and ι does not belong to $\mathscr{C}_{\mathbb{N}}$.

We consider two cases: 1) $n_{\varepsilon}^{\mathbf{d}} = n_{\iota}^{\mathbf{d}}$; and 2) $n_{\varepsilon}^{\mathbf{d}} \neq n_{\iota}^{\mathbf{d}}$.

Suppose case $n_{\varepsilon}^{\mathbf{d}} = n_{\iota}^{\mathbf{d}}$ holds. Since $\varepsilon \neq \iota$ without loss of generality we may assume that there exists a positive integer $n_0 < n_{\varepsilon}^{\mathbf{d}}$ such that $n_0 \in \operatorname{dom}\varepsilon \setminus \operatorname{dom}\iota$. Then $n_0 = n_{\varepsilon}^{\mathbf{d}} - (k+1)$ for some positive integer k.

For every positive integer $j < n_{\varepsilon}^{\mathbf{d}} - 1$ we define a partial bijection $\alpha_j \colon \mathbb{N} \rightharpoonup \mathbb{N}$ in the following way:

$$\operatorname{dom}\alpha_j = \{j\} \cup \left\{n \in \mathbb{N} \colon n \geqslant n_{\varepsilon}^{\mathbf{d}}\right\}, \qquad \operatorname{ran}\alpha_j = \{j+1\} \cup \left\{n \in \mathbb{N} \colon n \geqslant n_{\varepsilon}^{\mathbf{d}}\right\}$$

and

$$(n)\alpha_j = \begin{cases} n, & \text{if } n \geqslant n_{\varepsilon}^{\mathbf{d}}; \\ n+1, & \text{if } n = j. \end{cases}$$

Simple verifications show that

$$\varepsilon_{n_{\varepsilon}^{\mathbf{d}}-1} = \alpha_{n_{\varepsilon}^{\mathbf{d}}-2}^{-1} \cdots \alpha_{n_0+1}^{-1}\alpha_{n_0}^{-1}\varepsilon\alpha_{n_0}\alpha_{n_0+1}\cdots\alpha_{n_{\varepsilon}^{\mathbf{d}}-2}$$

and

$$\varepsilon_{n_{\varepsilon}^{\mathbf{d}}} = \alpha_{n_0}^{-1}\iota\alpha_{n_0} = \alpha_{n_0+1}^{-1}\alpha_{n_0}^{-1}\iota\alpha_{n_0}\alpha_{n_0+1} = \ldots = \alpha_{n_{\varepsilon}^{\mathbf{d}}-2}^{-1}\cdots\alpha_{n_0+1}^{-1}\alpha_{n_0}^{-1}\iota\alpha_{n_0}\alpha_{n_0+1}\cdots\alpha_{n_{\varepsilon}^{\mathbf{d}}-2}$$

are identity maps of the sets $\{n \in \mathbb{N} \colon n \geqslant n_{\varepsilon}^{\mathbf{d}} - 1\}$ and $\{n \in \mathbb{N} \colon n \geqslant n_{\varepsilon}^{\mathbf{d}}\}$, respectively, and hence $\varepsilon_{n_{\varepsilon}^{\mathbf{d}}-1}$ and $\varepsilon_{n_{\varepsilon}^{\mathbf{d}}}$ are distinct $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. By Theorem 2 all idempotents of the sebsemigroup $\mathbf{I}\mathbb{N}_{\infty}$ are $\mathfrak{C}$-equivalent, and hence $\mathfrak{C}$ is a group congruence on the semigroup S, because $E(\mathbf{I}\mathbb{N}_{\infty}) = E(S) = E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))$.

Suppose case $n_{\varepsilon}^{\mathbf{d}} \neq n_{\iota}^{\mathbf{d}}$ holds. Without loss of generality we may assume that $n_{\varepsilon}^{\mathbf{d}} > n_{\iota}^{\mathbf{d}}$. Put $\varepsilon_{n_{\iota}^{\mathbf{d}}-1} \colon \mathbb{N} \rightharpoonup \mathbb{N}$ is the identity map of the set $\{n \in \mathbb{N} \colon n \geqslant n_{\varepsilon}^{\mathbf{d}} - 1\}$. Simple verifications show that $\varepsilon_{n_{\iota}^{\mathbf{d}}-1} = \varepsilon_{n_{\iota}^{\mathbf{d}}-1}\varepsilon$ and $\overrightarrow{\iota} = \varepsilon_{n_{\iota}^{\mathbf{d}}-1}\iota$ are distinct $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{C}_{\mathbb{N}}$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. By Theorem 2 all idempotents of the sebsemigroup $\mathbf{I}\mathbb{N}_{\infty}$ are $\mathfrak{C}$-equivalent, and hence $\mathfrak{C}$ is a group congruence on the semigroup S, because $E(\mathbf{I}\mathbb{N}_{\infty}) = E(S) = E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))$. □

Theorem 3. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{[\underline{1}]}$ as a submonoid. Then every non-identity congruence $\mathfrak{C}$ on S is a group congruence.*

Proof. Let α and β be two distinct $\mathfrak{C}$-equivalent elements of the semigroup S.

We consider two cases:

(i) $\alpha\mathscr{H}\beta$ in S;

(ii) α and β belong to distinct two $\mathscr{H}$-classes in S.

Suppose that $\alpha\mathscr{H}\beta$ in S. Then Proposition 1.1(ix) of [15] and Proposition 3.2.11 of [35] imply that $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$, and hence there exists a positive integer $n_0 \in \operatorname{dom}\alpha$ such that $(n_0)\alpha \neq (n_0)\beta$. Let $\varepsilon_{n_0} \colon \mathbb{N} \rightharpoonup \mathbb{N}$ be the identity map of the set $\{n_0\} \cup \{n \in \mathbb{N} \colon n \geqslant m_0\}$, where $m_0 \in \operatorname{dom}\alpha$ is an arbitrary positive integer such that $m_0 \geqslant n_0 + n_{\alpha}^{\mathbf{d}}$. By Proposition 3$(i)$ of [32] and Proposition 3(i) of [15], $E(\mathbf{I}\mathbb{N}_{\infty}) = E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))$ and hence $\varepsilon_{n_0} \in E(S)$. Since S is an inverse semigroup Proposition 2.3.4 from [35] and $\alpha\mathfrak{C}\beta$ imply that $\alpha^{-1}\mathfrak{C}\beta^{-1}$, and hence we have that $(\alpha^{-1}\varepsilon_{n_0}\alpha)\mathfrak{C}(\beta^{-1}\varepsilon_{n_0}\beta)$. Then the definition of ε_{n_0} implies that $\alpha^{-1}\varepsilon_{n_0}\alpha$ and $\beta^{-1}\varepsilon_{n_0}\beta$ are distinct idempotents of the semigroup S, and hence by Lemma 6, $\mathfrak{C}$ is a group congruence on S.

If case (ii) holds then at least one of the following conditions holds

$$\alpha\alpha^{-1} \neq \beta\beta^{-1} \qquad \text{or} \qquad \alpha^{-1}\alpha \neq \beta^{-1}\beta.$$

Then by Proposition 2.3.4 of [35] the semigroup S has two distinct $\mathfrak{C}$-equivalent idempotents. Next we apply Lemma 6. □

Every inverse semigroup S admits the *least group congruence* $\mathfrak{C}_{\mathbf{mg}}$ (see [38, Section III]):

$$s\mathfrak{C}_{\mathbf{mg}}t \qquad \text{if and only if there exists an idempotent} \quad e \in S \quad \text{such that} \quad se = te.$$

Later we shall describe the least group congruence on any inverse subsemigroup S of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid.

Definitions of inverse semigroups $\mathscr{C}_{\mathbb{N}}$, $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ and the congruence $\mathfrak{C}_{\mathbf{mg}}$ imply the following lemma.

Lemma 7. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid. Then the following conditions hold:*

(i) *$\alpha\mathfrak{C}_{\mathbf{mg}}\overrightarrow{\alpha}$ for every $\alpha \in S$;*

(ii) *if α and β are elements of S such that $\alpha = \overrightarrow{\alpha}$ and $\beta = \overrightarrow{\beta}$, then $\alpha\mathfrak{C}_{\mathbf{mg}}\beta$ if and only if $(n)\alpha = (n)\beta$ for all $n \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$.*

Theorem 4. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid. Then the quotient semigroup $S/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$.*

Proof. We define a map $\mathfrak{F}\colon S \to \mathbb{Z}(+)$, $\alpha \mapsto \mathrm{i}_{\alpha}$ in the following way. Put $\mathrm{i}_{\alpha} = (n)\overrightarrow{\alpha} - n$, where $n \in \operatorname{dom}\overrightarrow{\alpha}$. Simple verification implies that so defined map $\mathfrak{F}$ is correct and it is a homomorphism. Also, Lemma 7 implies that $\alpha\mathfrak{C}_{\mathbf{mg}}\beta$ if and only if $(\alpha)\mathfrak{F} = (\beta)\mathfrak{F}$ for $\alpha, \beta \in S$. $\square$

Theorems 3 and 4 imply the following corollary.

Corollary 3. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{[1]}$ as a submonoid. Then for any non-injective homomorphism $\mathfrak{F}\colon S \to T$ into an arbitrary semigroup T there exists a unique homomorphism $\mathfrak{H}\colon \mathbb{Z}(+) \to T$ such that the following diagram*

$$\begin{tikzcd} S \arrow[r, "\mathfrak{F}"] \arrow[d, "\mathfrak{C}^{\sharp}_{\mathbf{mg}}"'] & T \\ \mathbb{Z}(+) \arrow[ur, "\mathfrak{H}"'] & \end{tikzcd}$$

commutes.

The semigroups $\mathscr{C}_{\mathbb{N}}$, $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ are bisimple (see [16], [29], [15]). But the semigroup $\mathbf{I}\mathbb{N}_{\infty}$ is not bisimple whereas it is simple. A very amazing property about some inverse subsemigroups of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ illustrates the following theorem.

Theorem 5. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid. Then S is simple.*

Proof. Since $\alpha = \alpha\mathbb{I} = \mathbb{I}\alpha$ for any element α of S, it is sufficient to show that for every $\beta \in S$ there exist $\gamma, \delta \in S$ such that $\gamma\beta\delta = \mathbb{I}$.

Fix an arbitrary element β in S. Simple verifications show that $\beta\overrightarrow{\beta}^{-1} = \overrightarrow{\beta}\,\overrightarrow{\beta}^{-1}$ and $\beta\overrightarrow{\beta}^{-1}$ is an idempotent of S, where $\overrightarrow{\beta}^{-1}$ is inverse of $\overrightarrow{\beta}$ in S, because $\overrightarrow{\beta}$ and $\overrightarrow{\beta}^{-1}$ are elements of the sebsemigroup $\mathscr{C}_{\mathbb{N}}$ in S. Next we define a partial maps $\gamma\colon \mathbb{N} \rightharpoonup \mathbb{N}$ in the following way

$$\operatorname{dom}\gamma = \mathbb{N}, \qquad \operatorname{ran}\gamma = \left\{n \in \mathbb{N}\colon n \geqslant n_{\gamma}^{\mathbf{d}}\right\} \qquad \text{and} \qquad (i)\gamma = i - 1 + n_{\gamma}^{\mathbf{d}} \quad \text{for} \quad i \in \operatorname{dom}\gamma.$$

Then $\gamma\beta(\overrightarrow{\beta}^{-1}\gamma^{-1}) = \mathbb{I}$. $\square$

3 ON SHIFT-CONTINUOUS TOPOLOGIES ON INVERSE SUBSEMIGROUPS OF $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$

A subset A of a topological space X is said to be *co-dense* in X if $X \setminus A$ is dense in X.
We recall that a topological space X is said to be:

compact if every open cover of X contains a finite subcover;

countably compact if each closed discrete subspace of X is finite;

feebly compact if each locally finite open cover of X is finite;

pseudocompact if X is Tychonoff and each continuous real-valued function on X is bounded;

locally compact if each point of X has an open neighbourhood with the compact closure;

Čech-complete if X is Tychonof and there exists a compactifcation cX of X such that the remainder $cX \setminus c(X)$ is an F_σ-set in cX;

a Baire space if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of X the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of X.

According to Theorem 3.10.22 of [20], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space is countably compact and every countably compact space is feebly compact (see [4]). Also, every compact space is locally compact, every locally compact space is Čech-complete, and every Čech-complete space is a Baire space (see [20]).

By the Eberhart-Selden theorem every Hausdorff semigroup topology on the bicyclic semigroup is discrete. It is natural to ask: *Do there exists non-discrete semigroup topology on the semigroup* $\mathbf{I}\mathbb{N}_\infty$*?*

Theorem 6. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. Then every Baire shift-continuous Hausdorff topology τ on S is discrete.*

Proof. If no point in S is isolated, then since the space (S, τ) is Hausdorff, it follows that $\{\alpha\}$ is nowhere dense for all $\alpha \in S$. But, if this is the case, then since the semigroup S is countable it cannot be a Baire space. Hence the space (S, τ) contains an isolated point μ. If $\gamma \in S$ is arbitrary, then by Theorem 5, there exist $\alpha, \beta \in S$ such that $\alpha \cdot \gamma \cdot \beta = \mu$. The map $f \colon \chi \mapsto \alpha \cdot \chi \cdot \beta$ is continuous and so the full preimage $(\{\mu\})f^{-1}$ is open. By Proposition 1.2 from [15] for every $\alpha, \beta \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N})$, both sets $\{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N}) \mid \alpha \cdot \chi = \beta\}$ and $\{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N}) \mid \chi \cdot \alpha = \beta\}$ are finite, and hence the same holds for the subsemigroup S of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$. This implies that the set $(\{\mu\})f^{-1}$ is finite and since (S, τ) is Hausdorff, $\{\gamma\}$ is open, and hence isolated. $\square$

Since every Čech complete space (and hence every locally compact space) is Baire, Theorem 6 implies Corollary 4.

Corollary 4. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. Then every Hausdorff Čech complete (locally compact) shift-continuous topology τ on S is discrete.*

The following example shows that there exists a non-discrete Tychonoff inverse semigroup topology τ_W on the semigroup $\mathbf{I}\mathbb{N}_\infty$.

Example 1. *We define a topology τ_W on the semigroup $\mathbf{I}\mathbb{N}_\infty$ as follows. For every $\alpha \in \mathbf{I}\mathbb{N}_\infty$ we define a family*

$$\mathscr{B}_W(\alpha) = \{U_\alpha(F) \mid F \textit{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathbf{I}\mathbb{N}_\infty \mid \operatorname{dom}\beta \subseteq \operatorname{dom}\alpha \textit{ and } (x)\beta = (x)\alpha \textit{ for all } x \in F\}.$$

It is straightforward to verify that $\{\mathscr{B}_W(\alpha)\}_{\alpha \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})}$ forms a basis for a topology τ_W on the semigroup $\mathbf{I}\mathbb{N}_\infty$.

Proposition 1. *$(\mathbf{I}\mathbb{N}_\infty, \tau_W)$ is a Tychonoff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathbf{I}\mathbb{N}_\infty$. We put $\gamma = \alpha \cdot \beta$ and let $F = \{n_1, \ldots, n_i\}$ be a finite subset of $\operatorname{dom}\gamma$. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1, \ldots, n_i\}) \cdot U_\beta(\{m_1, \ldots, m_i\}) \subseteq U_\gamma(\{n_1, \ldots, n_i\})$$

and

$$\left(U_\gamma(\{n_1, \ldots, n_i\})\right)^{-1} \subseteq U_{\gamma^{-1}}(\{k_1, \ldots, k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathbf{I}\mathbb{N}_\infty, \tau_W)$.

Let $N = \mathbb{N} \cup \{a\}$ for some $a \notin \mathbb{N}$. Then N^N with the operation composition is a semigroup and the map $\Psi \colon \mathbf{I}\mathbb{N}_\infty \to N^N$ defined by the formula

$$(x)(\alpha)\Psi = \begin{cases} (x)\alpha, & \text{if } x \in \operatorname{dom}\alpha; \\ a, & \text{if } x \notin \operatorname{dom}\alpha \end{cases}$$

is a monomorphism. Hence N^N is a topological semigroup with the product topology if N has the discrete topology. Obviously, this topology generates topology τ_W on $\mathbf{I}\mathbb{N}_\infty$. Therefore by Theorem 2.3.11 from [20] topological space N^N is Tychonoff and hence by Theorem 2.1.6 from [20] so is $(\mathbf{I}\mathbb{N}_\infty, \tau_W)$. This completes the proof of the proposition. □

Theorem 7. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. Let T be a T_1 semitopological semigroup which contains S as a dense discrete subsemigroup. If $I = T \setminus S \neq \varnothing$ then I is an ideal of T.*

Proof. By Lemma 3 [31], S is an open subspace of the topological space T.

Fix an arbitrary element $y \in I$. If $x \cdot y = z \notin I$ for some $x \in S$ then there exists an open neighbourhood $U(y)$ of the point y in the space T such that $\{x\} \cdot U(y) = \{z\} \subset S$. By Proposition 1.2 from [15] the open neighbourhood $U(y)$ should contain finitely many elements of the semigroup S which contradicts our assumption. Hence $x \cdot y \in I$ for all $x \in S$ and $y \in I$. The proof of the statement that $y \cdot x \in I$ for all $x \in S$ and $y \in I$ is similar.

Suppose to the contrary that $x \cdot y = w \notin I$ for some $x, y \in I$. Then $w \in S$ and the separate continuity of the semigroup operation in T yields open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in the space T, respectively, such that $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup S, equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ do not hold, because $\{x\} \cdot (U(y) \cap S) \subseteq I$. The obtained contradiction implies that $x \cdot y \in I$. □

Theorem 7 implies the following corollary.

Corollary 5. *Let T be a T_1 semitopological semigroup which contains $\mathbf{I}\mathbb{N}_\infty$ as a dense discrete submonoid. If $I = T \setminus \mathbf{I}\mathbb{N}_\infty \neq \varnothing$, then I is an ideal of T.*

Proposition 2. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. Let T be a Hausdorff topological semigroup which contains S as a dense discrete subsemigroup. Then for every $\gamma \in S$ the set*

$$D_\gamma = \{(\chi, \varsigma) \in S \times S \mid \chi \cdot \varsigma = \gamma\}$$

is a closed-and-open subset of $T \times T$.

Proof. Since S is a discrete subspace of T by Lemma 3 [31] we have that D_γ is an open subset of $T \times T$.

Suppose that there exists $\gamma \in S$ such that D_γ is a non-closed subset of $T \times T$. Then there exists an accumulation point $(\alpha, \beta) \in T \times T$ of the set D_γ. The continuity of the semigroup operation in T implies that $\alpha \cdot \beta = \gamma$. But $S \times S$ is a discrete subspace of $T \times T$ and hence by Theorem 7, the points α and β belong to the ideal $I = T \setminus S$ and hence $\alpha \cdot \beta \in T \setminus S$ cannot be equal to γ. ☐

Theorem 8. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. If a T_1 topological semigroup T contains S as a dense discrete subsemigroup then the square $T \times T$ cannot be feebly compact.*

Proof. By Proposition 2, for every $c \in S$ the square $T \times T$ contains an open-and-closed discrete subspace D_c. If we identify the elements of the semigroup $\mathscr{C}_\mathbb{N}$ with the elements the bicyclic monoid $\mathscr{C}(p,q)$ by an isomorphism $\mathfrak{h} \colon \mathscr{C}(p,q) \to \mathscr{C}_\mathbb{N}$, then the subspace D_c contains an infinite subset

$$\left\{ \left((q^i)\mathfrak{h}, (p^i)\mathfrak{h} \right) : i \in \mathbb{N}_0 \right\}$$

and hence the set D_c is infinite. This implies that the square $S \times S$ is not feebly compact. ☐

A topological semigroup S is called Γ-*compact* if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is compact in S (see [33]). The results obtained in [3], [5], [6], [28], [33] imply the following

Corollary 6. *Let S be an inverse subsemigroup of $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_\mathbb{N}$ as a submonoid. If a Hausdorff topological semigroup T satisfies one of the following conditions:*

(i) *T is compact;*

(ii) *T is Γ-compact;*

(iii) *T is a countably compact topological inverse semigroup;*

(iv) *the square $T \times T$ is countably compact;*

(v) *the square $T \times T$ is a Tychonoff pseudocompact space,* *then T does not contain the semigroup S and for every homomorphism $\mathfrak{h}\colon S \to T$ the image $(S)\mathfrak{h}$ is a cyclic subgroup of T. Moreover, for every homomorphism $\mathfrak{h}\colon S \to T$ there exists a unique homomorphism $\mathfrak{u}_{\mathfrak{h}}\colon \mathbb{Z}(+) \to T$ such that the following diagram*

$$\begin{tikzcd} S \arrow[r, "\mathfrak{h}"] \arrow[d, "\mathfrak{C}^{\sharp}_{\mathbf{mg}}"'] & T \\ \mathbb{Z}(+) \arrow[ur, "\mathfrak{u}_{\mathfrak{h}}"'] & \end{tikzcd}$$

commutes.

Recall [17] that a *Bohr compactification* of a topological semigroup S is a pair $(\beta, B(S))$ such that $B(S)$ is a compact topological semigroup, $\beta\colon S \to B(S)$ is a continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f\colon B(S) \to T$ such that the diagram

$$\begin{tikzcd} S \arrow[rr, "\beta"] \arrow[dr, "g"'] & & B(S) \arrow[dl, "f"] \\ & T & \end{tikzcd}$$

commutes. Then Corollary 6 and Proposition 2 from [2] imply the following:

Corollary 7. *Let S be an inverse subsemigroup of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid. The Bohr compactification of the discrete semigroup S is topologically isomorphic to the Bohr compactification of discrete group $\mathbb{Z}(+)$.*

REFERENCES

[1] Andersen O. *Ein Bericht uber die Struktur abstrakter Halbgruppen*. PhD Thesis, Hamburg, 1952.

[2] Anderson L.W., Hunter R.P. *On the compactification of certain semigroups*. Contrib. Extens. Theory Topol. Struct. Proc. Sympos. Berlin 1967. VEB Deutscher Verlag der Wissenschaften, Berlin, 1969, 21–27.

[3] Anderson L.W., Hunter R.P., Koch R.J. *Some results on stability in semigroups*. Trans. Amer. Math. Soc. 1965, **117**, 521–529. doi:10.2307/1994222

[4] Arkhangel'skii A.V. *Topological function spaces*. Kluwer Publ., Dordrecht, 1992.

[5] Banakh T.O., Dimitrova S., Gutik O.V. *The Rees-Suschkiewitsch Theorem for simple topological semigroups*. Mat. Stud. 2009, **31** (2), 211–218.

[6] Banakh T., Dimitrova S., Gutik O. *Embedding the bicyclic semigroup into countably compact topological semigroups*. Topology Appl. 2010, **157** (18), 2803–2814. doi:10.1016/j.topol.2010.08.020

[7] Bardyla S. *Classifying locally compact semitopological polycyclic monoids*. Math. Bull. Shevchenko Sci. Soc. 2016, **13**, 21–28.

[8] Bardyla S. *On a semitopological α-bicyclic monoid*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2016, **81**, 9–22.

[9] Bardyla S. *On locally compact shift-continuous topologies on the α-bicyclic monoid*. Topol. Algebra Appl. 2018, **6**, 34–42. doi:10.1515/taa-2018-0003

[10] Bardyla S. *On locally compact semitopological graph inverse semigroups*. Mat. Stud. 2018, **49** (1), 19–28. doi: 10.15330/ms.49.1.19-28

[11] Bardyla S. *On locally compact topological graph inverse semigroups*. Topology Appl. 2019, **267**, 106873. doi: 10.1016/j.topol.2019.106873

[12] Bardyla S., Gutik O. *On a semitopological polycyclic monoid*. Algebra Discr. Math. 2016, **21** (2), 163–183.

[13] Bertman M.O., West T.T. *Conditionally compact bicyclic semitopological semigroups*. Proc. Roy. Irish Acad. 1976, **A76** (21–23), 219–226.

[14] Carruth J.H., Hildebrant J.A., Koch R.J. *The theory of topological semigroups*. Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.

[15] Chuchman I.Ya., Gutik O.V. *Topological monoids of almost monotone injective co-finite partial selfmaps of the set of positive integers*. Carpathian Math. Publ. 2010, **2** (1) 119–132.

[16] Clifford A.H., Preston G.B. *The algebraic theory of semigroups*. Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.

[17] DeLeeuw K., Glicksberg I. *Almost-periodic functions on semigroups*. Acta Math. 1961, **105** (1-2), 99–140. doi:10.1007/BF02559536

[18] Doroshenko V. *Generators and relations for the semigroups of increasing functions on* $\mathbb{N}$ *and* $\mathbb{Z}$. Algebra Discr. Math. 2005 (4), 1–15.

[19] Doroshenko V.V. *On semigroups of order-preserving transformations of countable linearly ordered sets*. Ukrainian Math. J. 2009, **61** (6), 859–872. doi:10.1007/s11253-009-0256-3 (translation of Ukrainian Math. J. 2009, **61** (6), 723–732 (in Ukrainian))

[20] Engelking R. *General topology*. 2nd ed., Heldermann, Berlin, 1989.

[21] Fihel I.R., Gutik O.V. *On the closure of the extended bicyclic semigroup*. Carpathian Math. Publ. 2011, **3** (2), 131–157.

[22] Gutik O. *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2015, **80**, 33–41.

[23] Gutik O. *Topological properties of Taimanov semigroups*. Math. Bull. Shevchenko Sci. Soc. 2016, **13**, 29–34.

[24] Gutik O. *On locally compact semitopological* 0*-bisimple inverse* ω*-semigroups*. Topol. Algebra Appl. 2018, **6**, 77–101. doi:10.1515/taa-2018-0008

[25] Gutik O., Maksymyk K. *On semitopological interassociates of the bicyclic monoid*. Visn. L'viv. Univ., Ser. Mekh.-Mat. 2016, **82**, 98–108.

[26] Gutik O.V., Maksymyk K.M. *On semitopological bicyclic extensions of linearly ordered groups*. J. Math. Sci. 2019, **238** (1), 32–45. doi:10.1007/s10958-019-04216-x

[27] Gutik O., Pozdnyakova I. *On monoids of monotone injective partial selfmaps of* $L_n \times_{\mathrm{lex}} \mathbb{Z}$ *with co-finite domains and images*. Algebra Discr. Math. 2014, **17** (2), 256–279.

[28] Gutik O., Repovš D. *On countably compact* 0*-simple topological inverse semigroups*. Semigroup Forum 2007, **75** (2), 464–469. doi:10.1007/s00233-007-0706-x

[29] Gutik O., Repovš D. *Topological monoids of monotone, injective partial selfmaps of* $\mathbb{N}$ *having cofinite domain and image*. Studia Sci. Math. Hungar. 2011, **48** (3), 342–353. doi:10.1556/SScMath.48.2011.3.1176

[30] Gutik O., Repovš D. *On monoids of injective partial selfmaps of integers with cofinite domains and images*. Georgian Math. J. 2012, **19** (3), 511–532. doi:10.1515/gmj-2012-0022

[31] Gutik O., Savchuk A. *On the semigroup* $\mathbf{ID}_\infty$. Visn. Lviv. Univ., Ser. Mekh.-Mat. 2017, **83**, 5–19. (in Ukrainian)

[32] Gutik O., Savchuk A. *The semigroup of partial co-finite isometries of positive integers*. Bukovyn. Mat. Zh. 2018, **6** (1-2), 42–51. doi:10.31861/bmj2018.01.042 (in Ukrainian)

[33] Hildebrant J.A., Koch R.J. *Swelling actions of* Γ*-compact semigroups*. Semigroup Forum 1986, **33** (1), 65–85. doi:10.1007/BF02573183

[34] Hogan J.W. *The* α*-bicyclic semigroup as a topological semigroup*. Semigroup forum 1984, **28** (1), 265–271. doi:10.1007/BF02572488

[35] Lawson M. V. *Inverse semigroups. The theory of partial symmetries*. World Scientific, Singapore, 1998.

[36] Mesyan Z., Mitchell J.D., Morayne M., Péresse Y.H. *Topological graph inverse semigroups*. Topology Appl. 2016, **208**, 106–126. doi:10.1016/j.topol.2016.05.012

[37] Munn W.D. *The algebra of a combinatorial inverse semigroup*. J. Lond. Math. Soc., II. Ser. 1983, **27** (1), 35–38. doi:10.1112/jlms/s2-27.1.35

[38] Petrich M. *Inverse semigroups*. John Wiley & Sons, New York, 1984.

[39] Ruppert W. *Compact semitopological semigroups: an intrinsic theory*. Lect. Notes Math., **1079**, Springer, Berlin, 1984. doi:10.1007/BFb0073675

[40] Selden A.A. *A nonlocally compact nondiscrete topology for the α-bicyclic semigroup*. Semigroup Forum 1985, **31** (1), 372–374. doi:10.1007/BF02572664

[41] Taimanov A.D. *An example of a semigroup which admits only the discrete topology*. Algebra and Logic 1973, **12** (1), 64–65. doi:10.1007/BF02218642 (translation of Algebra i Logika 1973, **12** (1), 114–116 (in Russian))

[42] Taimanov A.D. *On the topologization of commutative semigroups*. Math. Notes 1975, **17** (5), 443–444. doi: 10.1007/BF01155800 (translation of Mat. Zametki 1975, **17** (5), 745–748 (in Russian))

[43] Wagner V.V. *Generalized groups*. Dokl. Akad. Nauk SSSR 1952, **84**, 1119–1122 (in Russian).

Received 02.05.2019

Гутік О.В., Савчук А.С. *Про інверсні підмоноїди моноїда майже монотонних ін'єктивних коскінченних часткових перетворень натуральних чисел* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 296–310.

У праці вивчаються інверсні підмоноїди моноїда $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ майже монотонних ін'єктивних коскінченних часткових перетворень множини натуральних чисел $\mathbb{N}$. Нехай $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ — підмоноїд в $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$, який складається з коскінченних монотонних часткових біекцій множини $\mathbb{N}$ і $\mathscr{C}_{\mathbb{N}}$ — підмоноїд в $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$, який породжений частковим зсувом $n \mapsto n+1$ натуральних чисел і до його оберненим частковим відображенням. Доведено, що кожен автоморфізм повної інверсної піднапівгрупи моноїда $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$, який містить напівгрупу $\mathscr{C}_{\mathbb{N}}$ є тотожнім відображенням. Побудовано піднапівгрупу $\mathbf{I}\mathbb{N}_{\infty}^{[1]}$ моноїда $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ з такою властивістю: якщо S — інверсна піднапівгрупа в $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$, що містить напівгрупу $\mathbf{I}\mathbb{N}_{\infty}^{[1]}$, як підмоноїд, то кожна відмінна від тотожної конгруенція $\mathfrak{C}$ на S є груповою. Доведено, якщо S — інверсна піднапівгрупа в $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$, що містить $\mathscr{C}_{\mathbb{N}}$ як підмоноїд, то напівгрупа S є простою і фактор-напівгрупа $S/\mathfrak{C}_{\mathbf{mg}}$, де $\mathfrak{C}_{\mathbf{mg}}$ – найменша групова конгруенція на S, ізоморфна адитивній групі цілих чисел. Також досліджуються топологізації інверсних піднапівгруп напівгрупи $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$, як містять напівгрупу $\mathscr{C}_{\mathbb{N}}$ і занурення таких напівгруп у близькі до компактних топологічні напівгрупи.

Ключові слова і фрази: інверсна напівгрупа, ізометрія, часткова біекція, конгруенція, біциклічна напівгрупа, напівтопологічна напівгрупа, топологічна напівгрупа, дискретна топологія, занурення, компактифікація Бора.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 311–320
doi:10.15330/cmp.11.2.311-320

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.311–320

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SPECTRA OF SOME ALGEBRAS OF ENTIRE FUNCTIONS OF BOUNDED TYPE, GENERATED BY A SEQUENCE OF POLYNOMIALS

In this work, we investigate the properties of the topological algebra of entire functions of bounded type, generated by a countable set of homogeneous polynomials on a complex Banach space.

Let X be a complex Banach space. We consider a subalgebra $H_{b\mathbb{P}}(X)$ of the Fréchet algebra of entire functions of bounded type $H_b(X)$, generated by a countable set of algebraically independent homogeneous polynomials $\mathbb{P}$. We show that each term of the Taylor series expansion of entire function, which belongs to the algebra $H_{b\mathbb{P}}(X)$, is an algebraic combination of elements of $\mathbb{P}$. We generalize the theorem for computing the radius function of a linear functional on the case of arbitrary subalgebra of the algebra $H_b(X)$ on the space X. Every continuous linear multiplicative functional, acting from $H_{b\mathbb{P}}(X)$ to $\mathbb{C}$ is uniquely determined by the sequence of its values on the elements of $\mathbb{P}$. Consequently, there is a bijection between the spectrum (the set of all continuous linear multiplicative functionals) of the algebra $H_{b\mathbb{P}}(X)$ and some set of sequences of complex numbers. We prove the upper estimate for sequences of this set. Also we show that every function that belongs to the algebra $H_{b\mathbb{P}}(X)$, where X is a closed subspace of the space ℓ_∞ such that X contains the space c_{00}, can be uniquely analytically extended to ℓ_∞ and algebras $H_{b\mathbb{P}}(X)$ and $H_{b\mathbb{P}}(\ell)$ are isometrically isomorphic. We describe the spectrum of the algebra $H_{b\mathbb{P}}(X)$ in this case for some special form of the set $\mathbb{P}$.

Results of the paper can be used for investigations of the algebra of symmetric analytic functions on Banach spaces.

Key words and phrases: n-homogeneous polynomial, analytic function, spectrum of algebra.

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INTRODUCTION

The theory of analytic functions is an important section of nonlinear functional analysis. In many modern investigations topological algebras of analytic functions and spectra of such algebras are studied.

The existence of algebraic basis plays an important role in the description of the spectrum (the set of all continuous complex-valued linear multiplicative functionals) of the algebra, since every continuous linear multiplicative functional is uniquely defined by its values on elements of the algebraic basis.

The problem of description of spectra of algebras of analytic functions of bounded type was considered by many authors (see, e.g., [2,3,10]). In the general case the problem of description of spectra of algebras with the countable algebraic bases is not solved. However for some of

УДК 517.98
2010 *Mathematics Subject Classification:* 46J20, 46E15.
The publication contains the results of studies conducted by President's of Ukraine grant for competitive projects 0119U103204.

these algebras descriptions of spectra were constructed. Algebras of symmetric analytic functions of bounded type on spaces with symmetric structures are typical examples of algebras generated by countable sets of polynomials and were studied in [1,4–8].

We generalize the theorem [2] for computing the radius function of a linear functional on case of arbitrary subalgebra of the Fréchet algebra of entire functions of bounded type $H_b(X)$ with the topology of uniform convergence on a complex Banach space X. We consider the subalgebra $H_{b\mathbb{P}}(X)$ of the algebra $H_b(X)$ of entire functions, generated by a countable set of algebraically independent polynomials $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$, such that P_n is an n-homogeneous polynomial for every $n \in \mathbb{N}$. We show that each term of the Taylor series expansion of entire function, which belongs to the algebra $H_{b\mathbb{P}}(X)$, is an algebraic combination of elements of $\mathbb{P}$. Accordingly, every $f \in H_{b\mathbb{P}}(X)$ can be uniquely represented in the form

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} (P_1(x))^{k_1} (P_2(x))^{k_2} \cdots (P_n(x))^{k_n},$$

where $x \in X$, $a_{k_1,k_2,\ldots,k_n} \in \mathbb{C}$ and $k_1, k_2, \ldots, k_n$ are non-negative integers. Therefore every continuous linear multiplicative functional φ acting from $H_{b\mathbb{P}}(X)$ to $\mathbb{C}$ is uniquely determined by the sequence $(\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n))$ of its values on elements of $\mathbb{P}$. Consequently, the spectrum of the algebra $H_{b\mathbb{P}}(X)$ is in one-to-one correspondence with some set of sequences of complex numbers. We prove the upper estimate for sequences of this set. Also we show that every function that belongs to the algebra $H_{b\mathbb{P}}(X)$, where X is a closed subspace of the space ℓ_∞ such that X contains the space c_{00}, can be uniquely analytically extended to ℓ_∞ and algebras $H_{b\mathbb{P}}(X)$ and $H_{b\mathbb{P}}(\ell_\infty)$ are isometrically isomorphic. We describe the spectrum of the algebra $H_{b\mathbb{P}}(X)$ in this case for the set $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ such that

$$P_n((x_1, x_2, \ldots, x_n, \ldots)) = x_n^n$$

for $n \in \mathbb{N}$.

In the first section we recall some basic notions on the theory of analytic functions on a Banach space and the theory of the Fréchet algebras which are necessary for a full comprehension of the paper.

In the second section we generalize the theorem for computing the radius function of a linear functional on case of arbitrary subalgebra of the Fréchet algebra of entire functions of bounded type $H_b(X)$ on a complex Banach space X. We also prove that every term of the Taylor series expansion of entire function, generated by the countable set of algebraically independent polynomials, is an algebraic combination of these polynomials.

In the third section of the paper we describe spectra of the Fréchet algebras of entire functions, generated by the sequence of polynomials $\mathbb{P}$ on complex spaces, which are the closed subspaces of the space ℓ_∞ and contain the linear space c_{00}.

1 PRELIMINARIES

In this section we will review the formal definition of polynomial on a Banach space and introduce the necessary background. To begin with, we establish some notation. Throughout the whole article the letter X will always stand for a complex Banach space. The set of all positive integers will be denoted by $\mathbb{N}$, whereas the set $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. We denote by $\mathbb{Q}^+$ the set of all positive rationals. We also denote by ℓ_∞ the complex Banach space of all bounded sequences $x = (x_1, x_2, \ldots)$ of complex numbers with the norm $\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$ and by c_{00} the linear space of eventually zero sequences $x = (x_1, x_2, \ldots, x_n, 0, \ldots)$ of complex numbers with the norm $\|x\| = \sup_{i\in\mathbb{N}} |x_i|$.

Definition 1. *For $n \in \mathbb{N}$ a mapping $P : X \to \mathbb{C}$ is said to be an n-homogeneous polynomial if there exists some n-linear form $A_P : X^n \to \mathbb{C}$ such that $P(x) = A_P(\underbrace{x, \ldots, x}_{n})$ for every $x \in X$.*

We shall denote by $\mathcal{P}^n(X)$ the vector space of all n-homogeneous polynomials from X to $\mathbb{C}$. Also let $\mathcal{P}^0(X)$ be the vector space of all constant mappings from X to $\mathbb{C}$. For each $P \in \mathcal{P}^n(X)$ we shall set

$$\|P\| = \sup\{|P(x)| : x \in X, \|x\| \le 1\}.$$

It is known that a polynomial $P \in \mathcal{P}^n(X)$ is continuous if and only if $\|P\| < \infty$.

Definition 2. *A mapping $P : X \to \mathbb{C}$ is said to be a polynomial of degree at most n, where $n \in \mathbb{N}_0$, if it can be represented as a sum $P = P_0 + P_1 + \cdots + P_n$, where $P_j \in \mathcal{P}^j(X)$ for $j = \overline{0, n}$.*

Definition 3. *Polynomials $P_1, P_2, \ldots$, where $P_j \in \mathcal{P}^j(X)$, $j \in \mathbb{N}$, are called algebraically independent polynomials, when for all $n \in \mathbb{N}$ and every polynomial $q : \mathbb{C}^n \to \mathbb{C}$ if the equality $q(P_1(x), P_2(x), \ldots, P_n(x)) = 0$ holds for every $x \in X$, then $q \equiv 0$.*

Definition 4. *A polynomial $P : X \to \mathbb{C}$ is called an algebraic combination of elements of $\mathbb{P} = \{P_1, P_2, \ldots\}$ if there exists $n \in \mathbb{N}$ and a polynomial $q : \mathbb{C}^n \to \mathbb{C}$ such that $P(x) = q(P_1(x), \ldots, P_n(x))$ for every $x \in X$.*

Let us denote by $B(a, r)$ and $\bar{B}(a, r)$ an open and a closed balls of radius r and center $a \in X$ respectively.

Definition 5. *Let U be the open subset of X. A mapping $f : U \to \mathbb{C}$ is said to be holomorphic or analytic on U if for each $a \in U$ there exists an open ball $B(a; r) \subset U$ and a sequence of polynomials $f_0, f_1, \ldots$, where $f_0 \in \mathbb{C}$ and f_j is a j-homogeneous polynomial for each $j \in \mathbb{N}$, such that*

$$f(x) = \sum_{n=0}^{\infty} f_n(x - a)$$

uniformly for $x \in B(a; r)$.

Note that the power series $\sum_{n=0}^{\infty} f_n(x - a)$ is called the Taylor series of the function f at the point a. If $U = X$ the function f is called an entire function.

According to [9, p. 47, Corollary 7.3] the terms of the Taylor series of an entire function $f : X \to \mathbb{C}$ can be found using the Cauchy's integral formula

$$f_n(x) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta x)}{\zeta^{n+1}}\, d\zeta, \qquad \text{where } r > 0. \tag{1}$$

Also recall that the radius of convergence ρ_a of the power series $\sum_{n=0}^{\infty} f_n(x - a)$ is the supremum of all $r \ge 0$ such that the series converges uniformly on the ball $\overline{B}(a, r)$. According to [4, p. 27, Theorem 4.3], the radius of convergence of the power series is given by the Cauchy-Hadamard formula

$$\frac{1}{\rho_a} = \limsup_{n\to\infty} \|f_n\|_1^{\frac{1}{n}}.$$

Denote by $H_b(X)$ the algebra of $\mathbb{C}$-valued entire functions of bounded type on X, that is, the space of all entire mappings from X to $\mathbb{C}$, which are bounded on bounded subsets. We endow the algebra $H_b(X)$ with the system of uniform norms

$$\|f\|_r = \sup\{|f(x)| : x \in X, \|x\| < r\}, \qquad \text{where } r \in \mathbb{Q}^+.$$

It is known that the topology on a countably-normed space can be given by some metric. Note that the algebra $H_b(X)$ is complete with respect to this metric. Hence $H_b(X)$ is a Frechet algebra.

Let $H_b^{(0)}(X)$ be an arbitrary subalgebra of $H_b(X)$. For every continuous linear functional $\varphi \in \left(H_b^{(0)}(X)\right)^*$ there exists $r \in \mathbb{Q}^+$ such that φ is continuous with respect to the norm $\|\cdot\|_r$, where $\left(H_b^{(0)}(X)\right)^*$ is the space of all continuous linear functionals on $H_b^{(0)}(X)$.

Analogically to [2, Section 2] let us define the radius function on $\left(H_b^{(0)}(X)\right)^*$ as follows.

Definition 6. *For $\varphi \in \left(H_b^{(0)}(X)\right)^*$ let the radius function $R(\varphi)$ be the infimum of all $r > 0$ such that φ is continuous with respect to the norm $\|\cdot\|_r$.*

Thus,

$$0 \le R(\varphi) < \infty.$$

For $n \in \mathbb{N}_0$ let $\widetilde{\mathcal{P}^n}(X) = \mathcal{P}^n(X) \bigcap H_b^{(0)}(X)$ denote the space of n-homogeneous polynomials on X, which belong to $H_b^{(0)}(X)$. For each $P \in \widetilde{\mathcal{P}^n}(X)$ we shall set

$$\|P\| = \|P\|_1 = \sup\{|P(x)| : x \in X, \|x\| \le 1\}.$$

Each $f \in H_b^{(0)}(X)$ has a Taylor series expansion

$$f = \sum_{n=0}^{\infty} f_n, \tag{2}$$

where $f_n \in \widetilde{\mathcal{P}^n}(X)$ for $n \in \mathbb{N}_0$, and the series (2) converges in $H_b^{(0)}(X)$, that is, the series (2) converges uniformly to f on each bounded subset of X.

Let $\varphi \in \left(H_b^{(0)}(X)\right)^*$. Taking into account the continuity and the linearity of φ, we obtain

$$\varphi(f) = \sum_{n=0}^{\infty} \varphi(f_n). \tag{3}$$

We denote by φ_n the restriction of $\varphi \in \left(H_b^{(0)}(X)\right)^*$ to $\widetilde{\mathcal{P}^n}(X)$. Then φ_n is continuous. Its norm on $\widetilde{\mathcal{P}^n}(X)$ will be denoted by

$$\|\varphi_n\| = \sup\{|\varphi(P)| : P \in \widetilde{\mathcal{P}^n}(X), \|P\| \le 1\}.$$

Definition 7. *The spectrum of the topological algebra A is the set of all continuous complex-valued linear multiplicative functionals = continuous complex-valued homomorphisms = continuous characters.*

2 ALGEBRAS, GENERATED BY A COUNTABLE SET OF POLYNOMIALS

Let $\mathcal{P}_{\mathbb{P}}(X)$ be the algebra, consisting of all polynomials, which are algebraic combinations of elements of the set $\mathbb{P}$. Let us denote by $H_{b\mathbb{P}}(X)$ the closure of $\mathcal{P}_{\mathbb{P}}(X)$ in the metric of the algebra $H_b(X)$. It can be checked that $H_{b\mathbb{P}}(X)$ is a subalgebra of $H_b(X)$ and that $H_{b\mathbb{P}}(X)$ is a Frechet algebra with respect to the metric of the algebra $H_b(X)$.

Proposition 1. *Each term of the Taylor series of a function $f \in H_{b\mathbb{P}}(X)$ can be uniquely represented as an algebraic combination of elements of the set $\mathbb{P}$. Consequently,*

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} (P_1(x))^{k_1} (P_2(x))^{k_2} \cdots (P_n(x))^{k_n},$$

where $x \in X$, $a_{k_1,k_2,\ldots,k_n} \in \mathbb{C}$ and $k_1, k_2 \ldots, k_n \in \mathbb{N}_0$.

Proof. For $n \in \mathbb{N}$ let f_n be the nth term of the Taylor series of f. Let us show that f_n can be uniquely represented as an algebraic combination of polynomials $P_1, \ldots, P_n$.

Let us denote by $\mathcal{P}^n_{\mathbb{P}_n}(X)$ the space of all n-homogeneous polynomials, which are algebraic combinations of polynomials $P_1, \ldots, P_n$. Note that the set of polynomials of the form $P_1^{k_1} P_2^{k_2} \ldots P_n^{k_n}$, where $k_1, k_2, \ldots, k_n \in \mathbb{N}_0$ and $k_1 + 2k_2 + \ldots + nk_n = n$, is a Hamel basis for the space $\mathcal{P}^n_{\mathbb{P}_n}(X)$. Since there is a finite number of such polynomials the space $\mathcal{P}^n_{\mathbb{P}_n}(X)$ is a finite-dimensional. Therefore $\mathcal{P}^n_{\mathbb{P}_n}(X)$ is complete with respect to each of the norms. In particular $\mathcal{P}^n_{\mathbb{P}_n}(X)$ is complete with respect to the norm $\|\cdot\|_1$.

Since $H_{b\mathbb{P}}(X)$ is the closure of the algebra $\mathcal{P}_{\mathbb{P}}(X)$, then there exists a sequence $\{a_l\}_{l=1}^{\infty} \subset \mathcal{P}_{\mathbb{P}}(X)$, which converges to the function f with respect to the metric of $H_b(X)$. Let a_{ln} be the nth member of the Taylor series of the polynomial a_l. Note that $a_{ln} \in \mathcal{P}^n_{\mathbb{P}_n}(X)$ for each $l \in \mathbb{N}$. Let us show that the sequence $\{a_{ln}\}_{l=1}^{\infty}$ converges to f_n with respect to the norm $\|\cdot\|_1$. According to the Cauchy's integral formula (1), in which we take $r = 1$,

$$f_n(x) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta x)}{\zeta^{n+1}}\, d\zeta \qquad \text{and} \qquad a_{ln}(x) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{a_l(\zeta x)}{\zeta^{n+1}}\, d\zeta.$$

Therefore

$$\begin{aligned} |f_n(x) - a_{ln}(x)| &= \left| \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta x) - a_l(\zeta x)}{\zeta^{n+1}}\, d\zeta \right| \\ &\le \frac{1}{2\pi} \int_{|\zeta|=1} \frac{|f(\zeta x) - a_l(\zeta x)|}{|\zeta|^{n+1}}\, d\zeta = \frac{1}{2\pi} \int_{|\zeta|=1} |f(\zeta x) - a_l(\zeta x)|\, d\zeta. \end{aligned}$$

When $x \in X$ is such that $\|x\| \le 1$ and $\zeta \in \mathbb{C}$ is such that $|\zeta| = 1$, we obtain $\|\zeta x\| \le 1$. So when $\|x\| \le 1$ we have

$$|f(\zeta x) - a_l(\zeta x)| \le \|f - a_l\|_1.$$

It follows that

$$\|f_n - a_{ln}\|_1 = \sup_{\|x\| \le 1} |f_n(x) - a_{ln}(x)| \le \frac{1}{2\pi} \|f - a_l\|_1 \int_{|\zeta|=1} d\zeta = \|f - a_l\|_1.$$

Since $a_l \to f$ as $l \to \infty$, then $\|f - a_l\|_1 \to 0$ as $l \to \infty$. Therefore $\|f_n - a_{ln}\|_1 \to 0$ as $l \to \infty$. Hence $a_{ln} \to f_n$ as $l \to \infty$ with respect to the norm $\|\cdot\|_1$. Since $\mathcal{P}^n_{\mathbb{P}_n}(X)$ is complete with respect to the norm $\|\cdot\|_1$, then $f_n \in \mathcal{P}^n_{\mathbb{P}_n}(X)$. Hence f_n can be represented as an algebraic combination of polynomials $P_1, P_2, \ldots, P_n$. Such representation is unique, since polynomials $P_1, P_2, \ldots, P_n$ are algebraic independent. ☐

Let $M_{b\mathbb{P}}$ be the spectrum of the algebra $H_{b\mathbb{P}}(X)$. According to the Proposition 1 every function $f \in H_{b\mathbb{P}}(X)$ can be uniquely represented in the form

$$f = \sum_{n=0}^{\infty} f_n = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1,k_2\ldots,k_n} P_1^{k_1} P_2^{k_2} \cdots P_n^{k_n},$$

where $a_{k_1,k_2\ldots,k_n} \in \mathbb{C}$ and $k_1, k_2 \ldots, k_n \in \mathbb{N}_0$. Consequently, for every non-trivial character $\varphi \in M_{b\mathbb{P}}$, taking into account that $\varphi(1) = 1$, we have the following:

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1,k_2\ldots,k_n} (\varphi(P_1))^{k_1} (\varphi(P_2))^{k_2} \cdots (\varphi(P_n))^{k_n}.$$

Thus we can see that φ is completely defined by its values on polynomials P_j, where $j \in \mathbb{N}$. Hence we can identify every $\varphi \in M_{b\mathbb{P}}$ with the sequence $\{\varphi(P_j)\}_{j=1}^{\infty}$.

Let us prove the following analog of [2, Theorem 2.3] on case of arbitrary subalgebra $H_b^{(0)}(X)$ of algebra $H_b(X)$. Let us recall that we denote by φ_n the restriction of $\varphi \in \left(H_b^{(0)}(X)\right)^*$ to $\widetilde{\mathcal{P}^n}(X)$, where $\widetilde{\mathcal{P}^n}(X) = \mathcal{P}^n(X) \cap H_b^{(0)}(X)$.

Theorem 1. *The radius function R on $\left(H_b^{(0)}(X)\right)^*$ is given by*

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}}.$$

Proof. Suppose that $0 < t < \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}}$. Then there is a sequence of homogeneous polynomials P_j of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\varphi(P_j)| > t^{n_j}$. If $0 < r < t$, then by homogeneity, $\|P_j\|_r = r^{n_j}$, so that

$$|\varphi(P_j)| > \left(\frac{t}{r}\right)^{n_j} \|P_j\|_r,$$

and φ is not continuous with respect to the norm of uniform convergence on rB. It follows that $R(\varphi) \geq r$, and on account of the arbitrary choices of r and t we obtain $R(\varphi) \geq \limsup\limits_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}}$.

For the reverse inequality, let s be strictly larger than the supremum above, so that $\|\varphi_n\| \leq s^n$ for n large. Then there is $c \geq 1$ such that $\|\varphi_n\| \leq cs^n$, $n \geq 0$. Let $r > s$ is arbitrary, and a function $f \in H_b^{(0)}$ has Taylor series (2). Then the Cauchy estimates yield

$$r^n \|f_n\| = \|f_n\|_r \leq \|f\|_r, \; n \geq 0.$$

Hence, $|\varphi(f_n)| \leq \|\varphi_n\| \|f_n\| \leq \frac{cs^n}{r^n} \|f\|_r$, so that in view of (3) we obtain

$$|\varphi(f)| \leq c\Big(\sum_{n=0}^{\infty} \frac{s^n}{r^n}\Big) \|f\|_r.$$

Thus φ is continuous with respect to the norm of uniform convergence on the ball rB and $R(\varphi) \leq r$. On account of the arbitrary choices of s and r, we can see that

$$R(\varphi) \leq \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}}.$$

□

Proposition 2. *For every $\varphi \in M_{b\mathbb{P}}$ there exists $r \in \mathbb{Q}^+$, such that the estimate*

$$|\varphi(P_n)| \le r^n \|P_n\|_1$$

holds for all $P_n \in \mathcal{P}^n_{\mathbb{P}_n}(X)$.

Proof. Each $\varphi \in M_{b\mathbb{P}}$ is continuous with respect to the norm of uniform convergence on some ball in X. Let φ be continuous with respect to the norm $\|\cdot\|_r$, where $r \in \mathbb{Q}^+$. Since the norm of every non-trivial continuous complex valued homomorphism is equal to 1, the estimate

$$|\varphi(P_n)| \le \|P_n\|_r, \; P_n \in \mathcal{P}^n_{\mathbb{P}_n}(X)$$

holds for all $r > R(\varphi)$.

So, $|\varphi(P_n)| \le \sup\limits_{\|x\|\le r} |P_n(x)|$. Let us make the following replacement $x = ry$. Thus we obtain $|\varphi(P_n)| \le r^n \sup\limits_{\|y\|\le 1} |P_n(y)|$, $|\varphi(P_n)| \le r^n \|P_n\|_1$. □

3 THE CASE OF SUBSPACE X, $c_{00} \subset X \subset \ell_\infty$

Let X be a closed subspace of ℓ_∞ such that X contains c_{00} and $\mathbb{P}$ be a sequence of continuous polynomials $P_1, \dots, P_n, \dots$ such that

1. P_n is an n-homogeneous polynomial;
2. P_n's are algebraically independent;
3. every P_n depends only on a finite number of coordinates.

Lemma 1. *Let us define the mapping $J : H_{b\mathbb{P}}(\ell_\infty) \to H_{b\mathbb{P}}(X)$ by $J(g) = g|_X$, where $g \in H_{b\mathbb{P}}(\ell_\infty)$. Let $g_n \in H_{b\mathbb{P}}(\ell_\infty)$ be an n-homogeneous polynomial. Then the following equality holds:*

$$\|g_n\|_1 = \|J(g_n)\|_1.$$

Proof. According to the Proposition 1 each term of the Taylor series g_n can be uniquely represented as an algebraic combination of polynomials $P_1, \dots, P_n$. Since in our case every polynomial P_n depends on a finite number of coordinates, then polynomials g_n depend on a finite number of variables. Let us denote by $\kappa(j)$ the maximum among indices of elements of the sequence x on which the polynomial P_j depends on, $j = \overline{1,n}, n \in \mathbb{N}$. Obviously, $\kappa(j) \in \mathbb{N}$. Also let us denote by $\kappa_{max} = \max\{\kappa(j) : 1 \le j \le n\}$. Then we can write down the following chain of equalities:

$$\begin{aligned}\|g_n\|_1 &= \sup\{|g_n(x)| : x = (x_1, \dots, x_m, \dots) \in \ell_\infty, x_m = 0 \,\forall m > \kappa_{max}, m \in \mathbb{N}, \|x\| \le 1\}\\ &= \sup\{|g_n(x)| : x \in c_{00}, \|x\| \le 1\}.\end{aligned}$$

Thinking analogically we obtain the following chain of equalities for norms of $J(g_n) \in H_{b\mathbb{P}}(X)$:

$$\begin{aligned}\|J(g_n)\|_1 &= \sup\{|J(g_n(x))| : x = (x_1, \dots, x_m, \dots) \in X, \|x\| \le 1\}\\ &= \sup\{|g_n(x)| : x = (x_1, \dots, x_m, \dots) \in X, \|x\| \le 1\}\\ &= \sup\{|g_n(x)| : x = (x_1, \dots, x_m, \dots) \in X, x_m = 0 \,\forall m > \kappa_{max}, m \in \mathbb{N}, \|x\| \le 1\}\\ &= \sup\{|g_n(x)| : x \in c_{00}, \|x\| \le 1\}.\end{aligned}$$

Thus the equality $\|g_n\|_1 = \|J(g_n)\|_1$ is established. □

Theorem 2. *Every function that belongs to $H_{b\mathbb{P}}(X)$ can be uniquely analytically extended to ℓ_∞ and algebras $H_{b\mathbb{P}}(X)$ and $H_{b\mathbb{P}}(\ell_\infty)$ are isometrically isomorphic.*

Proof. Let us consider a mapping $J : H_{b\mathbb{P}}(\ell_\infty) \to H_{b\mathbb{P}}(X)$ such that $J(f) = f|_X$ for every function $f \in H_{b\mathbb{P}}(\ell_\infty)$. It is easy to check that J is a homomorphism from $H_{b\mathbb{P}}(\ell_\infty)$ onto $H_{b\mathbb{P}}(X)$.

Next we will show that the mapping J is a bijection. Firstly, let us prove that for all $f_1, f_2 \in H_{b\mathbb{P}}(\ell_\infty)$ whenever $J(f_1) = J(f_2)$, then $f_1 = f_2$, that is J is an injection. Let us consider $g \in H_{b\mathbb{P}}(\ell_\infty)$ such that $g = f_1 - f_2$ and g has a Taylor series representation $g = \sum_{n=0}^{\infty} g_n$. By assumption, $J(f_1) = J(f_2)$, and so $J(g) = J(f_1 - f_2) = J(f_1) - J(f_2) = 0$. On the other hand,

$$J(g) = J(\sum_{n=0}^{\infty} g_n) = \sum_{n=0}^{\infty} J(g_n)$$

and the Cauchy estimate yields $\|J(g_n)\|_1 \le \|J(g)\|_1$, $n \in \mathbb{N}_0$. Since $J(g) = 0$, then $\|J(g)\|_1 = 0$ and $\|J(g_n)\|_1 = 0$. According to Lemma 1 we have $\|g_n\|_1 = \|J(g_n)\|_1$. Therefore $\|g_n\|_1 = 0$ and it follows that $g_n(x) = 0$ for all $x \in \ell_\infty$.

Thus we obtain the chain of equalities

$$f_1(x) - f_2(x) = g(x) = \sum_{n=0}^{\infty} g_n(x) = 0$$

for all $x \in \ell_\infty$. It follows that $f_1 = f_2$. Hence, the mapping J is injective.

Now let us show that J is a surjection, that is for every $h \in H_{b\mathbb{P}}(X)$ there is at least one $\widetilde{h} \in H_{b\mathbb{P}}(\ell_\infty)$, such that $J(\widetilde{h}) = h$. Since $h \in H_{b\mathbb{P}}(X)$, it has a Taylor series representation $h = \sum_{n=0}^{\infty} h_n$ with the radius of convergence

$$\rho_0(h) = \frac{1}{\limsup_{n\to\infty} \|h_n\|_1^{\frac{1}{n}}} = \infty$$

for all $x \in X$. Let us show that the last equality also holds for all $x \in \ell_\infty$. According to Proposition 1 each term h_n of the Taylor series of h can be uniquely represented as an algebraic combination of polynomials $P_1, \ldots, P_n$. Since every polynomial P_n depends on a finite number of coordinates, then polynomials h_n depend on a finite number of variables. Let us denote by $\kappa(j)$ the maximum among indices of elements of the sequence x on which the polynomial P_j depends on, $j = \overline{1,n}, n \in \mathbb{N}$. Obviously, $\kappa(j) \in \mathbb{N}$. Also let us denote by $\kappa_{max} = \max\{\kappa(j) : 1 \le j \le n\}$. Then we have the following chain of equalities

$$\begin{aligned}
\|h_n\|_1 &= \sup\{|h_n(x)| : x = (x_1, \ldots, x_k, \ldots) \in X, \|x\| \le 1\} \\
&= \sup\{|h_n(x)| : x = (x_1, \ldots, x_k, \ldots) \in X, x_k = 0 \ \forall k > \kappa_{max}, k \in \mathbb{N}, \|x\| \le 1\} \\
&= \sup\{|h_n(x)| : x \in c_{00}, \|x\| \le 1\} \\
&= \sup\{|h_n(x)| : x = (x_1, \ldots, x_k, \ldots) \in \ell_\infty, x_k = 0 \ \forall k > \kappa_{max}, k \in \mathbb{N}, \|x\| \le 1\} \\
&= \sup\{|h_n(x)| : x = (x_1, \ldots, x_k, \ldots) \in \ell_\infty, \|x\| \le 1\} = \|\widetilde{h}_n\|_1.
\end{aligned}$$

Therefore $\limsup_{n\to\infty} \|\widetilde{h}_n\|_1^{\frac{1}{n}} = 0$ and respectively $\rho_0(\widetilde{h}) = \frac{1}{\limsup_{n\to\infty} \|\widetilde{h}_n\|_1^{\frac{1}{n}}} = \infty$ for all $x \in \ell_\infty$. Hence every function $h \in H_{b\mathbb{P}}(X)$ can be uniquely analytically extended to ℓ_∞. This extension is a desired function $\widetilde{h}$. Thus the mapping J is surjective.

It remains to prove that the given function J is an isometry between algebras $H_{b\mathbb{P}}(\ell_\infty)$ and $H_{b\mathbb{P}}(X)$. For this it is sufficient to show that for all $h \in H_{b\mathbb{P}}(X)$, $\widetilde{h} \in H_{b\mathbb{P}}(\ell_\infty)$ such that $J(\widetilde{h}) = h$, and $r \in \mathbb{Q}^+$ the following equality $\|J(\widetilde{h})\|_r = \|\widetilde{h}\|_r$ holds, that is $\|h\|_r = \|\widetilde{h}\|_r$.

Let $h = \sum_{n=0}^\infty h_n$ and $\widetilde{h} = \sum_{n=0}^\infty \widetilde{h}_n$ be the Taylor series representations of the functions $h \in H_{b\mathbb{P}}(X)$ and $\widetilde{h} \in H_{b\mathbb{P}}(\ell_\infty)$ respectively. Also let $S_{n+1} = h_0 + \ldots + h_n$ and $\widetilde{S}_{n+1} = \widetilde{h}_0 + \ldots + \widetilde{h}_n$ be the partial sums of the given Taylor series. Then the following equalities hold:

$$\lim_{n\to\infty} \|h - S_{n+1}\|_r = 0, \tag{4}$$

$$\lim_{n\to\infty} \|\widetilde{h} - \widetilde{S}_{n+1}\|_r = 0. \tag{5}$$

Besides, by the continuity of a norm we have the following inequalities:

$$|\|h\|_r - \|S_{n+1}\|_r| \le \|h - S_{n+1}\|_r, \tag{6}$$

$$|\|\widetilde{h}\|_r - \|\widetilde{S}_{n+1}\|_r| \le \|\widetilde{h} - \widetilde{S}_{n+1}\|_r. \tag{7}$$

Taking into account (4) and (6) we obtain

$$\lim_{n\to\infty} \|S_{n+1}\|_r = \|h\|_r.$$

Analogically, by (5) and (7) we have

$$\lim_{n\to\infty} \|\widetilde{S}_{n+1}\|_r = \|\widetilde{h}\|_r.$$

Therefore $\lim_{n\to\infty} \|S_{n+1}\|_r = \lim_{n\to\infty} \|\widetilde{S}_{n+1}\|_r$ and so $\|h\|_r = \|\widetilde{h}\|_r$. Thus, the mapping J is the isometry and the algebras $H_{b\mathbb{P}}(X)$ and $H_{b\mathbb{P}}(\ell_\infty)$ are isometrically isomorphic. □

Theorem 3. *Let $P_n : \ell_\infty \to \mathbb{C}$ be defined by*

$$P_n(x) = x_n^n$$

for $x = (x_1, x_2, \ldots) \in \ell_\infty$ and $\|P_n\| = 1, n \in \mathbb{N}$. Then the spectrum $M_{b\mathbb{P}}$ of the algebra $H_{b\mathbb{P}}(\ell_\infty)$ coincides with the set of all point-evaluation functionals at points of ℓ_∞.

Proof. Let $\varphi \in M_{b\mathbb{P}}$ be a character that belongs to the spectrum of the algebra $H_{b\mathbb{P}}(\ell_\infty)$. Let us denote by δ_x the point-evaluation functional at a point $x \in \ell_\infty$. Let us show that $\varphi = \delta_x$ for some $x \in \ell_\infty$.

Every $\varphi \in M_{b\mathbb{P}}$ is uniquely determined by the sequence $(\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n), \ldots)$. Let us put $x = (\varphi(P_1), \sqrt[2]{\varphi(P_2)}, \ldots, \sqrt[n]{\varphi(P_n)}, \ldots)$. Since $\|P_n\| = 1, n \in \mathbb{N}$, then according to the Proposition 2 the sequence $(\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n), \ldots)$ grows no faster than some geometric progression. Thus the sequence $x = (\varphi(P_1), \sqrt[2]{\varphi(P_2)}, \ldots, \sqrt[n]{\varphi(P_n)}, \ldots)$ is bounded and so, $x \in l_\infty$.

Besides, the following chain of equalities holds

$$\delta_x(P_n) = P_n(x) = x_n^n = (\sqrt[n]{\varphi(P_n)})^n = \varphi(P_n).$$

Hence $\varphi = \delta_x$ and every character $\varphi \in M_{b\mathbb{P}}$ is a point-evaluation functional at some point of ℓ_∞. □

Corollary 1. *Let $c_{00} \subset X \subset \ell_\infty$ polynomials $P_n : X \to \mathbb{C}$ be defined by*

$$P_n(x) = x_n^n$$

for every $x = (x_1, x_2, \ldots) \in X$. Then the spectrum $M_{b\mathbb{P}}$ of the algebra $H_{b\mathbb{P}}(X)$ coincides with the set of all point-evaluation functionals at points of ℓ_∞.

REFERENCES

[1] Alencar R., Aron R.M., Galindo P., Zagorodnyuk A.V. *Algebras of symmetric holomorphic functions on ℓ_p*. Bull. London Math. Soc. 2003, **35** (2), 55–64. doi:10.1112/S0024609302001431

[2] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space*. J. Reine Angew. Math. 1991, **415**, 51–93.

[3] Aron R.M., Cole B.J., Gamelin T.W. *Weak-star continuous analytic functions*. Can. J. Math. 1995, **47**, 673–683.

[4] Chernega I.V., Galindo P., Zagorodnyuk A.V. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinb. Math. Soc. 2012, **55**, 125–142. doi:10.1017/S0013091509001655

[5] Galindo P., Vasylyshyn T.V., Zagorodnyuk A.V. *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0, +\infty]$*. Math. Nachr. 2018, **291** (11-12), 1712–1726. doi:10.1002/mana.201700314

[6] Galindo P., Vasylyshyn T.V., Zagorodnyuk A.V. *The algebra of symmetric analytic functions on L_∞*. Proc. Roy. Soc. Edinburgh 2017, **147** (4), 743–761. doi:10.1017/S0308210516000287

[7] Jawad F., Zagorodnyuk A.V. *Supersymmetric Polynomials on the Space of Absolutely Convergent Series*. Symmetry 2019, **11** (9), 1–16. doi:10.3390/sym11091111

[8] Kravtsiv V.V., Vasylyshyn T.V., Zagorodnyuk A.V. *On Algebraic Basis of the Algebra of Symmetric Polynomials on $l_p(\mathbb{C}^n)$*. J. Funct. Spaces 2017, **2017** (7), 8 p. doi:10.1155/2017/4947925

[9] Mujica J. Complex Analysis in Banach Spaces. North Holland 1986.

[10] Zagorodnyuk A.V. *Spectra of Algebras of Entire Functions on Banach Spaces*. Proc. Amer. Math. Soc. 2006, **134**, 2559–2569. doi:10.1090/S0002-9939-06-08260-8

Received 10.09.2019

Галущак С.І. *Спектри алгебр цілих функцій породжених деякою послідовністю поліномів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 311–320.

У даній роботі досліджено властивості топологічної алгебри цілих функцій, породженої зліченною множиною однорідних поліномів на комплексному банаховому просторі.

Нехай X є комплексним банаховим простором. Розглянуто підалгебру $H_{b\mathbb{P}}(X)$ алгебри Фреше цілих функцій обмеженого типу $H_b(X)$, породжену зліченною множиною алгебраїчно незалежних однорідних поліномів $\mathbb{P}$. Показано, що кожен член ряду Тейлора цілої функції, яка належить алгебрі $H_{b\mathbb{P}}(X)$, є алгебраїчною комбінацією елементів $\mathbb{P}$. Узагальнено теорему про обчислення радіус функції лінійного функціонала на випадок довільної підалгебри алгебри $H_b(X)$ на просторі X. Кожен неперервний лінійний мультиплікативний функціонал, який діє з $H_{b\mathbb{P}}(X)$ у $\mathbb{C}$ однозначно визначається послідовністю своїх значень на елементах $\mathbb{P}$. Як наслідок, існує взаємно однозначна відповідність між спектром (множиною всіх неперервних лінійних мультиплікативних функціоналів) алгебри $H_{b\mathbb{P}}(X)$ та деякою множиною послідовностей комплексних чисел. Встановлено оцінку зверху для послідовностей з цієї множини. Також доведено, що кожну функцію, яка належить алгебрі $H_{b\mathbb{P}}(X)$, де X є замкненим підпростором простору ℓ_∞ і містить простір c_{00}, можна єдиним чином аналітично продовжити на ℓ_∞ і алгебри $H_{b\mathbb{P}}(X)$ та $H_{b\mathbb{P}}(\ell_\infty)$ є ізометрично ізоморфними. Описано спектр алгебри $H_{b\mathbb{P}}(X)$ у даному випадку для деякого спеціального вигляду елементів множини $\mathbb{P}$.

Результати даної роботи можуть бути використані для дослідження алгебри симетричних аналітичних функцій на банахових просторах.

Ключові слова і фрази: n-однорідний поліном, аналітична функція, спектр алгебри.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 321–334
doi:10.15330/cmp.11.2.321-334

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.321–334

HRABOVA U.Z., KAL'CHUK I.V.

APPROXIMATION OF THE CLASSES $W^r_{\beta,\infty}$ BY THREE-HARMONIC POISSON INTEGRALS

In the paper, we solve one extremal problem of the theory of approximation of functional classes by linear methods. Namely, questions are investigated concerning the approximation of classes of differentiable functions by λ-methods of summation for their Fourier series, that are defined by the set $\Lambda = \{\lambda_\delta(\cdot)\}$ of continuous on $[0,\infty)$ functions depending on a real parameter δ. The Kolmogorov-Nikol'skii problem is considered, that is one of the special problems among the extremal problems of the theory of approximation. That is, the problem of finding of asymptotic equalities for the quantity $\mathcal{E}(\mathfrak{N}; U_\delta)_X = \sup\limits_{f\in\mathfrak{N}} \|f(\cdot) - U_\delta(f;\cdot;\Lambda)\|_X$, where X is a normalized space, $\mathfrak{N} \subseteq X$ is a given function class, $U_\delta(f;x;\Lambda)$ is a specific method of summation of the Fourier series. In particular, in the paper we investigate approximative properties of the three-harmonic Poisson integrals on the Weyl-Nagy classes. The asymptotic formulas are obtained for the upper bounds of deviations of the three-harmonic Poisson integrals from functions from the classes $W^r_{\beta,\infty}$. These formulas provide a solution of the corresponding Kolmogorov-Nikol'skii problem. Methods of investigation for such extremal problems of the theory of approximation arised and got their development owing to the papers of A.N. Kolmogorov, S.M. Nikol'skii, S.B. Stechkin, N.P. Korneichuk, V.K. Dzyadyk, A.I. Stepanets and others. But these methods are used for the approximations by linear methods defined by triangular matrices. In this paper we modified the mentioned above methods in order to use them while dealing with the summation methods defined by a set of functions of a natural argument.

Key words and phrases: Kolmogorov-Nikol'skii problem, three-harmonic Poisson integral, Weyl-Nagy classes.

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1 INTRODUCTION

Let L be a space of 2π-periodic summable on a period functions f equipped with the norm $\|f\|_L = \int\limits_{-\pi}^{\pi} |f(t)|dt$; C be a space of 2π-periodic continuous functions f in which the norm is set by means of the equality $\|f\|_C = \max\limits_t |f(t)|$; L_∞ be a space of 2π-periodic measurable essentially bounded functions f with the norm $\|f\|_\infty = \operatorname*{ess\,sup}\limits_t |f(t)|$.

Assume that $f \in L$ and $S[f] = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ is the corresponding Fourier series. Let, further, $r > 0$ and $\beta \in \mathbb{R}$. If the series

$$\sum_{k=1}^{\infty} k^r \left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

УДК 517.5
2010 *Mathematics Subject Classification:* 42A05, 41A60.

is the Fourier series of a summable function φ, then we call the function φ a (r,β)-derivative of f in the Weyl–Nagy sense and denote it by f^r_β (see, e.g., [14], p. 130). A set of functions for which this condition is satisfied is denoted by W^r_β. If $f \in W^r_\beta$ and, besides, $\left\|f^r_\beta(\cdot)\right\|_\infty \le 1$, then f belongs to the class $W^r_{\beta,\infty}$.

Let $f \in L$, $\delta > 0$. Functions of the following form

$$P_1(\delta; f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} \left(a_k \cos kx + b_k \sin kx\right),$$

$$P_2(\delta; f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(1 + \frac{k}{2}(1 - e^{-\frac{2}{\delta}})\right) e^{-\frac{k}{\delta}} \left(a_k \cos kx + b_k \sin kx\right),$$

$$P_3(\delta; f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(1 + \frac{1}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}})k + \frac{1}{8}(1 - e^{-\frac{2}{\delta}})^2 k^2\right) e^{-\frac{k}{\delta}} \left(a_k \cos kx + b_k \sin kx\right),$$

are called the Poisson integral [10], the biharmonic Poisson integral [16] and the three-harmonic Poisson integral [2] of the function f, respectively.

The paper is devoted to investigation of asymptotic behavior as $\delta \to \infty$ of the quantity

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \sup_{f \in W^r_{\beta,\infty}} \|f(\cdot) - P_3(\delta; f; \cdot)\|_C. \tag{1}$$

If the function $\varphi(\delta)$ is found in an explicit form, such that $\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \varphi(\delta) + o(\varphi(\delta))$ as $\delta \to \infty$, then according to Stepanets [14, p. 198] we say that the Kolmogorov-Nikol'skii problem is solved for the class $W^r_{\beta,\infty}$ and the three-harmonic Poisson integral in the uniform metric.

The Kolmogorov-Nikol'skii problem for the Poisson integral on classes of differentiable functions have been solved in [7,9,12,15,18,19]. The papers [5,11,20] are devoted to an investigation of analogous problem for the biharmonic Poisson integral. Asymptotic properties of the three-harmonic Poisson integrals were considered in [2], [17]. Nevertheless, the Kolmogorov-Nikol'skii problem have not been solved for the three-harmonic Poisson integral on the classes $W^r_{\beta,\infty}$. Therefore a question arose of finding asymptotic equalities for the quantities (1).

2 Asymptotic equalities for upper bounds of deviations of three-harmonic Poisson integrals from functions from the class $W^r_{\beta,\infty}$.

For the three-harmonic Poisson integral, analogous to the relation (6) from [8], let us rewrite a sum function $\tau(u)$ in the following form

$$\tau(u) = \begin{cases} \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) \delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) u^{-r}, & u \ge \frac{1}{\delta}, \end{cases} \tag{2}$$

where $\gamma = \gamma(\delta) = \frac{1}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}})\delta$, $\theta = \theta(\delta) = \frac{1}{8}(1 - e^{-\frac{2}{\delta}})^2\delta^2$, $\delta > 0$.

The following statement is true.

Theorem 1. *Let $0 < r \le 3$. Then the asymptotic equality holds as $\delta \to \infty$*

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \frac{1}{\delta^r} A(\tau) + O\left(\frac{1}{\delta^3} + \frac{1}{\delta^{r+1}}\right),$$

where the quantity $A(\tau)$ is defined by

$$A(\tau) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \Big| \int\limits_{0}^{\infty} \tau(u) \cos\Big(ut + \frac{\beta\pi}{2}\Big) du \Big| dt \tag{3}$$

and the estimate

$$A(\tau) = \begin{cases} O(1), & 0 < r < 3, \\ O(\ln\delta), & r = 3, \end{cases} \tag{4}$$

takes place.

Proof. To conduct the proof let us use theorem A from [1]. We now check if its conditions are fulfilled. For that reason let us show a summability of the Fourier transform $\widehat{\tau}_\beta(t)$ of function $\tau(u)$ of the form

$$\widehat{\tau}_\beta(t) = \frac{1}{\pi} \int\limits_{0}^{\infty} \tau(u) \cos\Big(ut + \frac{\beta\pi}{2}\Big) du, \tag{5}$$

i.e., a convergence of integral $A(\tau)$ of the form (3). According to theorem 1 from [1], for proving a convergence of the integral (3) it is necessary and sufficient to show that the following integrals are convergent

$$\int\limits_{0}^{\frac{1}{2}} u\, |d\tau'(u)|, \quad \int\limits_{\frac{1}{2}}^{\infty} |u-1|\, |d\tau'(u)|, \quad \int\limits_{0}^{\infty} \frac{|\tau(u)|}{u} du, \quad \int\limits_{0}^{1} \frac{|\tau(1-u) - \tau(1+u)|}{u} du. \tag{6}$$

As while investigating the first integral of (10) from [6] let us estimate the first integral of (6) on each segment $\left[0, \frac{1}{\delta}\right]$ and $\left[\frac{1}{\delta}, \frac{1}{2}\right]$ (assume, that $\delta > 3$). Taking into account that $\tau''(u) \geq 0$ if $u \in \left[0, \frac{1}{\delta}\right]$, $\delta > 3$, and the inequalities

$$e^{-u} \leq 1, \quad e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad u \geq 0, \tag{7}$$

we get

$$\int\limits_{0}^{\frac{1}{\delta}} u\, |d\tau'(u)| = \big(u\tau'(u) - \tau(u)\big)\Big|_{0}^{\frac{1}{\delta}} \leq \delta^r \left(\frac{1}{\delta^2}(\frac{1}{2} - \theta) + \frac{1}{\delta^3}(\frac{\gamma}{2} + \theta)\right).$$

In view of estimates $\frac{1}{2} - \theta \leq \frac{1}{\delta}$, $\;\frac{\gamma}{2} + \theta \leq \frac{3}{2}$, we obtain

$$\int\limits_{0}^{\frac{1}{\delta}} u\, |d\tau'(u)| = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty. \tag{8}$$

Let further $u \in \left[\frac{1}{\delta}, \frac{1}{2}\right]$. We set

$$\begin{aligned} \tau_1(u) &= \left(1 - (1 + \gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3\right)u^{-r}, \\ \tau_2(u) &= \frac{4}{3\delta^2}u^{1-r} + \frac{1}{\delta}u^{2-r} + \frac{1}{6}u^{3-r}, \end{aligned} \tag{9}$$

then $\tau(u) = \tau_1(u) + \tau_2(u)$ and

$$\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau'(u)\right| \leq \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau_1'(u)\right| + \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau_2'(u)\right|. \tag{10}$$

To estimate the first integral from the right-hand side of inequality (10), we first investigate the following function

$$\widetilde{\mu}(u) = 1 - (1 + \gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3. \tag{11}$$

Taking into account, that

$$\widetilde{\mu}'(u) = (1 + \gamma + \theta u^2)e^{-u} - (\gamma + 2\theta u)e^{-u} - \frac{4}{3\delta^2} - \frac{2}{\delta}u - \frac{1}{2}u^2,$$

$$\widetilde{\mu}''(u) = -(1 + \gamma + \theta u^2)e^{-u} - 2(\gamma + 2\theta u)e^{-u} - 2\theta e^{-u} - \frac{2}{\delta} - u,$$

$$\widetilde{\mu}(0) = 0,\ \widetilde{\mu}'(0) = 1 - \gamma - \frac{4}{3\delta^2} < 0,$$

we can show that if $u \geq 0$, then

$$\widetilde{\mu}(u) \leq 0,\ \ \widetilde{\mu}'(u) < 0,\ \ \widetilde{\mu}''(u) < 0. \tag{12}$$

In view of (12) and the inequalities (7) and

$$e^{-u} \leq 1 - u + \frac{u^2}{2} - \frac{u^3}{6} + \frac{u^4}{24},\quad e^{-u} \geq 1 - u + \frac{u^2}{2} - \frac{u^3}{6},\quad e^{-u} \geq 1 - u,\quad u \geq 0,$$

we have

$$|\widetilde{\mu}(u)| \leq u\Big(\gamma - 1 + \frac{4}{3\delta^2}\Big) + u^2\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + u^3\Big(\frac{\gamma}{2} - \theta\Big) + u^4\Big(\frac{1}{24} + \frac{\theta}{2}\Big),$$

$$\left|\widetilde{\mu}'(u)\right| \leq \Big(\gamma - 1 + \frac{4}{3\delta^2}\Big) + 2u\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + 3u^2\Big(\frac{\gamma}{2} - \theta\Big) + u^3\Big(\frac{1}{6} + 2\theta\Big),$$

$$\left|\widetilde{\mu}''(u)\right| \leq 2\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + 6u\Big(\frac{\gamma}{2} - \theta\Big) + u^2\Big(\frac{1}{2} + 6\theta\Big).$$

Further, using the estimates

$$\gamma - 1 + \frac{4}{3\delta^2} \leq \frac{3}{\delta^3},\ \frac{1}{2} - \gamma + \theta + \frac{1}{\delta} \leq \frac{3}{\delta^2},\ \frac{\gamma}{2} - \theta \leq \frac{2}{\delta},\ \frac{1}{24} + \frac{\theta}{2} \leq 1,\ \frac{1}{2} + 6\theta \leq 3,\ \frac{1}{6} + 2\theta \leq 2,$$

we obtain

$$\begin{aligned} &|\widetilde{\mu}(u)| \leq \frac{3}{\delta^3}u + \frac{3}{\delta^2}u^2 + \frac{2}{\delta}u^3 + u^4,\ \left|\widetilde{\mu}'(u)\right| \leq \frac{3}{\delta^3} + \frac{6}{\delta^2}u + \frac{6}{\delta}u^2 + 2u^3,\\ &\left|\widetilde{\mu}''(u)\right| \leq \frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2. \end{aligned} \tag{13}$$

Taking into account (9), (11) and relation (13), in the case $u \geq \frac{1}{\delta}$ we get

$$\begin{aligned}\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\tau_1'(u)| \leq \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{6}{\delta^2}+\frac{12}{\delta}u+3u^2\Big)u^{1-r}du + r\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{6}{\delta^3}+\frac{12}{\delta^2}u+\frac{12}{\delta}u^2+4u^3\Big)u^{-r}du \\ + r(r+1)\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{3}{\delta^3}u+\frac{3}{\delta^2}u^2+\frac{2}{\delta}u^3+u^4\Big)u^{-r-1}du \leq K_1.\end{aligned} \tag{14}$$

One can easily verify that the estimate

$$\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\tau_2'(u)| = O(1) \quad \text{as } \delta \to \infty \tag{15}$$

is true. Combining (14) and (15), we have

$$\int\limits_{0}^{\frac{1}{2}} u|d\tau'(u)| = O(1) \quad \text{as } \delta \to \infty. \tag{16}$$

Now we move to an estimation of the second integral from (6). If $u \geq \frac{1}{\delta}$ from a representation of function $\tau(u)$ of the form (2) we obtain

$$\begin{aligned}\tau''(u) = e^{-u}\big((2\gamma-2\theta-1)+u(4\theta-\gamma)-\theta u^2)\big)u^{-r} - 2re^{-u}\big((1-\gamma)+u(\gamma-2\theta) \\ + \theta u^2\big)u^{-r-1} + r(r+1)(1-(1+\gamma u+\theta u^2)e^{-u})u^{-r-2}.\end{aligned} \tag{17}$$

The relaton (17) yields

$$\begin{aligned}\int\limits_{\frac{1}{2}}^{\infty} |u-1||d\tau'(u)| \leq \int\limits_{\frac{1}{2}}^{\infty} u|d\tau'(u)| \leq \int\limits_{\frac{1}{2}}^{\infty} e^{-u}\big((2\gamma-2\theta-1)+u(4\theta-\gamma)-\theta u^2)\big)u^{1-r}du \\ + 2r\int\limits_{\frac{1}{2}}^{\infty} e^{-u}\big((1-\gamma)+u(\gamma-2\theta)+\theta u^2\big)u^{-r}du + r(r+1)\int\limits_{\frac{1}{2}}^{\infty}(1-(1+\gamma u+\theta u^2))e^{-u}u^{-r-1}du.\end{aligned} \tag{18}$$

Further, taking into account the following estimates for $u \geq 0$

$$\begin{aligned}&1-(1+\gamma u+\theta u^2)e^{-u} \leq 1, \\ &ue^{-u}((1-\gamma)+u(\gamma-2\theta)+\theta u^2) \leq 2, \\ &(2\gamma-2\theta-1)+u(4\theta-\gamma)-\theta u^2 \leq 8,\end{aligned} \tag{19}$$

from (18) we have

$$\int\limits_{\frac{1}{2}}^{\infty} |u-1||d\tau'(u)| = O(1) \quad \text{as } \delta \to \infty. \tag{20}$$

Let us estimate the third integral from (6) on each segment $[0, \frac{1}{\delta}]$, $[\frac{1}{\delta}, 1]$ and $[1, \infty)$. In view of (2) and the inequality

$$1 - e^{-u} - \gamma u e^{-u} - \theta u^2 e^{-u} \le \frac{2}{\delta^2} u + \frac{2}{\delta} u^2 + u^3, \quad u \ge 0, \tag{21}$$

we get

$$\int_0^{\frac{1}{\delta}} \frac{|\tau(u)|}{u} du = \delta^r \int_0^{\frac{1}{\delta}} (1 - e^{-u} - \gamma u e^{-u} - \theta u^2 e^{-u}) \frac{du}{u} \le \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{2}{\delta^2} + \frac{2}{\delta} u + u^2 \right) du \le \frac{K_1}{\delta^{3-r}}. \tag{22}$$

From relations (2), (11), (13) we obtain

$$\left| \int_{\frac{1}{\delta}}^{1} \frac{\tau(u)}{u} du - \frac{4}{3\delta^2} \int_{\frac{1}{\delta}}^{1} u^{-r} du - \frac{1}{\delta} \int_{\frac{1}{\delta}}^{1} u^{1-r} du - \frac{1}{6} \int_{\frac{1}{\delta}}^{1} u^{2-r} du \right|$$

$$\le \int_{\frac{1}{\delta}}^{1} \frac{|\widetilde{\mu}(u)|}{u} du \le \int_{\frac{1}{\delta}}^{1} \left(\frac{3}{\delta^3} + \frac{3}{\delta^2} u + \frac{2}{\delta} u^2 + u^3 \right) u^{-r-1} du \le \begin{cases} K_1, & r < 3, \\ K_2 \ln \delta, & r = 3. \end{cases}$$

Therefore

$$\begin{aligned} \int_{\frac{1}{\delta}}^{1} \frac{|\tau(u)|}{u} du = \frac{4}{3\delta^2} \int_{\frac{1}{\delta}}^{1} u^{-r} du + \frac{1}{\delta} \int_{\frac{1}{\delta}}^{1} u^{1-r} du + \frac{1}{6} \int_{\frac{1}{\delta}}^{1} u^{2-r} du + \begin{cases} O(1), & r < 3, \\ O(\ln \delta), & r = 3, \end{cases} \\ = \begin{cases} O(1), & r < 3, \\ O(\ln \delta), & r = 3, \end{cases} \quad \text{as } \delta \to \infty. \end{aligned} \tag{23}$$

Taking into account the formula (2) and the first inequality from (19), we get

$$\int_1^{\infty} \frac{\tau(u)}{u} du = \int_1^{\infty} (1 - (1 + \gamma u + \theta u^2) e^{-u}) u^{-r-1} du \le \int_1^{\infty} u^{-r-1} du = \frac{1}{r}. \tag{24}$$

From (22)–(24) the estimate follows

$$\int_0^{\infty} \frac{|\tau(u)|}{u} du = \begin{cases} O(1), & r < 3, \\ O(\ln \delta), & r = 3, \end{cases} \quad \text{as } \delta \to \infty. \tag{25}$$

Now we estimate the fourth integral from (6). Similarly as to obtain the formula (39) form [3], we can get the equalities

$$\int_0^1 \frac{|\tau(1-u) - \tau(1+u)|}{u} du = \int_0^1 \frac{|\lambda(1-u) - \lambda(1+u)|}{u} du + O\left(H(\tau)\right), \tag{26}$$

where $H(\tau)$ is defined by equality

$$H(\tau) = |\tau(0)| + |\tau(1)| + \int_0^{\frac{1}{2}} u \left| d\tau'(u) \right| + \int_{\frac{1}{2}}^{\infty} |u-1| \left| d\tau'(u) \right|, \tag{27}$$

and $\lambda(u) = (1+\gamma u+\theta u^2)e^{-u}$. Taking into account, that $\int\limits_0^1 |\lambda(1-u)-\lambda(1+u)|\frac{du}{u} = O(1)$ and using the estimates (16), (20), we have

$$\int\limits_0^1 \frac{|\tau(1-u)-\tau(1+u)|}{u}du = O(1),\ \delta\to\infty. \tag{28}$$

Therefore, in view of theorem 1 from [1], integral $A(\tau)$ of the form (3) is convergent. Using inequalities (2.14) and (2.15) from [1] and the formulas (16), (20), (25) and (28) we obtain the estimate (4).

Hence, we proved that for the function $\tau(u)$ defined by (2) the conditions of theorem A from [1] are fulfilled. Then, as $\delta\to\infty$, the equality

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \frac{1}{\delta^r}A(\tau) + O\Big(\frac{1}{\delta^r}a(\tau)\Big) \tag{29}$$

holds, where

$$a(\tau) = \int\limits_{|t|\ge\frac{\delta\pi}{2}} \big|\widehat{\tau}_\beta(t)\big|dt. \tag{30}$$

Let us estimate the integral (30). First, we represent a transform $\widehat{\tau}_\beta(t)$ in the form

$$\widehat{\tau}_\beta(t) = \frac{1}{\pi}\Big(\int\limits_0^{\frac{1}{\delta}} + \int\limits_{\frac{1}{\delta}}^{\infty}\Big)\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{31}$$

Integrating both integrals from the right-hand side of the equality (31) twice by parts and taking into account that $\tau(0)=0$ and $\lim\limits_{u\to\infty}\tau(u) = \lim\limits_{u\to\infty}\tau'(u) = 0$, we have

$$\widehat{\tau}_\beta(t) = -\frac{1}{\pi t^2}\bigg((1-\gamma)\delta^r\cos\frac{\beta\pi}{2} - r\delta^{r+1}\Big(1-(1+\frac{\gamma}{\delta}+\frac{\theta}{\delta^2})e^{-\frac{1}{\delta}}\Big)\cos\Big(\frac{t}{\delta}+\frac{\beta\pi}{2}\Big)$$
$$+\int\limits_0^{\frac{1}{\delta}}\tau''(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du + \int\limits_{\frac{1}{\delta}}^{\infty}\tau''(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du\bigg).$$

Further, in view of inequalities (21) and $1-\gamma\le\frac{2}{\delta^2}$, we obtain

$$\big|\widehat{\tau}_\beta(t)\big| \le \frac{K_1}{t^2\delta^{2-r}} + \frac{1}{\pi t^2}\Big(\int\limits_0^{\frac{1}{\delta}} + \int\limits_{\frac{1}{\delta}}^{1} + \int\limits_1^{\infty}\Big)|\tau''(u)|du. \tag{32}$$

Taking into account that $\tau''(u)\ge 0$ if $u\in[0,\frac{1}{\delta}]$ ($\delta>3$) and using inequalities $\gamma-2\theta\le\frac{3}{\delta}$, $\theta\le\frac{1}{2}$, we get

$$\int\limits_0^{\frac{1}{\delta}}|\tau''(u)|du = \delta^r e^{-\frac{1}{\delta}}\left((1-\gamma)+\frac{\gamma-2\theta}{\delta}+\frac{\theta}{\delta^2}\right) - \delta^r(1-\gamma) \le \frac{K_2}{\delta^{2-r}}. \tag{33}$$

Let $u \in [\frac{1}{\delta}, 1]$. Repeating the argumentations used to estimate the first integral from (6) on the segment $[\frac{1}{\delta}, \frac{1}{2}]$, we can easily verify that the estimate

$$\int\limits_{\frac{1}{\delta}}^{1} |\tau''(u)|du = O\left(1 + \frac{1}{\delta^{2-r}}\right), \ \delta \to \infty, \tag{34}$$

holds.

Consider now $u \in [1, \infty)$. Taking into account the relation (17), we get

$$\int\limits_{1}^{\infty} |\tau''(u)|du \leq \int\limits_{1}^{\infty} e^{-u}u^{-r}((2\gamma - 2\theta - 1) + u(4\theta - \gamma) - \theta u^2)du$$

$$+2r\int\limits_{1}^{\infty} e^{-u}u^{-r-1}((1-\gamma) + u(\gamma - 2\theta) + \theta u^2)du$$

$$+r(r+1)\int\limits_{1}^{\infty}(1 - (1 + \gamma u + \theta u^2)e^{-u})u^{-r-2}du.$$

In view of the first and the third inequalities from (19) and the inequality

$$e^{-u}((1-\gamma) + u(\gamma - 2\theta) + \theta u^2) \leq 2, \ \ u \geq 1,$$

the last relation yields

$$\int\limits_{1}^{\infty} |\tau''(u)|du \leq K_3. \tag{35}$$

Combining formulas (32)–(35), we obtain

$$\left|\widehat{\tau}_\beta(t)\right| = O\left(1 + \frac{1}{\delta^{2-r}}\right)\frac{1}{t^2}.$$

Therefore,

$$a(\tau) = \int\limits_{|t| \geq \frac{\delta\pi}{2}} |\widehat{\tau}_\beta(t)|dt = O\left(\frac{1}{\delta} + \frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty. \tag{36}$$

From the relations (29) and (36) the equality follows. Theorem 1 is proved.

□

Theorem 2. *If $r > 3$ the following asymptotic equality holds as $\delta \to \infty$*

$$\mathcal{E}\left(W^r_{\beta,\infty}; P_3(\delta)\right)_C = \frac{1}{\delta^3}\sup_{f \in W^r_{\beta,\infty}}\left\|\frac{4}{3}f_0^{(1)}(\cdot) + f_0^{(2)}(\cdot) + \frac{1}{6}f_0^{(3)}(\cdot)\right\|_C + O(\Upsilon(\delta; r)), \tag{37}$$

where $f_0^{(r)}$, $r = 1, 2, 3$, are (r, β)-derivatives in the Weyl-Nagy sense for $\beta = 0$, and

$$\Upsilon(\delta; r) = \begin{cases} \frac{1}{\delta^r}, & 3 < r < 4, \\ \frac{\ln\delta}{\delta^4}, & r = 4, \\ \frac{1}{\delta^4}, & r > 4. \end{cases}$$

Proof. As in the paper [4], let us represent function $\tau(u)$ defined by the relation (2) in the form $\tau(u) = \varphi(u) + \mu(u)$, where

$$\varphi(u) = \begin{cases} (\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3)\delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3\right)u^{-r}, & u \ge \frac{1}{\delta}, \end{cases} \tag{38}$$

$$\mu(u) = \begin{cases} \left(1 - (1+\gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3\right)\delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(1 - (1+\gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3\right)u^{-r}, & u \ge \frac{1}{\delta}. \end{cases} \tag{39}$$

Now we show a convergence of the integrals $A(\varphi)$ and $A(\mu)$ of the form (3).

To prove a convergence of the integral $A(\varphi)$, in view of theorem 1 from [1], let us show a convergence of the integrals

$$\int_0^{\frac{1}{2}} u\left|d\varphi'(u)\right|, \quad \int_{\frac{1}{2}}^{\infty} |u-1|\left|d\varphi'(u)\right|, \quad \int_0^{\infty} \frac{|\varphi(u)|}{u}du, \quad \int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u}du \tag{40}$$

and find their upper estimates.

From (38) we get that for $u \in \left[0, \frac{1}{\delta}\right], \delta > 2$,

$$\int_0^{\frac{1}{\delta}} u|d\varphi'(u)| = \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{2}{\delta}u + u^2\right)du \le \frac{K_1}{\delta^{3-r}}. \tag{41}$$

Since $\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\varphi'(u)| \le \int\limits_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$ and $\int\limits_{\frac{1}{2}}^{\infty} |u-1||d\varphi'(u)| \le \int\limits_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$, then it is sufficient to get an estimate of the integral $\int\limits_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$. If $u \ge \frac{1}{\delta}$ we have

$$\begin{aligned} \int_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|du \le \int_{\frac{1}{\delta}}^{\infty} \left(\frac{2}{\delta} + u\right)u^{-r+1}du + 2r\int_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2} + \frac{2}{\delta}u + \frac{1}{2}u^2\right)u^{-r}du \\ + r(r+1)\int_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3\right)u^{-r-1}du \le \frac{K_2}{\delta^{3-r}}. \end{aligned} \tag{42}$$

Combining (41) and (42), we get

$$\int_0^{\frac{1}{2}} u|d\varphi'(u)| = O\left(\frac{1}{\delta^{3-r}}\right), \int_{\frac{1}{2}}^{\infty} |u-1||d\varphi'(u)| = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty. \tag{43}$$

From (38) we easily derive that

$$\int_0^{\frac{1}{\delta}} \frac{|\varphi(u)|}{u}du = \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{4}{3\delta^2} + \frac{1}{\delta}u + \frac{1}{6}u^2\right)du \le \frac{K_3}{\delta^{3-r}},$$

$$\int_{\frac{1}{\delta}}^{\infty} \frac{|\varphi(u)|}{u} du = \int_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2} + \frac{1}{\delta} u + \frac{1}{6} u^2 \right) u^{-r} du \leq \frac{K_4}{\delta^{3-r}}.$$

Hence,

$$\int_{0}^{\infty} \frac{|\varphi(u)|}{u} du = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty.$$

Analogous to (26), the formula

$$\int_{0}^{1} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = \int_{0}^{1} \frac{|\lambda(1-u) - \lambda(1+u)|}{u} du + O(H(\varphi)) \tag{44}$$

is true, where $\lambda(u) = 1 - \frac{4}{3\delta^2} u - \frac{1}{\delta} u^2 - \frac{1}{6} u^3$, and $H(\varphi)$ is defined by formula (27). In view of the relation $\int\limits_0^1 \frac{|\lambda(1-u)-\lambda(1+u)|}{u} du = O(1)$ and (43), from (44) we have

$$\int_{0}^{1} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty.$$

Therefore, all integrals from (40) are convergent. Further, applying Theorem 1 from the paper [1] we conclude that the integral $A(\varphi)$ converges and the estimate

$$A(\varphi) = O\Big(\frac{1}{\delta^{3-r}}\Big) \quad \text{as } \delta \to \infty$$

holds.

Now we prove a convergence of the integral $A(\mu)$. For this reason, according to Theorem 1 from [1], let us show a convergence of the integrals

$$\int_{0}^{\frac{1}{2}} u \left|d\mu'(u)\right|, \quad \int_{\frac{1}{2}}^{\infty} |u-1| \left|d\mu'(u)\right|, \quad \int_{0}^{\infty} \frac{|\mu(u)|}{u} du, \quad \int_{0}^{1} \frac{|\mu(1-u) - \mu(1+u)|}{u} du. \tag{45}$$

Repeating the argumentations used to estimate the first integral of (24) from [4], we divide the segment $[0, \frac{1}{2}]$ into two parts: $[0, \frac{1}{\delta}]$ and $[\frac{1}{\delta}, \frac{1}{2}]$, $\delta > 2$. From the representation (39) of function $\mu(u)$, for $u \in [0, \frac{1}{\delta}]$ we have $\mu''(u) = \widetilde{\mu}''(u)\delta^r$, where $\widetilde{\mu}(u)$ is defined by equality (11). Then, taking into account the third inequality from (13), we get

$$\int_{0}^{\frac{1}{\delta}} u|d\mu'(u)| \leq \delta^r \int_{0}^{\frac{1}{\delta}} \left(\frac{6}{\delta^2} u + \frac{12}{\delta} u^2 + 3u^3 \right) du = \frac{K_1}{\delta^{4-r}}. \tag{46}$$

Analogous to (14), we obtain

$$\int_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\mu'(u)| = \begin{cases} O(1), & 3 < r < 4, \\ O(\ln \delta), & r = 4, \\ O\Big(\frac{1}{\delta^{4-r}}\Big), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{47}$$

Combining (46) and (47) we get the estimate

$$\int_0^{\frac{1}{2}} u|d\mu'(u)| = \begin{cases} O(1), & 3 < r < 4, \\ O\big(\ln\delta\big), & r = 4, \\ O\Big(\frac{1}{\delta^{4-r}}\Big), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{48}$$

Let us move to an estimation of the second integral from (45). In view of (39), for $u \geq \frac{1}{\delta}$ holds

$$|\mu''(u)| \leq \frac{r(r+1)|\widetilde{\mu}(u)|}{u^{r+2}} + \frac{2r|\widetilde{\mu}'(u)|}{u^{r+1}} + \frac{|\widetilde{\mu}''(u)|}{u^r}. \tag{49}$$

To make further estimations, we take into account inequalities (12) and

$$e^{-u} \leq 1, \quad e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad e^{-u} \geq 1 - u, \quad u \geq 0,$$

and, hence, get

$$|\widetilde{\mu}(u)| \leq u\big(-1 + \gamma + \frac{4}{3\delta^2}\big) + u^2\big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\big) + u^3\big(\frac{\gamma}{2} + \frac{1}{6}\big),$$

$$|\widetilde{\mu}'(u)| \leq \big(-1 + \gamma + \frac{4}{3\delta^2}\big) + u\big(1 - 2\gamma + 2\theta + \frac{2}{\delta}\big) + u^2\big(\frac{3}{2}\gamma + \theta + \frac{1}{2}\big),$$

$$|\widetilde{\mu}''(u)| \leq \big(1 - 2\gamma + 2\theta + \frac{2}{\delta}\big) + u\big(3\gamma + 1\big) + \big(\theta u^2 + 4\theta u\big)e^{-u}.$$

Then, using estimates

$$-1 + \gamma + \frac{4}{3\delta^2} \leq \frac{2}{\delta^2}, \ \frac{1}{2} - \gamma + \theta + \frac{1}{\delta} \leq \frac{2}{\delta}, \ \frac{\gamma}{2} + \frac{1}{6} \leq 1, \ \frac{3}{2}\gamma + \theta + \frac{1}{2} \leq 4,$$

$$3\gamma + 1 \leq 6, \ \big(4\theta u + \theta u^2\big)e^{-u} \leq 2u, \ u \geq 0,$$

we obtain

$$|\widetilde{\mu}(u)| \leq \frac{2}{\delta^2}u + \frac{2}{\delta}u^2 + u^3, \ |\widetilde{\mu}'(u)| \leq \frac{2}{\delta^2} + \frac{4}{\delta}u + 4u^2, \ |\widetilde{\mu}''(u)| \leq \frac{4}{\delta} + 8u, \ u \geq 0. \tag{50}$$

In view of (49), (50), we have

$$\begin{aligned} \int_{\frac{1}{2}}^{\infty} |u-1||d\mu'(u)| &\leq \int_{\frac{1}{2}}^{\infty} u|d\mu'(u)| \leq r(r+1) \int_{\frac{1}{2}}^{\infty} \left(\frac{2}{\delta^2}u + \frac{2}{\delta}u^2 + u^3\right) u^{-r-1}du \\ &+ 2r\int_{\frac{1}{2}}^{\infty} \left(\frac{2}{\delta^2} + \frac{4}{\delta}u + 4u^2\right) u^{-r}du + \int_{\frac{1}{2}}^{\infty} \left(\frac{4}{\delta} + 8u\right) u^{-r+1}du \leq K_1, \ \ r > 3. \end{aligned} \tag{51}$$

Let us estimate the third integral from (45). We devide the segment $[0, \infty)$ into three parts: $[0, \frac{1}{\delta}]$, $[\frac{1}{\delta}, 1]$, $[1, \infty)$. From formula (11) using the first inequality from (13) and (50) we obtain

$$\int_0^{\frac{1}{\delta}} \frac{|\mu(u)|}{u}du = \delta^r \int_0^{\frac{1}{\delta}} |\widetilde{\mu}(u)|\frac{du}{u} \leq \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{3}{\delta^3} + \frac{3}{\delta^2}u + \frac{2}{\delta}u^2 + u^3\right) du \leq \frac{K_1}{\delta^{4-r}};$$

$$\int\limits_{\frac{1}{\delta}}^{1}\frac{|\mu(u)|}{u}du\leq\int\limits_{\frac{1}{\delta}}^{1}\left(\frac{3}{\delta^3}+\frac{3}{\delta^2}u+\frac{2}{\delta}u^2+u^3\right)u^{-r}du\leq\begin{cases}K_2, & 3<r<4,\\ K_3\ln\delta, & r=4,\\ \frac{K_4}{\delta^{4-r}}, & r>4,\end{cases}$$

$$\int\limits_{1}^{\infty}\frac{|\mu(u)|}{u}du\leq\int\limits_{1}^{\infty}\left(\frac{2}{\delta^2}+\frac{2}{\delta}u+u^2\right)u^{-r}du\leq K_5,\quad r>3.$$

Combining last relations, we have

$$\int\limits_{0}^{\infty}\frac{|\mu(u)|}{u}du=\begin{cases}O(1), & 3<r<4,\\ O(\ln\delta), & r=4,\\ O(\frac{1}{\delta^{4-r}}), & r>4,\end{cases}\qquad \text{as } \delta\to\infty. \tag{52}$$

To estimate the fourth integral from (45) we use the formula

$$\int\limits_{0}^{1}|\mu(1-u)-\mu(1+u)|\frac{du}{u}=\int\limits_{0}^{1}|\lambda(1-u)-\lambda(1+u)|\frac{du}{u}+O(H(\mu)), \tag{53}$$

where $\lambda(u)=e^{-u}(1+\gamma u+\theta u^2)+\frac{4}{3\delta^2}u+\frac{1}{\delta}u^2+\frac{1}{6}u^3$, and $H(\mu)$ is defined by formula (27).

In view of $\int\limits_0^1|\lambda(1-u)-\lambda(1+u)|\frac{du}{u}=O(1)$, using relations (48) and (51), from (53) we get

$$\int\limits_{0}^{1}|\mu(1-u)-\mu(1+u)|\frac{du}{u}=\begin{cases}O(1), & 3<r<4,\\ O(\ln\delta), & r=4,\\ O(\frac{1}{\delta^{4-r}}), & r>4,\end{cases}\qquad \text{as } \delta\to\infty. \tag{54}$$

Hence, taking into account Theorem 1 from [1], according to formulas (48), (51), (52) and (54) we can verify that the integral $A(\mu)$ is convergent and the following estimate holds

$$A(\mu)=\begin{cases}O(1), & 3<r<4,\\ O(\ln\delta), & r=4,\\ O(\frac{1}{\delta^{4-r}}), & r>4,\end{cases}\qquad \text{as } \delta\to\infty. \tag{55}$$

In view of the fact, that the Fourier transform $\widehat{\tau}_\beta(t)$ of the form (5) is summable on a whole real axis, for an arbitrary function $f\in W^r_{\beta,\infty}$ and $x\in\mathbb{R}$ the equality

$$f(x)-P_3(\delta;f;x)=\delta^{-r}\int\limits_{-\infty}^{\infty}f^r_\beta\left(x+\frac{t}{\delta}\right)\widehat{\tau}_\beta(t)dt,\ \ \delta>0, \tag{56}$$

is true.

Using (56), (38), (39), for the quantity (1) we get

$$\begin{aligned}\mathcal{E}\left(W^r_{\beta,\infty};P_3(\delta)\right)_C&=\sup_{f\in W^r_{\beta,\infty}}\left\|\delta^{-r}\int\limits_{-\infty}^{\infty}f^r_\beta\left(x+\frac{t}{\delta}\right)\widehat{\tau}_\beta(t)dt\right\|_C\\ &=\sup_{f\in W^r_{\beta,\infty}}\left\|\delta^{-r}\int\limits_{-\infty}^{\infty}f^r_\beta\left(x+\frac{t}{\delta}\right)(\widehat{\varphi}_\beta(t)+\widehat{\mu}_\beta(t))dt\right\|_C\\ &=\sup_{f\in W^r_{\beta,\infty}}\left\|\delta^{-r}\int\limits_{-\infty}^{\infty}f^r_\beta\left(x+\frac{t}{\delta}\right)\widehat{\varphi}_\beta(t)dt\right\|_C+O\left(\delta^{-r}A(\mu)\right).\end{aligned} \tag{57}$$

It is easy to show, that the Fourier series of a continuous function

$$f_\varphi(x) = \int_{-\infty}^{\infty} f^r_\beta\left(x + \frac{t}{\delta}\right) \widehat{\varphi}_\beta(t)dt$$

takes the form

$$S\left[f_\varphi\right] = \sum_{k=1}^{\infty} \varphi\left(\frac{k}{\delta}\right)k^r\left(a_k(f)\cos kx + b_k(f)\sin kx\right), \tag{58}$$

(see speculations used in proving Theorem 1.3.1 from the paper of A.I. Stepanets [13], p. 54). Due to (58), taking into account (38), we obtain the equality

$$S\left[f_\varphi\right] = \frac{1}{\delta^{3-r}} \sum_{k=1}^{\infty} \left(\frac{4}{3}k + k^2 + \frac{1}{6}k^3\right)\left(a_k(f)\cos kx + b_k(f)\sin kx\right).$$

On the other hand,

$$S\left[\frac{4}{3}f_0^{(1)}(x) + f_0^{(2)}(x) + \frac{1}{6}f_0^{(3)}(x)\right] = \frac{1}{\delta^{3-r}} \sum_{k=1}^{\infty} \left(\frac{4}{3}k + k^2 + \frac{1}{6}k^3\right)\left(a_k(f)\cos kx + b_k(f)\sin kx\right).$$

In view of (58), we get for all $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f^r_\beta\left(x + \frac{t}{\delta}\right) \widehat{\varphi}_\beta(t)dt = \frac{1}{\delta^{3-r}}\left(\frac{4}{3}f_0^{(1)}(x) + f_0^{(2)}(x) + \frac{1}{6}f_0^{(3)}(x)\right). \tag{59}$$

Therefore, from (57), in view of formulas (55) and (59), we get the equality (37). Theorem 2 is proved. □

REFERENCES

[1] Bausov L.I. *Linear methods of summation of Fourier series with prescribed rectangular matrices. I.* Izv. Vyssh. Uchebn. Zaved. Mat. 1965, **55** (6), 3–17. (in Russian)

[2] Hrabova U.Z. *Approximative properties of the threeharmonic Poisson integrals on the Hölder classes*. Journal of Automation and Information Sciences 2018, **50** (8), 77–86. doi:10.1615/jautomatinfscien.v50.i8.70

[3] Hrabova U.Z., Kal'chuk I.V., Stepanyuk T.A. *Approximation of functions from the classes $W^r_\beta H^\alpha$ by Weierstrass integrals*. Ukrainian Math. J. 2017, **69** (4), 598–608. doi:10.1007/s11253-017-1383-x

[4] Hrabova U.Z., Kal'chuk I.V., Stepaniuk T.A. *Approximative properties of the Weierstrass integrals on the classes $W^r_\beta H^\alpha$*. J. Math. Sci. (N.Y.) 2018, **231** (1), 41–47. doi:10.1007/s10958-018-3804-2

[5] Hrabova U.Z., Kal'chuk I.V., Stepaniuk T.A. *On the approximation of the classes $W^r_\beta H^\alpha$ by biharmonic Poisson integrals*. Ukrainian Math. J. 2018, **70** (5), 719–729. doi:10.1007/s11253-018-1528-6

[6] Kal'chuk I.V. *Approximation of (ψ,β)–differentiable functions defined on the real axix by Weierstrass integrals*. Ukrainian Math. J. 2007, **59** (9), 1342–1363. doi:10.1007/s11253-007-0091-3

[7] Kal'chuk I.V., Kharkevych Yu.I. *Complete asymptotics of the approximation of function from the Sobolev classes by the Poisson integrals*. Acta Comment. Univ. Tartu. Math. 2018, **22** (1), 23–36. doi:10.12697/ACUTM.2018.22.03

[8] Kharkevych Yu.I., Kal'chuk I.V. *Approximation of (ψ,β)-differentiable functions by Weierstrass integrals*. Ukrainian Math. J. 2007, **59** (7), 1059–1087. doi: 10.1007/s11253-007-0069-1

[9] Kharkevych Yu.I., Pozharska K.V. *Asymptotics of approximation of conjugate functions by Poisson integrals*. Acta Comment. Univ. Tartu. Math. 2018, **22** (2), 235–243. doi:10.12697/ACUTM.2018.22.19

[10] Kharkevych Yu.I., Stepanyuk T.A. *Approximation properties of Poisson integrals for the classes $C^{\psi}_{\beta}H^{\alpha}$*. Math. Notes 2014, **96** (5-6), 1008–1019. doi:10.1134/S0001434614110406

[11] Kharkevych Yu.I., Zhyhallo T.V. *Approximation of functions from the class $\widehat{C}^{\psi}_{\beta,\infty}$ by Poisson biharmonic operators in the uniform metric*. Ukrainian Math. J. 2008, **60** (5), 769–798. doi:10.1007/s11253-008-0093-9

[12] Kharkevych Yu.I., Zhyhallo T.V. *Approximation of (ψ,β)-differentiable functions defined on the real axis by Abel–Poisson operators*. Ukrainian Math. J. 2005, **57** (8), 1297–1315. doi:10.1007/s11253-005-0262-z

[13] Stepanets A.I. Classification and approximation of periodic functions. Kiev: Naukova Dumka, 1987. (in Russian)

[14] Stepanets A.I. Methods of Approximation Theory. Part 1. Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 2002. (in Russian)

[15] Zhyhallo K.M., Kharkevych Yu.I. *Approximation of conjugate differentiable functions by their Abel–Poisson integrals*. Ukrainian Math. J. 2009, **61** (1), 86–98. doi:10.1007/s11253-009-0196-y

[16] Zhyhallo K.M., Kharkevych Yu.I. *Approximation of (ψ,β)-differentiable functions of low smoothness by biharmonic Poisson integrals*. Ukrainian Math. J. 2012, **63** (12), 1820–1844. doi:10.1007/s11253-012-0616-2

[17] Zhyhallo K.M., Kharkevych Yu.I. *On the approximation of functions of the Holder class by triharmonic Poisson integrals*. Ukrainian Math. J. 2001, **53** (6), 1012–1018. doi:10.1023/A:1013364321249

[18] Zhyhallo T.V., Kharkevych Yu.I. *Approximation of functions from the class $C^{\psi}_{\beta,\infty}$ by Poisson integrals in the uniform metric*. Ukrainian Math. J. 2009, **61** (12), 1893–1914. doi:10.1007/s11253-010-0321-y

[19] Zhyhallo T.V., Kharkevych Yu.I. *Approximation of (ψ,β)-differentiable functions by Poisson integrals in the uniform metric*. Ukrainian Math. J. 2009, **61** (11), 1757–1779. doi:10.1007/s11253-010-0311-0

[20] Zhyhallo T.V., Kharkevych Yu.I. *Approximating properties of biharmonic Poisson operators in the clasess $\widehat{L}^{\psi}_{\beta,1}$*. Ukrainian Math. J. 2017, **69** (5), 757–765. doi:10.1007/s11253-017-1393-8

Received 14.09.2019

Revised 22.10.2019

Грабова У.З., Кальчук І.В. *Наближення класів $W^r_{\beta,\infty}$ тригармонійними інтегралами Пуассона* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 321–334.

Робота присвячена розв'язанню однієї з екстремальних задач теорії наближення функціональних класів лінійними методами, а саме дослідженню питань про наближення класів диференційовних функцій λ-методами підсумовування їх рядів Фур'є, заданими сукупністю $\Lambda = \{\lambda_\delta(\cdot)\}$ неперервних на $[0,\infty)$ функцій, залежних від дійсного параметра δ. Розглянуто задачу Колмогорова-Нікольського, що займає особливе місце серед екстремальних задач теорії наближення, тобто задачу про знаходження асимптотичних рівностей для величини $\mathcal{E}(\mathfrak{N};U_\delta)_X = \sup\limits_{f\in\mathfrak{N}} \|f(\cdot) - U_\delta(f;\cdot;\Lambda)\|_X$, де X — нормований простір, $\mathfrak{N} \subseteq X$ — заданий клас функцій, $U_\delta(f;x;\Lambda)$ — конкретний метод підсумовування рядів Фур'є. Зокрема, в роботі досліджуються апроксимативні властивості тригармонійних інтегралів Пуассона на класах Вейля-Надя. Отримано асимптотичні формули для верхніх граней відхилень тригармонійних інтегралів Пуассона від функцій з класів $W^r_{\beta,\infty}$, які забезпечують розв'язок відповідної задачі Колмогорова-Нікольського. Методи дослідження екстремальних задач наближення такого типу виникли і отримали свій розвиток завдяки роботам А.М. Колмогорова, С.М. Нікольського, С.Б. Стечкина, М.П. Корнейчука, В.К. Дзядика, О.І. Степанця та інших, але вони використовуються для наближень лінійними методами підсумовування, що задаються трикутними числовими матрицями. В даній же роботі згадані методи модифіковано для методів підсумовування, що задаються множиною функцій натурального аргументу.

Ключові слова і фрази: задача Колмогорова-Нікольського, тригармонійний інтеграл Пуассона, класи Вейля-Надя.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 335–344
doi:10.15330/cmp.11.2.335-344

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.335–344

JAWAD F., KARPENKO H., ZAGORODNYUK A.

ALGEBRAS GENERATED BY SPECIAL SYMMETRIC POLYNOMIALS ON ℓ_1

Let X be a weighted direct sum of infinity many copies of complex spaces $\ell_1 \bigoplus \ell_1$. We consider an algebra consisting of polynomials on X which are supersymmetric on each term $\ell_1 \bigoplus \ell_1$. Point evaluation functionals on such algebra gives us a relation of equivalence '$\sim$' on X. We investigate the quotient set $X/\sim$ and show that under some conditions, it has a real topological algebra structure.

Key words and phrases: symmetric and supersymmetric polynomials on Banach spaces, algebras of analytic functions on Banach spaces, spectra algebras of analytic functions.

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INTRODUCTION AND PRELIMINARIES

Let X be a complex Banach space and (P_α) a family of continuous complex valued polynomials on X. Often, it is interesting to consider algebras of analytic functions on X, generated by the family of polynomials (see e. g. [6,12,16]). If the family (P_α) does not separate points of X, then the same is true for any function, generated by (P_α). So, we have a natural relation of equivalence on X: $z \sim w$ if and only if $P_\alpha(z) = P_\alpha(w)$ for every α. If X is finite-dimensional, then from the Algebraic Geometry is well known that the quotient set $X/\sim$ is dens in an algebraic variety. The same is true for infinite-dimensional case, if the family (P_α) is finite [2]. But in the general case, the situation may be more complicated.

Let S be the group of all permutations on the set of natural numbers $\mathbb{N}$. A polynomial $P\colon \ell_1 \to \mathbb{C}$ is said to be *symmetric* if $P(\sigma(x)) = P(x)$ for every $X \in \ell_1$ and $\sigma \in S$. It is known [15] that polynomials

$$F_k(X) = \sum_{n=1}^{\infty} x_n^k, \quad k = 1, 2, \ldots,$$

form an algebraic basis in the algebra of all continuous symmetric polynomials $\mathcal{P}_s(\ell_1)$. In other words, $\{F_k\}_{k=1}^{\infty}$ are algebraically independent and $\mathcal{P}_s(\ell_1)$ is the minimal unital algebra containing $\{F_k\}_{k=1}^{\infty}$. In [1] it was shown that two vectors with finite supports $x, y \in \ell_1$ are equivalent in the means $F_k(x) = F_k(y)$ for every k, if and only if $x = \sigma(y)$ for some $\sigma \in S$. Some algebraic operations on $\ell_1/\sim$ which form a semi-ring structure [4] were considered in [5,7]. Composition operators, associated with these operations, on analytic functions were investigated in [8]. Algebras of analytic functions generated by symmetric polynomials on ℓ_p were investigated in [1,3,5–7,13,14].

УДК 517.98
2010 *Mathematics Subject Classification:* 46J15, 46J20.
The work was partially supported by Ministry of Education and Science of Ukraine Grant 0119U100063

Let $X = \ell_1 \bigoplus \ell_1$. We represent each element z of X by $z = (y|x)$, $x, y \in \ell_1$. Let us consider polynomials $T_m\colon X \to \mathbb{C}$,

$$T_m(z) = F_m(x) - F_m(y) = \sum_{k=1}^{\infty}(x_k^m - y_k^m).$$

Polynomials T_m, $m \in \mathbb{N}$ are algebraically independent and form an algebraic basis on the algebra of *supersymmetric* polynomials on X. In [11] the algebra of supersymmetric polynomials was investigated and a commutative ring structure on the corresponding quotient set $X/\sim$ was described.

For a given complex Banach space E with an unconditional basis $\{e_n\}_{n=0}^{\infty}$ we denote by $\ell_1^{(E)}$ a Banach space defined by the following way. If $x \in \ell_1^{(E)}$, then

$$x = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots), \tag{1}$$

where each $x^{(n)} = (x_1^{(n)}, \ldots, x_k^{(n)}, \ldots) \in \ell_1$ and

$$\sum_{n=0}^{\infty} \|x^{(n)}\|_{\ell_1} e_n \in E \qquad \text{with} \qquad \|x\|_{\ell_1^{(E)}} = \left\| \sum_{n=0}^{\infty} \|x^{(n)}\|_{\ell_1} e_n \right\|_E .$$

A polynomial P on $\ell_1^{(E)}$ is *separately symmetric* [10] if for every sequence of permutations on $\mathbb{N}$, $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$, $\sigma_n \in S$ we have $P(\sigma(x)) = P(\sigma_0(x^{(0)}), \ldots, \sigma_n(x^{(n)}), \ldots) = P(x)$ for all $x \in \ell_1^{(E)}$. Polynomials

$$F_m^{(j)}(x) = \sum_{k=1}^{\infty} (x_k^{(j)})^m, \quad j \in \mathbb{Z}_+, \quad m \in \mathbb{N}$$

are separately symmetric and algebraically independent.

In this paper we consider a complex Banach space X which is a weighted direct sum of infinity copies of $\ell_1 \bigoplus \ell_1$ and polynomials which are supersymmetric on each term of this sum. We show that under some assumptions, $X/\sim$ is a real locally convex algebra which contains a normed subalgebra. This is an extension of results on supersymmetric polynomials, obtained in [11]. For details about analytic mappings on Banach spaces we refer the reader to [9].

1 THE RING $\mathcal{M}^{\omega}$

Let ω be a positive number, $0 < \omega \leq 1$. We denote by $\ell_{1,\infty}^{\omega}$ a "weighted" version of the space ℓ_1^E. Namely, if $x \in \ell_{1,\infty}^{\omega}$, then

$$x = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots), \quad x^{(n)} = (x_k^{(n)}) \in \ell_1$$

and

$$\|x\| = \|x\|_{\ell_{1,\infty}^{\omega}} = \max\left(\sum_{n=1}^{\infty} \omega^n \|x^{(n)}\|_{\ell_1}, \sup_{n,k} |x_k^{(n)}| \right).$$

We denote by Λ_1^{ω} the direct sum of two copies of $\ell_{1,\infty}^{\omega}$, $\Lambda_1^{\omega} = \ell_{1,\infty}^{\omega} \bigoplus \ell_{1,\infty}^{\omega}$. Elements of Λ_1^{ω} will be denoted by $(y|x)$, $y \in \ell_{1,\infty}^{\omega}$, $x \in \ell_{1,\infty}^{\omega}$ and $\|(y|x)\| = \|y\|_{\ell_{1,\infty}^{\omega}} + \|x\|_{\ell_{1,\infty}^{\omega}}$. In other words, any element $z \in \Lambda^{\omega}$ can be represented as

$$z = (y|x) = \left(\begin{array}{c|c} \ldots y_k^{(0)} \ldots y_1^{(0)} & x_1^{(0)} \ldots x_k^{(0)} \ldots \\ \ldots & \ldots \\ \ldots y_k^{(n)} \ldots y_1^{(n)} & x_1^{(n)} \ldots x_k^{(n)} \ldots \\ \ldots & \ldots \end{array} \right)$$

or

$$z = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} x_k^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} y_k^{(n)} e_k^{-(n)}, \tag{2}$$

where

$$x_k^{(n)} e_k^{(n)} = \left(\begin{array}{c|c} \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \\ \ldots 0 \ldots 0 & 0 \ldots 0\, x_k^{(n)}\, 0 \ldots \\ \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \end{array} \right)$$

and

$$y_k^{(n)} e_k^{-(n)} = \left(\begin{array}{c|c} \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \\ \ldots\ 0\, y_k^{(n)}\, 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \end{array} \right).$$

Note that the expansion (2) is formal, that is, the series on the right is not convergent in general.

We denote by $\Lambda_1^{\omega+}$ and $\Lambda_1^{\omega-}$ subspaces $\{(0|x) \colon x \in \ell_{1,\infty}^{\omega}\}$ and $\{(y|0) \colon y \in \ell_{1,\infty}^{\omega}\}$ respectively. If $z = (y|x)$ we will use also notations $z_+ = x$ and $z_- = y$ when it will be convenient.

Let us define the following polynomials on Λ_1^{ω}

$$\begin{aligned} T_m^{\omega}(y|x) &= \sum_{n=0}^{\infty} \omega^n F_m^{(n)}(x^{(n)}) - \sum_{n=0}^{\infty} \omega^n F_m^{(n)}(y^{(n)}) \\ &= \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} (x_k^{(n)})^m - \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} (y_k^{(n)})^m, \qquad (y|x) \in \Lambda_1^{\omega}. \end{aligned} \tag{3}$$

Proposition 1. *For every $m \in \mathbb{N}$ the polynomial T_m^{ω} is continuous on Λ_1^{ω} and $\|T_m\| = 1$.*

Proof. Let $\|(y|x)\| \le 1$. Then $\|y\|_{\ell_1^{\omega}} + \|x\|_{\ell_1^{\omega}} \le 1$, and $|x_k^{(n)}| \le 1$ and $|y_k^{(n)}| \le 1$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Thus

$$|T_m^{\omega}(x)| \le \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} \left(|x_k^{(n)}|^m + |y_k^{(n)}|^m \right) \le \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} \left(|x_k^{(n)}| + |y_k^{(n)}| \right) \le \|(y|x)\|.$$

So $\|T_m\| \le 1$. Let now $(y|x)$ be such that $y = 0$, $x^{(0)} = (1, 0, 0, \ldots)$, $x^{(n)} = 0$ for $n > 0$. Then $\|(y|x)\| = 1$ and $T_m(y|x) = 1$. Thus $\|T_m\| = 1$. $\square$

Definition 1. *Let us say that a polynomial $P \colon \Lambda_1^{\omega} \to \mathbb{C}$ is ω-supersymmetric if it is an algebraic combination of polynomials T_m^{ω}, $m \in \mathbb{N}$. We denote by $\mathcal{P}_s^{\omega} = \mathcal{P}_s^{\omega}(\Lambda_1^{\omega})$ the algebra of all ω-supersymmetric polynomials on Λ_1^{ω}.*

Theorem 1. *Let $\omega = 1/N$ for some $N \in \mathbb{N}$, $N > 1$. For every number $a \in \mathbb{R}$ there exists $z_{\{a\}} \in \Lambda_1^\omega$ such that*

$$\|z_{\{a\}}\| = \begin{cases} |a| & \text{if } |a| \geq 1 \\ 1 & \text{if } |a| < 1 \end{cases}$$

and $T_m^\omega(z_{\{a\}}) = a$ for every $m \in \mathbb{N}$.

Proof. Let $a > 0$. Then we can write

$$a = \sum_{j=0}^{\infty} \frac{a_j}{N^j}, \quad a_j \in \mathbb{N}, \tag{4}$$

that is, $a_0 = [a]$ the integer part of a and $(0.a_1a_2\ldots)_N$ is the representation of $a - [a]$ in the positional base N numeral system. Let $z_{\{a\}}$ be of the form $z_{\{a\}} = (0|x_{\{a\}})$, where

$$x_{\{a\}} = \sum_{n=0}^{\infty} x_{\{a\}}^{(n)}$$

and

$$x_{\{a\}}^{(n)} = (\underbrace{1, \ldots, 1}_{a_n}, 0, 0, \ldots) = e_1^{(n)} + e_2^{(n)} + \cdots + e_{a_n}^{(n)}, \quad n = 0, 1, 2, \ldots.$$

Then for $|a| \geq 1$,

$$\|z_{\{a\}}\| = \max\left(\sum_{n=0}^{\infty} \frac{a_n}{N^n}, 1\right) = \sum_{n=0}^{\infty} \frac{a_n}{N^n} = T_m^\omega(z_{\{a\}}) = a, \quad m \in \mathbb{N}$$

and $\|z_{\{a\}}\| = 1$ for $|a| < 1$. If $a < 0$ we can consider $b = -a > 0$. By the same way, using (4) for b, we can find the vector $x_{\{b\}}$. Let us define now $z_{\{a\}} = (x_{\{b\}}|0)$. Then

$$\|z_{\{a\}}\| = \begin{cases} \mu = |a| & \text{if } |a| \geq 1, \\ 1 & \text{if } |a| < 1, \end{cases}$$

and $T_m^\omega(z_{\{a\}}) = a$ for every $m \in \mathbb{N}$. □

Let us recall that two operations on ℓ_1 "$\bullet$" and "$\diamond$" which preserve symmetric polynomials were introduced in [7] and [5]. Namely, let $x = (x_1, x_2, \ldots, x_k, \ldots)$ and $x = (y_1, y_2, \ldots, y_k, \ldots)$ are in ℓ_1, then

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k, \ldots)$$

and $x \diamond y$ is the resulting sequence of ordering the set $\{x_iy_j \colon i, j \in \mathbb{N}\}$ with one single index in some fixed order. It is easy to check that for every symmetric polynomial P on ℓ_1 and fixed $y \in \ell_1$, polynomials $P(x \bullet y)$ and $P(x \diamond y)$ are symmetric. In [11] these operations were extended to $\ell_1 \bigoplus \ell_1$ with preserving supersymmetric polynomials. Now we propose natural extensions of these operations to Λ_1^ω.

Definition 2. *Let $z = (z_-|z_+)$ and $r = (r_-|r_+)$ are in Λ_1^ω. We say that $h = z \bullet r$ if $h_-^{(n)} = z_-^{(n)} \bullet r_-^{(n)}$ and $h_+^{(n)} = z_+^{(n)} \bullet r_+^{(n)}$ for every $n \in \mathbb{Z}_+$. We also say that $s = z \diamond r$ if*

$$s_+^{(n)} = (z_+^{(0)} \diamond r_+^{(n)}) \bullet (z_+^{(1)} \diamond r_+^{(n-1)}) \bullet \cdots \bullet (z_+^{(n)} \diamond r_+^{(0)}) \bullet (z_-^{(0)} \diamond r_-^{(n)}) \bullet (z_-^{(1)} \diamond r_-^{(n-1)}) \bullet \cdots \bullet (z_-^{(n)} \diamond r_-^{(0)})$$

and

$$s_-^{(n)} = (z_+^{(0)} \diamond r_-^{(n)}) \bullet (z_+^{(1)} \diamond r_-^{(n-1)}) \bullet \cdots \bullet (z_+^{(n)} \diamond r_-^{(0)}) \bullet (z_-^{(0)} \diamond r_+^{(n)}) \bullet (z_-^{(1)} \diamond r_+^{(n-1)}) \bullet \cdots \bullet (z_-^{(n)} \diamond r_+^{(0)}).$$

Proposition 2. *$T_m^\omega(z \bullet r) = T_m^\omega(z) + T_m^\omega(r)$ and $T_m^\omega(z \diamond r) = T_m^\omega(z)T_m^\omega(r)$ for all $z, r \in \Lambda_1^\omega$ and $m \in \mathbb{N}$.*

Proof. The first equality directly follows from the definition of T_m^ω (3). Also, in [5] it is proved that $F_m(x \diamond y) = F_m(x)F_m(y)$, $x, y \in \ell_1$, $m \in \mathbb{N}$. So, using (3) and Definition 2, we have for $s = z \diamond r$

$$\begin{aligned}
T_m^\omega(s) &= T_m^\omega(z \diamond r) = \sum_{n=0}^\infty \omega^n F_m^{(n)}(s_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(s_-^{(n)}) \\
&= \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(n)}(z_+^{(j)} \diamond r_+^{(n-j)}) + \sum_{j=0}^n F_m^{(n)}(z_-^{(j)} \diamond r_-^{(n-j)}) \right) \\
&\quad - \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(n)}(z_+^{(j)} \diamond r_-^{(n-j)}) + \sum_{j=0}^n F_m^{(n)}(z_-^{(j)} \diamond r_+^{(n-j)}) \right) \\
&= \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(j)}(z_+^{(j)}) F_m^{(n-j)}(r_+^{(n-j)}) + \sum_{j=0}^n F_m^{(j)}(z_-^{(j)}) F_m^{(n-j)}(r_-^{(n-j)}) \right) \\
&\quad - \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(j)}(z_+^{(j)}) F_m^{(n-j)}(r_-^{(n-j)}) + \sum_{j=0}^n F_m^{(j)}(z_-^{(j)}) F_m^{(n-j)}(r_+^{(n-j)}) \right) \\
&= \left(\sum_{n=0}^\infty \omega^n F_m^{(n)}(z_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(z_-^{(n)}) \right) \left(\sum_{n=0}^\infty \omega^n F_m^{(n)}(r_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(r_-^{(n)}) \right) \\
&= T_m^\omega(z) T_m^\omega(r).
\end{aligned}$$

□

Corollary 1. *Let $P(z) \in \mathcal{P}_s^\omega$. Then, for every fixed $r \in \Lambda_1^\omega$ polynomials $P(z \bullet r)$ and $P(z \diamond r)$ are in $\mathcal{P}_s^\omega$.*

For a given $z = (y|x) \in \Lambda_1^\omega$ we denote $z^- = (x|y)$. Clearly, the map $z \mapsto z^-$ is a continuous involution in $r \in \Lambda_1^\omega$ and $T_m^\omega(z^-) = -T_m^\omega(z)$.

Let us introduce the following relation of equivalence on Λ_1^ω. We say that $z \sim r$ if and only if $T_m^\omega(z) = T_m^\omega(r)$ for every $m \in \mathbb{N}$. Let us denote by $\mathcal{M}^\omega$ the quotient set $\Lambda_1^\omega / \sim$ and by $[z]$ the class of equivalence which contains z.

Proposition 3. *The following operations $[z] + [r] := [z \bullet r]$; $[z][r] := [z \diamond r]$, $z, r \in \Lambda_1^\omega$, of addition and multiplication are well-defined on $\mathcal{M}^\omega \times \mathcal{M}^\omega$ and $(\mathcal{M}^\omega, +, \cdot)$ is a unital commutative ring.*

Proof. Let $z' \in [z]$ and $r' \in [r]$. By Proposition 2 and the definition of the equivalence we have that for every $m \in \mathbb{N}$,

$$T_m^\omega(z) + T_m^\omega(r) = T_m^\omega(z') + T_m^\omega(r') = T_m^\omega(z' \bullet r')$$

and

$$T_m^\omega(z)T_m^\omega(r) = T_m^\omega(z')T_m^\omega(r') = T_m^\omega(z' \diamond r').$$

So the operations on $\mathcal{M}^\omega$ do not depend on representatives. Let $[u] = [z]([r] + [s])$ and $[v] = [z][r] + [z][s]$. Since for every $m \in \mathbb{N}$

$$T_m^\omega(u) = T_m^\omega(z)(T_m^\omega(r) + T_m^\omega(s)) = T_m^\omega(z)T_m^\omega(r) + T_m^\omega(z)T_m^\omega(s) = T_m^\omega(v),$$

so $[u] = [v]$ and we have the distributive law. Clearly that the associativity and commutativity of the addition and multiplication can be proved by the same way. Also, $-[z] = [z^-]$ and $\mathbb{I} = [e_1^{(0)}]$ is the identity. Thus $\mathcal{M}^\omega$ is a unital commutative ring. $\square$

For any $\lambda \in \mathbb{C}$ and $z \in \mathcal{M}^\omega$ we set $\lambda * [z] = [\lambda z]$. Since, $T_m^\omega(\lambda z) = \lambda^m T_m^\omega(z)$, the operation "$*$" is well defined on $\mathbb{C} \times \mathcal{M}^\omega$. But $(\mathcal{M}^\omega, +, *)$ is not a linear space. Indeed, if $z \in \Lambda_1^\omega$ and $z \neq 0$, then $[z] + [z] = [z \bullet z] \neq 2 * [z]$ because $T_m^\omega([z \bullet z]) = 2T_m^\omega(z)$ but $T_m^\omega(2z) = 2^m T_m^\omega(z)$.

2 Operators and seminorms on $\mathcal{M}^{1/N}$

For a given $z = (y|x) \in \Lambda_1^\omega$, we denote by $\operatorname{supp} z$ the *support* of z, that is, the following pair of sets of indexes

$$\operatorname{supp} z = (\{i \in \mathbb{N}, j \in \mathbb{Z}_+ \colon y_i^{(j)} \neq 0\}, \{k \in \mathbb{N}, n \in \mathbb{Z}_+ \colon x_k^{(n)} \neq 0\}).$$

Let us define the following maps on $\Lambda_1^{1/N}$:

$$S_k^{+(n,m)}(z) = (z - x_k^{(n)} e_k^{(n)}) \bullet (\underbrace{x_k^{(m)} e_k^{(m)} \bullet \cdots \bullet (x_k^{(m)} e_k^{(m)}}_{N^{m-n}})$$

and

$$S_k^{-(n,m)}(z) = (z - y_k^{(n)} e_k^{-(n)}) \bullet (\underbrace{y_k^{(m)} e_k^{-(m)} \bullet \cdots \bullet (y_k^{(m)} e_k^{-(m)}}_{N^{m-n}}),$$

where $m \geq n$ and $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a permutation. We denote by $S_\sigma^{+(i)}$ and $S_\sigma^{-(i)}$ linear operators on $\Lambda_1^{1/N}$ such that

$$S_\sigma^{+(i)}(e_k^{(j)}) = e_{\sigma(k)}^{(i)} \text{ if } i = j \text{ and } S_\sigma^{+(i)}(e_k^{\pm(j)}) = e_k^{\pm(j)} \text{ otherwise,}$$

and

$$S_\sigma^{-(i)}(e_k^{-(i)}) = e_{\sigma(k)}^{-(i)} \text{ if } i = j \text{ and } S_\sigma^{-(i)}(e_k^{\pm(j)}) = e_k^{\pm(j)} \text{ otherwise.}$$

Lemma 1. *For every $z = (y|x) \in \Lambda_1^{1/N}$, permutation σ on $\mathbb{N}$ and $m \geq n$ we have*

$$[z] = [S_\sigma^{+(i)}(z)] = [S_\sigma^{-(i)}(z)] = [S_k^{+(n,m)}(z)] = [S_k^{-(n,m)}(z)].$$

Proof. The proof follows from the definitions and direct calculations. $\square$

Proposition 4. *Let $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$ and z has a finite support. If $[z] = [0]$, then there is a number $j \in \mathbb{N}$ and a composition S of a finite set of mappings $\{S_k^{\pm(n,m)}, S_\sigma^{\pm(j)}\}$ defined above such that*

$$S(z) = (y'|x') = \left(\begin{array}{c|c} \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \\ \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots y_k^{\prime(j)} \ldots y_1^{\prime(j)} & x_1^{\prime(j)} \ldots x_k^{\prime(j)} \ldots \\ \ldots 0 \ldots 0 & 0 \ldots 0 \ldots \\ \ldots & \ldots \end{array} \right) = \sum_{k=1}^\infty x_k^{\prime(j)} e_k^{(j)} + \sum_{k=1}^\infty y_k^{\prime(j)} e_k^{-(j)} \tag{5}$$

and $x_k^{\prime(j)} = y_k^{\prime(j)}$ for every $k \in \mathbb{N}$.

Proof. Let j be a minimal number such that $x_k^{(j)} = 0$ and $y_k^{(j)}$ for every $k \in \mathbb{N}$. Using a finite number of mappings $S_k^{\pm(n,m)}$ and Lemma 1 we can find $z' = (y'|x')$, $z' \sim z$ which satisfies (5). So, for every $m \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \left(y_k^{\prime(j)}\right)^m = \sum_{k=1}^{\infty} \left(x_k^{\prime(j)}\right)^m.$$

From [1] it follows that vectors $\left(y_k^{\prime(j)}\right)_k$ and $\left(x_k^{\prime(j)}\right)_k$ coincide up to a permutation σ of coordinates $(x_1, \ldots, x_k, \ldots)$. So, applying $S_\sigma^{(j)}$ to z' we have $x_k^{\prime(j)} = y_k^{\prime(j)}$ for every $k \in \mathbb{N}$. ☐

Corollary 2. *Let $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$, and z has a finite support. Then there is an element $z' = (y'|x') \in \Lambda_1^{1/N}$ such that $z \sim z'$ and z' has the following property: if $y'^{(j)}_i \neq 0$, then $x'^{(n)}_k \neq y'^{(j)}_i$ for all $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$.*

Proof. To get a proof it is enough to apply Proposition 4 to $z \bullet z'^{-} = (y \bullet x'|x \bullet y')$. ☐

Due to Theorem 1, we can introduce an alternative multiplication by *real* constants in $\mathcal{M}^\omega$, at least for the case $\omega = 1/N$, $N \in \mathbb{N}$, $N > 1$.

Theorem 2. *Let $N \in \mathbb{N}$, $N > 1$. Then $\mathcal{M}^{1/N}$ is a real linear commutative unital algebra with respect to the operations of addition and multiplication defined in Proposition 3 and the following multiplication by constants:*

$$a[z] := [z_{\{a\}}][z] = [z_{\{a\}} \diamond z], \quad a \in \mathbb{R},$$

where $z_{\{a\}}$ is as in Theorem 1.

Proof. Note first that from Theorem 1 and Proposition 2 it follows that for every $m \in \mathbb{N}$, $T_m^\omega(z_{\{a\}} \diamond z) = aT_m^\omega(z)$. So $\mathbb{I} = z_{\{1\}}$ is the unity in $\mathcal{M}^{1/N}$ and $[z_{\{a_1+a_2\}}] = [z_{\{a_1\}}] + [z_{\{a_2\}}]$, $a_1, a_2 \in \mathbb{R}$. Thus,

$$a([z] + [r]) = a[z] + a[r] \qquad \text{and} \qquad (a_1 + a_2)[z] = a_1[z] + a_2[z],$$

where $a, a_1, a_2 \in \mathbb{R}$ and $[z], [r] \in \mathcal{M}^{1/N}$. ☐

Let us denote by Ω the class of functions $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that the mappings $\Phi_\gamma\colon \Lambda_1^\omega \to \Lambda_1^\omega$ defined by

$$\Phi_\gamma(z) = \Phi_\gamma(y|x) = \begin{pmatrix} \ldots \gamma(y_k^{(0)}) \ldots \gamma(y_1^{(0)}) & | & \gamma(x_1^{(0)}) \ldots \gamma(x_k^{(0)}) \ldots \\ \ldots & | & \ldots \\ \ldots \gamma(y_k^{(n)}) \ldots \gamma(y_1^{(n)}) & | & \gamma(x_1^{(n)}) \ldots \gamma(x_k^{(n)}) \ldots \\ \ldots & | & \ldots \end{pmatrix}$$

are well defined and $z \sim z'$ implies $\Phi_\gamma(z) = \Phi_\gamma(z')$. Such class is nonempty, for example, $\gamma(t) = t^m \in \Omega$, $m \in \mathbb{N}$.

Theorem 3. *Let $\gamma \in \Omega$. Then Φ_γ generates a linear operator $\widehat{\Phi}_\gamma\colon \mathcal{M}^{1/N} \to \mathcal{M}^{1/N}$ defined by $\widehat{\Phi}_\gamma([z]) = \Phi_\gamma(z)$.*

Proof. From the definition of Ω it follows that $\widehat{\Phi}_\gamma$ is well defined. Also, it is clear

$$\widehat{\Phi}_\gamma([z]+[r]) = \Phi_\gamma(z \bullet r) = \Phi_\gamma(z) \bullet \Phi_\gamma(r) = \widehat{\Phi}_\gamma([z]) + \widehat{\Phi}_\gamma([r]),$$

$z, r \in \Lambda_1^{1/N}$. Let now $z_{\{a\}} = (y_{\{a\}}|x_{\{a\}})$ be as in Theorem 1, that is,

$$x_{\{a\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{a_n} e_i^{(n)}, \quad y_{\{a\}} = 0 \text{ if } a \geq 0 \qquad \text{and} \qquad y_{\{a\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{a_n} e_i^{-(n)}, \quad x_{\{a\}} = 0 \text{ if } a < 0,$$

where

$$|a| = \sum_{j=0}^{\infty} \frac{a_j}{N^j}, \quad a_j \in \mathbb{N}.$$

If $a \geq 0$, then $[z_{\{a\}}][z] = a[z]$, $a \in \mathbb{R}$, $z = (y|x) \in \Lambda_1^{1/N}$ and

$$\Phi_\gamma(z_{\{a\}} \diamond z) = \Phi_\gamma\big((\underbrace{z \bullet \ldots \bullet z}_{a_0}) \diamond e_1^{(0)} \bullet \ldots \bullet (\underbrace{z \bullet \ldots \bullet z}_{a_n}) \diamond e_1^{(n)} \bullet \ldots\big)$$

$$= \big(\underbrace{\Phi_\gamma(z) \bullet \ldots \bullet \Phi_\gamma(z)}_{a_0}\big) \diamond e_1^{(0)} \bullet \ldots \bullet \big(\underbrace{\Phi_\gamma(z) \bullet \ldots \bullet \Phi_\gamma(z)}_{a_n}\big) \diamond e_1^{(n)} \bullet \ldots = z_{\{a\}} \diamond \Phi_\gamma(z).$$

If $a < 0$, we have to replace $e_1^{(n)}$ by $e_1^{-(n)}$, $n \in \mathbb{Z}_+$. So $\widehat{\Phi}_\gamma\big(a[z]\big) = a\widehat{\Phi}_\gamma\big([z]\big)$. Therefore, $\widehat{\Phi}_\gamma$ is a linear operator. $\square$

Let us denote $\tau_m\big([z]\big) = T_m^{1/N}(z)$, $[z] \in \mathcal{M}^{1/N}$, $m \in \mathbb{N}$. Clearly, τ_m are complex valued real-linear and multiplicative functions, that is, τ_m are homomorphisms from $\mathcal{M}^{1/N}$ to $\mathbb{C}$. By the definition of $\mathcal{M}^{1/N}$ we have that functionals $\tau_m \colon m \in \mathbb{N}$ separate points of $\mathcal{M}^{1/N}$. Let us denote by $\bar{z} = \Phi_\gamma(z)$, where $\gamma(t) = \bar{t}$ is the complex conjugate of t. It is easy to check that $\tau_m\big([\bar{z}]\big) = \overline{\tau_m\big([z]\big)}$ and so $\gamma(t) = \bar{t}$ belongs to Ω. So $[z] \mapsto \tau_m\big([\bar{z}]\big)$ is a complex valued functional for every $m \in \mathbb{N}$. Thus $\tau_m + \overline{\tau}_m$ and $-i\big(\tau_m - \overline{\tau}_m\big)$ are real valued linear functionals on $\mathcal{M}^{1/N}$.

Corollary 3. *If $\gamma \in \Omega$ is multiplicative, then $\widehat{\Phi}_\gamma$ is an algebra homomorphism.*

Proof. Let $[z], [r] \in \mathcal{M}^{1/N}$,

$$z = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z_{+k}^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z_{-k}^{(n)} e_k^{-(n)}$$

and

$$r = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} r_{+k}^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} r_{-k}^{(n)} e_k^{-(n)}.$$

Since $\Phi_\gamma\big(z_{+k}^{(n)} e_k^{(n)}\big) = \gamma\big(z_{+k}^{(n)}\big) e_k^{(n)}$, we have

$$\Phi_\gamma\big(z_{\pm k}^{(n)} e_k^{\pm(n)} \diamond r_{\pm i}^{(j)} e_i^{\pm(j)}\big) = \gamma\big(z_{\pm k}^{(n)} r_{\pm i}^{(j)}\big) e_k^{\pm(n)} \diamond e_i^{\pm(j)},$$

$k, i \in \mathbb{N}$, $n, j \in \mathbb{Z}_+$. From the linearity and multiplicativity of τ_m it follows

$$\tau_m\big(\widehat{\Phi}_\gamma([z])\big)\tau_m\big(\widehat{\Phi}_\gamma([r])\big) = \tau_m\big(\widehat{\Phi}_\gamma([z])\widehat{\Phi}_\gamma([r])\big) = \tau_m\big(\widehat{\Phi}_\gamma([z][r])\big).$$

Since it is true for every m, we have

$$\widehat{\Phi}_\gamma([z])\widehat{\Phi}_\gamma([r]) = \widehat{\Phi}_\gamma([z][r]).$$

$\square$

Proposition 5. *Let $\gamma \in \Omega$ and $\gamma(0) = 0$. Then the following formula defines a seminorm on $\mathcal{M}^{1/N}$:*

$$p_\gamma([z]) = \inf_{(y|x)\in[z]} \sum_{n=0}^{\infty} \frac{1}{N^n} \sum_{k=1}^{\infty} \Big(|\gamma(x_k^{(n)})| + |\gamma(y_k^{(n)})|\Big).$$

Proof. Since the infimum is taken over all representations $(y|x) \in [z]$, the norm is well defined. It is easy to check that p_γ is nonnegative and satisfies the triangle inequality and is homogeneous. □

Definition 3. *Let us define the following seminorms on $\mathcal{M}^{1/N}$:*

$$p_m([z]) = p_{\gamma_m}([z]) \text{ for } \gamma_n(t) = t^m.$$

It is clear that $|\tau_m([z])| \le p_m([z])$, $[z] \in \mathcal{M}^{1/N}$ and so, if $[z] \neq 0$, then there is $m \in \mathbb{N}$ such that $p_m([z]) > 0$.

Let us denote $(\mathcal{M}^{1/N}, (p_m))$ the linear space $\mathcal{M}^{1/N}$ endowed with the projective topology, generated by seminorms (p_m). So we have the following proposition.

Proposition 6. *The space $(\mathcal{M}^{1/N}, (p_m))$ is a locally convex metrisable topological vector space and each functional τ_m is continuous on $(\mathcal{M}^{1/N}, (p_m))$.*

Let us denote by $\mathcal{D}$ the following subset of $\mathcal{M}^{1/N}$:

$$\mathcal{D} = \Big\{u \in \mathcal{M}^{1/N} : \text{ there is } z \in u \text{ such that } |z_k^{(n)}| \le 1,\ n \in \mathbb{Z}_+,\ k \in \mathbb{N}\Big\}.$$

Theorem 4. *$\mathcal{D}$ is a subalgebra in $\mathcal{M}^{1/N}$ and the restriction of the topology of $(\mathcal{M}^{1/N}, (p_n))$ to $\mathcal{D}$ is generated by a norm on $\mathcal{D}$.*

Proof. From the definition of addition and multiplication in $\mathcal{M}^{1/N}$ it follows that $u + v \in \mathcal{D}$ and $uv \in \mathcal{D}$ for all $u, v \in \mathcal{D}$. Also, for every $a \in \mathbb{R}$, $[z_{\{a\}}] \in \mathcal{D}$ and so $au = [z_{\{a\}}]u \in \mathcal{D}$. Hence, $\mathcal{D}$ is a subalgebra in $\mathcal{M}^{1/N}$. Note that for every $u \in \mathcal{D}$ and $m \in \mathbb{N}$, $p_m(u) \le p_1(u)$. Also, p_1 is a norm on $\mathcal{D}$. Indeed, if $u \neq 0$, then there is $m \in \mathbb{N}$ such that $\tau_m(u) \neq 0$. So

$$0 \neq |\tau_m(u)| \le p_m(u) \le p_1(u).$$

So $(\mathcal{D}, p_1)$ is a normed space and all p_m are continuous with respect to p_1. So the restriction of topology of $(\mathcal{M}^{1/N}, (p_n))$ to $\mathcal{D}$ coincides with the norm topology of $(\mathcal{D}, p_1)$. □

REFERENCES

[1] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebra of symmetric holomorphic functions on ℓ_p*. Bull. Lond. Math. Soc. 2003, **35**, 55–64. doi:10.1112/S0024609302001431

[2] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space*. J. Reine Angew. Math. 1991, **415**, 51–93.

[3] Aron R., Galindo P., Pinasco D., Zalduendo I. *Group-symmetric holomorphic functions on a Banach space*. Bull. Lond. Math. Soc. 2016, **48** (5), 779–796. doi:10.1112/blms/bdw043

[4] Chernega I.V. *A semiring in the spectrum of the algebra of symmetric analytic functions in the space ℓ_1*. J. Math. Sci. 2016, **212**, 38–45. doi:10.1007/s10958-015-2647-3

[5] Chernega I., Galindo P., Zagorodnyuk A. *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*. Rev. Mat. Complut. 2014, **27** (2), 575–585. doi:10.1007/s13163-013-0128-0

[6] Chernega I., Galindo P., Zagorodnyuk A. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinb. Math. Soc. 2012, **55**, 125–142. doi:10.1017/S0013091509001655

[7] Chernega I., Galindo P., Zagorodnyuk A. *The convolution operation on the spectra of algebras of symmetric analytic functions*. J. Math. Anal. Appl. 2012, **395** (2), 569–577. doi:10.1016/j.jmaa.2012.04.087

[8] Chernega I., Holubchak O., Novosad Z., Zagorodnyuk A. *Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces*. Eur. J. Math. 2019, 12 P. doi:10.1007/s40879-019-00390-z

[9] Dineen S. Complex analysis on infinite-dimensional spaces. Springer-Verlag, London, 1999.

[10] Jawad F. *Note on separately symmetric polynomials on the Cartesian product of ℓ_1*. Mat. Stud. 2018, **50** (2), 204–210. doi:10.15330/ms.50.2.204-210

[11] Jawad F., Zagorodnyuk A. *Supersymmetric Polynomials on the Space of Absolutely Convergent Series*. Symmetry 2019, **11** (9), 1111 (19 p.). doi:10.3390/sym11091111

[12] Galindo P., Vasylyshyn T., Zagorodnyuk A. *The algebra of symmetric analytic functions on L_∞*. Proc. Roy. Soc. Edinburgh Sect. A. 2107, **147** (4), 743–761. doi:10.1017/S0308210516000287

[13] Galindo P., Vasylyshyn T., Zagorodnyuk A. *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0,+\infty)$*. Math. Nachr. 2018, **291** (11-12), 1712–1726. doi:10.1002/mana.201700314

[14] García D., Maestre M., Zalduendo I. *The spectra of algebras of group-symmetric functions*. Proc. Edinb. Math. Soc. 2019, **62** (3), 609–623. doi:10.1017/S0013091518000603

[15] Gonzaléz M., Gonzalo R., Jaramillo J. *Symmetric polynomials on rearrangement invariant function spaces*. J. London Math. Soc. 1999, **59**, 681–697.

[16] Kravtsiv V., Vasylyshyn T., Zagorodnyuk A. *On Algebraic Basis of the Algebra of Symmetric Polynomials on $\ell_p(\mathbf{C^n})$*. J. Funct. Spaces 2017, **2017** (7), Article ID 4947925, 8 p. doi:10.1155/2017/4947925

Received 04.08.2019

Джавад Ф., Карпенко Г., Загороднюк А. *Алгебри, породжені спеціальними симетричними поліномами на ℓ_1* // Карпатські матем. публ. — 2019. — Т.11, №2. — C. 335–344.

Нехай X — зважена пряма сума нескінченної кількості копій комплексного простору $\ell_1 \bigoplus \ell_1$. Ми розглядаємо алгебру, яка складається з поліномів на X, котрі є суперсиметричними на кожному доданку $\ell_1 \bigoplus \ell_1$. Функціонали значень в точках на цій алгебрі задають відношення еквівалентності '$\sim$' на X. У роботі досліджено фактор-множину $X/\sim$ і показано, що за деяких умов на цій множині є структура дійсної топологічної алгебри.

Ключові слова і фрази: симетричні і суперсиметричні поліноми на банахових просторах, алгебри аналітичних функцій на банахових просторах, спектри алгебр аналітичних функцій.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 345–349
doi:10.15330/cmp.11.2.345-349

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.345–349

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A NEW FACTOR THEOREM FOR GENERALIZED ABSOLUTE RIESZ SUMMABILITY

The aim of this paper is to consider an absolute summability method and generalize a theorem concerning $|\bar{N}, p_n|_k$ summability of infinite series to $\varphi-\mid \bar{N}, p_n; \delta \mid_k$ summability of infinite series by using almost increasing sequence. Furthermore, it is explained that a well known result dealing with $|\bar{N}, p_n|_k$ summability is obtained when this generalization is restricted under special conditions.

Key words and phrases: summability factors, almost increasing sequence, infinite series, Hölder inequality, Minkowski inequality.

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INTRODUCTION

A positive sequence (z_n) is said to be almost increasing if there exists a positive increasing sequence (d_n) and two positive constants L and M such that $Ld_n \leq z_n \leq Md_n$ (see [1]).

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid w_n - w_{n-1} \mid^k < \infty.$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [16])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} \mid w_n - w_{n-1} \mid^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid \bar{N}, p_n; \delta \mid_k$ summability (see [4]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $\delta = 0$, then we get $|\bar{N}, p_n|_k$ summability.

УДК 517.521.7
2010 *Mathematics Subject Classification:* 26D15, 40D15, 40F05, 40G99.

1 THE KNOWN RESULT

A well known theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series is given below.

Theorem 1 ([3]). *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (λ_n) and (β_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} \mid s_n \mid^k = O(X_m) \quad as \quad m \to \infty \tag{5}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

2 THE MAIN RESULT

Some works dealing with generalized absolute summability methods have been done (see [5–7, 9, 10, 13–19]). The aim of this paper is to generalize Theorem 1 to $\varphi - |\bar{N}, p_n; \delta|_k$ summability using almost increasing sequence in place of positive non-decreasing sequence.

Theorem 2. *Let (φ_n) be a sequence of positive real numbers such that*

$$\varphi_n p_n = O(P_n), \tag{8}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k - 1} \frac{1}{P_{n-1}} = O(\varphi_v^{\delta k} \frac{1}{P_v}) \quad as \quad m \to \infty. \tag{9}$$

Let (X_n) be an almost increasing sequence. If conditions (1)–(4), (6)–(7) of the Theorem 1 and

$$\sum_{n=1}^{m} \varphi_n^{\delta k} \frac{\mid s_n \mid^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{10}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

We need the following lemmas for the proof of Theorem 2.

Lemma 1 ([11]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, we have that*

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{12}$$

Lemma 2 ([12]). *If the conditions (6) and (7) of Theorem 1 are satisfied, then* $\Delta\left(\frac{P_n}{np_n}\right) = O(\frac{1}{n})$.

Remark 1 ([3]). *It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$.*

3 PROOF OF THEOREM 2

Proof. Let (J_n) indicate $(\bar{N}, p_n)$ means of the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$. Then, for $n \geq 1$, we obtain

$$\bar{\Delta} J_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's formula, we get

$$\begin{aligned}\bar{\Delta} J_n &= \frac{s_n\lambda_n}{n} + \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v+1}P_v\Delta\lambda_v}{(v+1)p_{v+1}} s_v + \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v\lambda_v s_v \Delta(\frac{P_v}{vp_v}) - \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v\lambda_v s_v \frac{1}{v} \\ &= J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4}.\end{aligned}$$

For the proof of Theorem 2, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid J_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4.$$

By using Abel's formula, we have

$$\begin{aligned}\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \mid J_{n,1} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \frac{1}{n^k} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k = O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} |\lambda_n| \frac{|s_n|^k}{n} \\ &= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{\delta k} \frac{|s_v|^k}{v} + O(1)|\lambda_m| \sum_{n=1}^{m} \varphi_n^{\delta k} \frac{|s_n|^k}{n} \\ &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by virtue of (1), (4), (6), (8), (10) and (12).

Now, using Hölder's inequality and (1), (6), (8), we obtain

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid J_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} (\frac{p_n}{P_nP_{n-1}})^k \left(\sum_{v=1}^{n-1} P_v \, |\Delta\lambda_v| \, |s_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \, |\Delta\lambda_v| \, |s_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v\beta_v \, |s_v|^k \right) \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v\beta_v \right)^{k-1}.\end{aligned}$$

Again, using Abel's formula and (3), (9)–(12), we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\mid J_{n,2}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\beta_v\,|s_v|^k = O(1)\sum_{v=1}^{m}P_v\beta_v|s_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{|s_v|^k}{v}v\beta v = O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{|s_r|^k}{r}\\ &\quad + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{|s_v|^k}{v} = O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)X_v + O(1)m\beta_m X_m\\ &= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m = O(1) \text{ as } m\to\infty.\end{aligned}$$

Since $\Delta\left(\frac{P_v}{vp_v}\right) = O(\frac{1}{v})$, as in $J_{n,1}$, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\mid J_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}(\frac{p_n}{P_nP_{n-1}})^k\left(\sum_{v=1}^{n-1}P_v\,|s_v|\,|\lambda_v|\,\frac{1}{v}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v\,|s_v|\,|\lambda_v|\,\frac{1}{v}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\left(\sum_{v=1}^{n-1}(\frac{P_v}{vp_v})^kp_v\,|s_v|^k\,|\lambda_v|^k\right)\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}(\frac{P_v}{vp_v})^{k-1}\frac{P_v}{vp_v}p_v\,|s_v|^k\,|\lambda_v|^k\\ &= O(1)\sum_{v=1}^{m}\frac{P_v}{vp_v}p_v\,|s_v|^k\,|\lambda_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\\ &= O(1)\sum_{v=1}^{m}\frac{P_v}{vp_v}p_v\,|s_v|^k\,|\lambda_v|^{k-1}\,|\lambda_v|\,\varphi_v^{\delta k}\frac{1}{P_v}\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{|s_v|^k}{v}\,|\lambda_v| = O(1) \quad\text{as } m\to\infty,\end{aligned}$$

by means of (1), (4), (6), (8)–(10) and (12).

Finally, as in $J_{n,3}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\mid J_{n,4}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}(\frac{p_n}{P_nP_{n-1}})^k\left(\sum_{v=1}^{n-1}P_v\,|s_v|\,|\lambda_v|\,\frac{1}{v}\right)^k\\ &= O(1) \quad\text{as } m\to\infty,\end{aligned}$$

in view of (1), (4), (6), (8)–(10) and (12).

Thus, the proof of Theorem 2 is completed. □

4 CONCLUSION

If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $\delta = 0$ in Theorem 2, then we get Theorem 1. In this case, condition (10) reduces to condition (5). Also, the conditions (8) and (9) are automatically satisfied.

REFERENCES

[1] Bari N. K., Stečkin S. B. *Best approximations and differential properties of two conjugate functions*. Trudy Moskov. Mat. Obšč. 1956, **5**, 483–522. (in Russian)

[2] Bor H. *On two summability methods*. Math. Proc. Camb. Philos. Soc. 1985, **97**, 147–149.

[3] Bor H. *A note on* $\mid \bar{N}, p_n \mid_k$ *summability factors of infinite series*. Indian J. Pure Appl. Math. 1987, **18**, 330–336.

[4] Bor H. *On local property of* $\mid \bar{N}, p_n; \delta \mid_k$ *summability of factored Fourier series*. J. Math. Anal. Appl. 1993, **179** (2), 646–649.

[5] Bor H., Seyhan H. *On almost increasing sequences and its applications*. Indian J. Pure Appl. Math. 1999, **30**, 1041–1046.

[6] Bor H., Özarslan H. S. *On absolute Riesz summability factors*. J. Math. Anal. Appl. 2000, **246**, 657–663.

[7] Bor H., Özarslan H. S. *A note on absolute summability factors*. Adv. Stud. Contemp. Math. (Kyungshang) 2003, **6** (1), 1–11.

[8] Hardy G. H. Divergent Series. Oxford University Press, Oxford, 1949.

[9] Karakaş A. *A note on absolute summability method involving almost increasing and δ-quasi-monotone sequences*. Int. J. Math. Comput. Sci. 2018, **13** (1), 73–81.

[10] Kartal B. *On generalized absolute Riesz summability method*. Commun. Math. Appl. 2017, **8** (3), 359–364.

[11] Mazhar S. M. *A note on absolute summability factors*. Bull. Inst. Math. Acad. Sinica 1997, **25**, 233–242.

[12] Mishra K. N., Srivastava R. S. L. *On* $|\bar{N}, p_n|$ *summability factors of infinite series*. Indian J. Pure Appl. Math. 1984, **15**, 651–656.

[13] Özarslan H. S. *On almost increasing sequences and its applications*. Int. J. Math. Math. Sci. 2001, **25**, 293–298.

[14] Özarslan H. S. *A note on* $|\bar{N}, p_n; \delta|_k$ *summability factors*. Indian J. Pure Appl. Math. 2002, **33**, 361–366.

[15] Özarslan H. S. *On* $|\bar{N}, p_n; \delta|_k$ *summability factors*. Kyungpook Math. J. 2003, **43**, 107–112.

[16] Seyhan H. *On the local property of* $\varphi - \mid \bar{N}, p_n; \delta \mid_k$ *summability of factored Fourier series*. Bull. Inst. Math. Acad. Sinica 1997, **25**, 311–316.

[17] Seyhan H. *A note on absolute summability factors*. Far East J. Math. Sci. 1998, **6** (1), 157-162.

[18] Seyhan H. *On the absolute summability factors of type (A,B)*. Tamkang J. Math. 1999, **30** (1), 59–62.

[19] Seyhan H., Sönmez A. *On* $\varphi - |\bar{N}, p_n; \delta|_k$ *summability factors*. Portugaliae Math. 1997, **54**, 393–398.

Received 21.02.2019

Каракас А. *Нова факторизаційна теорема для узагальненого абсолютного підсумовування за Рісом* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 345–349.

Метою цієї статті є розгляд методу абсолютного підсумовування і узагальнення теореми про $|\bar{N}, p_n|_k$ сумовність нескінченного ряду до $\varphi- \mid \bar{N}, p_n; \delta \mid_k$ сумовності, використовуючи майже зростаючі послідовності. Більше того, показано, що добре відомі результати для $|\bar{N}, p_n|_k$ сумовності випливають з цих узагальнень за деяких обмежень.

Ключові слова і фрази: сумовні дільники, майже зростаюча послідовність, ряди, нерівність Гьольдера, нарівність Мінковського.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 350–360
doi:10.15330/cmp.11.2.350-360

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.350–360

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ON SINGLE-LAYER POTENTIALS FOR A CLASS OF PSEUDO-DIFFERENTIAL EQUATIONS RELATED TO LINEAR TRANSFORMATIONS OF A SYMMETRIC α-STABLE STOCHASTIC PROCESS

In this article an arbitrary invertible linear transformations of a symmetric α-stable stochastic process in d-dimensional Euclidean space $\mathbb{R}^d$ are investigated. The result of such transformation is a Markov process in $\mathbb{R}^d$ whose generator is the pseudo-differential operator defined by its symbol $(-(Q\xi,\xi)^{\alpha/2})_{\xi\in\mathbb{R}^d}$ with some symmetric positive definite $d\times d$-matrix Q and fixed exponent $\alpha\in(1,2)$. The transition probability density of this process is the fundamental solution of some parabolic pseudo-differential equation. The notion of a single-layer potential for that equation is introduced and its properties are investigated. In particular, an operator is constructed whose role in our consideration is analogous to that the gradient in the classical theory. An analogy to the classical theorem on the jump of the co-normal derivative of the single-layer potential is proved. This result can be applied for solving some boundary-value problems for the parabolic pseudo-differential equations under consideration. For $\alpha = 2$, the process under consideration is a linear transformation of Brownian motion, and all the investigated properties of the single-layer potential are well known.

Key words and phrases: pseudo-differential equation, single-layer potential, α-stable stochastic process, jump theorem.

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INTRODUCTION

Let us consider a symmetric α-stable process $(x_0(t))_{t\geq 0}$ in the d-dimensional Euclidean space $\mathbb{R}^d$ (we denote by $(\cdot,\cdot)$ the inner product in this space), that is, a Markov process with its transition probability density given by the equality

$$g_0(t,x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(\xi,x-y)-t|\xi|^\alpha}d\xi, \quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d,$$

where the exponent $\alpha\in(1,2)$ is fixed. The class of all symmetric α-stable processes can be obtained from the process $(x_0(t))_{t\geq 0}$ by multiplying it on some positive constants. More complex processes can be obtained in the following way.

УДК 519.21
2010 *Mathematics Subject Classification:* 60G52, 35S11.

Let P be some invertible $d \times d$-matrix and $x(t) = Px_0(t)$, $t \geq 0$. This process is obviously Markov process and its transition probability density is given by the equality

$$g(t,x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi,x-y)-t(Q\xi,\xi)^{\alpha/2}} d\xi, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{1}$$

where $Q = PP^T$. It is clear that the process $(x(t))_{t\geq 0}$ is stochastically equivalent to the Markov process $(Lx_0(t))_{t\geq 0}$, where L is some lower triangular matrix which satisfies the equality $Q = LL^T$.

The function g is the fundamental solution of the pseudo-differential equation

$$\frac{\partial u(t,x)}{\partial t} = \mathbf{A}u(t,\cdot)(x), \quad t > 0,\ x \in \mathbb{R}^d, \tag{2}$$

where operator $\mathbf{A}$ is a pseudo-differential operator whose symbol is given by the function $(-(Q\xi,\xi)^{\alpha/2})_{\xi \in \mathbb{R}^d}$. The operator $\mathbf{A}$ is the generator of Markov process $(x(t))_{t\geq 0}$.

For a given surface S, which separates $\mathbb{R}^d$ into two open sets D_- and D_+ ($\mathbb{R}^d = D_- \cup S \cup D_+$) and a given continuous function $(\psi(t,x))_{t\geq 0, x\in S}$, we consider a function

$$v(t,x) = \int_0^t d\tau \int_S g(t-\tau,x,y)\psi(\tau,y)d\sigma_y, \quad t > 0,\ x \in \mathbb{R}^d,$$

where the inner integral is a surface one. The function v is called a single-layer potential on the surface S with the density ψ for equation (2).

In this article, we determine the existence conditions of the single-layer potential and investigate its properties. The case of $Q = c^{2/\alpha}I$ ($c > 0$ and I is a unit $d \times d$-matrix) was considered in article [3]. We will use several methods from [3]. In the case of $\alpha = 2$, the theory of single-layer potentials is well-known (see, for example, [2]).

1 SOME AUXILIARY RESULTS

1.1 The function g

The function g defined above by formula (1) is continuous on the domain $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, and is uniformly continuous on each set of the type $(t,x,y) \in [\tau,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with $\tau > 0$. The following estimations of g and its derivatives are known (see [1, Ch.4]):

$$|\mathbf{D}^k g(t,\cdot,y)(x)| \leq N_k \frac{t}{(t^{1/\alpha} + |y-x|)^{d+\alpha+k}}, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d; \tag{3}$$

$$|\mathbf{D}^{\varkappa} g(t,\cdot,y)(x)| \leq \tilde{N}_{\varkappa} \frac{1}{(t^{1/\alpha} + |y-x|)^{d+\varkappa}}, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d.$$

Here $\mathbf{D}^k$ means a differential operator of the order k ($k = 0,1,2,\ldots$), $\mathbf{D}^{\varkappa}$ means a pseudo-differential operator with a homogeneous symbol $(p_{\varkappa}(\xi))_{\xi \in \mathbb{R}^d}$ of the order $\varkappa$ which has all derivatives of the orders $l < M$ with some $M \geq 2d + \varkappa + \alpha + 1$ and $|p_{\varkappa}^{(l)}(\xi)| \leq C_M|\xi|^{\varkappa - 1}$ with some constant $C_M \geq 0$ for all $\xi \neq 0$, and N_k and $\tilde{N}_{\varkappa}$ are some positive constants.

1.2 The operator A

An action of the operator $\mathbf{A}$ defined in Introduction on a smooth (with at least Lipschitz continuous gradient) and bounded together with its derivatives function $\varphi(x)_{x\in\mathbb{R}^d}$ is given by the expression

$$\mathbf{A}\varphi(x) = \frac{q_\alpha}{(\det Q)^{1/2}} \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x) - (\nabla\varphi(x), y))(Q^{-1}y, y)^{-(d+\alpha)/2} dy, \tag{4}$$

where

$$q_\alpha = \frac{\alpha\Gamma((3-\alpha)/2)\Gamma((d+\alpha)/2)}{\pi^{(d+1)/2}\Gamma(2-\alpha)}.$$

The value of the constant q_α can be obtained by applying the operator $\mathbf{A}$ to the function $\varphi_\xi(x) = e^{i(\xi,x)}$, $x \in \mathbb{R}^d$ with some fixed $\xi \in \mathbb{R}^d$.

1.3 An operator B

Let us introduce the operator $\mathbf{B}$ using its symbol $(i(Q\xi,\xi)^{\alpha/2-1}\xi)_{\xi\in\mathbb{R}^d}$. Some simple calculations lead us to the relation $\mathbf{A} = (\nabla, Q\mathbf{B})$. The action of the operator $\mathbf{B}$ on a bounded Lipschitz continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$ is defined by the following formula

$$\mathbf{B}\varphi(x) = \frac{q_\alpha}{\alpha(\det Q)^{1/2}} \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x))(Q^{-1}y, y)^{-(d-\alpha)/2} Q^{-1}y dy,$$

where q_α has the above meaning.

Let ν be some fixed ort in $\mathbb{R}^d$. Consider the operator $\mathbf{B}_\nu = 2(Q\nu, \mathbf{B})$. We denote the result of its action on the function g with respect to the second argument by $g_\nu(t,x,y)$. Using representation (1) of the function g and the integration by parts, it is easy to obtain the relation

$$g_\nu(t,x,y) = \frac{2}{\alpha}\frac{(y-x,\nu)}{t} g(t,x,y), \quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d. \tag{5}$$

1.4 A surface of the class $H^{1+\gamma}$

Let some surface S in $\mathbb{R}^d$ (a manifold of dimension $d-1$) divide the set $\mathbb{R}^d$ into two open sets: outer D_+ and inner D_- (i.e., $\mathbb{R}^d = D_- \cup S \cup D_+$). Suppose that this surface has a tangent hyperplane at each point $x \in S$. We will denote $\nu(x)$ the unit vector of the outer normal to the surface S at the point $x \in S$. Choose the point $x \in S$ and consider a local orthogonal coordinate system with the origin at this point, such that $\nu(x)$ is the ort of its last axis. Assume the surface S is such that for some $\delta > 0$ each part $S_\delta(x) = S \cap B_\delta(x)$, $x \in S$, of the surface S (here $B_\delta(x)$ is a ball with the radius $\delta > 0$ and the center at the point x) can be described in the mentioned above local coordinate system by the equation $y_d = F_x(y_1, ..., y_{d-1})$ with a single-valued function F_x.

The bounded closed surface S belongs to the class $H^{1+\gamma}$ if the function F_x has all partial derivatives $\frac{\partial F_x}{\partial y_k}$, $k = 1,2,...,d-1$, satisfy Hölder's condition with a power $\gamma \in (0;1)$ and the constant does not dependent on x.

Among the properties of the surface S which belongs to the class $H^{1+\gamma}$ we will use the following one (see [2, Ch.5]): there are some positive numbers δ_0 and r_0, and a finite set of points x_1, x_2,..., x_m on the surface S, such that $S \setminus S_{r_0/2}(x) \subset \cup_{k\in I_x} S_{r_0/2}(x_k)$ for each $x \in S$, and $\min_{k\in I_x} \inf_{y\in S_{r_0/2}(x_k)} |y-x| \geq \delta_0$, where I_x is some subset of the indices $\{1,2,...,m\}$.

2 A SINGLE-LAYER POTENTIAL

2.1 Existence conditions

Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$. Consider some continuous function $(\psi(t,x))_{t\geq 0, x\in S}$ and define the function $(v(t,x))_{t\geq 0, x\in\mathbb{R}^d}$ by the following equality

$$v(t,x) = \int_0^t d\tau \int_S g(t-\tau,x,y)\psi(\tau,y)d\sigma_y, \quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d, \tag{6}$$

where an inner integral is a surface one. This function is called a single-layer potential on the surface S with the density ψ. The following statement contains the conditions under which a single-layer potential is well defined.

Lemma 1. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$ and $(\psi(t,x))_{t\geq 0, x\in S}$ be a continuous function, which satisfies the inequality $|\psi(t,x)| \leq C_T t^{-\beta}$ in each set of $(t;x) \in (0;T] \times S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on $T > 0$). Then, the single-layer potential (6) is finite for all $t > 0$ and $x \in \mathbb{R}^d$.*

Proof. Estimation (3) with $k = 0$ and the fact that (see [3])

$$\int_S \frac{d\sigma_y}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \leq Kt^{-1-1/\alpha}, t>0, x\in\mathbb{R}^d$$

with some constant $K > 0$ imply to the inequality

$$|v(t,x)| \leq KN_0C_T \int_0^t \frac{d\tau}{\tau^\beta(t-\tau)^{1/\alpha}} = KN_0C_T B(1-\beta, 1-1/\alpha)t^{1-\beta-1/\alpha}$$

for all $t \in (0;T]$, $x \in \mathbb{R}^d$ and each $T > 0$. □

2.2 Properties of the single-layer potential

Classically (when $\alpha = 2$), a single-layer potential satisfies the appropriate parabolic differential equation in the domain $(0;+\infty) \times (\mathbb{R}^d \setminus S)$ (see [2, Ch.5]). Let us prove an analogous statement in our case ($1 < \alpha < 2$).

Theorem 1. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$, and $(\psi(t,x))_{t\geq 0, x\in S}$ be a continuous function satisfying the inequality $|\psi(t,x)| \leq C_T t^{-\beta}$ in each set of $(t;x) \in (0;T] \times S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on $T > 0$). Then the single-layer potential (3) satisfies the equation*

$$\frac{\partial v(t,x)}{\partial t} = \mathbf{A}v(t,\cdot)(x), \quad t>0,\ x\in\mathbb{R}^d$$

in the domain $(t;x) \in (0;\infty) \times (\mathbb{R}^d \setminus S)$.

Proof. It has already been mentioned that the function g is the fundamental solution of equation (2), and therefore, for all $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ the equality $\frac{\partial g(t,x,y)}{\partial t} = \mathbf{A}g(t,\cdot,y)(x)$ holds true. So, we only have to prove that the operator $\mathbf{A}$ with respect to the variable x can be moved under the integral symbol in the right-hand part of (6) and the equality

$$\lim_{\varepsilon\to 0+}\int_S g(\varepsilon,x,y)\psi(t,y)d\sigma_y = 0,$$

holds true for $t > 0$, $x \in \mathbb{R}^d \setminus S$. The last is due to estimation (3) with $k = 0$ and the inequality

$$\left|\int_S g(\varepsilon,x,y)\psi(t,y)d\sigma_y\right| \le N_0\varepsilon\frac{|S|}{(\rho(x,S))^{d+\alpha}}C_T t^{-\beta},$$

where $|S|$ is the area of the surface S, and $\rho(x,S)$ is the distance from the point x to the surface S.

Next, we will take presentation (4) of the operator $\mathbf{A}$ and prove the possibility to change the order of integrating in the integral

$$\int_0^t d\tau\int_S \psi(\tau,y)d\sigma_y\int_{\mathbb{R}^d}\frac{g(t-\tau,x+z,y)-g(t-\tau,x,y)-(\nabla g(t-\tau,\cdot,y)(x),z)}{(Q^{-1}z,z)^{(d+\alpha)/2}}dz.$$

Take into account that the following inequalities $\frac{1}{M}|z|^2 \le (Q^{-1}z,z) \le M|z|^2$ hold true with some constant $M > 0$ for all $x \in \mathbb{R}^d$. Divide the last integral into the sum of two integrals I_1 and I_2 taken from the same function: the first of them is by $(0;t) \in S \times B_\varepsilon$, and another is by $(0;t) \in S \times (\mathbb{R}^d \setminus B_\varepsilon)$, where B_ε is a ball of some small enough radius $\varepsilon > 0$ centered at the origin.

Since for $0 < \tau < t$, $x \in \mathbb{R}^d$, $y \in S$, $z \in B_\varepsilon$ the following equality

$$g(t-\tau,x+z,y)-g(t-\tau,x,y)-(\nabla g(t-\tau,\cdot,y)(x),z) = \frac{1}{2}\sum_{i,j=1}^d\frac{\partial^2 g(t-\tau,x+\theta z,y)}{\partial x_i\partial x_j}z_iz_j$$

is true, where $\theta = \theta(\tau;y) \in (0;1)$, the absolute value of the integrand in I_1 is estimated by the expression

$$C_t\tau^{-\beta}\frac{1}{2}d^2N_2\frac{t-\tau}{((t-\tau)^{1/\alpha}+|y-x-\theta z|)^{d+\alpha+2}}|z|^2M^{(d+\alpha)/2}|z|^{-d-\alpha},$$

where estimation (3) for $k = 2$ is used. Take a sufficiently small $\varepsilon > 0$ such that the inequality $\inf_{y\in S,z\in B_\varepsilon,\theta\in(0,1)}|y-x-\theta z| = \rho_0 > 0$ holds true. Therefore, I_1 is absolutely convergent.

The absolute value of the integrand in I_2 is estimated by the expression (estimation (3) is also taken into account)

$$\begin{aligned}&C_t\tau^{-\beta}M^{(d+\alpha)/2}|z|^{-d-\alpha}\left(\frac{N_0(t-\tau)}{((t-\tau)^{1/\alpha}+|y-x-z|)^{d+\alpha}}+\frac{N_0(t-\tau)}{((t-\tau)^{1/\alpha}+|y-x|)^{d+\alpha}}\right.\\&\qquad\left.+\frac{N_1(t-\tau)}{((t-\tau)^{1/\alpha}+|y-x|)^{d+\alpha+1}}|z|\right)\\&\le C_t\tau^{-\beta}M^{(d+\alpha)/2}|z|^{-d-\alpha}\left(\frac{N_0(t-\tau)}{((t-\tau)^{1/\alpha}+|y-x-z|)^{d+\alpha}}+\frac{N_0(t-\tau)}{(\rho(x,S))^{d+\alpha}}+\frac{N_1(t-\tau)}{(\rho(x,S))^{d+\alpha+1}}|z|\right).\end{aligned}$$

Observe that the second and the third terms in this expression are integrable. Consider the integral of the first term and change the variable z into u using the equality $y-x-z = (t-\tau)^{1/\alpha}u$. We have got

$$N_0C_tM^{(d+\alpha)/2}\int_0^t\frac{(t-\tau)^{1+d/\alpha}}{\tau^\beta}d\tau\int_S d\sigma_y\int_{D(\tau,y)}|y-x-(t-\tau)^{1/\alpha}u|^{-d-\alpha}(1+|u|)^{-d-\alpha}du,$$

where $D(\tau,y) = \{u \in \mathbb{R}^d : |y-x-(t-\tau)^{1/\alpha}u| > \varepsilon\}$. Surely, this integral converges, and for completing the proof of the theorem, we have to use the Fubini's theorem. □

2.3 The jump theorem

In a classical theory (with $\alpha = 2$) of a single-layer potential the jump theorem takes an essential place. It is the theorem on the jump of the co-normal derivative of a single-layer potential. This section is devoted to an analogue of that theorem in our situation ($1 < \alpha < 2$).

Lemma 2. *Let the surface S and the function $(\psi(t,x))_{t\geq 0, x\in S}$ satisfy the conditions of Theorem 1. Then for each $t > 0$ and $x \in S$ the following integral*

$$\int_0^t d\tau \int_S \mathbf{B}_{\nu(x)} g(t-\tau,\cdot,y)(x)\psi(\tau,y)d\sigma_y \tag{7}$$

is finite.

Proof. By equality (5), we can rewrite integral (7) in the following form

$$\frac{2}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S (y-x,\nu(x))g(t-\tau,x,y)\psi(\tau,y)d\sigma_y.$$

Taking into account estimation (3) and the properties of the surface S (see Section 1.1), we can obtain for $\tau < t$, $x \in S$, the inequality

$$\begin{aligned}\left|\int_S (y-x,\nu(x))g(t-\tau,x,y)\psi(\tau,y)d\sigma_y\right| &\leq C_t\tau^{-\beta}K((t-\tau)^{\gamma/2}+((t-\tau)^{1/2}+\delta_0)^{-d-\alpha+1}(t-\tau))\\ &\leq const_t\tau^{-\beta}((t-\tau)^{\gamma/2}+(t-\tau)),\end{aligned}$$

where $K > 0$ is some constant and $const_t$ is some positive constant, which probably depends on t. Hence the statement of the lemma is proved. □

Remark 1. *Integral (7) is called a direct value of the action result of the operator $\mathbf{B}_{\nu(x)}$, $x \in S$ on single-layer potential (6) at the point $x \in S$. We will denote it by $\mathbf{B}^{(dv)}_{\nu(x)}v(t,\cdot)(x)$.*

The next statement is the jump theorem mentioned above.

Theorem 2. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$ in $\mathbb{R}^d$, and $(\psi(t,x))_{t\geq 0,x\in S}$ be a continuous function satisfying the inequality $|\psi(t,x)| \leq C_T t^{-\beta}$, $0 < t \leq T$, $x \in S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on T) for each $T > 0$. Then for each $t \geq 0$, $x \in S$ the following equality*

$$\lim_{y\to x\pm}\mathbf{B}_{\nu(x)}v(t,\cdot)(y) = \mp\psi(t,x) + \mathbf{B}^{(dv)}_{\nu(x)}v(t,\cdot)(x),$$

holds true, where $y \to x\pm$ means that y approaches x staying in some closed bounded cone $\mathcal{K} \subset \mathbb{R}^d$ with the vertex at the point x and $\mathcal{K} \subset D_\pm \cup \{x\}$.

Proof. Similar to the classic case it is sufficient to consider only the case of $y = x + \delta\nu(x)$ and $\delta \to 0\pm$. Therefore, taking into account formula (5) we will obtain

$$\begin{aligned}\mathbf{B}_{\nu(x)}v(t,\cdot)(y) &= \frac{2}{\alpha}\int_0^t \frac{d\tau}{t-\tau}\int_S (z-x,\nu(x))g(t-\tau,y,z)\psi(\tau,z)d\sigma_z\\ &\quad -\delta\frac{2}{\alpha}\int_0^t\frac{d\tau}{t-\tau}\int_S g(t-\tau,y,z)\psi(\tau,z)d\sigma_z\\ &= \mathbf{B}^{(dv)}_{\nu(x)}v(t,\cdot)(x) + \frac{2}{\alpha}\int_0^t\frac{d\tau}{t-\tau}\int_S (z-x,\nu(x))(g(t-\tau,y,z)-g(t-\tau,x,z))\psi(\tau,z)d\sigma_z\\ &\quad -\delta\frac{2}{\alpha}\int_0^t\frac{d\tau}{t-\tau}\int_S g(t-\tau,y,z)\psi(\tau,z)d\sigma_z.\end{aligned}$$

Denote the integrals on the right-hand side of this equality by I_1 and I_2 accordingly.

First, let us prove that $\lim_{\delta\to 0} I_1 = 0$. In order to get this proof, rewrite I_1 in the form of the sum of the following expressions

$$J_1^{(1)} = \frac{2}{\alpha}\int_0^{t-\rho}\frac{d\tau}{t-\tau}\int_S (z-x,\nu(x))(g(t-\tau,x+\delta\nu(x),z)-g(t-\tau,x,z))\psi(\tau,z)d\sigma_z,$$

$$J_2^{(1)} = \frac{2}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S_{r_{0/2}}(x)} (z-x,\nu(x))(g(t-\tau,x+\delta\nu(x),z)-g(t-\tau,x,z))\psi(\tau,z)d\sigma_z,$$

$$J_3^{(1)} = \frac{2}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S\setminus S_{r_{0/2}}(x)} (z-x,\nu(x))(g(t-\tau,x+\delta\nu(x),z)-g(t-\tau,x,z))\psi(\tau,z)d\sigma_z,$$

where $0<\rho<t$ is some constant (t is fixed), which should be chosen. We estimate each of these expressions. Taking into account the properties of the surface S, we can obtain $|(z-x,\nu(x))| \le |z-x|^{1+\gamma}$ for $z\in S_{r_{0/2}}(x)$. As a result, we have

$$|J_2^{(1)}| \le C_t\frac{2}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}\int_{S_{r_{0/2}}(x)}\frac{|z-x|^{1+\gamma}}{((t-\tau)^{1/\alpha}+|z-x-\delta\nu(x)|)^{d+\alpha}}d\sigma_z$$
$$+C_t\frac{2}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}\int_{S_{r_{0/2}}(x)}\frac{|z-x|^{1+\gamma}}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}}d\sigma_z.$$

Let $\tilde z$ be the orthogonal projection of the point $z\in S_{r_{0/2}}(x)$ on the tangent hyperplane to S at the point x. Hence, taking into account the inequalities $|z-x|\ge|\tilde z-x|$, $|z-x-\delta\nu(x)|\ge|\tilde z-x|$, $|z-x|\ge const|\tilde z-x|$, where $const$ is some positive constant that does not depend on the point x (see [2, Ch.5]), we will obtain to the inequalities

$$|J_2^{(1)}| \le \hat C_t\frac{2}{\alpha}\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}\int_{\triangle K_{r_{0/2}}(x)}\frac{|\tilde z|d\tilde z}{((t-\tau)^{1/\alpha}+|\tilde z|)^{d+\alpha}}d\sigma_z \le \frac{\tilde C_t}{(t-\rho)^\beta}\rho^{\gamma/2},$$

where $\hat C_t$, $\tilde C_t$ are positive constants that probably depend on t, and $\triangle_{r_{0/2}}(x)\subset\mathbb{R}^{d-1}$ is some bounded set. It means that $\triangle_{r_{0/2}}(x)$ is the orthogonal projection of $S_{r_{0/2}}(x)$ on the tangent hyperplane to S at the point $x\in S$ in the coordinate system of this hyperplane.

Next, we will estimate $J_3^{(1)}$:

$$|J_3^{(1)}| \le C_t\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}\int_{S\setminus S_{r_{0/2}}(x)}\frac{|z-x|}{((t-\tau)^{1/\alpha}+|z-x-\delta\nu(x)|)^{d+\alpha}}d\sigma_z$$
$$+C_t\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}\int_{S\setminus S_{r_{0/2}}(x)}\frac{|z-x|}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}}d\sigma_z.$$

Taking into account that for $z\in S\setminus S_{r_{0/2}}(x)$ the inequalities $|z-x|\ge\delta_0$, $|z-x-\delta\nu(x)|\ge|z-x|-|\delta|\ge\delta_0-|\delta|$ are true (choose δ to be the one that $|\delta|<\delta_0$), we will have

$$|J_3{}^{(1)}| \le \hat C_t(\delta_0-|\delta|)^{-d-\alpha}\int_{t-\rho}^{t}\frac{d\tau}{\tau^\beta}.$$

Thus, the sum $J_2^{(1)}+J_3^{(1)}$ can be made as small as we want by choosing $\rho>0$.

Now, consider $J_1^{(1)}$. Since the function $g(t-\tau,x,z)$ is uniformly continuous in the sets of the type $(\tau,x,z)\in[0;t-\rho]\times K_1\times K_2$, where K_1 and K_2 are some compacta in $\mathbb{R}^d$, and taking into account the integrability of the function ψ on $[0;t-\rho]\times S$ and the boundary of the function $(z-x,\nu(x))$ as the function of z on S, we will obtain that $\lim_{\delta\to0}J_1^{(1)}=0$. Therefore, $I_1\to0$ as $\delta\to0$.

Now, consider the behavior of I_2 as $\delta\to0$. Put I_2 in the form of the sum of the following expressions

$$J_1^{(2)}=\frac{2}{\alpha}\delta\psi(t,x)\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),z)d\sigma_z,$$

$$J_2^{(2)}=\frac{2}{\alpha}\delta\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),z)(\psi(\tau,z)-\psi(t,x))d\sigma_z,$$

$$J_3^{(2)}=\frac{2}{\alpha}\delta\int_{0}^{t-\rho}\frac{d\tau}{t-\tau}\int_{S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),z)\psi(\tau,z)d\sigma_z,$$

$$J_4^{(2)}=\frac{2}{\alpha}\delta\int_{0}^{t}\frac{d\tau}{t-\tau}\int_{S\setminus S_\varepsilon(x)}g(t-\tau,x+\delta\nu(x),z)\psi(\tau,z)d\sigma_z,$$

where $\rho>0$, $\varepsilon>0$ are rather small constants.

Let us estimate each of these terms. We start with the last one. Taking into account the properties of the surface S, there are numbers l_0 (natural) and $p_0>0$, such that we can find points $x_k\in S\setminus S_\varepsilon(x)$, $k=1,2\dots,l_0$ that $S\setminus S_\varepsilon(x)\subset\cup_{k=1}^{l_0}S_{r_{0/2}}(x_k)$ and $\inf_{|\zeta|\le|\delta|}\inf_{z\in S_{r_{0/2}}(x_k)}|z-x-\zeta\nu(x)|\ge p_0$. Then estimation (3) implies

$$|J_4^{(2)}|\le\frac{2}{\alpha}C_tN_0|S|l_0|\delta|\int_0^t\tau^{-\beta}((t-\tau)^{1/\alpha}+p_0)^{-d-\alpha}d\tau\to0,\quad\delta\to0.$$

Similarly, using inequality (3) we will get

$$\begin{aligned}|J_3^{(2)}|&\le\frac{2}{\alpha}|\delta|C_tN_0\int_0^{t-\rho}\frac{d\tau}{t^\beta}\int_{S_\varepsilon(x)}\frac{d\sigma_z}{((t-\tau)^{1/\alpha}+|z-x-\delta\nu(x)|)^{d+\alpha}}\\&\le\frac{2}{\alpha}|\delta|C_tN_0|S|\int_0^{t-\rho}\frac{d\tau}{t^\beta(t-\tau)^{1+d/\alpha}}\to0,\quad\delta\to0.\end{aligned}$$

Here $|S|$ means the area of the surface S like mentioned above.

Now, prove the existence of a limit of $J_1^{(2)}$ as $\delta\to0$. By the way, it means that by choosing $\rho>0$ and $\varepsilon>0$ the term $J_2^{(2)}$ can be made as small as you like. It is sufficiently to note that the function ψ is uniformly continuous on the set $[t-\rho;t]\times S_\varepsilon(x)$. Denote the tangent hyperplane to the surface S at the point $x\in S$ by Π_x, and consider

$$R=\frac{2}{\alpha}\delta\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_{\Pi_x}g(t-\tau,x+\delta\nu(x),z)d\sigma_z.$$

To prove that $\lim_{\delta\to0}(J_1^{(2)}-\psi(t,x)R)=0$, consider

$$\begin{aligned}&\delta\int_0^\rho\frac{d\tau}{\tau}\left(\int_{S_\varepsilon(x)}-\int_{\Pi_x}\right)g(\tau,x+\delta\nu(x),z)d\sigma_z\\&\qquad=\delta\int_0^\rho\frac{d\tau}{\tau}\left(\int_{S_\varepsilon(x)}-\int_{\Pi_x}\right)g(\tau,x+\delta\nu(x),z)d\sigma_z\\&\qquad\qquad-\delta\int_0^\rho\frac{d\tau}{\tau}\int_{\Pi_x\setminus\Pi_\varepsilon(x)}g(\tau,x+\delta\nu(x),z)d\sigma_z\\&\qquad=J'+J'',\end{aligned}$$

where $\Pi_\varepsilon(x)$ is the orthogonal projection $S_\varepsilon(x)$ on Π_x. Taking into account the properties of the surface S, it is easy to see that there is a constant $\theta > 0$ such that for all $z \in \Pi_x \setminus \Pi_\varepsilon(x)$ the inequality $|z| \geq \theta$ is true. Therefore the following estimation

$$|J''| \leq C|\delta|\rho \int_\theta^\infty \frac{r^{d-2}dr}{(\delta^2 + r^2)^{(d+\alpha)/2}} = C\rho \int_{\theta/|\delta|}^\infty \frac{r^{d-2}}{(1+r^2)^{(d+\alpha)/2}} \frac{dr}{|\delta|^\alpha}$$

holds true with some constant $C > 0$. As a result, taking L'Hôpital's rule, we will obtain that $J'' \to 0$ as $\delta \to 0$.

In order to estimate J' let us transfer it to the local coordinate system with its origin at the point x and the vector $\nu(x)$ as the ort of its last axis. We have

$$S_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = F_x(u^{<d>}), u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

$$\Pi_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = 0, u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

where $D_\varepsilon(x)$ is some bounded closed set depended only on properties of the surface S, $u^{<d>}$ is the vector $(u_1, u_2, ..., u_{d-1})$, and F_x is some single-valued function with Hölder continuous gradient (see Section 1.1) with power $\gamma \in (0;1)$. Talking into account inequality (3), it is not difficult to state that

$$|J'| \leq K|\delta| \int_0^\rho \frac{d\tau}{\tau} \int_{D_\varepsilon(x)} \frac{\tau|u|^\gamma(1+|u|^\gamma)du}{(\tau^{1/\alpha} + k\sqrt{|u|^2+\delta^2})^{d+\alpha}}$$
$$\leq \hat{K}|\delta| \int_0^\rho d\tau \int_0^{\varepsilon_0} \frac{r^{d-2+\gamma}dr}{(\tau^{1/\alpha} + k\sqrt{r^2+\delta^2})^{d+\alpha}},$$

where $K > 0, \hat{K} > 0, k > 0, \varepsilon_0 > 0$ are some constants. Changing the order of integration in the last integral and taking into account the equality $\int_0^\infty \frac{d\tau}{(\tau^{1/\alpha}+a)^{d+\alpha}} = \alpha B(d,\alpha)a^{-d}$ that is correct for all $a > 0$, we will obtain the estimation

$$|J'| \leq \tilde{K}|\delta| \int_0^{\varepsilon_0} \frac{r^{d-2+\gamma}dr}{(\sqrt{r^2+\delta^2})^d} \leq \tilde{K} \int_0^\infty \frac{r^{d-2+\gamma}dr}{(\sqrt{r^2+1})^d} |\delta|^\delta$$

with some constant $\tilde{K} > 0$. Therefore $J' \to 0$ as $\delta \to 0$ and $\lim_{\delta\to 0}(J_1^{(2)} - \psi(t,x)R) = 0$. Thus, we have to prove the existence of $\lim_{\delta\to 0\pm} R$ and to find it. In order to prove this we will take the equality proved in [3] ($\hat{v}$ is a fixed ort in $\mathbb{R}^d$)

$$\int_\Pi h_d(x + \lambda\hat{v})d\sigma_x = \frac{1}{\pi}\int_0^\infty e^{-cr^\alpha}\cos\lambda r dr, \tag{8}$$

where $\lambda \in \mathbb{R}^d$, $\Pi = \{x \in \mathbb{R}^d : (x,\hat{v}) = 0\}$ and $h_d(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x;\xi)-c|\xi|^\alpha}d\xi$, $x \in \mathbb{R}^d$. Therefore, after simple transformations related with the changing of the variables in the surface integral, we will obtain the equality

$$I = \int_{\Pi_x} g(t, x + \delta\nu(x), y)d\sigma_y = t^{-1/\alpha}\int_\Pi h_d(z - \delta t^{-1/\alpha}\hat{v})d\sigma_z,$$

where $\Pi = \{z \in \mathbb{R}^d : (z,\hat{v}) = 0\}$, $\hat{v}$ is some ort in $\mathbb{R}^d$. Equality (8) implies

$$I = t^{-1/\alpha}\frac{1}{\pi}\int_0^\infty e^{-cr^\alpha}\cos r\delta t^{-1/\alpha}dr = \frac{1}{\pi}\int_0^\infty e^{-ctr^\alpha}\cos r\delta dr.$$

Thus,

$$R = \frac{2}{\alpha\pi}\delta\int_{t-\rho}^{t}\frac{d\tau}{t-\tau}\int_0^\infty e^{-c(t-\tau)r^\alpha}\cos\delta r dr.$$

By changing the order of integration in this integral, we obtain the equality

$$R = \mathrm{sign}\delta - \frac{2}{\pi}\int_0^\infty e^{-c\rho r^\alpha}\frac{\sin\delta r}{r}dr$$

and, therefore, we have that $\lim_{\delta\to 0\pm} R = \pm 1$. Hence, $\lim_{\delta\to 0\pm} J_1^{(2)} = \pm\psi(t,x)$, $t>0$, $x\in S$ and the theorem has been proved. □

2.4 The single-layer potential with a hyperplane as a carrier

Let S be a hyperplane defined by the equation $(x,\nu) = r$, where $\nu\in\mathbb{R}^d$ is some unit vector, and $r\in\mathbb{R}$ is a fixed real number. Let the function $(\psi(t,x))_{t\geq 0,x\in S}$ be continuous as mentioned above, and the inequality $|\psi(t,x)| \leq C_T t^{-\beta}$ be true for all $0<t\leq T$, $x\in S$ and each $T>0$. Here the constant $C_T>0$ probably depends on T and $\beta<1$. Taking into account inequality (3), we obtain the estimation $\int_S g(t,x,y)d\sigma_y \leq Kt^{-1/\alpha}$, $t>0$, $x\in\mathbb{R}^d$ and for the fixed $x\in\mathbb{R}^d\setminus S$ we have $\int_S g(t,x,y)d\sigma_y \leq Mt$, $t>0$, where $K>0$ is some constant, and $M>0$ is the constant depending on x. This, is analogous to the previous, one can state that the statements of Lemma 1 and Theorem 1 are true in this case (S is a hyperplane) as well.

Furthermore, formula (5) shows that $g_\nu(t,x,y) = 0$ for all $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. Therefore, the equality $\mathbf{B}_{\nu(x)}^{(dv)}v(t,\cdot)(x) = 0,\quad t>0,\ x\in S$ holds true. Thus, an analogue of Theorem 2 is valid.

Theorem 3. *Let S be a hyperplane with a unit normal vector $\nu\in\mathbb{R}^d$ and $(\psi(t,x))_{t\geq 0,x\in S}$ be a continuous function satisfying the following inequality $|\psi(t,x)|\leq C_T t^{-\beta}$, $0<t\leq T$, $x\in S$ with some constants $\beta<1$ and $C_T>0$ (the last one can depend on T) for each $T>0$. Then for all $t>0$, $x\in S$ the following relations*

$$\lim_{y\to x\pm}\mathbf{B}_\nu v(t,\cdot)(y) = \mp\psi(t,x)$$

hold true, where $y\to x+$, (or $y\to x-$) means that $y\to z$ in the way that $(y-x,\nu)>0$ (or $(y-x,\nu)<0$).

Proof. The proof of this theorem repeats the proof of Theorem 2 with some simplification. □

Acknowledgment. The authors thank to Prof. Mykola Portenko for useful discussions on the topic of the paper and an anonymous referee for valuable comments that helped to improve the article.

REFERENCES

[1] Eidelman S. D., Ivasyshen S. D., Kochubei A. N. Analytic methods in the theory of differential and pseudo-differential equations of parabolic type. In: Ball J.A., Böttcher A., Dym H., Langer H., Tretter C. (Eds.) Operator Theory: Advances and Applications, 152. Birkhäuser, Basel, 2004. doi:10.1007/978-3-0348-7844-9

[2] Fridman A. A. Partial differential equations of parabolic type. Prentige-Hall Inc., Englewood Cliffs, N. J., 1964.

[3] Osypchuk M. M., Portenko M. I. *On simple-layer potentials for one class of pseudo-differential equations*. Ukrainian Math. J. 2016, **67** (11), 1704–1720. doi:10.1007/s11253-016-1184-7 (translation of Ukrainian Math. J. 2015, **67** (11), 1512-1524. (in Ukrainian))

Received 25.05.2019

Revised 28.11.2019

Мамалига Х.В., Осипчук М.М. *Потенціали простого шару для одного класу псевдодиференціальних рівнянь пов'язаних з лінійними перетвореннями симетричного α-стійкого випадкового процесу* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 350–360.

Стаття присвячена дослідженню невиродженого лінійного перетворення симетричного α-стійкого випадкового процесу в евклідовому просторі $\mathbb{R}^d$. Результат цього перетворення є процесом Маркова в $\mathbb{R}^d$, чий твірний оператор задається символом $(-(Q\xi,\xi)^{\alpha/2})_{\xi\in\mathbb{R}^d}$ з деякою симетричною додатно визначеною $d\times d$-матрицею Q та фіксованим $\alpha\in(1,2)$. Щільність ймовірності переходу цього процесу є фундаментальним розв'язком деякого параболічного псевдодиференціального рівняння. Вводиться поняття потенціалу простого шару та досліджуються його властивості. Зокрема встановлено оператор, який відіграє роль градієнта в класичній теорії. Доведено аналог класичної теореми про стрибок конормальної похідної потенціалу простого шару. Ця властивість потенціалу простого шалу може бути використана для побудови розв'язків деяких крайових задач для розглянутих параболічних псевдодиференціальних рівнянь. Якщо $\alpha=2$, розглянутий процес є лінійним перетворенням процесу броунівського руху і всі досліджені властивості потенціалу простого шару добре відомі.

Ключові слова і фрази: псевдодиференціальне рівняння, потенціал простого шару, α-стійкий випадковий процес, теорема про стрибок.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 361–378
doi:10.15330/cmp.11.2.361-378

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.361–378

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FILTERING OF MULTIDIMENSIONAL STATIONARY SEQUENCES WITH MISSING OBSERVATIONS

The problem of mean-square optimal linear estimation of linear functionals which depend on the unknown values of a multidimensional stationary stochastic sequence is considered. Estimates are based on observations of the sequence with an additive stationary stochastic noise sequence at points which do not belong to some finite intervals of a real line. Formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of the functionals are proposed under the condition of spectral certainty, where spectral densities of the sequences are exactly known. The minimax (robust) method of estimation is applied in the case where spectral densities are not known exactly while some sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and minimax spectral characteristics are proposed for some special sets of admissible densities.

Key words and phrases: stationary sequence, minimax-robust estimate, mean square error, least favorable spectral density, minimax spectral characteristic.

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INTRODUCTION

The problem of estimation of the unknown values of stochastic processes is of constant interest in the theory and applications of stochastic processes. The formulation of the estimation problems (interpolation, extrapolation and filtering) for stationary stochastic sequences with known spectral densities and reducing these problems to the corresponding problems of the theory of functions belongs to Kolmogorov [17]. Effective methods of solution of the estimation problems for stationary stochastic sequences and processes were developed by Wiener [41] and Yaglom [42,43]. Further results are described in the books by Rozanov [38], Hannan [12], Box et al. [3], Brockwell and Davis [4]. The crucial assumption of most of the methods developed for estimating the unobserved values of stochastic processes is that the spectral densities of the involved stochastic processes are exactly known. In practice, however, complete information on the spectral densities is impossible in most cases. In this situation one finds parametric or non-parametric estimates of the unknown spectral densities and then apply one of the traditional estimation methods provided that the selected spectral densities are true. This procedure can result in significant increasing of the value of the error of estimate as Vastola and Poor [40] have demonstrated with the help of some examples. To avoid this effect one can search estimates which are optimal for all densities from a certain given class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error of

УДК 519.21
2010 *Mathematics Subject Classification:* 60G10, 60G25, 60G35, 62M20, 93E10, 93E11.

estimates. The paper by Grenander [11] was the first one where this approach to extrapolation problem for stationary processes was proposed. Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by Kassam and Poor [16]. Franke [7, 8], Franke and Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for some classes of admissible densities.

In the papers by Moklyachuk [23, 25, 26] results of investigation of the extrapolation, interpolation and filtering problems for functionals which depend on the unknown values of stationary processes and sequences are described. The problem of estimation of functionals which depend on the unknown values of multivariate stationary stochastic processes is the aim of the papers by Moklyachuk and Masyutka [28, 29]. In the book by Moklyachuk and Golichenko [27] results of investigation of the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences are proposed. In their papers Luz and Moklyachuk [18–22] deal with the problems of estimation of functionals which depend on the unknown values of stochastic sequences with stationary increments. Prediction problem for stationary sequences with missing observations is investigated in papers by Bondon [1, 2], Cheng, Miamee and Pourahmadi [5], Cheng and Pourahmadi [6], Kasahara, Pourahmadi and Inoue [15], Pourahmadi, Inoue and Kasahara [35], Pelagatti [34]. In papers by Moklyachuk and Sidei [30–33] an approach is developed to investigation of the interpolation, extrapolation and filtering problems for stationary stochastic sequences with missing observations.

In this paper we investigate the problem of the mean-square optimal estimation of the functional $A\vec{\xi} = \sum\limits_{j\in Z^S} \vec{a}(j)^\top \vec{\xi}(-j)$ which depends on the unknown values of a multidimensional stationary sequence $\{\vec{\xi}(j), j \in \mathbb{Z}\}$ from the observations of the sequence $\{\vec{\xi}(j) + \vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_-\backslash S$, where $\{\vec{\eta}(j), j \in \mathbb{Z}\}$ is uncorrelated with $\{\vec{\xi}(j), j \in \mathbb{Z}\}$ multidimensional stationary sequence, $S = \bigcup\limits_{l=1}^{s} \{-(M_l + N_l), \dots, -M_l\}$, $Z^S = \{1, 2, \dots\}\backslash S^+$, $S^+ = \bigcup\limits_{l=1}^{s} \{M_l, \dots, M_l + N_l\}$, $M_0 = 0$, $N_0 = 0$. The problem is investigated in the case where both spectral densities of the sequences $\{\vec{\xi}(j), j \in \mathbb{Z}\}$ and $\{\vec{\eta}(j), j \in \mathbb{Z}\}$ are known. In this case we derive formulas for calculating the spectral characteristic and the mean-square error of the optimal estimate using the method of projection in the Hilbert space of random variables with finite second moments proposed by Kolmogorov (see, for example, selected works by Kolmogorov [17]). In the case of spectral uncertainty, where the spectral densities of the sequences are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas that determine the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functional are proposed for some specific classes of admissible spectral densities.

1 HILBERT SPACE PROJECTION METHOD OF FILTERING

Consider multidimensional stationary stochastic sequences $\vec{\xi}(j) = \{\xi_k(j)\}_{k=1}^{T}$, $j \in \mathbb{Z}$, and $\vec{\eta}(j) = \{\eta_k(j)\}_{k=1}^{T}$, $j \in \mathbb{Z}$, with absolutely continuous spectral functions and correlation functions of the form

$$R_{\xi}(n) = E\vec{\xi}(j+n)(\vec{\xi}(j))^{*} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\lambda}F(\lambda)d\lambda,$$

$$R_{\eta}(n) = E\vec{\eta}(j+n)(\vec{\eta}(j))^{*} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\lambda}G(\lambda)d\lambda,$$

where $F(\lambda) = \{f_{kl}(\lambda)\}_{k,l=1}^{T}$, $G(\lambda) = \{g_{kl}(\lambda)\}_{k,l=1}^{T}$ are the spectral densities of the sequences $\{\vec{\xi}(j), j \in \mathbb{Z}\}$ and $\{\vec{\eta}(j), j \in \mathbb{Z}\}$ respectively. We will suppose that the spectral densities $F(\lambda)$ and $G(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi} \mathrm{Tr}\ (F(\lambda) + G(\lambda))^{-1}\, d\lambda < \infty. \tag{1}$$

This condition is necessary and sufficient in order that the error-free filtering of unknown values of the sequences is impossible (see, for example, Rozanov [38]).
The stationary stochastic sequences $\{\vec{\xi}(j)\}$ and $\{\vec{\eta}(j)\}$ admit the following spectral decomposition (see, for example, Gikhman and Skorokhod [10]; Karhunen [14])

$$\xi(j) = \int_{-\pi}^{\pi} e^{ij\lambda} Z_{\xi}(d\lambda), \qquad \eta(j) = \int_{-\pi}^{\pi} e^{ij\lambda} Z_{\eta}(d\lambda),$$

where $Z_{\xi}(d\lambda)$ and $Z_{\eta}(d\lambda)$ are orthogonal stochastic measures defined on $[-\pi, \pi)$ such that the following relations hold true

$$EZ_{\xi}(\Delta_1)(Z_{\xi}(\Delta_2))^{*} = \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} F(\lambda)d\lambda,$$

$$EZ_{\eta}(\Delta_1)(Z_{\eta}(\Delta_2))^{*} = \frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} G(\lambda)d\lambda.$$

Suppose that we have observations of the sequence $\{\vec{\xi}(j) + \vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_{-}\backslash S$, where $S = \bigcup_{l=1}^{s}\{-(M_l + N_l), \dots, -M_l\}$. The problem is to find the mean-square optimal linear estimate of the functional

$$A\vec{\xi} = \sum_{j\in Z^S} \vec{a}(j)^{\top}\vec{\xi}(-j),$$

which depends on the unknown values of the sequence $\{\vec{\xi}(j)\}$, $Z^S = \{1, 2, \dots\}\backslash S^{+}$, $S^{+} = \bigcup_{l=1}^{s}\{M_l, \dots, M_l + N_l\}$.

Suppose that coefficients $\{\vec{a}(j), j = 0, 1, \dots\}$ defining the functional $A\vec{\xi}$ satisfy the condition

$$\sum_{j\in Z^S}\sum_{k=1}^{T} |a_k(j)| < \infty.$$

This condition ensures that the functional $A\vec{\xi}$ has a finite second moment, since

$$E\,|A\xi|^2 \le \max_{1\le k\le T} E\,|\xi_k(0)|^2 \left(\sum_{j\in Z^S}\sum_{k=1}^{T}|a_k(j)|\right)^2.$$

It follows from the spectral decomposition of the sequence $\{\vec{\xi}(j)\}$ that the functional $A\vec{\xi}$ can be represented in the following form

$$A\vec{\xi} = \int_{-\pi}^{\pi} (A(e^{i\lambda}))^{\top} Z_{\xi}(d\lambda), \quad A(e^{i\lambda}) = \sum_{j\in Z^S} \vec{a}(j)e^{-ij\lambda}.$$

Consider values $\xi_k(j), k = 1,\dots,T; j \in \mathbb{Z}$ and $\eta_k(j), k = 1,\dots,T; j \in \mathbb{Z}$ as elements of the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ generated by random variables ξ with zero mathematical expectations, $E\xi = 0$, finite variations, $E|\xi|^2 < \infty$, and the inner product $(\xi, \eta) = E\xi\overline{\eta}$. Denote by $H^s(\xi + \eta)$ the closed linear subspace generated by elements $\{\xi_k(j) + \eta_k(j) : j \in \mathbb{Z}_-\backslash S, k = \overline{1,T}\}$ in the Hilbert space H.

Denote by $L_2(F + G)$ the Hilbert space of vector-valued functions $\vec{a}(\lambda) = \{a_k(\lambda)\}_{k=1}^{T}$ such that

$$\int_{-\pi}^{\pi} \vec{a}(\lambda)^{\top} \left(F(\lambda) + G(\lambda)\right) \overline{\vec{a}(\lambda)} d\lambda < \infty.$$

Denote by $L_2^s(F + G)$ the subspace of $L_2(F + G)$ generated by functions of the form

$$e^{in\lambda}\delta_k,\ \delta_k = \{\delta_{kl}\}_{l=1}^{T},\ k = 1,\dots,T,\ n \in Z_-\backslash S,$$

where δ_{kl} are Kronecker symbols.

The mean square optimal linear estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ from observations of the sequence $\{\vec{\xi}(j) + \vec{\eta}(j)\}$ can be represented in the form

$$\hat{A}\vec{\xi} = \int_{-\pi}^{\pi} (h(e^{i\lambda}))^{\top} (Z_{\xi}(d\lambda) + Z_{\eta}(d\lambda)),$$

where $h(e^{i\lambda}) = \left\{h_k(e^{i\lambda})\right\}_{k=1}^{T} \in L_2^s(F + G)$ is the spectral characteristic of the estimate.

The mean square error $\Delta(h; F, G)$ of the estimate $\hat{A}\vec{\xi}$ is given by the formula

$$\Delta(h; F, G) = E\left|A\vec{\xi} - \hat{A}\vec{\xi}\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(A(e^{i\lambda}) - h(e^{i\lambda})\right)^{\top} F(\lambda)\overline{(A(e^{i\lambda}) - h(e^{i\lambda}))}d\lambda$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(h(e^{i\lambda})\right)^{\top} G(\lambda)\overline{(h(e^{i\lambda}))}d\lambda.$$

The Hilbert space projection method proposed by Kolmogorov [17] makes it possible to find the spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(h; F, G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ in the case where spectral densities $F(\lambda)$ and $G(\lambda)$ of the sequences are exactly known and the minimality condition (1) is satisfied. According to this method the optimal estimation of the functional $A\vec{\xi}$ is a projection of the element $A\vec{\xi}$ of the space H on the space $H^s(\xi + \eta)$. It can be found from the following conditions:

$$1) \hat{A}\vec{\xi} \in H^s(\xi + \eta),$$
$$2) A\vec{\xi} - \hat{A}\vec{\xi} \perp H^s(\xi + \eta).$$

It follows from the second condition that the spectral characteristic $h(e^{i\lambda})$ for any $j \in \mathbb{Z}_-\backslash S$ satisfies the equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(A(e^{i\lambda}) - h(e^{i\lambda})\right)^{\top} F(\lambda)e^{-ij\lambda}d\lambda - \frac{1}{2\pi}\int_{-\pi}^{\pi} (h(e^{i\lambda}))^{\top} G(\lambda)e^{-ij\lambda}d\lambda = \vec{0}.$$

The last relation is equivalent to equations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[(A(e^{i\lambda}))^{\top}F(\lambda)-(h(e^{i\lambda}))^{\top}(F(\lambda)+G(\lambda))\right]e^{-ij\lambda}d\lambda=\vec{0},\ j\in\mathbb{Z}_{-}\backslash S.$$

Hence the function $\left[(A(e^{i\lambda}))^{\top}F(\lambda)-(h(e^{i\lambda}))^{\top}(F(\lambda)+G(\lambda))\right]$ is of the form

$$(A(e^{i\lambda}))^{\top}F(\lambda)-(h(e^{i\lambda}))^{\top}(F(\lambda)+G(\lambda))=(C(e^{i\lambda}))^{\top},$$

where

$$C(e^{i\lambda})=\sum_{j\in S}\vec{c}(j)e^{ij\lambda}+\sum_{j=0}^{\infty}\vec{c}(j)e^{ij\lambda}.$$

Here $\vec{c}(j),\ j\in U=S\cup\{0,1,2,\dots\}$ are unknown coefficients that we have to find.

From the last relation we deduce that the spectral characteristic of the optimal linear estimate $\hat{A}\vec{\xi}$ is of the form

$$(h(e^{i\lambda}))^{\top}=(A(e^{i\lambda}))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(e^{i\lambda}))^{\top}(F(\lambda)+G(\lambda))^{-1}.$$

It follows from the first condition, $\hat{A}\vec{\xi}\in H^{s}(\xi+\eta)$, which determine the optimal linear estimate of the functional $A\vec{\xi}$, that the Fourier coefficients of the function $h(e^{i\lambda})$ are equal to zero for $k\in U$, namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left((A(e^{i\lambda}))^{\top}F(\lambda)(F(\lambda)+G(\lambda))^{-1}-(C(e^{i\lambda}))^{\top}(F(\lambda)+G(\lambda))^{-1}\right)e^{-ik\lambda}d\lambda=\vec{0},\ k\in U.$$

We will use the last equality to find equations which determine the unknown coefficients $\vec{c}(j), j\in U$. After disclosing the brackets we get the relation

$$\begin{aligned}&\sum_{j\in Z^{S}}\vec{a}(j)^{\top}\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{-i(k+j)\lambda}d\lambda-\sum_{j\in S}\vec{c}(j)^{\top}\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\lambda)\\&+G(\lambda))^{-1}e^{-i(k-j)\lambda}d\lambda-\sum_{j=0}^{\infty}\vec{c}(j)^{\top}\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\lambda)+G(\lambda))^{-1}e^{-i(k-j)\lambda}d\lambda=\vec{0},\ k\in U.\end{aligned}\tag{2}$$

For the functions

$$(F(\lambda)+G(\lambda))^{-1},\quad F(\lambda)(F(\lambda)+G(\lambda))^{-1},\quad F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)$$

we introduce the Fourier coefficients

$$B(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(\lambda)+G(\lambda))^{-1}e^{-i(k-j)\lambda}d\lambda,$$

$$R(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{-i(k+j)\lambda}d\lambda,$$

$$Q(k,j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\lambda)(F(\lambda) + G(\lambda))^{-1} G(\lambda) e^{-i(k-j)\lambda} d\lambda.$$

Using the introduced notations we can verify that the equality (2) is equivalent to the following system of equations:

$$\sum_{j \in Z^S} R(k,j)\vec{a}(j) = \sum_{j \in S} B(k,j)\vec{c}(j) + \sum_{j=0}^{\infty} B(k,j)\vec{c}(j), \qquad k \in U.$$

Let us introduce notations $\vec{a}(j) = \vec{0},\ j \in S$, $\vec{a}(0) = \vec{0}$ and $\vec{a}(j) = \vec{0},\ j \in S^+$. Thus, we can write

$$\sum_{j \in U} R(k,j)\vec{a}(j) = \sum_{j \in S} B(k,j)\vec{c}(j) + \sum_{j=0}^{\infty} B(k,j)\vec{c}(j), \qquad k \in U.$$

The last equations can be rewritten in the following form

$$\mathbf{R}\vec{\mathbf{a}} = \mathbf{B}\vec{\mathbf{c}}, \tag{3}$$

where $\vec{\mathbf{c}}$ is a vector constructed from the unknown coefficients $\vec{c}(j), j \in U$, vector $\vec{\mathbf{a}}$ has the same with the vector $\vec{\mathbf{c}}$ dimension, it is of the form

$$\vec{\mathbf{a}}^\top = (\vec{0}_0^\top, \vec{a}_1^\top, \vec{0}_1^\top, \vec{a}_2^\top, \vec{0}_2^\top, \ldots \vec{a}_i^\top, \vec{0}_i^\top, \ldots, \vec{a}_s^\top, \vec{0}_s^\top, \vec{a}_{s+1}^\top),$$

where $\vec{0}_0$ is the vector which consists of $(|S|+1)T$ zeros, where $|S| = \sum_{k=1}^{s}(N_k+1)$ is the amount of missing values, vectors $\vec{0}_i$, $i = 1, 2, \ldots, s$, consist of $(N_i+1)T$ zeros, vectors

$$\vec{a}_1^\top = (\vec{a}(1)^\top, \ldots, \vec{a}(M_1-1)^\top),$$

$$\vec{a}_i^\top = (\vec{a}(M_{i-1}+N_{i-1}+1)^\top, \ldots, \vec{a}(M_i-1)^\top), \quad i = 2, \ldots, s,$$

$$\vec{a}_{s+1}^\top = (\vec{a}(M_s+N_s+1)^\top, \vec{a}(M_s+N_s+2)^\top, \ldots),$$

are constructed from the coefficients that determine the functional $A\vec{\xi}$.
Here $\mathbf{B}$ is a linear operator in the space ℓ_2 which is defined by the matrix

$$B = \begin{pmatrix} B_{s,s} & B_{s,s-1} & \ldots & B_{s,1} & B_{s,n} \\ B_{s-1,s} & B_{s-1,s-1} & \ldots & B_{s-1,1} & B_{s-1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{1,s} & B_{1,s-1} & \ldots & B_{1,1} & B_{1,n} \\ B_{n,s} & B_{n,s-1} & \ldots & B_{n,1} & B_{n,n} \end{pmatrix},$$

where elements in the last column and the last row are compound matrices constructed from the block-matrices

$$B_{l,n}(k,j) = B(k,j),\ l = 1, 2, \ldots, s,\ k = -M_l - N_l, \ldots, -M_l,\ j = 0, 1, 2, \ldots,$$
$$B_{n,m}(k,j) = B(k,j),\ m = 1, 2, \ldots, s,\ k = 0, 1, 2, \ldots,\ j = -M_m - N_m, \ldots, -M_m,$$
$$B_{n,n}(k,j) = B(k,j),\ k, j = 0, 1, 2, \ldots$$

and other elements of matrix B are the compound matrices with elements of the form

$$B_{l,m}(j,k) = B(k,j), \quad l,m = 1,2,\dots,s,$$
$$k = -M_l - N_l, \dots, -M_l, \quad j = -M_m - N_m, \dots, -M_m.$$

The linear operator $\mathbf{R}$ in the space ℓ_2 is defined by the corresponding matrix in the same manner.

The unknown coefficients $\vec{c}(k), k \in U$, which are defined by the equations (3), can be calculated by the formula

$$\vec{c}(k) = (\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(k),$$

where $(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(k)$ is the k-th component of the vector $\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}$. (see paper by Salehi [39] for more details).

The formula for calculating the spectral characteristic $h(e^{i\lambda})$ of the estimate $\hat{A}\vec{\xi}$ is of the form

$$\begin{aligned}(h(e^{i\lambda}))^\top &= (A(e^{i\lambda}))^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1} \\ &\quad - \left(\sum_{k\in U}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(k)e^{ik\lambda}\right)^\top (F(\lambda)+G(\lambda))^{-1}.\end{aligned} \tag{4}$$

The mean square error of the estimate $\hat{A}\vec{\xi}$ can be calculated by the formula

$$\begin{aligned}\Delta(F,G) = E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(r_G(\lambda))^\top F(\lambda)\overline{r_G(\lambda)}d\lambda \\ &\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi}(r_F(\lambda))^\top G(\lambda)\overline{r_F(\lambda)}d\lambda = \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle,\end{aligned} \tag{5}$$

where

$$(r_F(\lambda))^\top = \left((A(e^{i\lambda}))^\top F(\lambda) - \left(\sum_{k\in U}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(k)e^{ik\lambda}\right)^\top\right)(F(\lambda)+G(\lambda))^{-1},$$

$$(r_G(\lambda))^\top = \left((A(e^{i\lambda}))^\top G(\lambda) + \left(\sum_{k\in U}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})(k)e^{ik\lambda}\right)^\top\right)(F(\lambda)+G(\lambda))^{-1},$$

and $\mathbf{Q}$ is the linear operator in the space ℓ_2 defined by matrix with coefficients $Q(k,j)$, $k,j \in U$ in the same way as the operator $\mathbf{B}$ is defined.

Let us summarize the obtained results and present them in the form of a theorem.

Theorem 1. *Let $\{\vec{\xi}(j)\}$ and $\{\vec{\eta}(j)\}$ be uncorrelated multidimensional stationary sequences with spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the sequence $\vec{\xi}(j)$ based on observations of the sequence $\{\vec{\xi}(j)+\vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_-\backslash S$ can be calculated by formulas (4), (5).*

Consider the problem of the mean-square optimal linear estimation of the functional $A\vec{\xi}$, which depends on the unknown values of the sequence $\{\vec{\xi}(j)\}$ from observations of the sequence $\{\vec{\xi}(j)+\vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_-\backslash S$, $S = \{-(M+N),\dots,-M\}$, $Z^S = \{1,2,\dots\}\backslash S^+$, $S^+ = \{M,\dots,M+N\}$.

From Theorem 1 the following corollary can be derived for this problem.

Corollary 1. *Let $\{\vec{\xi}(j)\}$ and $\{\vec{\eta}(j)\}$ be uncorrelated multidimensional stationary sequences with spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the sequence $\vec{\xi}(j)$ based on observations of the sequence $\{\vec{\xi}(j)+\vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_-\backslash S$ can be calculated by formulas (4), (5), where $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are linear operators in the space ℓ_2 defined by compound matrices constructed of coefficients $B(k,j)$, $R(k,j)$, $Q(k,j)$, $k,j \in U$, $(U = S \cup \{0,1,2,\dots\})$. For example, the matrix B is of the form*

$$B = \begin{pmatrix} B_{s,s} & B_{s,n} \\ B_{n,s} & B_{n,n} \end{pmatrix},$$

where its components are matrices constructed from the block-matrices

$$\begin{aligned} B_{s,n}(k,j) &= B(k,j), \quad k = -M-N,\dots,-M, \quad j = 0,1,2,\dots, \\ B_{n,s}(k,j) &= B(k,j), \quad k = 0,1,2,\dots, \quad j = -M-N,\dots,-M, \\ B_{n,n}(k,j) &= B(k,j), \quad k,j = 0,1,2,\dots, \\ B_{s,s}(k,j) &= B(k,j), \quad k = -M-N,\dots,-M, \quad j = -M-N,\dots,-M. \end{aligned}$$

Consider the problem of the mean-square optimal linear estimation of the functional $A\vec{\xi}$ which depends on the unknown values of the sequence $\{\vec{\xi}(j)\}$ from observations of the sequence $\{\vec{\xi}(j)+\vec{\eta}(j)\}$ at points $j \in \mathbb{Z}_-\backslash\{-s\}$, $Z^S = \{1,2,\dots\}\backslash\{s\}$.

It follows from Theorem 1 that the following corollary holds true.

Corollary 2. *Let $\vec{\xi}(j)$ and $\vec{\eta}(j)$ be uncorrelated multidimensional stationary sequences with spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(F,G)$ of the optimal linear estimate of the functional $A\vec{\xi}$ which depends on the unknown values of the sequence $\vec{\xi}(j)$ based on observations of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$, $j \in \mathbb{Z}_-\backslash\{-s\}$ can be calculated by formulas (4), (5), where $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are linear operators in the space ℓ_2 defined by compound matrices constructed of coefficients $B(k,j)$, $R(k,j)$, $Q(k,j)$, $k,j \in U$, $(U = S \cup \{0,1,2,\dots\})$,*

$$B = \begin{pmatrix} B(-s,-s) & B_{-s,n} \\ B_{n,-s} & B_{n,n} \end{pmatrix},$$

where elements in the last column and the last row are the matrices with the elements

$$\begin{aligned} B_{-s,n}(k,j) &= B(k,j), \quad k = -s, \quad j = 0,1,2,\dots, \\ B_{n,-s}(k,j) &= B(k,j), \quad k = 0,1,2,\dots, \quad j = -s, \\ B_{n,n}(k,j) &= B(k,j), \quad k,j = 0,1,2,\dots. \end{aligned}$$

Consider the problem of the mean-square optimal linear estimation of the functional

$$A_N\vec{\xi} = \sum_{j \in Z^S \cap \{0,\dots,N\}} \vec{a}(j)^\top \vec{\xi}(-j),$$

which depends on the unknown values of the sequence $\vec{\xi}(j)$ from observations of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$ at points $j \in \mathbb{Z}_-\backslash S$ where S is defined in the introduction. The linear estimate of the functional $A_N\vec{\xi}$ has the representation

$$\hat{A}_N\vec{\xi} = \int_{-\pi}^{\pi} (h_N(e^{i\lambda})^\top (Z_\xi(d\lambda) + Z_\eta(d\lambda)).$$

Define the vector $\vec{\mathbf{a}}_N$ as follows: elements with indices from the set $U \cap (S \cup \{0,\ldots,N\})$ coincide with the elements of the vector $\vec{\mathbf{a}}$ with the same indices and elements with indices from the set $U\backslash(S \cup \{0,\ldots,N\})$ are zeros. $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are linear operators in the space ℓ_2 defined in the Theorem 1.

The spectral characteristic $h_N(e^{i\lambda})$ and the mean square error $\Delta(h_N;F,G)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ can be calculated by formulas (6), (7)

$$\begin{aligned}(h_N(e^{i\lambda}))^\top &= (A_N(e^{i\lambda}))^\top F(\lambda)(F(\lambda)+G(\lambda))^{-1} \\ &\quad - \left(\sum_{k\in U}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_N)(k)e^{ik\lambda}\right)^\top (F(\lambda)+G(\lambda))^{-1},\end{aligned} \tag{6}$$

$$\Delta(h_N;F,G) = \langle \mathbf{R}\vec{\mathbf{a}}_N, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_N\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}_N, \vec{\mathbf{a}}_N\rangle, \tag{7}$$

where $A_N(e^{i\lambda}) = \sum\limits_{j\in Z^S\cap\{0,\ldots,N\}} \vec{a}(j)e^{-ij\lambda}$.

The following corollary holds true.

Corollary 3. *Let $\vec{\xi}(j)$ and $\vec{\eta}(j)$ be multidimensional uncorrelated stationary sequences with the spectral densities $F(\lambda)$ and $G(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h_N(e^{i\lambda})$ and the mean square error $\Delta(h_N;F,G)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ which depends on the unknown values of the sequence $\vec{\xi}(j)$ from observation of the sequence $\{\vec{\xi}(j)+\vec{\eta}(j)\}$ at points of time $j \in \mathbb{Z}_-\backslash S$ can be calculated by formulas (6), (7).*

2 MINIMAX-ROBUST METHOD OF FILTERING

Theorem 1 and its corollaries can be applied to filtering of the functional in the cases when spectral densities of the sequences are exactly known. If complete information on the spectral densities is impossible but the class of admissible densities is given, it is reasonable to apply the minimax-robust method of filtering which consists in minimizing the value of the mean-square error for all spectral densities from the given class. For description of minimax method we propose the following definitions (see Moklyachuk and Masytka [29]).

Definition 1. *For a given class of spectral densities $D = D_F \times D_G$ the spectral densities $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ are called least favorable in the class D for the optimal linear filtering of the functional $A\vec{\xi}$ if the following relation holds true*

$$\Delta\left(F^0,G^0\right) = \Delta\left(h\left(F^0,G^0\right);F^0,G^0\right) = \max_{(F,G)\in D_F\times D_G} \Delta\left(h\left(F,G\right);F,G\right).$$

Definition 2. *For a given class of spectral densities $D = D_F \times D_G$ the spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear estimate of the functional $A\vec{\xi}$ is called minimax-robust if there are satisfied conditions*

$$h^0(e^{i\lambda}) \in H_D = \bigcap_{(F,G)\in D_F\times D_G} L_2^s(F+G),$$

$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G) = \max_{(F,G)\in D}\Delta\left(h^0;F,G\right).$$

From the introduced definitions and formulas derived above we can obtain the following statement.

Lemma 1. *Spectral densities $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ satisfying the minimality condition (1) are the least favorable in the class $D = D_F \times D_G$ for the optimal linear filtering of the functional $A\vec{\xi}$ if operators B^0, R^0, Q^0 determined by the Fourier coefficients of the functions*

$$(F^0(\lambda)+G^0(\lambda))^{-1},\ F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1},\ F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda)$$

determine a solution to the constrain optimization problem

$$\max_{(F,G)\in D_F\times D_G}\langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}},\vec{\mathbf{a}}\rangle = \langle \mathbf{R}^0\vec{\mathbf{a}}, (\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}^0\vec{\mathbf{a}},\vec{\mathbf{a}}\rangle. \tag{8}$$

The minimax spectral characteristic $h^0 = h(F^0, G^0)$ is determined by the formula (4) if $h(F^0, G^0) \in H_D$.

The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ and the minimax spectral characteristic $h^0 = h(F^0, G^0)$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta\left(h;F^0,G^0\right) \geq \Delta\left(h^0;F^0,G^0\right) \geq \Delta\left(h^0;F,G\right)$$
$$\forall\quad h\in H_D, \forall\quad F\in D_F, \forall\quad G\in D_G$$

hold true if $h^0 = h(F^0, G^0)$ and $h(F^0, G^0) \in H_D$, where (F^0, G^0) is a solution to the constrained optimization problem

$$\sup_{(F,G)\in D_F\times D_G}\Delta\left(h(F^0,G^0);F,G\right) = \Delta\left(h(F^0,G^0);F^0,G^0\right), \tag{9}$$

where the functional $\Delta\left(h(F^0, G^0); F, G\right)$ is calculated by the formula

$$\Delta\left(h(F^0,G^0);F,G\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(r_G^0(\lambda))^\top F(\lambda)\overline{r_G^0(\lambda)}d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi}(r_F^0(\lambda))^\top G(\lambda)\overline{r_F^0(\lambda)}d\lambda,$$

$$(r_F^0(\lambda))^\top = \left((A(e^{i\lambda}))^\top F^0(\lambda) - \left(\sum_{k\in U}((\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}})(k)e^{ik\lambda}\right)^\top\right)(F^0(\lambda)+G^0(\lambda))^{-1},$$

$$(r_G^0(\lambda))^\top = \left((A(e^{i\lambda}))^\top G^0(\lambda) + \left(\sum_{k\in U}((\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}})(k)e^{ik\lambda}\right)^\top\right)(F^0(\lambda)+G^0(\lambda))^{-1}.$$

The constrained optimization problem (9) is equivalent to the unconstrained optimization problem (see, for example, Pshenichnyj [36]):

$$\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)\,|D_F \times D_G) \to \inf, \tag{10}$$

where $\delta((F,G)\,|D_F \times D_G)$ is the indicator function of the set $D = D_F \times D_G$. Solution of the problem (10) is characterized by the condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\Delta_D(F,G)$ at point (F^0,G^0). This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities D (see books by Ioffe and Tihomirov [13], Pshenichnyj [36], Rockafellar [37]).

Note, that the form of the functional $\Delta\left(h^0;F,G\right)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (10). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books by Moklyachuk [24, 25], Moklyachuk and Masyutka [29] for additional details).

3 LEAST FAVORABLE SPECTRAL DENSITIES IN THE CLASS $D = D_0 \times D_{2\delta}$

Consider the problem of filtering of the functional $A\vec{\xi}$ in the case where spectral densities $F(\lambda)$, $G(\lambda)$ belong to the set of admissible spectral densities $D_0 \times D_{2\delta}$, where

$$D_0^1 = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Tr} F(\lambda)d\lambda = p \right.\right\},$$

$$D_{2\delta}^1 = \left\{ G(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\operatorname{Tr}(G(\lambda)-G_1(\lambda))\right|^2 d\lambda \le \delta \right.\right\};$$

$$D_0^2 = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{kk}(\lambda)d\lambda = p_k, k=\overline{1,T} \right.\right\},$$

$$D_{2\delta}^2 = \left\{ G(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|^2 d\lambda \le \delta_k, k=\overline{1,T} \right.\right\};$$

$$D_0^3 = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle B_1, F(\lambda)\rangle\, d\lambda = p \right.\right\},$$

$$D_{2\delta}^3 = \left\{ G(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\langle B_2, G(\lambda)-G_1(\lambda)\rangle\right|^2 d\lambda \le \delta \right.\right\};$$

$$D_0^4 = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\lambda)d\lambda = P \right.\right\},$$

$$D_{2\delta}^4 = \left\{ G(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|^2 d\lambda \le \delta_i^j, i,j=\overline{1,T} \right.\right\}.$$

Here the spectral density $G_1(\lambda))$ is known and fixed, $p, \delta, p_k, \delta_k, k=\overline{1,T}, \delta_i^j, i,j=\overline{1,T}$, are fixed numbers, P, B_1, B_2 are fixed positive definite Hermitian matrices.

The classes D_0^k, $k=\overline{1,4}$ describe densities with the moment restrictions while the classes $D_{2\delta}^k$, $k=\overline{1,4}$ describe the "δ-neighborhood" models in the space L_2 of a fixed bounded spectral density $G_1(\lambda)$.

From the condition $0 \in \partial\Delta_D(F^0, G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair $D_0^1 \times D_{2\delta}^1$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = \alpha^2(F^0(\lambda)+G^0(\lambda))^2, \tag{11}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = \beta^2 \mathrm{Tr}\,(G^0(\lambda)-G_1(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{12}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{Tr}\,(G(\lambda)-G_1(\lambda))|^2\, d\lambda = \delta, \tag{13}$$

where α^2, β^2 are Lagrange multipliers.

For the second pair $D_0^2 \times D_{2\delta}^2$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{14}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{\beta_k^2(g_{kk}^0(\lambda)-g_{kk}^1(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{15}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g_{kk}(\lambda)-g_{kk}^1(\lambda)\right|^2 d\lambda = \delta_k,\ k=\overline{1,T}, \tag{16}$$

where α_k^2, β_k^2 are Lagrange multipliers.

For the third pair $D_0^3 \times D_{2\delta}^3$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = \alpha^2(F^0(\lambda)+G^0(\lambda))B_1^\top(F^0(\lambda)+G^0(\lambda)), \tag{17}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = \beta^2\left\langle B_2, G^0(\lambda)-G_1(\lambda)\right\rangle(F^0(\lambda)+G^0(\lambda))^2, \tag{18}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\langle B_2, G(\lambda)-G_1(\lambda)\rangle|^2\, d\lambda = \delta, \tag{19}$$

where α^2, β^2 are Lagrange multipliers.

For the fourth pair $D_0^4 \times D_{2\delta}^4$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\vec{\alpha}\cdot\vec{\alpha}^*(F^0(\lambda)+G^0(\lambda)), \tag{20}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{\beta_{ij}(g_{ij}^0(\lambda)-g_{ij}^1(\lambda))\right\}_{i,j=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{21}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g_{ij}(\lambda)-g_{ij}^1(\lambda)\right|^2 d\lambda = \delta_i^j,\ i,j=\overline{1,T}, \tag{22}$$

where $\vec{\alpha}, \beta_{ij}$ are Lagrange multipliers.

The following theorem and corollaries hold true.

Theorem 2. *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_0^k \times D_{2\delta}^k$, $k=\overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ are determined by relations (11) – (13) for the first pair $D_0^1 \times D_{2\delta}^1$ of sets of admissible spectral densities; (14) – (16) for the second pair $D_0^2 \times D_{2\delta}^2$ of sets of admissible spectral densities; (17) – (19) for the third pair $D_0^3 \times D_{2\delta}^3$ of sets of admissible spectral densities; (20) – (22) for the fourth pair $D_0^4 \times D_{2\delta}^4$ of sets of admissible spectral densities; the minimality condition (1), the constrained optimization problem (8) and restrictions on densities from the corresponding classes $D_0 \times D_{2\delta}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (4).*

Corollary 4. *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda)+G(\lambda)$ satisfies the minimality condition (1). The spectral density $F^0(\lambda)$ is the least favorable in the classes D_0^k, $k=\overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (11), (14), (17), (20), respectively, and the pair $(F^0(\lambda),G(\lambda))$ is a solution of the optimization problem (8). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (4).*

Corollary 5. *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda)+G^0(\lambda)$ satisfies the minimality condition (1). The spectral density $G^0(\lambda)$ is the least favorable in the classes $D_{2\delta}^k$, $k=\overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (12) – (13), (15) – (16), (18) – (19), (21) – (22), respectively, and the pair $(F(\lambda),G^0(\lambda))$ is a solution of the optimization problem (8). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (4).*

4 Least favorable spectral densities in the class $D=D_{1\delta}\times D_V^U$

Consider the problem of filtering of the functional $A\vec{\xi}$ in the case where spectral densities $F(\lambda)$, $G(\lambda)$ belong to the set of admissible spectral densities $D_{1\delta}\times D_V^U$, where

$$D_{1\delta}^1=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mathrm{Tr}(F(\lambda)-F_1(\lambda))|\,d\lambda\le\delta\right.\right\},$$

$$D_V^{U^1}=\left\{G(\lambda)\left|\mathrm{Tr}\,V(\lambda)\le\mathrm{Tr}\,G(\lambda)\le\mathrm{Tr}\,U(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}\,G(\lambda)d\lambda=q\right.\right\},$$

$$D_{1\delta}^2=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f_{kk}(\lambda)-f_{kk}^1(\lambda)\right|d\lambda\le\delta_k,k=\overline{1,T}\right.\right\},$$

$$D_V^{U^2}=\left\{G(\lambda)\left|v_{kk}(\lambda)\le g_{kk}(\lambda)\le u_{kk}(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}g_{kk}(\lambda)d\lambda=q_k,k=\overline{1,T}\right.\right\},$$

$$D_{1\delta}^3=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}|\langle B_1,F(\lambda)-F_1(\lambda)\rangle|\,d\lambda\le\delta\right.\right\},$$

$$D_V^{U^3}=\left\{G(\lambda)\left|\ \langle B_2,V(\lambda)\rangle\le\langle B_2,G(\lambda)\rangle\le\langle B_2,U(\lambda)\rangle,\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle B_2,G(\lambda)\rangle\,d\lambda=q\right.\right\},$$

$$D_{1\delta}^4=\left\{F(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f_{ij}(\lambda)-f_{ij}^1(\lambda)\right|d\lambda\le\delta_i^j,i,j=\overline{1,T}\right.\right\},$$

$$D_V^{U^4}=\left\{G(\lambda)\left|V(\lambda)\le G(\lambda)\le U(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\lambda)d\lambda=Q\right.\right\}.$$

Here the spectral densities $F_1(\lambda),V(\lambda),U(\lambda)$ are known and fixed, $\delta,q,\delta_k,q_k,k=\overline{1,T}$, $\delta_i^j,i,j=\overline{1,T}$, are fixed numbers, Q,B_1,B_2 are fixed positive definite Hermitian matrices.

The classes $D_V^{U^k}$, $k=\overline{1,4}$ describe the "strip" models of spectral densities while the classes $D_{1\delta}^k$, $k=\overline{1,4}$ describe "δ-neighborhood" model in the space L_1 of a fixed bounded spectral density $F_1(\lambda)$.

From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these sets of admissible spectral densities.

For the first pair $D_{1\delta}^{1} \times D_{V}^{U1}$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = \alpha^2\gamma(\lambda)(F^0(\lambda)+G^0(\lambda))^2, \tag{23}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{Tr}\,(F^0(\lambda)-F_1(\lambda))\right|d\lambda=\delta, \tag{24}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (\beta^2+\gamma_1(\lambda)+\gamma_2(\lambda))(F^0(\lambda)+G^0(\lambda))^2, \tag{25}$$

where α^2, β^2 are Lagrange multipliers, $|\gamma(\lambda)| \leq 1$ and

$$\gamma(\lambda) = \mathrm{sign}\,(\mathrm{Tr}\,(F^0(\lambda)-F_1(\lambda))) :\ \mathrm{Tr}\,(F^0(\lambda)-F_1(\lambda)) \neq 0,$$

$\gamma_1(\lambda) \leq 0$ and $\gamma_1(\lambda) = 0$ if $\mathrm{Tr}\,G^0(\lambda) > \mathrm{Tr}\,V(\lambda)$, $\gamma_2(\lambda) \geq 0$ and $\gamma_2(\lambda) = 0$ if $\mathrm{Tr}\,G^0(\lambda) < \mathrm{Tr}\,U(\lambda)$.

For the second pair $D_{1\delta}^{2} \times D_{V}^{U2}$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{\alpha_k^2\gamma_k(\lambda)\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{26}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f_{kk}^0(\lambda)-f_{kk}^1(\lambda)\right|d\lambda=\delta_k,\ k=\overline{1,T}, \tag{27}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{(\beta_k^2+\gamma_{1k}(\lambda)+\gamma_{2k}(\lambda))\delta_{kl}\right\}_{k,l=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{28}$$

where α_k^2, β_k^2 are Lagrange multipliers, $|\gamma_k(\lambda)| \leq 1$ and

$$\gamma_k(\lambda) = \mathrm{sign}\,(f_{kk}^0(\lambda)-f_{kk}^1(\lambda)) :\ f_{kk}^0(\lambda)-f_{kk}^1(\lambda) \neq 0,\ k=\overline{1,T},$$

$\gamma_{1k}(\lambda) \leq 0$ and $\gamma_{1k}(\lambda) = 0$ if $g_{kk}^0(\lambda) > v_{kk}(\lambda)$, $\gamma_{2k}(\lambda) \geq 0$ and $\gamma_{2k}(\lambda) = 0$ if $g_{kk}^0(\lambda) < u_{kk}(\lambda)$.

For the third pair $D_{1\delta}^{3} \times D_{V}^{U3}$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = \alpha^2\gamma'(\lambda)(F^0(\lambda)+G^0(\lambda))B_1^\top(F^0(\lambda)+G^0(\lambda)), \tag{29}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left\langle B_1, F^0(\lambda)-F_1(\lambda)\right\rangle\right|d\lambda=\delta, \tag{30}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (\beta^2+\gamma_1'(\lambda)+\gamma_2'(\lambda))(F^0(\lambda)+G^0(\lambda))B_2^\top(F^0(\lambda)+G^0(\lambda)), \tag{31}$$

where α^2, β^2 are Lagrange multipliers, $|\gamma'(\lambda)| \leq 1$ and

$$\gamma'(\lambda) = \mathrm{sign}\,\left\langle B_1, F^0(\lambda)-F_1(\lambda)\right\rangle :\ \left\langle B_1, F^0(\lambda)-F_1(\lambda)\right\rangle \neq 0,$$

$\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, G^0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, $\gamma_2'(\lambda) \geq 0$ and $\gamma_2'(\lambda) = 0$ if $\langle B_2, G^0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$.

For the fourth pair $D_{1\delta}^{4} \times D_{V}^{U4}$ we have equations

$$(r_G^0(\lambda))^*(r_G^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))\left\{\alpha_{ij}\gamma_{ij}(\lambda)\right\}_{i,j=1}^{T}(F^0(\lambda)+G^0(\lambda)), \tag{32}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f_{ij}^0(\lambda)-f_{ij}^1(\lambda)\right|d\lambda=\delta_i^j,\ i,j=\overline{1,T}, \tag{33}$$

$$(r_F^0(\lambda))^*(r_F^0(\lambda))^\top = (F^0(\lambda)+G^0(\lambda))(\vec{\beta}\cdot\vec{\beta}^*+\Gamma_1(\lambda)+\Gamma_2(\lambda))(F^0(\lambda)+G^0(\lambda)) \tag{34}$$

where $\vec{\beta}, \alpha_{ij}$ are Lagrange multipliers, $\left|\gamma_{ij}(\lambda)\right| \leq 1$ and

$$\gamma_{ij}(\lambda) = \frac{f_{ij}^0(\lambda)-f_{ij}^1(\lambda)}{\left|f_{ij}^0(\lambda)-f_{ij}^1(\lambda)\right|} :\ f_{ij}^0(\lambda)-f_{ij}^1(\lambda) \neq 0,\ i,j=\overline{1,T},$$

$\Gamma_1(\lambda) \leq 0$ and $\Gamma_1(\lambda) = 0$ if $G^0(\lambda) > V(\lambda)$, $\Gamma_2(\lambda) \geq 0$ and $\Gamma_2(\lambda) = 0$ if $G^0(\lambda) < U(\lambda)$.

The following theorem and corollaries hold true.

Theorem 3. *The least favorable spectral densities $F^0(\lambda)$, $G^0(\lambda)$ in the classes $D_{1\delta}^k \times D_V^{U^k}$, $k = \overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ are determined by relations (23) – (25) for the first pair $D_{1\delta}^4 \times D_V^{U^1}$ of sets of admissible spectral densities; (26) – (28) for the second pair $D_{1\delta}^4 \times D_V^{U^2}$ of sets of admissible spectral densities; (29) – (31) for the third pair $D_{1\delta}^4 \times D_V^{U^3}$ of sets of admissible spectral densities; (32) – (34) for the fourth pair $D_{1\delta}^4 \times D_V^{U^4}$ of sets of admissible spectral densities; the minimality condition (1), the constrained optimization problem (8) and restrictions on densities from the corresponding classes $D_{1\delta} \times D_V^U$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by the formula (4).*

Corollary 6. *Assume that the spectral density $G(\lambda)$ is known. Let the function $F^0(\lambda) + G(\lambda)$ satisfies the minimality condition (1). The spectral density $F^0(\lambda)$ is the least favorable in the classes $D_{1\delta}^k$, $k = \overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (23) – (24), (26) – (27), (29) – (30), (32) – (33), respectively, and the pair $(F^0(\lambda), G(\lambda))$ is a solution of the optimization problem (8). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (4).*

Corollary 7. *Assume that the spectral density $F(\lambda)$ is known. Let the function $F(\lambda) + G^0(\lambda)$ satisfies the minimality condition (1). The spectral density $G^0(\lambda)$ is the least favorable in the classes $D_V^{U^k}$, $k = \overline{1,4}$, for the optimal linear filtering of the functional $A\vec{\xi}$ if it satisfies relations (25), (28), (31), (34), respectively, and the pair $(F(\lambda), G^0(\lambda))$ is a solution of the optimization problem (8). The minimax-robust spectral characteristic of the optimal estimate of the functional $A\vec{\xi}$ is determined by formula (4).*

5 CONCLUSIONS

In the article we propose methods of the mean-square optimal linear filtering of functionals which depend on the unknown values of a multidimensional stationary stochastic sequence. Estimates are based on observations of the sequence with an additive stationary noise sequence. We develop methods of finding the optimal estimates of the functionals in the case of missing observations. The problem is investigated in the case of spectral certainty, where the spectral densities of the sequences are exactly known. In this case we propose an approach based on the Hilbert space projection method. We derive formulas for calculating the spectral characteristic and the mean-square error of the optimal estimate of the functionals. In the case of spectral uncertainty, where the spectral densities of the stationary sequences are not exactly known while some special sets of admissible spectral densities are given, we apply the minimax-robust estimation method of estimation. This method allows us to find estimates that minimize the maximum values of the mean-square errors of the estimates for all spectral density matrices from a given class of admissible spectral density matrices. We derive formulas that determine the least favorable spectral densities and the minimax spectral characteristics for some special sets of admissible spectral densities.

These least favourable spectral density matrices are solutions of the optimization problem $\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)\,|D_F \times D_G) \to \inf$, which is characterized by the condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional

$\Delta_D(F,G)$ at point (F^0,G^0). The form of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application of the Lagrange method of indefinite multipliers for finding solution to the optimization problem. The complexity of solution of the problem is determined by the complexity of calculating of subdifferentials of the indicator functions $\delta((f,g)|D_f \times D_g)$ of sets $D_f \times D_g$. Making use of the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine the least favourable spectral densities in some special classes of spectral densities. These are: classes D_0 of densities with the moment restrictions, classes $D_{1\delta}$ which describe the "δ-neighborhood" models in the space L_1 of a fixed bounded spectral density, classes $D_{2\delta}$ which describe the "δ-neighborhood" models in the space L_2 of a fixed bounded spectral density, classes D_V^U which describe the "strip" models of spectral densities.

REFERENCES

[1] Bondon P. *Prediction with incomplete past of a stationary process*. Stochastic Process. Appl. 2002, **98**, 67–76. doi:10.1016/S0304-4149(01)00116-8

[2] Bondon P. *Influence of missing values on the prediction of a stationary time series*. J. Time Ser. Anal. 2005, **26** (4), 519-525. doi:10.1111/j.1467-9892.2005.00433.x

[3] Box G.E.P., Jenkins G.M., Reinsel G.C., Ljung G.M. Time series analysis. Forecasting and control. 5th ed. Wiley, 2016.

[4] Brockwell P.J., Davis R.A. Time series: Theory and methods. 2nd ed. Springer, New York, 1998.

[5] Cheng R., Miamee A.G., Pourahmadi M. *Some extremal problems in $L^p(w)$*. Proc. Amer. Math. Soc. 1998, **126**, 2333–2340. doi:10.1090/S0002-9939-98-04275-0

[6] Cheng R., Pourahmadi M. *Prediction with incomplete past and interpolation of missing values*. Stat. Probab. Lett. 1996, **33**, 341–346. doi:10.1016/S0167-7152(96)00146-0

[7] Franke J. *On the robust prediction and interpolation of time series in the presence of correlated noise*. J. Time Ser. Anal. 1984, **5** (4), 227–244. doi:10.1111/j.1467-9892.1984.tb00389.x

[8] Franke J. *Minimax robust prediction of discrete time series*. Z. Wahrscheinlichkeitstheor. Verw. Geb. 1985, **68**, 337–364. doi:10.1007/BF00532645

[9] Franke J., Poor H.V. *Minimax-robust filtering and finite-length robust predictors*. Robust and Nonlinear Time Series Analysis. Lecture Notes in Statistics, Springer-Verlag, 1984, **26**, 87–126.

[10] Gikhman I.I., Skorokhod A.V. The theory of stochastic processes. I. Springer, Berlin, 2004.

[11] Grenander U. *A prediction problem in game theory*. Ark. Mat. 1957, **26**, 371–379.

[12] Hannan E.J. Multiple time series. Wiley, New York, 1970. doi:10.1002/9780470316429

[13] Ioffe A.D., Tihomirov V.M. Theory of extremal problems. North-Holland, Amsterdam, New York, Oxford, 1979.

[14] Karhunen K. *Über lineare Methoden in der Wahrscheinlichkeitsrechnung*. Ann. Acad. Sci. Fenn., Ser. A I, 1947, **37**, 1–79.

[15] Kasahara Y., Pourahmadi M., Inoue A. *Duals of random vectors and processes with applications to prediction problems with missing values*. Stat. Probab. Lett. 2009, **79** (14), 1637–1646. doi:10.1016/j.spl.2009.04.005

[16] Kassam S.A., Poor H.V. *Robust techniques for signal processing: A survey*. Proc. IEEE 1985, **73** (3), 433–481. doi:10.1109/PROC.1985.13167

[17] Kolmogorov A.N. In: Shiryayev A. N. (Ed.) Selected works by A. N. Kolmogorov. Vol. II: Probability theory and mathematical statistics, Kluwer, Dordrecht etc., 1992.

[18] Luz M. M., Moklyachuk M.P. *Minimax-robust filtering problem for stochastic sequences with stationary increments*. Theory Probab. Math. Statist. 2014, **89**, 127–142. doi:10.1090/S0094-9000-2015-00940-6 (translation of Teor. Imovir. ta Matem. Statist. 2013, **89**, 115–129. (in English))

[19] Luz M. M., Moklyachuk M. P. *Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences*. Stat., Optim. Inf. Comput. 2014, **2** (3), 176–199. doi:10.19139/soic.v2i3.56

[20] Luz M. M., Moklyachuk M. P. *Minimax-robust filtering problem for stochastic sequences with stationary increments and cointegrated sequences*. Cogent Mathematics 2016, **3**, 1–21. doi:10.1080/23311835.2016.1167811

[21] Luz M.M., Moklyachuk M.P. *Filtering problem for functionals of stationary sequences*. Stat., Optim. Inf. Comput. 2016, **4** (1), 68–83. doi:10.19139/soic.v4i1.172

[22] Luz M. M., Moklyachuk M. P. Estimates of functionals from processes with stationary increments and cointegrated sequences. NVP "Interservis", Kyiv, 2016. (in Ukrainian)

[23] Moklyachuk M.P. *On a filtering problem for vector-valued sequences*. Theory Probab. Math. Statist. 1992, **47**, 107–118. (translation of Teor. Imovir. ta Matem. Statist. 1992, **47**, 104–117. (in Ukrainian))

[24] Moklyachuk M.P. Nonsmooth analysis and optimization. Kyiv University, Kyiv, 2008. (in Ukrainian)

[25] Moklyachuk M.P. Robust estimations of functionals of stochastic processes. Kyiv University, Kyiv, 2008. (in Ukrainian)

[26] Moklyachuk M.P. *Minimax-robust estimation problems for stationary stochastic sequences*. Stat., Optim. Inf. Comput. 2015, **3** (4), 348–419. doi:10.19139/soic.v3i4.173

[27] Moklyachuk M.P., Golichenko I.I. Periodically correlated processes estimates. LAP Lambert Academic Publishing, Saarbrücken, 2016.

[28] Moklyachuk M.P., Masyutka O.Yu. *Robust filtering of stochastic processes*. Theory Stoch. Process. 2007, **13** (1-2), 166–181.

[29] Moklyachuk M.P., Masyutka O.Yu. Minimax-robust estimation technique for stationary stochastic processes. LAP Lambert Academic Publishing, Saarbrücken, 2012.

[30] Moklyachuk M.P., Sidei M.I. *Interpolation problem for stationary sequences with missing observations*. Stat., Optim. Inf. Comput. 2015, **3** (3), 259–275. doi:10.19139/soic.v3i3.149

[31] Moklyachuk M.P., Sidei M. *Filtering problem for stationary sequences with missing observations*. Stat., Optim. Inf. Comput. 2016, **4** (4), 308–325. doi:10.19139/soic.v4i4.241

[32] Moklyachuk M.P., Sidei M.I. *Filtering Problem for functionals of stationary processes with missing observations*. Commun. Optim. Theory 2016, Article ID 21, 1–18.

[33] Moklyachuk M.P., Sidei M.I. *Extrapolation problem for stationary sequences with missing observations*. Stat., Optim. Inf. Comput. 2017, **5** (3), 212–233. doi:10.19139/soic.v5i3.284

[34] Pelagatti M.M. Time series modelling with unobserved components. CRC Press, New York, 2015.

[35] Pourahmadi M., Inoue A., Kasahara Y. *A prediction problem in* $L^2(w)$. Proc. Amer. Math. Soc. 2007, **135** (4), 1233–1239. doi:10.1090/S0002-9939-06-08575-3

[36] Pshenichnyj B. N. Necessary conditions of an extremum. Marcel Dekker, New York, 1971.

[37] Rockafellar R. T. Convex Analysis. Princeton University Press, Princeton, 1997.

[38] Rozanov Yu.A. Stationary stochastic processes. Holden-Day, San Francisco-Cambridge-London-Amsterdam, 1967.

[39] Salehi H. *Algorithms for linear interpolator and interpolation error for minimal stationary stochastic processes*. Ann. Probab. 1979, **7** (5), 840–846.

[40] Vastola S. K., Poor H. V. *An analysis of the effects of spectral uncertainty on Wiener filtering*. Automatica 1983, **19** (3), 289–293. doi:10.1016/0005-1098(83)90105-X

[41] Wiener N. Extrapolation, interpolation and smoothing of stationary time series. With engineering applications. The M. I. T. Press, Cambridge, 1966.

[42] Yaglom A.M. Correlation theory of stationary and related random functions. Vol. 1: Basic results. Springer-Verlag, New York etc., 1987.

[43] Yaglom A.M. Correlation theory of stationary and related random functions. Vol. 2: Supplementary notes and references. Springer-Verlag, New York etc., 1987.

Received 24.01.2019

Масютка О. Ю., Моклячук М.П., Сідей М. І. *Фільтрація багатовимірних стаціонарних послідовностей із пропусками спостережень* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 361–378.

Досліджується задача оптимального в середньоквадратичному сенсі оцінювання лінійних функціоналів, що залежать від невідомих значень багатовимірних стаціонарних послідовностей. Оцінки базуються на спостереженнях послідовності з адитивним стаціонарним шумом із пропусками спостережень. Знайдено формули для обчислення середньоквадратичних похибок та спектральних характеристик оптимальних оцінок функціоналів у тому випадку, коли спектральні щільності послідовностей точно відома. Мінімаксний (робасиний) метод оцінювання застосовано у тому випадку коли спектральні щільності послідовностей точно невідомі а задані множини допустимих спектральних щільностей. Формули, що визначають найменш сприятливі спектральні щільністі та мінімаксні спектральні характеристики оптимальних оцінок функціоналів, запропоновані для заданих множин допустимих спектральних щільностей.

Ключові слова і фрази: стаціонарні послідовності, мініміксна оцінка, робастна оцінка, середньоквадратична похибка, найменш сприятлива спектральна щільність, мінімаксна спектральна характеристика.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 379–398
doi:10.15330/cmp.11.2.379-398

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.379–398

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PROPERTIES OF SOLUTIONS OF A HETEROGENEOUS DIFFERENTIAL EQUATION OF THE SECOND ORDER

Suppose that a power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R[A] \in [1, +\infty]$. For a heterogeneous differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = A(z)$$

with complex parameters geometrical properties of its solutions (convexity, starlikeness and close-to-convexity) in the unit disk are investigated. Two cases are considered: if $\gamma_2 \neq 0$ and $\gamma_2 = 0$. We also consider cases when parameters of the equation are real numbers. Also we prove that for a solution f of this equation the radius of convergence $R[f]$ equals to $R[A]$ and the recurrent formulas for the coefficients of the power series of $f(z)$ are found. For entire solutions it is proved that the order of a solution f is not less then the order of A ($\varrho[f] \geq \varrho[A]$) and the estimate is sharp. The same inequality holds for generalized orders ($\varrho_{\alpha\beta}[f] \geq \varrho_{\alpha\beta}[A]$). For entire solutions of this equation the belonging to convergence classes is studied. Finally, we consider a linear differential equation of the endless order $\sum\limits_{n=0}^{\infty} \frac{a_n}{n!} w^{(n)} = \Phi(z)$, and study a possible growth of its solutions.

Key words and phrases: differential equation, convexity, starlikeness, close-to-convexity, generalized order, convergence class.

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INTRODUCTION

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [4, p.203] that the condition $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0 (z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [7] the function f is said to be close-to-convex in $\mathbb{D}$ (see also [4, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}(f'(z)/\Phi'(z)) > 0 \, (z \in \mathbb{D})$. A close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that the function f is close-to-convex

УДК 517.925.4
2010 *Mathematics Subject Classification:* 30B10, 30C45,35B08.

in $\mathbb{D}$ if and only if the function $(f(z) - f(0))/f'(0)$ is close-to-convex in $\mathbb{D}$. Therefore, f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark that a function defined by (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is a starlike domain with respect to the origin and the condition $\mathrm{Re}\,\{zg'(z)/g(z)\} > 0\,(z \in \mathbb{D})$ is necessary and sufficient for the starlikeness of g. It is clear that every starlike function is close-to-convex. We remark also that if the function g is starlike, then the function cg is starlike, where $c = const$.

S.M. Shah [9] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$$

under which there exits a transcendental solution given by (1) such that either all its derivatives or even derivatives or odd derivatives are close-to-convex functions in $\mathbb{D}$. The investigations of Shah are continued in the papers [12–15].

Here we consider a heterogeneous differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = \sum_{n=0}^{\infty} a_n z^n, \tag{3}$$

where parameters β_0, β_1, γ_0, γ_1, γ_2 are complex and the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R[A] \in (0, +\infty]$. We will investigate conditions such that equation (3) has convex or close-to-convex solutions, and in the case if a solution is entire function we will study its possible growth and belonging to convergence classes.

1 PRELIMINARY LEMMAS

At first we remark that an analytic in some neighborhood of the origin of coordinates function given by (1) is a solution of equation (3) if and only if

$$\sum_{n=2}^{\infty} n(n-1)f_n z^n + \beta_0 \sum_{n=2}^{\infty}(n-1)f_{n-1}z^n + \gamma_0 \sum_{n=2}^{\infty} f_{n-2}z^n$$
$$+ \beta_1 \sum_{n=1}^{\infty} nf_n z^n + \gamma_1 \sum_{n=1}^{\infty} f_{n-1}z^n + \gamma_2 \sum_{n=0}^{\infty} f_n z^n \equiv \sum_{n=0}^{\infty} a_n z^n,$$

i. e.

$$\gamma_2 f_0 = a_0, \quad (\beta_1 + \gamma_2)f_1 + \gamma_1 f_0 = a_1 \tag{4}$$

and for $n \geq 2$

$$(n(n + \beta_1 - 1) + \gamma_2)f_n + (\beta_0(n-1) + \gamma_1)f_{n-1} + \gamma_0 f_{n-2} = a_n. \tag{5}$$

Lemma 1. *If a function defined by* (1) *is a solution of equation* (3) *and* $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ *for all* $n \geq 2$*, then* $R[f] = R[A]$.

Proof. Suppose at first that $R[A] < +\infty$. From (5) for $n \geq 2$ we have

$$f_n = -\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2} f_{n-1} - \frac{\gamma_0}{n(n+\beta_1-1)+\gamma_2} f_{n-2} + \frac{a_n}{n(n+\beta_1-1)+\gamma_2}. \tag{6}$$

Let $n_0 = n_0(R[A])$ is such that for all $n \geq n_0$

$$R[A]\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| \leq \frac{1}{4}, \quad R[A]^2\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| \leq \frac{1}{4}. \tag{7}$$

Then for each $r < R[A]$

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |f_n| r^n &\leq \sum_{n=n_0}^{\infty} r\left|\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2}\right| |f_{n-1}| r^{n-1} \\
&+ \sum_{n=n_0}^{\infty} r^2\left|\frac{\gamma_0}{n(n+\beta_1-1)+\gamma_2}\right| |f_{n-2}| r^{n-2} + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} \\
&= r\sum_{n=n_0-1}^{\infty}\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| |f_n| r^n \\
&+ r^2\sum_{n=n_0-2}^{\infty}\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| |f_n| r^n + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} \\
&= r\sum_{n=n_0}^{\infty}\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| |f_n| r^n + r\left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0-1} \\
&+ r^2\sum_{n=n_0}^{\infty}\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right| |f_n| r^n + r^2\left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| r^{n_0-2} \\
&+ r^2\left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0-1} + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|},
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\left(1 - r\left|\frac{\beta_0 n+\gamma_1}{(n+1)(n+\beta_1)+\gamma_2}\right| - r^2\left|\frac{\gamma_0}{(n+2)(n+\beta_1+1)+\gamma_2}\right|\right) |f_n| r^n \\
&\leq \left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0} + \left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| r^{n_0} \\
&+ \left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| r^{n_0+1} + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|}.
\end{aligned}$$

In view of (7) hence we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{n=n_0}^{\infty} |f_n| r^n &\leq \left|\frac{\beta_0(n_0-1)+\gamma_1}{n_0(n_0-1+\beta_1)+\gamma_2}\right| |f_{n_0-1}| R[A]^{n_0} + \left|\frac{\gamma_0}{n_0(n_0+\beta_1-1)+\gamma_2}\right| |f_{n_0-2}| R[A]^{n_0} \\
&+ \left|\frac{\gamma_0}{(n_0+1)(n_0+\beta_1)+\gamma_2}\right| |f_{n_0-1}| R[A]^{n_0+1} + \sum_{n=n_0}^{\infty} \frac{|a_n| r^n}{|n(n+\beta_1-1)+\gamma_2|} < +\infty,
\end{aligned}$$

i. e. $R[f] \geq R[A]$. On the other hand, from (5) we get

$$\begin{aligned}
\sum_{n=2}^{\infty} |a_n| r^n &\leq \sum_{n=2}^{\infty} |(n(n+\beta_1-1)+\gamma_2)| |f_n| r^n \\
&+ r\sum_{n=2}^{\infty} |\beta_0(n-1)+\gamma_1| |f_{n-1}| r^{n-1} + r^2\sum_{n=2}^{\infty} |\gamma_0| |f_{n-2}| r^{n-2},
\end{aligned}$$

and, since the convergence of the series $\sum\limits_{n=n_0}^{\infty} |f_n|r^n$ implies the convergence of each series in right-hand side of the last inequality, we have $R[A] \geq R[f]$. In the case if $R[A] < +\infty$ the equality $R[A] = R[f]$ is proved.

If $R[A] = +\infty$, then the proof of the equality $R[A] = R[f]$ is similar. Now it is enough to choose $n_0 = n_0(R)$ for every $R \in (0, +\infty)$ so that inequality (7) holds with R instead of $R[A]$. Then instead of the inequality $R[f] \geq R[A]$ we obtain the inequality $R[f] \geq R$, whence in view of the arbitrariness of R we get the equality $R[f] = +\infty$. Lemma 1 is proved. □

For the investigation of the convexity and the starlikeness of solutions of differential equation (3) we will use the following lemma ([1,5,6]).

Lemma 2. *If* $\sum\limits_{n=2}^{\infty} n|g_n| \leq 1$, *then function* (2) *is starlike, and if* $\sum\limits_{n=2}^{\infty} n^2|g_n| \leq 1$, *then it is convex in* $\mathbb{D}$.

From Lemma 2 the following lemma follows.

Lemma 3. *If* $\sum\limits_{n=2}^{\infty} n|f_n| \leq |f_1|$, *then function* (1) *is close-to-convex, and if* $\sum\limits_{n=2}^{\infty} n^2|f_n| \leq |f_1|$, *then it is convex in* $\mathbb{D}$.

From the first equality (4) it is clear that the choice of coefficients f_n of solution (1) of equation (3) depends on the equality of the parameter γ_2 to zero.

2 CLOSE-TO-CONVEXITY AND CONVEXITY IN THE CASE $\gamma_2 \neq 0$

From (4) we get $f_0 = a_0/\gamma_2$ and $(\beta_1 + \gamma_2)f_1 = a_1 - \gamma_1 f_0$. Since we find univalent solutions, f_1 must be not equal to zero. In view of (4) two cases are possible:

2a) $a_1 - \gamma_1 f_0 \neq 0$ and $\beta_1 + \gamma_2 \neq 0$;

2b) $a_1 - \gamma_1 f_0 = \beta_1 + \gamma_2 = 0$.

By the conditions 2a) from (4) we get $f_1 = \dfrac{a_1 - \gamma_1 f_0}{\beta_1 + \gamma_2} = \dfrac{\gamma_2 a_1 - \gamma_1 a_0}{\gamma_2(\beta_1 + \gamma_2)}$, and thus the solution is of the form

$$f(z) = \frac{a_0}{\gamma_2} + \frac{\gamma_2 a_1 - \gamma_1 a_0}{\gamma_2(\beta_1 + \gamma_2)} z + \sum_{n=2}^{\infty} f_n z^n, \tag{8}$$

where the coefficients f_n are defined by the recurrent formula (5). Supposing that $n(n + \beta_1 - 1) + \gamma_2 \neq 0$ for all $n \geq 2$, this formula can be rewritten in the form (6).

Suppose that $|\beta_1| < 1$ and $|\gamma_2|/2 < (1 - |\beta_1|)$. Then $|n(n + \beta_1 - 1) + \gamma_2| \geq$ $\geq n(n - 1 - |\beta_1|) - |\gamma_2|$ and, since the function $x^2 - (1 + |\beta_1|)x - |\gamma_2|$ is increasing on $[2, +\infty)$, we have $n(n - 1 - |\beta_1|) - |\gamma_2| \geq 2(1 - |\beta_1|) - |\gamma_2| > 0$ for all $n \geq 2$. Therefore, (6) implies

$$|f_n| \leq \frac{|\beta_0|(n-1) + |\gamma_1|}{n(n - 1 - |\beta_1|) - |\gamma_2|}|f_{n-1}| + \frac{|\gamma_0|}{n(n - 1 - |\beta_1|) - |\gamma_2|}|f_{n-2}| + \frac{|a_n|}{n(n - 1 - |\beta_1|) - |\gamma_2|}. \tag{9}$$

Hence it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty} \frac{n}{n-1}\frac{|\beta_0|(n-1)+|\gamma_1|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-1)|f_{n-1}| \\
&+ \sum_{n=2}^{\infty} \frac{n}{n-2}\frac{|\gamma_0|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} \\
&= \sum_{n=1}^{\infty} \frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n|f_n| + \sum_{n=0}^{\infty} \frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+ \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|} = \sum_{n=2}^{\infty} \frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+ 2\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty} \frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n|f_n| \\
&+ \frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}.
\end{aligned} \tag{10}$$

Since for $n \ge 2$

$$\frac{n+1}{n}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|} = \frac{|\beta_0|+|\gamma_1|/n}{(n-|\beta_1|)-|\gamma_2|/(n+1)} \le \frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}$$

and

$$\frac{n+2}{n}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|} = \frac{|\gamma_0|/n}{(n+1-|\beta_1|)-|\gamma_2|/(n+2)} \le \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4},$$

from (10) it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty} \frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}n|f_n| + \sum_{n=2}^{\infty} \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}n|f_n| + \frac{2(|\beta_0|+|\gamma_1|)|f_1|}{2(1-|\beta_1|)-|\gamma_2|} \\
&+ \frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}
\end{aligned}$$

and by the condition

$$\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} + \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4} < 1 \tag{11}$$

we obtain

$$\begin{aligned}
&\left(1 - \frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} - \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)\sum_{n=2}^{\infty} n|f_n| \le 2\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1| \\
&+ \frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|},
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \left(\left(\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|} + \frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)|f_1| + \frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0| \right. \\
&\left. + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\right)\left(1 - \frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3} - \frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)^{-1}.
\end{aligned} \tag{12}$$

By Lemma 3 solution (1) of equation (3) is close-to-convex if the right-hand side of (12) is less than $|f_1|$, i. e.

$$\left(\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}+\frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)|f_1|+\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|$$
$$+\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right)|f_1|. \tag{13}$$

Thus, the following proposition is proved.

Proposition 1. *Let* $\gamma_2\ne 0$, $a_1\gamma_2-a_0\gamma_1\ne 0$, $\beta_1+\gamma_2\ne 0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ *and* $R[A]\ge 1$. *If*

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le\left(1-\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4}\right.$$
$$\left.-\frac{2(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}-\frac{3|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)\frac{|\gamma_2 a_1-\gamma_1 a_0|}{|\gamma_2(\beta_1+\gamma_2)|}-\frac{2|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}\frac{|a_0|}{|\gamma_2|}, \tag{14}$$

then there exists a solution given by (8) *of differential equation* (3) *with* $R[f]=R[A]$, *which is close-to-convex in* $\mathbb{D}$. *If moreover* $a_0=0$ *it is starlike.*

Indeed, the condition (14) is equivalent to condition (13), and (13) implies (11).

We will pass to the convexity. From (9) we get

$$\sum_{n=2}^{\infty}n^2|f_n|\le\sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{|\beta_0|(n-1)+|\gamma_1|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-1)^2|f_{n-1}|$$
$$+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{n(n-1-|\beta_1|)-|\gamma_2|}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}$$
$$=\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n^2|f_n|$$
$$+\sum_{n=0}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}$$
$$=\sum_{n=2}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}n^2|f_n|+4\frac{|\beta_0|+|\gamma_1|}{2(1-|\beta_1|)-|\gamma_2|}|f_1|$$
$$+\sum_{n=2}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}n^2|f_n|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|$$
$$+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}.$$

Since now for $n\ge 2$

$$\frac{(n+1)^2}{n^2}\frac{|\beta_0|n+|\gamma_1|}{(n+1)(n-|\beta_1|)-|\gamma_2|}\le\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}$$

and

$$\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+2)(n+1-|\beta_1|)-|\gamma_2|}\le 2\frac{|\gamma_0|/2}{(3-|\beta_1|)-|\gamma_2|/4},$$

by the condition

$$\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}+\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}<1,$$

as above we obtain

$$\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)\sum_{n=2}^{\infty}n^2|f_n|\le\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|$$
$$+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|},$$

i. e.

$$\begin{aligned}\sum_{n=2}^{\infty}n^2|f_n|\le&\left(\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|\right.\\&\left.+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\right)\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)^{-1},\end{aligned}\tag{15}$$

By Lemma 3 a solution given by (1) of equation (3) is convex if the right-hand side of (15) is less than $|f_1|$, i. e.

$$\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}|f_1|+\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}|f_0|+\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}|f_1|$$
$$+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right)|f_1|.$$

Thus, the following proposition is proved.

Proposition 2. *Let* $\gamma_2\neq 0$, $a_1\gamma_2-a_0\gamma_1\neq 0$, $\beta_1+\gamma_2\neq 0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ *and* $R[A]\ge 1$. *If*

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n-1-|\beta_1|)-|\gamma_2|}\le&\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{(2-|\beta_1|)-|\gamma_2|/3}-\frac{|\gamma_0|}{(3-|\beta_1|)-|\gamma_2|/4}\right.\\&\left.-\frac{4(|\beta_0|+|\gamma_1|)}{2(1-|\beta_1|)-|\gamma_2|}-\frac{9|\gamma_0|}{3(2-|\beta_1|)-|\gamma_2|}\right)\frac{|\gamma_2a_1-\gamma_1a_0|}{|\gamma_2(\beta_1+\gamma_2)|}-\frac{4|\gamma_0|}{2(1-|\beta_1|)-|\gamma_2|}\frac{|a_0|}{|\gamma_2|},\end{aligned}\tag{16}$$

then there exists a solution defined by (8) *of differential equation* (3) *with* $R[f]=R[A]$, *which is convex in* $\mathbb{D}$.

Uniting Propositions 1 and 2 we get such theorem.

Theorem 1. *Let* $\gamma_2\neq 0$, $a_1\gamma_2-a_0\gamma_1\neq 0$, $\beta_1+\gamma_2\neq 0$, $|\beta_1|<1$, $|\gamma_2|/2<(1-|\beta_1|)$ *and* $R[A]\ge 1$. *Then there exists a solution given by* (8) *of differential equation* (3) *with* $R[f]=R[A]$, *which by the condition* (14) *is close-to-convex and by the condition* (16) *is convex in* $\mathbb{D}$. *If* $a_0=0$ *and* (14) *holds then* (8) *is starlike.*

The conditions $|\beta_1|<1$ and $|\gamma_2|/2<(1-|\beta_1|)$ in Theorem 1 can be weakened if β_1 and γ_2 are real numbers. We will consider a simple case, when $\gamma_2>0$, $\beta_1>-1$ and $\gamma_2+\beta_1>0$.

Suppose also that $\gamma_2 a_1 - \gamma_1 a_0 \neq 0$. Then from recurrent formula (6) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n|f_n| \le \sum_{n=2}^{\infty} \frac{n}{n-1} \frac{|\beta_0|(n-1)+|\gamma_1|}{n(n+\beta_1-1)+\gamma_2}(n-1)|f_{n-1}| \\
&+ \sum_{n=2}^{\infty} \frac{n}{n-2} \frac{|\gamma_0|}{n(n+\beta_1-1)+\gamma_2}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2} \\
&\le \sum_{n=2}^{\infty} \frac{|\beta_0|+|\gamma_1|/(n-1)}{(n+\beta_1-1)+\gamma_2/n}(n-1)|f_{n-1}| + \sum_{n=2}^{\infty} \frac{|\gamma_0|/(n-2)}{(n+\beta_1-1)+\gamma_2/n}(n-2)|f_{n-2}| \\
&+ \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2} \le \sum_{n=2}^{\infty} \frac{|\beta_0|+|\gamma_1|/(n-1)}{(n+\beta_1-1)}(n-1)|f_{n-1}| \\
&+ \sum_{n=2}^{\infty} \frac{|\gamma_0|/(n-2)}{(n+\beta_1-1)}(n-2)|f_{n-2}| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2} \\
&= \sum_{n=1}^{\infty} \frac{|\beta_0|+|\gamma_1|/n}{n+\beta_1} n|f_n| + \sum_{n=0}^{\infty} \frac{|\gamma_0|/n}{(n+\beta_1+1)} n|f_n| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2} \\
&\le \frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| + \sum_{n=2}^{\infty} \frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} n|f_n| + \frac{|\gamma_0|}{\beta_1+1}|f_0| + \frac{|\gamma_0|}{2+\beta_1}|f_1| \\
&+ \sum_{n=2}^{\infty} \frac{|\gamma_0|/2}{3+\beta_1} n|f_n| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2},
\end{aligned}$$

whence by the condition

$$\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} + \frac{|\gamma_0|/2}{3+\beta_1} < 1$$

we obtain

$$\begin{aligned}
&\left(1 - \frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} - \frac{|\gamma_0|/2}{3+\beta_1}\right) \sum_{n=2}^{\infty} n|f_n| \le \frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| \\
&+ \frac{|\gamma_0|}{\beta_1+1}|f_0| + \frac{|\gamma_0|}{2+\beta_1}|f_1| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}.
\end{aligned} \tag{17}$$

Similarly we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2|f_n| \le \sum_{n=2}^{\infty} \frac{n}{n-1} \frac{|\beta_0|+|\gamma_1|/(n-1)}{n+\beta_1-1}(n-1)^2|f_{n-1}| \\
&+ \sum_{n=2}^{\infty} \frac{n|\gamma_0|/(n-2)^2}{(n+\beta_1-1)}(n-2)^2|f_{n-2}| + \sum_{n=2}^{\infty} \frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2} \\
&= \sum_{n=1}^{\infty} \frac{n+1}{n} \frac{|\beta_0|+|\gamma_1|/n}{n+\beta_1} n^2|f_n| + \sum_{n=0}^{\infty} \frac{(n+2)|\gamma_0|/n^2}{(n+\beta_1+1)} n^2|f_n| + \sum_{n=2}^{\infty} \frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2} \\
&\le 2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| + \sum_{n=2}^{\infty} \frac{3}{2} \frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} n^2|f_n| + \frac{2|\gamma_0|}{\beta_1+1}|f_0| + \frac{3|\gamma_0|}{2+\beta_1}|f_1| \\
&+ \sum_{n=2}^{\infty} \frac{|\gamma_0|}{3+\beta_1} n^2|f_n| + \sum_{n=2}^{\infty} \frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2},
\end{aligned}$$

whence by the condition

$$\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1} + \frac{|\gamma_0|}{3+\beta_1} < 1$$

we get

$$\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|}{3+\beta_1}\right)\sum_{n=2}^{\infty}n^2|f_n|\le 2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}|f_1| + \frac{2|\gamma_0|}{\beta_1+1}|f_0|+\frac{3|\gamma_0|}{2+\beta_1}|f_1|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}. \tag{18}$$

From (17) and (18) we obtain the following proposition.

Proposition 3. *Let $\gamma_2>0$, $\beta_1>-1$, $\gamma_2+\beta_1>0$, $\gamma_2 a_1-\gamma_1 a_0\neq 0$ and $R[A]\ge 1$. Then there exists a solution* (8) *of differential equation* (3) *with $R[f]=R[A]$, which by the condition*

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n(n+\beta_1-1)+\gamma_2}\le\left(1-\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|/2}{3+\beta_1}-\frac{|\beta_0|+|\gamma_1|}{1+\beta_1} -\frac{|\gamma_0|}{2+\beta_1}\right)\frac{|\gamma_2 a_1-\gamma_1 a_0|}{\gamma_2(\beta_1+\gamma_2)}-\frac{|\gamma_0|}{\beta_1+1}\frac{|a_0|}{|\gamma_2|}$$

is close-to-convex (starlike if $a_0=0$) and by the condition

$$\sum_{n=2}^{\infty}\frac{n^2|a_n|}{n(n+\beta_1-1)+\gamma_2}\le\left(1-\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{2+\beta_1}-\frac{|\gamma_0|}{3+\beta_1}-2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1} -\frac{3|\gamma_0|}{2+\beta_1}\right)\frac{|\gamma_2 a_1-\gamma_1 a_0|}{\gamma_2(\beta_1+\gamma_2)}-\frac{2|\gamma_0|}{1+\beta_1}\frac{|a_0|}{|\gamma_2|}$$

is a convex function in $\mathbb{D}$.

Now we suppose that the condition 2b) holds, that is, $\gamma_2\neq 0$ and $a_1-\gamma_1 f_0=\beta_1+\gamma_2=0$. Then $f_0=a_0/\gamma_2$ and f_1 can be arbitrary number, in particular we can choose $f_1=1$. Thus, the solution will have a form

$$f(z)=\frac{a_0}{\gamma_2}+z+\sum_{n=2}^{\infty}f_n z^n, \tag{19}$$

where the coefficients f_n are defined by the recurrent formula

$$(n-1)(n+\beta_1)f_n+(\beta_0(n-1)+\gamma_1)f_{n-1}+\gamma_0 f_{n-2}=a_n.$$

Supposing that $n+\beta_1\neq 0$ for all $n\ge 2$, this formula can be rewritten in the form

$$f_n=-\frac{\beta_0(n-1)+\gamma_1}{(n-1)(n+\beta_1)}f_{n-1}-\frac{\gamma_0}{(n-1)(n+\beta_1)}f_{n-2}+\frac{a_n}{(n-1)(n+\beta_1)},$$

whence by the condition $|\beta_1|<2$ we have

$$\begin{aligned}\sum_{n=2}^{\infty}n|f_n|&\le\sum_{n=2}^{\infty}\frac{n}{n-1}\frac{(n-1)|\beta_0|+|\gamma_1|}{(n-1)(n-|\beta_1|)}(n-1)|f_{n-1}|\\
&+\sum_{n=2}^{\infty}\frac{n}{n-2}\frac{|\gamma_0|}{(n-1)(n-|\beta_1|)}(n-2)|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)}\\
&=2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\sum_{n=2}^{\infty}\frac{n+1}{n}\frac{|\beta_0|+|\gamma_1|/n}{n+1-|\beta_1|}n|f_n|+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}\\
&+\sum_{n=2}^{\infty}\frac{n+2}{n}\frac{|\gamma_0|}{(n+1)(n+2-|\beta_1|)}n|f_n|+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)}\\
&\le\sum_{n=2}^{\infty}\frac{3}{2}\frac{|\beta_0|+|\gamma_1|/2}{3-|\beta_1|}n|f_n|+\sum_{n=2}^{\infty}\frac{2|\gamma_0|}{3(4-|\beta_1|)}n|f_n|+2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}\\
&+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)},\end{aligned}$$

i. e. by the condition

$$\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}+\frac{2|\gamma_0|}{3(4-|\beta_1|)}<1$$

we get

$$\begin{gathered}\left(1-\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}-\frac{2|\gamma_0|}{3(4-|\beta_1|)}\right)\sum_{n=2}^{\infty}n|f_n| \\ \le 2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{2|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{3|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)},\end{gathered}\tag{20}$$

Similarly,

$$\begin{aligned}\sum_{n=2}^{\infty}n^2|f_n| &\le \sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{(n-1)|\beta_0|+|\gamma_1|}{(n-1)(n-|\beta_1|)}(n-1)^2|f_{n-1}| \\ &+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{(n-1)(n-|\beta_1|)}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \\ &=4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\sum_{n=2}^{\infty}\frac{(n+1)^2}{n^2}\frac{|\beta_0|+|\gamma_1|/n}{n+1-|\beta_1|}n^2|f_n|+\frac{4|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{9|\gamma_0|}{2(3-|\beta_1|)} \\ &+\sum_{n=2}^{\infty}\frac{(n+2)^2}{n^2}\frac{|\gamma_0|}{(n+1)(n+2-|\beta_1|)}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \\ &\le \sum_{n=2}^{\infty}\frac{9}{4}\frac{|\beta_0|+|\gamma_1|/2}{3-|\beta_1|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{16}{4}\frac{|\gamma_0|}{3(4-|\beta_1)|}n^2|f_n| \\ &+4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{4|\gamma_0|}{2-|\beta_1|}|f_0|+\frac{9|\gamma_0|}{2(3-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)},\end{aligned}$$

i. e. by the condition

$$\frac{9(2|\beta_0|+|\gamma_1|)}{8(3-|\beta_1|)}+\frac{4|\gamma_0|}{3(4-|\beta_1)|}<1$$

we get

$$\begin{gathered}\left(1-\frac{9(2|\beta_0|+|\gamma_1|)}{8(3-|\beta_1|)}-\frac{4|\gamma_0|}{3(4-|\beta_1|)}\right)\sum_{n=2}^{\infty}n^2|f_n| \\ \le 4\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{9|\gamma_0|}{2(3-|\beta_1|)}+\frac{4|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}+\sum_{n=2}^{\infty}\frac{n^2|a_n|}{(n-1)(n-|\beta_1|)}.\end{gathered}\tag{21}$$

In view of Lemma 3 from (20) and (21), as in the proof of Proposition 1, we obtain the following theorem.

Theorem 2. *Let $\gamma_2\neq 0$, $a_1\gamma_2-a_0\gamma_1=\beta_1+\gamma_2=0$, $|\beta_1|<2$ and $R[A]\ge 1$. Then there exists a solution given by* (19) *of differential equation* (3) *with $R[f]=R[A]$, which by the condition*

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-1)(n-|\beta_1|)} &\le 1-\frac{3(2|\beta_0|+|\gamma_1|)}{4(3-|\beta_1|)}-\frac{2|\gamma_0|}{3(4-|\beta_1|)} \\ &-2\frac{|\beta_0|+|\gamma_1|}{2-|\beta_1|}-\frac{3|\gamma_0|}{2(3-|\beta_1|)}-\frac{2|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}\end{aligned}$$

is close-to-convex (if $a_0 = 0$ then starlike) and by the condition

$$\sum_{n=2}^{\infty} \frac{n^2|a_n|}{(n-1)(n-|\beta_1|)} \leq 1 - \frac{9(2|\beta_0| + |\gamma_1|)}{8(3-|\beta_1|)} - \frac{4|\gamma_0|}{3(4-|\beta_1|)} - 4\frac{|\beta_0| + |\gamma_1|}{2-|\beta_1|} - \frac{9|\gamma_0|}{2(3-|\beta_1|)} - \frac{4|\gamma_0|}{2-|\beta_1|}\frac{|a_0|}{|\gamma_2|}$$

is a convex function in $\mathbb{D}$.

In the case of real parameters γ_2 and β_1 as above it is easy to obtain following statement.

Proposition 4. *Let $\gamma_2 > 0$, $a_1\gamma_2 - a_0\gamma_1 = \beta_1 + \gamma_2 = 0$, $\beta_1 > -2$ and $R[A] \geq 1$. Then there exists a solution given by* (19) *of differential equation* (3) *with $R[f] = R[A]$, which by the condition*

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n-1)(n+\beta_1)} \leq 1 - \frac{3}{4}\frac{2|\beta_0| + |\gamma_1|}{4(3+\beta_1)} - \frac{2|\gamma_0|}{3(4+\beta_1)} - 2\frac{|\beta_0| + |\gamma_1|}{2+\beta_1} - \frac{3|\gamma_0|}{2(3+\beta_1)} - \frac{2|\gamma_0||a_0|}{(2+\beta_1)\gamma_2}$$

is close-to-convex (starlike if $a_0 = 0$) and by the condition

$$\sum_{n=2}^{\infty} \frac{n^2|a_n|}{(n-1)(n+\beta_1)} \leq 1 - \frac{9}{8}\frac{2|\beta_0| + |\gamma_1|}{3+\beta_1} - \frac{(4/3)|\gamma_0|}{4+\beta_1} - 4\frac{|\beta_0| + |\gamma_1|}{2+\beta_1} - \frac{(9/2)|\gamma_0|}{3+\beta_1} - \frac{4|\gamma_0||a_0|}{(2+\beta_1)\gamma_2}$$

is a convex function in $\mathbb{D}$.

3 Close-to-convexity and Convexity in the Case $\gamma_2 = 0$

In this case from (4) it follows that $a_0 = 0$, i. e. f_0 can be arbitrary number, and we choose $f_0 = 0$. Then $\beta_1 f_1 = a_1$. Since we are finding univalent solutions $f_1 \neq 0$. Therefore, two cases are possible:

3a) $a_1 \neq 0$ and $\beta_1 \neq 0$;

3b) $a_1 = \beta_1 = 0$.

By the condition 3a) a solution of equation (3) has the form

$$f(z) = \frac{a_1}{\beta_1}z + \sum_{n=2}^{\infty} f_n z^n, \tag{22}$$

where the coefficients f_n are defined by recurrent formula

$$n(n+\beta_1-1)f_n + (\beta_0(n-1)+\gamma_1)f_{n-1} + \gamma_0 f_{n-2} = a_n,$$

from which by the condition $n + \beta_1 - 1 \neq 0$ for all $n \geq 2$ it follows that

$$f_n = -\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)}f_{n-1} - \frac{\gamma_0}{n(n+\beta_1-1)}f_{n-2} + \frac{a_n}{n(n+\beta_1-1)},$$

whence by the condition $|\beta_1|<1$ we get

$$\begin{aligned}\sum_{n=2}^{\infty} n|f_n| &\le \sum_{n=2}^{\infty}\frac{n}{n-1}\frac{(n-1)|\beta_0|+|\gamma_1|}{n(n-|\beta_1|-1)}(n-1)|f_{n-1}|\\ &+\sum_{n=2}^{\infty}\frac{n}{n-2}\frac{|\gamma_0|}{n(n-|\beta_1|-1)}(n-2)|f_{n-2}|+\sum_{n=2}^{\infty}\frac{|a_n|}{n-|\beta_1|-1}\\ &=\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}|f_1|+\sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/n}{n-|\beta_1|}n|f_n|+\frac{|\gamma_0|}{2-|\beta_1|}|f_1|+\sum_{n=2}^{\infty}\frac{|\gamma_0|/n}{n+1-|\beta_1|}n|f_n|\\ &+\sum_{n=2}^{\infty}\frac{|a_n|}{n-|\beta_1|-1}\le\sum_{n=2}^{\infty}\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}n|f_n|+\sum_{n=2}^{\infty}\frac{|\gamma_0|/2}{3-|\beta_1|}n|f_n|\\ &+\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{|a_n|}{(n-|\beta_1|-1)},\end{aligned}$$

i. e. by the condition

$$\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}+\frac{|\gamma_0|/2}{3-|\beta_1|}<1$$

we obtain

$$\begin{aligned}&\left(1-\frac{|\beta_0|+|\gamma_1|/2}{2-|\beta_1|}-\frac{|\gamma_0|/2}{3-|\beta_1|}\right)\sum_{n=2}^{\infty}n|f_n|\\ &\le\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{|a_n|}{(n-|\beta_1|-1)}.\end{aligned}\tag{23}$$

Similarly,

$$\begin{aligned}\sum_{n=2}^{\infty} n^2|f_n| &\le \sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2}\frac{(n-1)|\beta_0|+|\gamma_1|}{n(n-|\beta_1|-1)}(n-1)^2|f_{n-1}|\\ &+\sum_{n=2}^{\infty}\frac{n^2}{(n-2)^2}\frac{|\gamma_0|}{n(n-|\beta_1|-1)}(n-2)^2|f_{n-2}|+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-|\beta_1|-1}\\ &=2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\sum_{n=2}^{\infty}\frac{n+1}{n}\frac{|\beta_0|+|\gamma_1|/n}{n-|\beta_1|}n^2|f_n|+\frac{3|\gamma_0|}{2-|\beta_1|}|f_1|\\ &+\sum_{n=2}^{\infty}\frac{n+2}{n^2}\frac{|\gamma_0|}{n+1-|\beta_1|}n^2|f_n|+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-|\beta_1|-1}\le\sum_{n=2}^{\infty}\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}n^2|f_n|\\ &+\sum_{n=2}^{\infty}\frac{|\gamma_0|}{3-|\beta_1|}n^2|f_n|+2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{3|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-|\beta_1|-1)},\end{aligned}$$

i. e. by the condition

$$\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}+\frac{|\gamma_0|}{3-|\beta_1|}<1$$

we get

$$\begin{aligned}&\left(1-\frac{3}{4}\frac{2|\beta_0|+|\gamma_1|}{2-|\beta_1|}-\frac{|\gamma_0|}{3-|\beta_1|}\right)\sum_{n=2}^{\infty}n^2|f_n|\\ &\le 2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\frac{|a_1|}{|\beta_1|}+\frac{3|a_1||\gamma_0|}{|\beta_1|(2-|\beta_1|)}+\sum_{n=2}^{\infty}\frac{n|a_n|}{(n-|\beta_1|-1)}.\end{aligned}\tag{24}$$

In view of Lemma 3 from (23) and (24) in the usual way we obtain the following theorem.

Theorem 3. *Let $\gamma_2 = 0$, $a_1 \neq 0$, $\beta_1 \neq 0$, $|\beta_1| < 1$ and $R[A] \geq 1$. Then there exists a solution given by* (22) *of differential equation* (3) *with $R[f] = R[A]$, which by the condition*

$$\sum_{n=2}^{\infty} \frac{|a_n|}{(n-|\beta_1|-1)} \leq \left(1 - \frac{|\beta_0| + |\gamma_1|/2 + |\gamma_0|}{2-|\beta_1|} - \frac{|\gamma_0|/2}{3-|\beta_1|} - \frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\right) \frac{|a_1|}{|\beta_1|}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n-|\beta_1|-1)} \leq \left(1 - \frac{3}{4}\frac{2|\beta_0| + |\gamma_1| + 4|\gamma_0|}{2-|\beta_1|} - \frac{|\gamma_0|}{3-|\beta_1|} - 2\frac{|\beta_0|+|\gamma_1|}{1-|\beta_1|}\right) \frac{|a_1|}{|\beta_1|}$$

is a convex function in $\mathbb{D}$.

For a real parameter β_1 in the usual way we obtain the following proposition.

Proposition 5. *Let $\gamma_2 = 0$, $a_1 \neq 0$, $\beta_1 \neq 0$, $\beta_1 > -1$ and $R[A] \geq 1$. Then there exists a solution given by* (22) *of differential equation* (3) *with $R[f] = R[A]$, which by the condition*

$$\sum_{n=2}^{\infty} \frac{|a_n|}{(n+\beta_1-1)} \leq \left(1 - \frac{|\beta_0| + |\gamma_1|/2 + |\gamma_0|}{2+\beta_1} - \frac{|\gamma_0|/2}{3+\beta_1} - \frac{|\beta_0|+|\gamma_1|}{1+\beta_1}\right) \frac{|a_1|}{|\beta_1|}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty} \frac{n|a_n|}{(n+\beta_1-1)} \leq \left(1 - \frac{3}{4}\frac{2|\beta_0| + |\gamma_1| + 4|\gamma_0|}{2+\beta_1} - \frac{|\gamma_0|}{3+\beta_1} - 2\frac{|\beta_0|+|\gamma_1|}{1+\beta_1}\right) \frac{|a_1|}{|\beta_1|}$$

is a convex function in $\mathbb{D}$.

If the condition 3b) holds then we can choose $f_1 = 1$ and search a solution in a form

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n, \tag{25}$$

where the coefficients f_n are defined by recurrent formula

$$f_n = -\frac{\beta_0(n-1)+\gamma_1}{n(n-1)} f_{n-1} - \frac{\gamma_0}{n(n-1)} f_{n-2} + \frac{a_n}{n(n-1)}. \tag{26}$$

Then

$$\begin{aligned}\sum_{n=2}^{\infty} n|f_n| &\leq |\beta_0| + |\gamma_1| + \sum_{n=2}^{\infty} \frac{n|\beta_0| + |\gamma_1|}{n^2} n|f_n| + \frac{|\gamma_0|}{2} + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{(n+1)n} n|f_n| + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1} \\ &\leq \sum_{n=2}^{\infty} \frac{2|\beta_0| + |\gamma_1|}{4} n|f_n| + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{6} n|f_n| + |\beta_0| + |\gamma_1| + \frac{|\gamma_0|}{2} + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1}\end{aligned}$$

and by the condition $(2|\beta_0| + |\gamma_1|)/4 + |\gamma_0|/6 < 1$ we get

$$(1 - (2|\beta_0| + |\gamma_1|)/4 - |\gamma_0|/6) \sum_{n=2}^{\infty} n|f_n| \leq |\beta_0| + |\gamma_1| + |\gamma_0|/2 + \sum_{n=2}^{\infty} \frac{|a_n|}{n-1}. \tag{27}$$

Similarly,

$$\begin{aligned}\sum_{n=2}^{\infty} n^2|f_n| &\leq \sum_{n=2}^{\infty} n^2 \frac{|\beta_0|(n-1) + |\gamma_1|}{n(n-1)} |f_{n-1}| + \sum_{n=2}^{\infty} n^2 \frac{|\gamma_0|}{n(n-1)} |f_{n-2}| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n-1} \\ &\leq 2(|\beta_0| + |\gamma_1|) + \sum_{n=2}^{\infty} \frac{3}{8}(2|\beta_0| + |\gamma_1|) n^2|f_n| + 3|\gamma_0|/2 + \sum_{n=2}^{\infty} \frac{|\gamma_0|}{3} n^2|f_n| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n-1},\end{aligned}$$

i. e. by the condition $3(2|\beta_0|+|\gamma_1|)/8+|\gamma_0|/3<1$

$$(1-3(2|\beta_0|+|\gamma_1|)/8-|\gamma_0|/3)\sum_{n=2}^{\infty} n^2|f_n| \le 2(|\beta_0|+|\gamma_1|)+3|\gamma_0|/2+\sum_{n=2}^{\infty}\frac{n|a_n|}{n-1}. \tag{28}$$

In view of Lemma 2 from (27) and (28) in the usual way we obtain the following theorem.

Theorem 4. *Let $\gamma_2=a_0=\beta_1=a_1=0$ and $R[A]\ge 1$. Then there exists a solution given by* (25) *of differential equation* (3) *with $R[f]=R[A]$, which by the condition*

$$\sum_{n=2}^{\infty}\frac{|a_n|}{n-1} \le 1-\frac{3}{2}|\beta_0|-\frac{5}{4}|\gamma_1|-\frac{2}{3}|\gamma_0| \tag{29}$$

is starlike, and by the condition

$$\sum_{n=2}^{\infty}\frac{n|a_n|}{n-1} \le 1-\frac{11}{4}|\beta_0|-\frac{19}{8}|\gamma_1|-\frac{11}{6}|\gamma_0| \tag{30}$$

is a convex function in $\mathbb{D}$.

4 GROWTH OF ENTIRE SOLUTIONS

If $n(n+\beta_1-1)+\gamma_2\ne 0$ for all $n\ge 2$ by Lemma 1 a function given by (1) can be an entire solution of equation (3) only if the function A is entire.

For an entire function (1) let $M_f(r)=\max\{|f(z)|:\,|z|=r\}$, and for the characteristic of the growth of $M_f(r)$ we will use generalized orders. To give a definition of generalized order we denote, as in [11], by L a class of continuous nonnegative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\ge 0$ for $x\le x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\le x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each fixed $c\in(0,+\infty)$, i. e. α is slowly increasing function. Clearly, $L_{si}\subset L^0$. The value

$$\varrho_{\alpha\beta}[f]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(\ln r)}\quad(\alpha\in L,\beta\in L)$$

is called [11] generalized order of f. The following lemma is true.

Lemma 4. *If $\alpha\in L_{si}$, $\beta\in L$, $\beta(x+O(1))=(1+o(1))\beta(x)$ as $x\to+\infty$ and f is an entire transcendental function then $\varrho_{\alpha\beta}[f']=\varrho_{\alpha\beta}[f]$.*

Proof. Indeed, from the integral formula of Cauchy it easily follows that $M_{f'}(r)\le M_f(r+1)$, whence we get $\varrho_{\alpha\beta}[f']\le\varrho_{\alpha\beta}[f]$. On the other hand, since $f(z)-f(0)=\int_0^z f'(t)dt$, we have $M_f(r)\le rM_{f'}(r)+|f(0)|$ and, thus, $\ln M_f(r)\le\ln M_{f'}(r)+\ln r+o(1)=(1+o(1))\ln M_{f'}(r)$ as $r\to+\infty$, because the function f is transcendental. Hence we get $\varrho_{\alpha\beta}[f]\le\varrho_{\alpha\beta}[f']$. Lemma 4 is proved. □

We will use the theory of the value distribution of Nevanlinna. For an entire function f we put

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+|f(re^{i\varphi})|d\varphi.$$

This function is said to be a characteristic function of Nevanlinna. It is known that

Lemma 5. *If $\alpha \in L_{si}$, $\beta \in L$, $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$ and f is an entire transcendental function then*

$$\varrho_{\alpha\beta}[f] = \overline{\lim_{r\to+\infty}} \frac{\alpha(T(r,f))}{\beta(\ln r)}. \tag{31}$$

Proof. Indeed, in [3, p. 54] it is proved that for $0 < r < r_1$

$$T(r,f) \le \ln^+ M_f(r) \le \frac{r_1 + r}{r_1 - r} T(r_1, f). \tag{32}$$

Choosing $r_1 = 2r$ and using (32), in view of the conditions $\alpha \in L_{si}$ and $\beta \in L^0$ hence we obtain

$$\begin{aligned}\overline{\lim_{r\to+\infty}} \frac{\alpha(T(r,f))}{\beta(\ln r)} &\le \overline{\lim_{r\to+\infty}} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)} \le \overline{\lim_{r\to+\infty}} \frac{\alpha(3T(2r,f))}{\beta(\ln r)} \\ &= \overline{\lim_{r\to+\infty}} \frac{\alpha(T(r,f))}{\beta(\ln r - \ln 2)} = \overline{\lim_{r\to+\infty}} \frac{\alpha(T(r,f))}{\beta(\ln r)}.\end{aligned}$$

Lemma 5 is proved. □

Now we prove the following theorem.

Theorem 5. *Let $\alpha \in L_{si}$, $\beta \in L$, $\alpha(\ln x) = o(\alpha(x))$, $\beta(x+O(1)) = (1+o(1))\beta(x)$, $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ and f be an entire transcendental solution of the differential equation*

$$a_0(z)w + a_1(z)w' + \cdots + a_m(z)w^{(m)} = A(z), \tag{33}$$

where a_j are polynomials, $0 \le j \le m$, and A is an entire function. Then $\varrho_{\alpha\beta}[f] \ge \varrho_{\alpha\beta}[A]$.

Proof. If $\varrho_{\alpha\beta}[f] = +\infty$ then theorem is obvious.

So we consider the case $\varrho_{\alpha\beta}[f] < +\infty$. At first we remark that if P_m is a polynomial of degree $m \ge 1$ then [3, p.47] $T(r, P_m) = m \ln r + O(1)$ as $r \to +\infty$. Further we put

$$\Omega_m(z,f) = a_0(z)f(z) + a_1(z)f'(z) + \cdots + a_m(z)f^{(m)}(z),$$

where $a_j (1 \le j \le m)$ are polynomials and f is an entire functions. Using well-known [3, p.44] inequalities

$$T\left(r, \prod_{j=1}^{q} f_j\right) \le \sum_{j=1}^{q} T(r, f_j), \quad T\left(r, \sum_{j=1}^{q} f_j\right) \le \sum_{j=1}^{q} T(r, f_j) + \ln q$$

we have

$$T(r, \Omega_m(\cdot, f)) \le T(r,f) + T(r,f') + \cdots + T(r, f^{(m)}) + O(\ln r), \quad r \to +\infty. \tag{34}$$

By the lemma about a logarithmic derivative [3, p.122] $T(r, f'/f) = Q(r,f)$ for each entire function f, where $Q(r,f)$ is denoting [3, p.122] an arbitrary function such that:

1) if f has a finite order then $Q(r,f) = O(\ln r)$ as $r \to +\infty$;

2) if f has an infinite order then $Q(r,f) = O(\ln T(r,f) + \ln r)$ as $r \to +\infty$ outside, possibly, some set of finite measure.

Clearly, $Q(r,f) \pm Q(r,f) = Q(r,f)$ and $AQ(r,f) = Q(r,f)$ [3, p.122]. We remak also that since f has a finite generalized order then in view of (31) $T(r,f) \leq \alpha^{-1}(\varrho\beta(\ln r))$ for $\varrho > \varrho_{\alpha\beta}[f]$ and $r \geq r_0$. Hence it follows that $Q(r,f) = O(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r)$ as $r \to +\infty$ and by Lemma 4 $Q(r,f') = O(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r)$ as $r \to +\infty$.

Therefore,

$$\begin{aligned} T(r,f') = T\left(r, f\frac{f'}{f}\right) &\leq T(r,f) + T\left(r, \frac{f'}{f}\right) = T(r,f) + Q(r,f) \\ &= T(r,f) + O(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r), \quad r \to +\infty. \end{aligned}$$

Similarly,

$$T(r,f'') = T\left(r, f'\frac{f''}{f'}\right) \leq T(r,f') + Q(r,f') = T(r,f) + O(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r),\ r \to +\infty,$$

et cetera. As a result from (34) we will get

$$T(r, \Omega_m(\cdot, f)) \leq (m+1)T(r,f) + O(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r), \quad r \to +\infty. \tag{35}$$

Since f is an entire solution of the differential equation (33), we have $\Omega_m(z,f) \equiv A(z)$. Therefore, since $\alpha \in L_{si}$, in view of (31) and (35) we obtain

$$\begin{aligned} \varrho_{\alpha\beta}[A] &= \varlimsup_{r\to+\infty} \frac{\alpha(T(r,A))}{\beta(\ln r)} \leq \varlimsup_{r\to+\infty} \frac{\alpha((m+1)T(r,f) + K_1(\ln \alpha^{-1}(\varrho\beta(\ln r)) + \ln r))}{\beta(\ln r)} \\ &\leq \varlimsup_{r\to+\infty} \frac{\alpha(K_2 \max\{T(r,f),\ \ln \alpha^{-1}(\varrho\beta(\ln r)),\ \ln r\})}{\beta(\ln r)} \\ &= \varlimsup_{r\to+\infty} \frac{\alpha(\max\{T(r,f),\ \ln \alpha^{-1}(\varrho\beta(\ln r)),\ \ln r\})}{\beta(\ln r)} \\ &= \varlimsup_{r\to+\infty} \frac{\max\{\alpha(T(r,f)),\ \alpha(\ln \alpha^{-1}(\varrho\beta(\ln r))),\ \alpha(\ln r)\}}{\beta(\ln r)} \\ &= \varlimsup_{r\to+\infty} \frac{\alpha(T(r,f)) + \alpha(\ln \alpha^{-1}(\varrho\beta(\ln r))) + \alpha(\ln r)}{\beta(\ln r)} \\ &\leq \varlimsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\ln r)} + \varlimsup_{r\to+\infty} \frac{\alpha(\ln \alpha^{-1}(\varrho\beta(\ln r)))}{\beta(\ln r)} + \varlimsup_{r\to+\infty} \frac{\alpha(\ln r)}{\beta(\ln r)}. \end{aligned}$$

Since $\alpha(x) = o(\beta(x))$ as $x \to +\infty$ we have $\dfrac{\alpha(\ln r)}{\beta(\ln r)} \to 0$ as $r \to +\infty$. Simultaneously,

$$\varlimsup_{r\to+\infty} \frac{\alpha(\ln \alpha^{-1}(\varrho\beta(\ln r)))}{\beta(\ln r)} = \varlimsup_{x\to+\infty} \frac{\alpha(\ln \alpha^{-1}(\varrho x))}{x} = \varrho \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\alpha(x)} = 0.$$

Therefore, $\varrho_{\alpha\beta}[A] \leq \varrho_{\alpha\beta}[f]$ and Theorem 5 is proved. $\square$

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \geq x_0$ then we come to the following statement.

Corollary 1. *If function f be an entire transcendental solution of the differential equation* (3) *then $\varrho[f] \geq \varrho[A]$, where $\varrho[f] = \varlimsup\limits_{r\to+\infty} \dfrac{\ln \ln M(r,f)}{\ln r}$ is the order of f.*

We remark that the contrary inequality is not true in general. Indeed, if for example $\beta_0 = -1$, $\beta_1 = \gamma_1 = \gamma_2$, $-1 \le \gamma_0 < 0$ and all $a_n = 0$, then [13] there exists an entire solution f of equation (3) such that

$$\ln M(r,f) = \frac{1+o(1)}{2}\left(|\beta_0| + \sqrt{|\beta_0|^2 + 4|\gamma_0|^2}\right) r, \quad r \to +\infty.$$

Clearly, in this case $\varrho[A] = 0 < 1 = \varrho[f]$.

Suppose that $\gamma_2 = a_0 = \beta_1 = a_1 = \beta_0 = \gamma_1 = \gamma_0 = 0$ and $A(z) = \sum_{n=2}^{\infty} a_n z^n$ is an entire function. Then equation (3) has the form $w'' = \sum_{n=2}^{\infty} a_n z^{n-2}$ and the function $f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{n(n-1)} z^n$ is a solution of this equation. Using the formula of Hadamard of the order it is easy to prove that $\varrho[A] = \varrho[f]$, i. e. the estimate $\varrho[A] \le \varrho[f]$ is sharp.

If $\varrho_{\alpha\beta}[f] = 0$ then for the characteristic of the growth of f it is used the belonging to generalized convergence classes. For $\alpha \in L$ and $\beta \in L$ we will say that an entire function f belongs to generalized convergence class if

$$\int_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr < +\infty, \tag{36}$$

Choosing $r_1 = 2r$ from (32) we get $T(r,f) \le \ln^+ M_f(r) \le 3T(2r,f)$. On the other hand, in [10] it is proved that if $\alpha \in L^0$ then α is RO-increasing [8], i. e. for every $h \in [1,a]$, $1 < a < +\infty$, and all $x \ge x_0$ the inequality $\alpha(hx)/\alpha(x) \le M(a) < +\infty$ is true. Therefore, if $\alpha \in L^0$, $\beta \in L$ and $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ then (36) holds if and only if

$$\int_{r_0}^{\infty} \frac{\alpha(T(r,f))}{r\beta(\ln r)} dr < +\infty. \tag{37}$$

Using (35) we prove the following theorem.

Theorem 6. *Let $\alpha \in L^0$, $\beta \in L$, $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ and*

$$\int_{x_0}^{\infty} \frac{\alpha(\ln \alpha^{-1}(\beta(x)))}{\beta(x)} dx < +\infty. \tag{38}$$

Suppose that f is an entire transcendental solution of the differential equation (33) *where a_j are polynomials, $0 \le j \le m$, A is an entire function and $\varrho_{\alpha\beta}[f] = 0$. Then in order that f belongs to generalized convergence class, it is necessary that A belongs to this class.*

Proof. Since $\varrho_{\alpha\beta}[f] = 0$, we have $Q(r,f) = O(\ln \alpha^{-1}(\beta(\ln r)) + \ln r)$ as $r \to +\infty$ and from (35) as above in view of the condition $\alpha \in L^0$ we obtain

$$\begin{aligned}
\int_{r_0}^{\infty} \frac{\alpha(T(r,A))}{r\beta(\ln r)} dr &= \int_{r_0}^{\infty} \frac{\alpha(T(r,\Omega_m(\cdot,f)))}{r\beta(\ln r)} dr \\
&\le \int_{r_0}^{\infty} \frac{\alpha((m+1)T(r,f) + K_1(\ln \alpha^{-1}(\beta(\ln r)) + \ln r))}{r\beta(\ln r)} dr \\
&\le \int_{r_0}^{\infty} \frac{\alpha(K_2 \max\{T(r,f),\, \ln \alpha^{-1}(\beta(\ln r)),\, \ln r\})}{\beta(\ln r)} \\
&\le M(K_2) \int_{r_0}^{\infty} \frac{\alpha(T(r,f)) + \alpha(\ln \alpha^{-1}(\beta(\ln r))) + \alpha(\ln r)}{r\beta(\ln r)} dr.
\end{aligned}$$

Since f is an entire function, from (36) it follows that $\int\limits_{r_0}^{\infty} \frac{\alpha(\ln r)}{r\beta(\ln r)}dr < +\infty$, and in view of (38)

$$\int\limits_{r_0}^{\infty} \frac{\alpha(\ln \alpha^{-1}(\beta(\ln r)))}{r\beta(\ln r)}dr = \int\limits_{x_0}^{\infty} \frac{\alpha(\ln \alpha^{-1}(\beta(x)))}{\beta(x)}dx < +\infty.$$

Therefore, (37) implies $\int\limits_{r_0}^{\infty} \frac{\alpha(T(r,A))}{r\beta(\ln r)}dr < +\infty$. Theorem 6 is proved. □

For entire functions of the minimal type of the order $\varrho \in (0,+\infty)$ G. Valiron [16, p.18] introduced the convergence class by the condition $\int\limits_{1}^{\infty} \frac{\ln M_f(r)}{r^{\varrho+1}}dr < +\infty$. If we choose $\alpha(x) = x$ and $\beta(x) = e^{\varrho x}$ for $x \geq x_0$, then from Theorem 6 we get the following statement.

Corollary 2. *If an entire function f is a solution of the differential equation* (3)*, then in order that f belongs to the convergence class of Valiron, it is necessary that A belongs to this class.*

Clearly, from the belonging of the function A to the convergence class of Valiron the belonging of the function f to this class does not follow. On the other hand, an entire solution of the differential equation $z^2w'' = A(z)$ belongs to the convergence class of Valiron if and only if A belongs to this class.

Finally we will consider a linear differential equation of the endless order

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} w^{(n)} = \Phi(z), \tag{39}$$

where the characteristic function $\varphi(t) = \sum\limits_{n=0}^{\infty} \frac{a_n}{n!} t^n$ is entire and has a growth not higher than the normal type of the first order, and Φ is an entire function.

A.O. Gelfond [2] proved that equation (39) for every $\theta > 1$ has an entire solution f such that

$$\ln \overline{M}_f(r) \leq C(\theta) \ln \overline{M}_\Phi(\theta r), \quad r \geq r_0, \tag{40}$$

where $C(\theta)$ does not depend on r and $\ln \overline{M}_f(r) = r\max\left\{\frac{\ln M_f(t)}{t} : 1 \leq t \leq r\right\}$. Using this result we prove the following statement.

Proposition 6. *Equation* (39) *has an entire solution f such that:*

1) *if $\alpha(e^x) \in L_{si}$, $\beta \in L$, $\beta(x + O(1)) \sim \beta(x)$ and $\alpha(x) = o(\beta(\ln x))$ as $x \to +\infty$, then $\varrho_{\alpha\beta}[f] \leq \varrho_{\alpha\beta}[\Phi]$;*

2) *if $\alpha(e^x) \in L^0$, $\beta \in L$, $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ and $\int\limits_{r_0}^{\infty} \frac{\alpha(r)}{r\beta(\ln r)}dr < +\infty$, then the belonging of Φ to the generalized convergence class implies the belonging of f to this class.*

Proof. Clearly, $\ln M_f(r) \leq \ln \overline{M}_f(r) \leq r \ln M_f(r)$ for $r \geq 1$. Therefore, if $\alpha(e^x) \in L_{si}$ and $\beta(x + O(1)) \sim \beta(x)$ as $x \to +\infty$ then from (40) we have

$$\begin{aligned}
\varrho_{\alpha\beta}[f] &= \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)} \leq \varlimsup_{r\to+\infty} \frac{\alpha(\ln \overline{M}_f(r))}{\beta(\ln r)} \leq \varlimsup_{r\to+\infty} \frac{\alpha(C(\theta) \ln \overline{M}_\Phi(\theta r))}{\beta(\ln (\theta r) - \ln \theta)} \\
&= \varlimsup_{r\to+\infty} \frac{\alpha(\ln \overline{M}_\Phi(r))}{\beta(\ln r)} \leq \varlimsup_{r\to+\infty} \frac{\alpha(r \ln M_\Phi(r))}{\beta(\ln r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{\ln r + \ln\ln M_\Phi(r)\})}{\beta(\ln r)} \\
&\leq \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{2\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{\beta(\ln r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{\beta(\ln r)} \\
&= \varlimsup_{r\to+\infty} \frac{\max\{\alpha(r), \alpha(\ln M_\Phi(r))\}}{\beta(\ln r)} \leq \varlimsup_{r\to+\infty} \frac{\alpha(r) + \alpha(\ln M_\Phi(r))}{\beta(\ln r)} = \varrho_{\alpha\beta}[\Phi].
\end{aligned}$$

The firs part of Proposition 6 is proved.

Similarly, if $\alpha(e^x) \in L^0$, $\beta(x + O(1)) = O(\beta(x))$ as $x \to +\infty$ and $\int\limits_{r_0}^{\infty} \frac{\alpha(\ln M_\Phi(r))}{r\beta(\ln r)} dr < +\infty$, then

$$\begin{aligned}
&\int\limits_{r_0}^{\infty} \frac{\alpha(\ln M_f(r))}{r\beta(\ln r)} dr \leq \int\limits_{r_0}^{\infty} \frac{\alpha(C(\theta) \ln \overline{M}_\Phi(\theta r))}{r\beta(\ln r)} dr \leq M_1 \int\limits_{r_0}^{\infty} \frac{\alpha(r \ln M_\Phi(r))}{r\beta(\ln r)} dr \\
&\leq M_1 \int\limits_{r_0}^{\infty} \frac{\alpha(\exp\{2\max\{\ln r, \ln\ln M_\Phi(r)\}\})}{r\beta(\ln r)} dr \leq M_1 M_2 \int\limits_{r_0}^{\infty} \frac{\alpha(r) + \alpha(\ln M_\Phi(r))}{r\beta(\ln r)} dr < +\infty,
\end{aligned}$$

where $M_1 = M_1(\theta)$ and $M_2 = M_2(2)$. The proof of Proposition 6 is completed. ☐

REFERENCES

[1] Alexander J.F. *Functions which map the interior of the unit circle upon simple regions*. Ann. of Math. 1915, **17**, 12–22. doi:10.2307/2007212

[2] Gelfond A.O. *Linear differential equations of infinite order with constant coefficients and asymptotic periods of entire functions*. Tr. Mat. Inst. Steklova 1951, **38**, 42–67. (in Russian)

[3] Goldberg A.A., Ostrovsky I.V. Value distribution of meromorphic functions. Nauka, Moscow, 1970. (in Russian)

[4] Golusin G.M. Geometric Theory of Functions of a Complex Variable. Amer. Math. Soc., Providence, 1969.

[5] Goodman A.W. Univalent function. Vol. II. Mariner Pub. Co., 1983.

[6] Goodman A.W. *Univalent functions and nonanalytic curves*. Proc. Amer. Math. Soc. 1957, **8**, 597–601. doi: 10.1090/S0002-9939-1957-0086879-9

[7] Kaplan W. *Close-to-convex schlicht functions*. Michigan Math. J. 1952, **1** (2), 169–185. doi:10.1307/mmj/1028988895

[8] Seneta E. Regularly varying functions. Lecture Notes in Mathematics, 508, Springer-Verlag, Berlin, 1976.

[9] Shah S.M. *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II*. J. Math. Anal. Appl. 1989, **142**, 422–430. doi:10.1016/0022-247X(89)90011-5

[10] Sheremeta M.M. *On two classes of positive functions and the belonging to them of main characteristics of entire functions*. Mat. Stud. 2003, **19** (1), 73–82.

[11] Sheremeta M.N. *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*. Izv. Vyssh. Uchebn. Zaved. Mat. 1967, (2), 100—108.(in Russian)

[12] Sheremeta Z.M. *Close-to-convexity of entire solutions of a differential equation*. Mat. Metodi Fiz.-Mekh. Polya 1999, **42** (3), 31–35. (in Ukrainian)

[13] Sheremeta Z.M. *On properties of entire solutions of a differential equation*. Diff. Equat. 2000, **36** (8), 1155–1161. doi:10.1007/BF02754183

[14] Sheremeta Z.M. *On entire solutions of a differential equation* . Mat. stud. 2000, **14** (1), 54–58.

[15] Sheremeta Z.M. *On the close-to-convexity of entire solutions of a differential equation*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2000, **57**, 88–91. (in Ukrainian)

[16] Valiron G. Lectures on the general theory of integral functions. Edouard Privat, Toulouse, 1923.

Received 27.05.2019

Мулява О.М., Шеремета М.М., Трухан Ю.С. *Властивості розв'язків одного неоднорідного диференціального рівняння другого порядку* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 379–398.

Нехай степеневий ряд $A(z) = \sum_{n=0}^{\infty} a_n z^n$ має радіус збіжності $R[A] \in [1, +\infty]$. Для неоднорідного диференціального рівняння

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = A(z)$$

з комплексними коефіцієнтами вивчаються геометричні властивості в одиничному крузі його розв'язків (опуклість, зірковість, близькість до опуклості). Розглядається два випадки: $\gamma_2 \neq 0$ і $\gamma_2 = 0$. Також ми розглядаємо випадки дійсних параметрів цього рівняння. Доведено, що для розв'язку f цього рівняння радіус збіжності $R[f]$ дорівнює $R[A]$ і знайдено рекурентні формули для знаходження коефіцієнтів степеневого розвинення $f(z)$. Для цілого розв'язку доведено, що порядок розв'язку f не менший ніж порядок функції A ($\varrho[f] \geq \varrho[A]$) і оцінка є точною. Аналогічна нерівність доведена для узагальнених порядків ($\varrho_{\alpha\beta}[f] \geq \varrho_{\alpha\beta}[A]$). Для цілого розв'язку цього рівняння вивчено належність до класу збіжності. Наприкінці розглядається лінійне диференціальне рівняння нескінченного порядку $\sum\limits_{n=0}^{\infty} \frac{a_n}{n!} w^{(n)} = \Phi(z)$, і вивчається можливе зростання його розв'язків.

Ключові слова і фрази: диференціальне рівняння, опуклість, зірковість, близькість до опуклості, узагальнений порядок, клас збіжності.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 399–406
doi:10.15330/cmp.11.2.399-406

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.399–406

PATTABIRAMAN K.[1,2]

INVERSE SUM INDEG COINDEX OF GRAPHS

The inverse sum indeg coindex $\overline{ISI}(G)$ of a simple connected graph G is defined as the sum of the terms $\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ over all edges uv not in G, where $d_G(u)$ denotes the degree of a vertex u of G. In this paper, we present the upper bounds on inverse sum indeg coindex of edge corona product graph and Mycielskian graph. In addition, we obtain the exact value of both inverse sum indeg index and its coindex of a double graph.

Key words and phrases: inverse sum indeg index, edge corona graph, Mycielskian graph, double graph.

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INTRODUCTION

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum vertex degrees of G, respectively. A *topological index* or molecular descriptor of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. Several types of such indices exist, especially those based on vertex and edge distances.

Molecular descriptors, results of functions mapping molecule's chemical information into a number [16], have found applications in modeling many physicochemical properties in QSAR and QSPR studies [8, 6]. A particularly common type of molecular descriptors are those that are defined as functions of the structure of the underlying molecular graph, such as the Wiener index [18], the Zagreb indices [4], the Randić index [14] or the Balaban J-index [5]. Damir Vukicević and Marija Gasperov [17] observed that many of these descritors are defined simply as the sum of individual bond contributions.

Among the 148 discrete Adriatic indices studied in [17], whose predictive properties were evaluated against the benchmark datasets of the Internation Academy of Mathematical Chemistry [7], 20 indices were selected as significant predictors of physicochemical properties. In this connection, Sedlar et al. [15] studied the properties of the inverse sum indeg index, the descriptor that was selected in [17] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of Math. Chem. have a particularly simple and elegant structure. The *inverse sum indeg index* is defined as

$$ISI(G) = \sum_{uv\in E(G)} \frac{1}{\frac{1}{d_G(u)} + \frac{1}{d_G(v)}} = \sum_{uv\in E(G)} \frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}.$$

УДК 519.17
2010 *Mathematics Subject Classification:* 05C12, 05C76.

The *first Zagreb index* $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(G) = \sum\limits_{u \in V(G)} d_G^2(u) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$, $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u) d_G(v)$, where $d_G(v)$ is a degree of a vertex v in G. For a connected graph G, the *harmonic index* $H(G)$ is defined as $H(G) = \sum\limits_{uv \in E(G)} \frac{2}{d_G(u)+d_G(v)}$.

The *first* and *second Zagreb coindices* are defined as $\overline{M}_1(G) = \sum\limits_{uv \notin E(G)} (d_G(u) + d_G(v))$, $\overline{M}_2(G) = \sum\limits_{uv \notin E(G)} d_G(u) d_G(v)$. Similarly, the *harmonic* coindex of G is defined as

$$\overline{H}(G) = \sum_{uv \notin E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

Motivated by the invariants like Zagreb and harmonic indices, we proposed the another invariant *inverse sum indeg coindex* as

$$\overline{ISI}(G) = \sum_{uv \notin E(G)} \frac{d_G(u) d_G(v)}{d_G(u) + d_G(v)}.$$

Extremal values of inverse sum indeg index across several graph classes, including connected graphs, chemical graphs, trees and chemical trees were determined in [15]. The bounds of a descriptor are important information of a molecular graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters. In [2], some sharp bounds for the inverse sum indeg index of connected graphs are given. The inverse sum indeg index of some nanotubes is computed in [3]. Several upper and lower bounds on the inverse sum indeg index in terms of some molecular structural parameters and relate this index to various well-known molecular descriptors are presented in [12]. In this paper, we present the upper bounds on the inverse sum indeg coindex of edge corona product graph and Mycielskian graph. In addition, we obtain the exact value of both inverse sum indeg index and its coindex of double graph.

1 EDGE CORONA

Hou and Shiu [5] introduced a kind of new graph operation, namely, edge corona product. The *edge corona product* $G \bullet H$ of G and H is defined as the graph obtained by taking one copy of G and $|E(G)|$ copies of H, and then joining two end vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H. The computation for some of the topological indices of edge corona product are resently studied in [1, 13, 5].

Lemma 1 ([9]). *Let f be a convex function on the interval I and $x_1, x_2, \ldots, x_n \in I$. Then $f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \leq \frac{f(x_1)+f(x_2)+\ldots f(x_n)}{n}$, with equality if and only if $x_1 = x_2 = \ldots = x_n$.*

Theorem 1. *Let G_1 and G_2 be two graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively.*

Then

$$\begin{aligned}\overline{ISI}(G_1 \bullet G_2) &\leq (n_2+1)\overline{ISI}(G_1) + \frac{m_1}{4}\Big(\overline{ISI}(G_2) + 2\overline{H}(G_2) + \frac{\overline{M_2}(G_2)}{4} + \frac{\overline{M_1}(G_2)}{2} \\ &+ \frac{3n_2(n_2-1)}{8} - \frac{3m_2}{4}\Big) + \frac{n_2(n_2+1)}{4}\Big(2m_1n_1 - M_1(G)\Big) \\ &+ \frac{(m_2+n_2)(n_1^2-2m_1)}{2} + \frac{m_1(m_1-1)n_2^2(\Delta(G_2)+2)^2}{4(\delta(G_2)+2)}.\end{aligned}$$

Proof. Let x_{ij} be the jth vertex in the ith copy of H, $i \in \{1, 2, \ldots, m_1\}$,$j \in \{1, 2, \ldots, n_2\}$, and let y_k be the kth in G_1,$k \in \{1, 2, \ldots, n_1\}$. Also let x_j be the jth vertex in G_2.

By the definition of edge corona of G_1 and G_2, for each vertex x_{ij}, we have $d_{G_1 \bullet G_2}(x_{ij}) = d_{G_2}(x_j) + 2$, and for every vertex y_k in G_1, $d_{G_1 \bullet G_2}(y_k) = d_{G_1}(y_k)n_2 + d_{G_1}(y_k) = (n_2+1)d_{G_1}(y_k)$.

Now, we consider the following four cases of nonadjacent vertex pairs in $G_1 \bullet G_2$.

Case 1: The nonadjacent vertex pairs $\{x_{ij}, x_{ih}\}$, $1 \leq i \leq m_1$,$1 \leq j < h \leq n_2$, and it is assumed that $x_j x_h \notin E(G_2)$.

$$\begin{aligned}C_1 &= \sum_{i=1}^{m_1} \sum_{x_{ij}x_{ih} \notin E(G_1 \bullet G_2)} \frac{d_{G_1 \bullet G_2}(x_{ij}) d_{G_1 \bullet G_2}(x_{ih})}{d_{G_1 \bullet G_2}(x_{ij}) + d_{G_1 \bullet G_2}(x_{ih})} \\ &= \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j) + d_{G_2}(x_h) + 4}.\end{aligned}$$

By Lemma 1, we have $\frac{1}{d_{G_2}(x_j)+d_{G_2}(x_h)+4} \leq \frac{1}{4(d_{G_2}(x_j)+d_{G_2}(x_h))} + \frac{1}{16}$ with equality if and only if $d_{G_2}(x_j) + d_{G_2}(x_h) = 4$. Thus,

$$\begin{aligned}C_1 &\leq \frac{1}{4}\sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \Big(\frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j) + d_{G_2}(x_h)} + \frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{4}\Big) \\ &= \frac{1}{4}\sum_{i=1}^{m_1} \Big(\overline{ISI}(G_2) + 3\Big(\frac{n_2(n_2-1)}{2} - m_2\Big) + 2\overline{H}(G_2) + \frac{\overline{M_2}(G_2)}{4} + \frac{\overline{M_1}(G_2)}{2}\Big) \\ &= \frac{m_1}{4}\overline{ISI}(G_2) + \frac{m_1}{2}\overline{H}(G_2) + \frac{m_1}{16}\overline{M_2}(G_2) + \frac{m_1}{8}\overline{M_1}(G_2) + \frac{3m_1n_2(n_2-1)}{8} - \frac{3m_1m_2}{4}.\end{aligned}$$

Case 2: The nonadjacent vertex pairs $\{y_k, y_s\}$,$1 \leq k < s \leq n_1$ and it is assumed that $y_k y_s \notin E(G_1)$. Thus,

$$\begin{aligned}C_2 &= \sum_{y_k y_s \notin E(G_1 \bullet G_2)} \frac{d_{G_1 \bullet G_2}(y_k) d_{G_1 \bullet G_2}(y_s)}{d_{G_1 \bullet G_2}(y_k) + d_{G_1 \bullet G_2}(y_s)} = \sum_{y_k y_s \notin E(G_1)} \frac{(n_2+1)^2 d_{G_1}(y_k) d_{G_1}(y_s)}{(n_2+1)(d_{G_1}(y_k) + d_{G_1}(y_s))} \\ &= (n_2+1) \sum_{y_k y_s \notin E(G_1)} \frac{d_{G_1}(y_k) d_{G_1}(y_s)}{d_{G_1}(y_k) + d_{G_1}(y_s)} = (n_2+1)\overline{ISI}(G_1).\end{aligned}$$

Case 3: The nonadjacent vertex pairs $\{x_{ij}, y_k\}$, $1 \leq i \leq m_1$,$1 \leq j \leq n_2$, $1 \leq k \leq n_1$, and it is assumed that the ith edge e_i $1 \leq i \leq m_1$ in G_1 does not pass through y_k.

Note that each vertex y_k is adjacent to all vertices of $d_{G_1}(y_k)$ copies of G_2, that is, each y_k is not adjacent to any vertex of $m_1 - d_{G_1}(y_k)$ copies of G_2. Hence

$$C_3 = \sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k)) \sum_{j=1}^{n_2} \frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{d_{G_2}(x_j) + 2 + (n_2+1)d_{G_1}(y_k)}$$

By Lemma 1, we obtain $\frac{1}{d_{G_2}(x_j)+2+(n_2+1)d_{G_1}(y_k)} \leq \frac{1}{4(d_{G_2}(x_j)+2)} + \frac{1}{4(n_2+1)d_{G_1}(y_k)}$. Thus,

$$\begin{aligned}
C_3 &\leq \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k))\sum_{j=1}^{n_2}\Big(\frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{d_{G_2}(x_j)+2} + \frac{(n_2+1)(d_{G_2}(x_j)+2)d_{G_1}(y_k)}{(n_2+1)d_{G_1}(y_k)}\Big)\\
&= \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_G(y_k))\sum_{j=1}^{n_2}\Big((n_2+1)d_{G_1}(y_k) + (d_{G_2}(x_j)+2)\Big)\\
&= \frac{1}{4}\sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k))\Big(n_2(n_2+1)d_{G_1}(y_k) + 2m_2 + 2n_2\Big)\\
&= \frac{n_2(n_2+1)}{4}\Big(2m_1n_1 - M_1(G)\Big) + \frac{(m_2+n_2)(n_1^2 - 2m_1)}{2}.
\end{aligned}$$

Case 4: The nonadjacent vertex pairs $\{x_{ij}, x_{\ell h}\}$, $1 \leq i < \ell \leq m_1$, $1 \leq j, h \leq n_2$.

$$C_4 = \sum_{x_{ij}x_{\ell h}\notin E(G_1\bullet G_2)} \frac{d_{G_1\bullet G_2}(x_{ij})d_{G_1\bullet G_2}(x_{\ell h})}{d_{G_1\bullet G_2}(x_{ij}) + d_{G_1\bullet G_2}(x_{\ell h})} = \frac{m_1(m_1-1)}{2}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\frac{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)}{d_{G_2}(x_j)+d_{G_2}(x_h)+4}$$

Since for any vertex $x_j \in V(G_2)$, $\delta(G_2) \leq d_{G_2}(x_j) \leq \Delta(G_2)$. Hence

$$C_4 \leq \frac{m_1(m_1-1)n_2^2(\Delta(G_2)+2)^2}{4(\delta(G_2)+2)}.$$

From the above four cases of nonadjacent vertex pairs, we can obtain the desired result. This completes the proof. □

1.1 Mycieskian graph

In a search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [8] developded an interesting graph transformation as follows: Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *Mycielskian graph*$\mu(G)$ of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. Some topological indices of Mycielskian graph were computed in [10, 11].

Lemma 2. *Let G be a connected graph on n vertices and m edges. Then for each $i \in \{1, \ldots, n\}$, we have $d_{\mu(G)}(v_i) = 2d_G(v_i)$, $d_{\mu(G)}(u_i) = d_G(v_i) + 1$ and $d_{\mu(G)}(w) = n$.*

By the definition of Mycielskian graph, for each edge v_iv_j of G, the Mycielskian graph includes two edges, u_iv_j and v_iu_j. Now we find the upper bound for inverse sum indeg coindex of Mycielskian graph.

Theorem 2. *Let G be a graph on n vertices and m edges. Then*

$$\begin{aligned}
\overline{ISI}(\mu(G)) &\leq \frac{n(n-1)-2m+16}{8}\overline{ISI}(G) + \frac{1}{4}\Big(\frac{n(n-1)}{2} - m\Big)\Big(\frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2}\\
&+ \frac{\overline{H}(G)}{2} + \frac{3n(n-1)}{4} - \frac{3m}{2}\Big) + \frac{m}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big)\\
&+ \frac{m+4}{4}M_1(G) + \Big(\frac{n(n-1)}{2} - m\Big)\frac{2\Delta(G)(\Delta(G)+1)}{3\delta G+1} + \frac{7m}{3} + \frac{n(3n+5)}{12}.
\end{aligned}$$

Proof. Let $V(\mu(G)) = \{v_1, \ldots, v_n\}$ and let $V(\mu(G)) = \{v_1, \ldots, v_n, u_1, \ldots, u_n, w\}$. By the structure of Mycielskian graph, if $v_iv_j \notin E(G)$, then $v_iu_j \notin E(G)$, and $v_ju_i \notin E(G)$.

Now we consider the following cases of nonadjacent vertex pairs in $\mu(G)$.

Case 1: The nonadjacent vertex pairs $\{v_i, v_j\}$ in $\mu(G)$.

$$\begin{aligned} C_1 &= \sum_{v_iv_j \notin E(\mu(G))} \frac{d_{\mu(G)}(v_i)d_{\mu(G)}(v_j)}{d_{\mu(G)}(v_i) + d_{\mu(G)}(v_j)} = \sum_{v_iv_j \notin E(G)} \frac{4d_G(v_i)d_G(v_j)}{2d_G(v_i) + 2d_G(v_j)}, \qquad \text{by Lemma 2} \\ &= 2\sum_{v_iv_j \notin E(G)} \frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} = 2\overline{ISI}(G). \end{aligned}$$

Case 2: The nonadjacent vertex pairs $\{u_i, u_j\}$ in $\mu(G)$.

Case 2.1: $u_iu_j \notin E(\mu(G))$ and $v_iv_j \notin E(G)$.

$$C_2' = \sum_{u_iu_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(u_j)} = \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)+1)(d_G(v_j)+1)}{d_G(v_i) + d_G(v_j) + 2}, \text{ by Lemma 2.}$$

By Lemma 1, we obtain

$$\begin{aligned} C_2' &\leq \frac{1}{4}\sum_{v_iv_j \notin E(G)} (d_G(v_i)+1)(d_G(v_j)+1)\Big(\frac{1}{d_G(v_i) + d_G(v_j)} + \frac{1}{2}\Big) \\ &= \frac{1}{4}\sum_{v_iv_j \notin E(G)}\Big(\frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} + \frac{d_G(v_i)d_G(v_j)}{2} + \frac{d_G(v_i) + d_G(v_j)}{2} + \frac{1}{d_G(v_i) + d_G(v_j)} + \frac{3}{2}\Big) \\ &= \frac{1}{4}\Big(\overline{ISI}(G) + \frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2} + \frac{\overline{H}(G)}{2} + \frac{3}{2}\Big(\frac{n(n-1)}{2} - m\Big)\Big). \end{aligned}$$

Case 2.2: $u_iu_j \notin E(\mu(G))$ and $v_iv_j \in E(G)$.

$$C_2'' = \sum_{u_iu_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(u_j)} = \sum_{v_iv_j \in E(G)} \frac{(d_G(v_i)+1)(d_G(v_j)+1)}{d_G(v_i) + d_G(v_j) + 2}, \text{ by Lemma 2.}$$

Apply Lemma 1, we have

$$\begin{aligned} C_2'' &\leq \frac{1}{4}\sum_{v_iv_j \in E(G)}\Big(\frac{d_G(v_i)d_G(v_j)}{d_G(v_i) + d_G(v_j)} + \frac{d_G(v_i)d_G(v_j)}{2} + \frac{d_G(v_i) + d_G(v_j)}{2} + \frac{1}{d_G(v_i) + d_G(v_j)} + \frac{3}{2}\Big) \\ &= \frac{1}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{M_1(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big). \end{aligned}$$

If $u_iu_j \notin E(\mu(G))$, then there are m edges $v_iv_j \in E(G)$ and $\frac{n(n-1)}{2} - m$ nonadjacent vertex pairs $\{v_i, v_j\}$ in G as well as $\mu(G)$. By Cases 2.1 and 2.2, we have the contribution of nonadjacent vertex pair of case 2 is given by

$$\begin{aligned} C_2 &= \Big(\frac{n(n-1)}{2} - m\Big)C_2' + mC_2'' \\ &= \frac{1}{4}\Big(\frac{n(n-1)}{2} - m\Big)\Big(\overline{ISI}(G) + \frac{\overline{M_2}(G)}{2} + \frac{\overline{M_1}(G)}{2} + \frac{\overline{H}(G)}{2} + \frac{3n(n-1)}{4} - \frac{3m}{2}\Big) \\ &\quad + \frac{m}{4}\Big(ISI(G) + \frac{M_2(G)}{2} + \frac{M_1(G)}{2} + \frac{H(G)}{2} + \frac{3m}{2}\Big). \end{aligned}$$

Case 3: The nonadjacent vertex pairs $\{u_i, v_i\}$ in $\mu(G)$ for each $i = 1, 2, \ldots, n$.

$$\begin{aligned} C_3 &= \sum_{i=1}^{n} \frac{d_{\mu(G)}(u_i) d_{\mu(G)}(v_i)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(v_i)} = \sum_{i=1}^{n} \frac{2(d_G(v_i)+1) d_G(v_i)}{3d_G(v_i)+1}, && \text{by Lemma 2} \\ &\leq \frac{1}{4} \sum_{i=1}^{n} \Big(2d_G^2(v_i) + 2d_G(v_i)\Big)\Big(\frac{1}{3d_G(v_i)} + 1\Big), && \text{by Lemma 1} \\ &= \frac{1}{4}\Big(2M_1(G) + \frac{16m}{3} + \frac{2n}{3}\Big). \end{aligned}$$

Case 4: The nonadjacent vertex pairs $\{u_i, v_j\}$ in $\mu(G)$.

$$C_4 = \sum_{u_i v_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i) d_{\mu(G)}(v_j)}{d_{\mu(G)}(u_i) + d_{\mu(G)}(v_j)} = \sum_{v_i v_j \notin E(G)} \frac{2(d_G(v_i)+1) d_G(v_j)}{d_G(v_i) + 2d_G(v_j) + 1}, \qquad \text{by Lemma 2.}$$

For any vertex $v_i \in V(G)$, we have $\delta(G) \leq d_G(v_i) \leq \Delta(G)$. Thus

$$C_4 \leq \Big(\frac{n(n-1)}{2} - m\Big) \frac{2\Delta(G)(\Delta(G)+1)}{3\delta(G)+1}.$$

Case 5: The nonadjacent vertex pairs $\{w, v_i\}$ in $\mu(G)$ for each $i = 1, 2, \ldots, n$.

$$\begin{aligned} C_5 &= \sum_{v_i w \notin E(\mu(G))} \frac{d_{\mu(G)}(v_i) d_{\mu(G)}(w)}{d_{\mu(G)}(v_i) + d_{\mu(G)}(w)} = \sum_{v_i \in V(G)} \frac{2(n+1) d_G(v_i)}{2d_G(v_i) + (n+1)}, && \text{by Lemma 2} \\ &\leq \frac{1}{4} \sum_{v_i \in V(G)} 2(n+1) d_G^2(v_i) \Big(\frac{1}{2d_G(v_i)} + \frac{1}{n+1}\Big), && \text{by Lemma 1} \\ &= \frac{1}{4}\Big(n(n+1) + 4m\Big). \end{aligned}$$

From the above five cases of nonadjacent vertex pairs, we can obtain the desired results. This completes the proof. ☐

1.2 Double graph

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The vertices of the double graph G^* are given by the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. The *double graph* G^* includes the initial edge set of each copies of G, and for any edge $v_i v_j \in E(G)$, two more edges $x_i y_j$ and $x_j y_i$ are added. For a given vertex v in G, let $D_G(v) = \sum\limits_{uv \notin E(G)} \frac{d_G(u) d_G(v)}{d_G(u) + d_G(v)}$. Now we find the exact value of the inverse sum indeg index and its coindex for double graph of a given graph.

Theorem 3. *The inverse sum indeg index of the double graph G^* of a graph G is given by $ISI(G^*) = 8\, ISI(G)$.*

Proof. From the definition of double graph it is clear that $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$, where $v_i \in V(G)$ and $x_i, y_i \in V(G^*)$ are corresponding clone vertices of v_i.

Thus from the definition of ISI, we have

$$\begin{aligned} ISI(G^*) &= \sum_{uv\in E(G^*)} \frac{d_{G^*}(u)d_{G^*}(v)}{d_{G^*}(u)+d_{G^*}(v)} = \sum_{x_ix_j\in E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} \\ &+ \sum_{y_iy_j\in E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} + \sum_{x_iy_j\in E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} \\ &+ \sum_{x_jy_i\in E(G^*)} \frac{d_{G^*}(x_j)d_{G^*}(y_i)}{d_{G^*}(x_j)+d_{G^*}(y_i)} = 4\sum_{v_iv_j\in E(G)} \frac{4\,d_G(v_i)d_G(v_j)}{2d_G(v_i)+2d_G(v_j)} = 8\,ISI(G). \end{aligned}$$

□

Theorem 4. *Let G be a connected graph with n vertices and m edges. Then $\overline{ISI}(G^*) = 8\overline{ISI}(G) + 2m$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in G^*, of v_i for each $i \in \{1, 2, \ldots, n\}$. For any given vertex v_i in G and its clone vertices x_i and y_i, $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$ by the definition of double graph.

For $v_i, v_j \in V(G)$, if $v_iv_j \notin E(G)$, then $x_ix_j \notin E(G)$,$y_iy_j \notin E(G)$,$x_iy_j \notin E(G)$and $y_ix_j \notin E(G)$.

Hence we only consider total contribution of the following three types of nonadjacent vertex pairs to calculate $\overline{ISI}(G)$.

Case 1: The nonadjacent vertex pairs $\{x_i, x_j\}$ and $\{y_i, y_j\}$, where $v_iv_j \notin E(G)$.

$$\begin{aligned} \sum_{y_iy_j\notin E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} &= \sum_{x_ix_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} = \sum_{v_iv_j\notin E(G)} \frac{4d_G(v_i)d_Gv_j)}{2d_G(v_i)+2d_Gv_j)} \\ &= 2\overline{ISI}(G). \end{aligned}$$

Case 2: The nonadjacent vertex pairs $\{x_i, y_i\}$ for each $i \in \{1, 2, \ldots, n\}$.

$$\sum_{i=1}^{n} \frac{d_{G^*}(x_i)d_{G^*}(y_i)}{d_{G^*}(x_i)+d_{G^*}(y_i)} = \sum_{i=1}^{n} \frac{4d_G(v_i)d_G(v_i)}{2d_G(v_i)+2d_G(v_i)} = \sum_{i=1}^{n} d_G(v_i) = 2m.$$

Case 3: The nonadjacent vertex pairs $\{x_i, y_j\}$ and $\{y_i, x_j\}$, where $v_iv_j \notin E(G)$.

For each x_i, there exist $n-1-d_G(v_i)$ vertices in the set $\{y_1, y_2, \ldots, y_n\}$, among which every vertex together with x_i compose a nonadjacent vertex pairs of G^*. The total contribution of these $n-1-d_G(v_i)$ nonadjacent vertex pairs to calculate $\overline{ISI}(G^*)$ is

$$\sum_{x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = \sum_{v_iv_j\notin E(G^*)} \frac{4d_G(v_i)d_G(v_j)}{2d_G(v_i)+2d_G(v_j)} = 2D_G(v_i).$$

Hence

$$\sum_{i\neq j,\ x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = \sum_{i=1}^{n} 2D_G(v_i) = 4\overline{ISI}(G).$$

Hence

$$\begin{aligned} \overline{ISI}(G^*) &= \sum_{x_ix_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(x_j)}{d_{G^*}(x_i)+d_{G^*}(x_j)} + \sum_{y_iy_j\notin E(G^*)} \frac{d_{G^*}(y_i)d_{G^*}(y_j)}{d_{G^*}(y_i)+d_{G^*}(y_j)} + \sum_{i=1}^{n} \frac{d_{G^*}(x_i)d_{G^*}(y_i)}{d_{G^*}(x_i)+d_{G^*}(y_i)} \\ &+ \sum_{i\neq j,\ x_iy_j\notin E(G^*)} \frac{d_{G^*}(x_i)d_{G^*}(y_j)}{d_{G^*}(x_i)+d_{G^*}(y_j)} = 8\overline{ISI}(G) + 2m. \end{aligned}$$

□

REFERENCES

[1] Abdolhosseinzadeh I. R., Rahbarnia F., Tavakoli M., Ashrafi A. R. *Some vertex-degree-based topological indices under edge corona product*. Ital. J. Pure Appl. Math. 2017, **38**, 81–91.

[2] Doslić T., Azari M., Falahati-Nezhad F. *Sharp bounds on the inverse sum indeg index*. Discrete Appl. Math. 2017, **217**, 185–195.

[3] Falahati-Nezhad F., Azari M. *The inverse sum indeg index of some nanotubes*. Studia Ubb Chemia, LXI 2016, **1**, 63–70.

[4] Gutman I., Trinajstić N., *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*. Chem. Phys. Lett. 1972, **17**, 535–538.

[5] Hou Y., Shiu W.C. *The spectrum of the edge corona of two graphs*. Electron. J. Linear Algebra 2010, **20**, 586–594.

[6] Karelson M. *Molecular Descriptors in QSAR/QSPR*. Wiley-Interscience, New York, 2000.

[7] Milano Chemometrics & QSAR research group, molecular descriptors dataset. http://www.moleculardescriptors.eu/dataset/dataset.htm (accessed 18.04.14).

[8] Mycielski J. *Sur le colouring des graphes*. Colloq. Math. 1955, **3**, 161–162.

[9] Niculescu C., Persson L.E. *Convex functions and their applications: a contemporary approach*. Springer, New York, 2006.

[10] Pattabiraman K. *On Topological Indices of Graph Transformation*. Int. J. Appl. Comput. Math. 2017, **3**, 1171–1184.

[11] Pattabiraman K. *Degree and Distance Based Topological Indices of Graphs*. Electron. Notes Discrete Math. 2017, **63**, 145–59.

[12] Pattabiraman K. *Inverse sum indeg index of graphs*. AKCE Int. J. Graphs Comb. 2017, **15** (2), 155–167 doi:10.1016/j.akcej.2017.06.001.

[13] Rinurwati, Slamin, Suprajitno H. *General results of local metric dimensions of edge-corona of graphs*. International Mathematical Forum 2016, **16** (11), 793–799. doi:10.12988/imf.2016.67102

[14] Randić M. *On characterization of molecular branching*. J. Am. Chem. Soc. 1975, **97**, 6609–6615.

[15] Sedlar J., Stevanović D., Vasilyev A. *On the inverse sum indeg index*. Discrete Appl. Math. 2015, **184**, 202–212.

[16] Todeschini R., Consonni V. *Handbook of Molecular Descriptors*. Wiley-VCH, Weinheim, 2000.

[17] Vukicević D., Gasperov M. *Bond aditive mdelling 1. Ariatic indices*. Croat. Chem. Acta 2010, **83**, 243–260.

[18] Wiener H. *Structural determination of paraffin boiling points*. J. Am. Chem. Soc. 1947, **69**, 17–20.

Received 15.10.2018

Паттабіраман К. *Оборотно підсумовуючий індег коіндекс графів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 399–406.

Оборотно підсумовуючий індег коіндекс $\overline{ISI}(G)$ простого зв'язного графу G визначено як сума доданків $\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ по всіх ребрах uv, які не лежать у G, де $d_G(u)$ позначає степінь вершини u в G. У статті встановлено верхні обмеження на оборотно підсумовуючий індег коіндекс графу добутку вершин корони та графу Мицелскіана. Крім того отримано точне значення оборотного підсумовуючого індег індексу і коіндексу для подвійного графу.

Ключові слова і фрази: оборотно підсумовуючий індег індекс, граф вершин корони, граф Мицелскіана, подвійний граф.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 407–417
doi:10.15330/cmp.11.2.407-417

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.407–417

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ON THE SUM OF SIGNLESS LAPLACIAN SPECTRA OF GRAPHS

For a simple graph $G(V,E)$ with n vertices, m edges, vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, the adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix associated to G, where $d_i = \deg(v_i)$, for all $i \in \{1,2,\dots,n\}$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called the Laplacian and the signless Laplacian matrices and their spectra (eigenvalues) are respectively called the Laplacian spectrum (L-spectrum) and the signless Laplacian spectrum (Q-spectrum) of the graph G. If $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ are the Laplacian eigenvalues of G, Brouwer conjectured that the sum of k largest Laplacian eigenvalues $S_k(G)$ satisfies $S_k(G) = \sum\limits_{i=1}^{k} \mu_i \leq m + \binom{k+1}{2}$ and this conjecture is still open. If $q_1, q_2, \dots, q_n$ are the signless Laplacian eigenvalues of G, for $1 \leq k \leq n$, let $S_k^+(G) = \sum_{i=1}^{k} q_i$ be the sum of k largest signless Laplacian eigenvalues of G. Analogous to Brouwer's conjecture, Ashraf et al. conjectured that $S_k^+(G) \leq m + \binom{k+1}{2}$, for all $1 \leq k \leq n$. This conjecture has been verified in affirmative for some classes of graphs. We obtain the upper bounds for $S_k^+(G)$ in terms of the clique number ω, the vertex covering number τ and the diameter of the graph G. Finally, we show that the conjecture holds for large families of graphs.

Key words and phrases: signless Laplacian spectra, Brouwer's conjecture, clique number, vertex covering number, diameter.

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Introduction

Let $G(V,E)$ be a simple graph with n vertices, m edges, having vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix associated to G, where $d_i = \deg(v_i)$, for all $i \in \{1,2,\dots,n\}$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called the Laplacian and the signless Laplacian matrices and their spectra (eigenvalues) are respectively called the Laplacian spectrum (L-spectrum) and the signless Laplacian spectrum (Q-spectrum) of the graph G. These matrices are real symmetric and positive semi-definite. We let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ to be the L-spectrum and Q-spectrum of G, respectively. It is well

УДК 519.17
2010 *Mathematics Subject Classification:* 05C30,05C50.
The research of S. Pirzada (first author) is supported by SERB-DST, New Delhi under the research project number MTR/2017/000084.

known that the multiplicity of the Laplacian eigenvalue $\mu_n = 0$ is equal to the number of connected components of G and also $\mu_{n-1} > 0$ if and only if G is connected. Moreover $\mu_i = q_i$, for all $i \in \{1, 2, \ldots, n\}$, if and only if G is bipartite [4].

For $k \in \{1, 2, \ldots, n\}$, let $S_k(G) = \sum\limits_{i=1}^{k} \mu_i$ be the sum of k largest Laplacian eigenvalues of G. Also, let $d_i^*(G) = |\{v \in V(G) : d_v \geq i\}|$, for $i \in \{1, 2, \ldots, n\}$. In 1994, Grone and Merris [12] observed that for any graph G and for any $k \in \{1, 2, \ldots, n\}$,

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G).$$

This observation was proved by Hua Bai [2] and is nowadays called as Grone-Merris theorem. As an analogue to Grone-Merris theorem, Andries Brouwer [3] conjectured that for a graph G with n vertices and m edges and for any $k \in \{1, 2, \ldots, n\}$,

$$S_k(G) = \sum_{i=1}^{k} \mu_i \leq m + \binom{k+1}{2}.$$

This conjecture is still open and is presently an active component of research. For the progress on this conjecture and related results, we refer to [8–11, 14] and the references therein.

For $k \in \{1, 2, \ldots, n\}$, let $S_k^+(G) = \sum_{i=1}^{k} q_i$ be the sum of k largest signless Laplacian eigenvalues of a graph G. Motivated by the definition of $S_k(G)$ and Brouwer's conjecture, Ashraf et al. [1] proposed the following conjecture about $S_k^+(G)$.

Conjecture 1. *If G is a graph with n vertices and m edges, then*

$$S_k^+(G) = \sum_{i=1}^{k} q_i \leq m + \binom{k+1}{2},$$

for all $k \in \{1, 2, \ldots, n\}$.

Using computations on a computer Ashraf et al. [1] verified the truth of this conjecture for all graphs with at most 10 vertices. For $k = 1$, the conjecture follows from the well-known inequality $q_1(G) \leq \frac{2m}{n-1} + n + 2$ and $m \geq n - 1$. Also, the cases $k = n$ and $k = n - 1$ are straightforward. The conjecture is true for trees. This follows from the fact that Brouwer's conjecture holds for trees and that both Laplacian and signless Laplacian eigenvalues are the same for trees. Ashraf et al. [1] showed that the conjecture is true for all graphs when $k = 2$ and is also true for regular graphs. Yang et al. [16] obtained various upper bounds for $S_k^+(G)$ and proved that the conjecture is also true for unicyclic graphs, bicyclic graphs and tricyclic graphs (except for $k = 3$). For the progress on this conjecture and related results, we refer to [1, 7, 16] and the references therein.

A *clique* of a graph G is the maximum complete subgraph of the graph G. The order of the maximum clique is called the *clique number* of the graph G and is denoted by ω. A subset S of the vertex set $V(G)$ is said to be a *covering set* of G if every edge of G is incident to at least one vertex in S. A covering set with minimum cardinality among all covering sets is called *minimum covering set* of G and its cardinality, denoted by τ, is called *vertex covering number* of G.

The *distance* between any two vertices u and v is defined as the length of shortest path between them and the *diameter* of a graph G is the maximum distance among all pair of vertices of G. If H is a subgraph of the graph G, we denote the graph obtained by removing the edges in H from G by $G \setminus H$ (that is, only the edges of H are removed from G).

Further, as usual P_n, K_n and $K_{s,t}$, respectively, denote the path on n vertices, the complete graph on n vertices and the complete bipartite graph on $s+t$ vertices. For other undefined notations and terminology from spectral graph theory, the readers are referred to [4, 13].

The paper is organized as follows. In Section 2, we obtain some upper bounds for $S_k^+(G)$ in terms of the clique number ω, the vertex covering number τ and the diameter of the graph G. As applications to the results obtained in Section 2, we prove that Conjecture 1 is true for some new classes of graphs in Section 3.

1 UPPER BOUNDS FOR $S_k^+(G)$

In this section, we obtain the upper bounds for $S_k^+(G)$, in terms of the clique number ω, the vertex covering number τ and the diameter of the graph G.

Yang et al. [16] obtained the following upper bound for $S_k^+(G)$, in terms of the clique number ω and the number of edges m:

$$S_k^+(G) \leq k(\omega - 2) + 2m - \omega(\omega - 2). \tag{1}$$

Das et al. [5] obtained an upper for $S_k(G)$ of a graph with n vertices, in terms of the vertex covering number τ and the number of edges m. Using similar analysis, the following upper bound can be obtained for $S_k^+(G)$, in terms of the vertex covering number τ and the number of edges m:

$$S_k^+(G) \leq m + k\tau, \tag{2}$$

with equality if and only if $G \cong K_{1,n-1}$.

The following observation is due to Fulton [6].

Lemma 1. *Let A and B be two real symmetric matrices of order n. Then for any $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where $\lambda_i(X)$ is the i^{th} eigenvalue of the matrix X.

Let Γ_1 be the family of all connected graphs except for the graphs G, where the vertices in the vertex covering set $S = \{v_1, v_2, \ldots, v_{\omega-1}\}$ of the subgraph K_ω have the property that there are pendent vertices incident to some $v_i \in S$ or any two vertices of S forms a triangle with a vertex $v \in V(G) \setminus C$, where C is the vertex covering set of G.

The following theorem gives an upper bound for $S_k^+(G)$ in terms of the clique number ω, the vertex covering number τ and the number of edges m of the graph G. The number of vertices in a graph G is denoted by $n(G)$ and the number of vertices adjacent to a vertex v is denoted by $N(v)$.

Theorem 2. *Let $G \in \Gamma_1$ be a connected graph of order $n \geq 2$ with m edges having clique number ω and vertex covering number τ. Then, for $1 \leq k \leq n$,*

$$S_k^+(G) \leq k(\tau - 1) + m - \frac{\omega(\omega-3)}{2}, \tag{3}$$

with equality if and only if $G \cong K_n$.

Proof. If $G \in \Gamma_1$ is a connected graph with clique number ω, vertex cover number τ and minimum vertex covering set $C = \{v_1, v_2, \ldots, v_\tau\}$, then K_ω is a subgraph of G. Further, the vertex covering number of a complete graph on ω vertices is $\omega - 1$. Without loss of generality, let $v_1, v_2, \ldots, v_{\omega-1}$ be the vertices in C, which belong to $V(K_\omega)$. The signless Laplacian spectrum of K_ω is $\{2\omega - 2, \omega - 2^{[\omega-1]}\}$. After removing the edges of K_ω from G, the signless Laplacian matrix of G isdecomposed as

$$Q(G) = Q(K_\omega \cup (n-\omega)K_1) + Q(G \setminus K_\omega),$$

where $G \setminus K_\omega$ is the graph obtained from G by removing the edges of K_ω. Using Lemma 1 and the fact $S_k^+(K_\omega \cup (n-\omega)K_1) = S_k^+(K_\omega)$, we have

$$\begin{aligned} S_k^+(G) &= \sum_{i=1}^{k} q_i(G) \leq \sum_{i=1}^{k} q_i(K_\omega) + \sum_{i=1}^{k} q_i(G \setminus K_\omega) \\ &= S_k^+(K_\omega) + S_k^+(G \setminus K_\omega) = \omega(k+1) - 2k + S_k^+(G \setminus K_\omega). \end{aligned}$$

To complete the proof, we need to estimate $S_k^+(G \setminus K_\omega)$. So let $G_\omega, G_{\omega+1}, \ldots, G_\tau$ be the spanning subgraphs of $H = G \setminus K_\omega$ corresponding to the vertices $v_\omega, v_{\omega+1}, \ldots, v_\tau$ of C, having vertex set same as H and edge sets defined as follows.

$$\begin{aligned} E(G_\omega) &= \{v_\omega v_t : v_t \in N(v_\omega) \setminus \{v_1, v_2, \ldots, v_{\omega-1}\}\} \\ E(G_{\omega+1}) &= \{v_{\omega+1} v_t : v_t \in N(v_{\omega+1}) \setminus \{v_1, v_2, \ldots, v_\omega\}\} \end{aligned}$$

and in general

$$E(G_i) = \{v_i v_t : v_t \in N(v_i) \setminus \{v_1, v_2, \ldots, v_{i-1}\}\}, \ \ i = \omega, \omega+1, \ldots, \tau.$$

For $i \in \{\omega, \omega+1, \ldots, \tau\}$, let $m_i = |E(G_i)|$. Clearly $E(H) = E(G_\omega) \cup E(G_{\omega+1}) \cup \cdots \cup E(G_\tau)$ and $G_i = K_{1,m_i} \cup (n(H) - m_i - 1)K_1$, for all $i \in \{\omega, \omega+1, \ldots, \tau\}$. Also, it is clear that

$$Q(H) = Q(G_\omega) + Q(G_{\omega+1}) + \cdots + Q(G_\tau). \tag{4}$$

The signless Laplacian spectrum of $G_i = K_{1,m_i} \cup (n(H) - m_i - 1)K_1$ is

$$\{m_i + 1, 1^{[n(G_i)-2]}, 0^{[n(H)-m_i]}\}.$$

Therefore,

$$S_k^+(G_i) = m_i + k, \ \ for\ all\ \ i = \omega, \omega+1, \ldots, \tau. \tag{5}$$

Now, applying Lemma 1 to Equation (4) and using Equation (5) and the fact that $\sum_{j=\omega}^{\tau} m_j = m(H) = m - \frac{\omega(\omega-1)}{2}$, we have

$$\begin{aligned} S_k^+(H) &= \sum_{i=1}^{k} q_i(H) \leq \sum_{j=\omega}^{\tau} \sum_{i=1}^{k} q_i(G_j) = \sum_{j=\omega}^{\tau} S_k^+(G_j) \\ &= \sum_{j=\omega}^{\tau} \Big(m_j + k\Big) = m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k. \end{aligned}$$

This shows that

$$S_k^+(G \setminus K_\omega) = S_k^+(H) \le m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k.$$

Therefore, it follows that

$$\begin{aligned} S_k^+(G) &\le \omega(k+1) - 2k + S_k^+(G \setminus K_\omega) \\ &\le \omega(k+1) - 2k + m - \frac{\omega(\omega-1)}{2} + (\tau - \omega + 1)k \\ &= k(\tau - 1) + m - \frac{\omega(\omega-3)}{2}. \end{aligned}$$

Equality occurs in (3) if and only if all the inequalities above become equalities. Since G is connected equality occurs in $S_k^+(G) \le S_k^+(K_\omega) + S_k^+(G \setminus K_\omega)$, only if $G \cong K_n$. Conversely, if $G \cong K_n$, then $\tau = n-1, \;\; \omega = n, \;\; m = \frac{n(n-1)}{2}$ and so equality holds in (3), completing the proof. □

Remark 1. *For a graph $G \in \Gamma_1$, it is easy to see that the upper bound given by* (3) *is better than the upper bound given by* (1) *for all $m \ge k(\tau - \omega + 1) + \frac{\omega(\omega-1)}{2}$. In particular, for the graph with $\tau = \omega$ and $k \le n - \omega$, the upper bound* (3) *is better than the upper bound* (1)*.*

Remark 2. *Clearly for the graph $G \in \Gamma_1$ the upper bound given by* (3) *is always better than the upper bound given by* (2)*.*

Let Γ_2 be the family of all connected graphs except for the graphs G, where the vertices in the vertex covering set $S = \{v_1, v_2, \ldots, v_{\lfloor \frac{d}{2} \rfloor}\}$ of the subgraph P_d has the property that there are pendent vertices incident at some $v_i \in S$ or any two vertices of S forms a triangle with a vertex $v \in V(G) \setminus C$, where C is the vertex covering set of G.

Rocha et al. [15] obtained an upper bound for $S_k(G)$ in terms of diameter of the graph G. Using similar analysis, the following upper bound can be obtained for $S_k^+(G)$, in terms of the diameter $d-1$ of the graph G.

$$S_k^+(G) \le 2(m-d) + 1 - n + 4k + p + \cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}, \tag{6}$$

where p is the number of isolated vertices in the graph obtained by removing the edges of P_d from G.

The following theorem gives an upper bound for $S_k^+(G)$, in terms of the diameter, the number of edges m and the vertex covering number τ of the graph G.

Theorem 3. *Let $G \in \Gamma_2$ be a connected graph of order $n \ge 3$ with m edges having diameter $d-1$ and vertex covering number τ. Then for $1 \le k \le n$,*

$$S_k^+(G) \le (\tau - \lfloor \frac{d}{2} \rfloor + 2)k + m - d + \cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}, \tag{7}$$

with equality if and only if $G \cong P_n$.

Proof. Let G be a connected graph with diameter $d-1$ and vertex cover number τ and let $C=\{v_1,v_2,\ldots,v_\tau\}$ be a minimum vertex covering set in G. Since the diameter of G is $d-1$, it follows that P_d is a subgraph of G. Also, the vertex covering number of a path graph P_n on n vertices is $\lfloor\frac{n}{2}\rfloor$. Let $v_1,v_2,\ldots,v_{\lfloor\frac{d}{2}\rfloor}$ be the vertices in C, which belong to $V(P_d)$. The signless Laplacian spectrum of P_d is $\{2-2\cos(\frac{\pi j}{d}),0 : j\in\{1,2,\ldots,d-1\}\}$. If we remove the edges of P_d from G, the signless Laplacian matrix of G can be decomposed as

$$Q(G)=Q(P_d\cup(n-d-1)K_1)+Q(G\setminus P_d),$$

where $G\setminus P_d$ is the graph obtained from G by removing the edges of P_d. Applying Lemma 1 and using the fact that $S_k^+(P_d\cup(n-d-1)K_1)=S_k^+(P_d)$, we have

$$\begin{aligned}
S_k^+(G)=\sum_{i=1}^{k}q_i(G)&\le\sum_{i=1}^{k}q_i(P_d)+\sum_{i=1}^{k}q_i(G\setminus P_d)=S_k^+(P_d)+S_k^+(G\setminus P_d)\\
&=\sum_{j=0}^{k-1}\left(2-2\cos(\frac{\pi(d-j-1)}{d})\right)+S_k^+(G\setminus P_d)\\
&=2k+\cos\left(\frac{k\pi}{d}\right)+\frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d})+\sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}-1+S_k^+(G\setminus P_d),
\end{aligned}$$

where we have used the well-known equality

$$\sum_{j=0}^{k-1}\cos(nj)=\frac{\sin(nk)\cos(n)+\sin(nk)}{2\sin(n)}-\frac{1}{2}\cos(nk)+\frac{1}{2}.$$

In order to establish the result, we need to estimate $S_k^+(G\setminus P_d)$.

Let $G_{\lfloor\frac{d}{2}\rfloor+1},G_{\lfloor\frac{d}{2}\rfloor+2},\ldots,G_\tau$ be the spanning subgraphs of $H=G\setminus P_d$ corresponding to the vertices $v_{\lfloor\frac{d}{2}\rfloor+1},v_{\lfloor\frac{d}{2}\rfloor+2},\ldots,v_\tau$ of C, having vertex set same as H and edge sets defined as follows.

$$E(G_i)=\{v_iv_t : v_t\in N(v_i)\setminus\{v_1,v_2,\ldots,v_{i-1}\}\},\ \ i=\lfloor\frac{d}{2}\rfloor+1,\lfloor\frac{d}{2}\rfloor+2,\ldots,\tau.$$

Now, proceeding similarly as in Theorem 2, we obtain

$$S_k^+(G\setminus P_d)\le k(\tau-\lfloor\frac{d}{2}\rfloor)+m-d+1.$$

Therefore, from above we have

$$\begin{aligned}
S_k^+(G)&\le 2k+\cos\left(\frac{k\pi}{d}\right)+\frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d})+\sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}-1+S_k^+(G\setminus P_d)\\
&\le(\tau-\lfloor\frac{d}{2}\rfloor+2)k+m-d+\cos\left(\frac{k\pi}{d}\right)+\frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d})+\sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})},
\end{aligned}$$

and hence the result follows.

Equality occurs in (7) if and only if all the inequalities above occur as equalities. Since G is connected, the equality in the inequality $S_k^+(G)\le S_k^+(P_d)+S_k^+(G\setminus P_d)$ can only occur if and only if $G\cong P_n$. Conversely, if $G\cong P_n$, then $\tau=\lfloor\frac{n}{2}\rfloor$, $m=n-1$, $d=n-1$ and so it can be seen that equality holds in (7), completing the proof. $\square$

Remark 3. *For the connected graphs $G \in \Gamma_2$, it is easy to see that the upper bound given by* (7) *is better than the upper bound given by* (6) *for all $k \leq \frac{m-n-d+1+p}{\tau-\lfloor\frac{d}{2}\rfloor-2}$. In particular, if $G \in \Gamma_2$ is such that $\tau \leq \lfloor\frac{d}{2}\rfloor + 2$ and $m \geq n+d-1-p$, the upper bound* (7) *is always better than the upper bound* (6).

Let Γ_3 be the family of all connected graphs except for the graphs G, where the vertices in the vertex set $S = \{v_1, v_2, \dots, v_{s_1}, u_1, u_2, \dots u_{s_2}\}$ of the subgraph K_{s_1,s_2}, $s_1 \leq s_2$, has the property that there are pendent vertices incident at some v_i or $u_j \in S$ or any two vertices of S forms a triangle with a vertex $v \in V(G) \setminus C$, where C is the vertex covering set of G.

Let K_{s_1,s_2} $s_1 \leq s_2$, be the maximal complete bipartite subgraph of a graph G. Using the fact that the vertex covering number of K_{s_1,s_2} $s_1 \leq s_2$, is s_1 and its signless Laplacian spectrum is $\{s_1+s_2, s_1^{[s_2-1]}, s_2^{[s_1-1]}, 0\}$, and proceeding similarly as in Theorem 2, we obtain the following upper bound for $S_k^+(G)$.

Theorem 4. *Let $G \in \Gamma_3$ be a connected graph of order $n \geq 2$ with m edges having vertex covering number τ. If K_{s_1,s_2} $s_1 \leq s_2$, is the maximal complete bipartite subgraph of the graph G, then*

$$S_k^+(G) \leq k(\tau + s_2 - s_1) + m - s_1(s_2 - 1), \tag{8}$$

with equality if and only if $G \cong K_{s_1,s_2}$ and $s_1 + s_2 = n$.

If $s_1 = s_2$, for the graphs $G \in \Gamma_3$, it is easy to see that the upper bound (8) is always better than the upper bound (2).

2 CONJECTURE 1 IS TRUE FOR SOME MORE CLASSES OF GRAPHS

In this section, we show that Conjecture 1 holds for some more classes of graphs.

Theorem 5. *If $G \in \Gamma_1$ is a connected graph of order $n \geq 12$ with m edges having clique number ω, then for $\omega \geq \frac{3+\sqrt{3n^2-14n+9}}{2}$,*

$$S_k^+(G) \leq m + \frac{k(k+1)}{2},$$

for all $k \in \{1, 2, \dots, \lfloor\frac{n}{2}\rfloor\}$.

Proof. Let G be a connected graph of order n having clique number ω and vertex covering number τ. If $\tau = n-1$, clearly $G \cong K_n$ and so Conjecture 1 always holds (this is due to the fact that Conjecture 1 holds for all regular graphs). So suppose that $\tau \leq n-2$. With this choice of τ, from inequality (3), we have

$$S_k^+(G) \leq k(n-3) + m - \frac{\omega(\omega-3)}{2} \leq m + \frac{k(k+1)}{2},$$

if $k(2n-6) \leq k^2 + k + \omega(\omega-3)$. That is, $k^2 - (2n-7)k + \omega(\omega-3) \geq 0$.

Consider the polynomial $f(k) = k^2 - (2n-7)k + \omega(\omega-3)$, $k \in [1, n-1]$. The roots of this polynomial are

$$\alpha = \frac{(2n-7) + \sqrt{4n^2 - 28n + 49 - 4\omega(\omega-3)}}{2}$$

and

$$\beta = \frac{(2n-7) - \sqrt{4n^2 - 28n + 49 - 4\omega(\omega-3)}}{2}.$$

Thus $f(k) \geq 0$, for all $k \in (-\infty, \beta] \cup [\alpha, +\infty)$. We will show $\beta \geq \frac{n}{2}$. We have $\beta \geq \frac{n}{2}$ implies

$$\frac{(2n-7) - \sqrt{4n^2 - 28n + 49 - 4\omega(\omega-3)}}{2} \geq \frac{n}{2},$$

which implies that $(n-7)^2 \geq 4n^2 - 28n + 49 - 4\omega(\omega-3)$, and further implies that $4\omega^2 - 12\omega - (3n^2 - 14n) \geq 0$, which gives $\omega \geq \frac{3+\sqrt{3n^2-14n+9}}{2}$.

Since $\alpha(\frac{3+\sqrt{3n^2-14n+9}}{2}) = \frac{3n-14}{2} \geq n-1$, for all $n \geq 12$, it follows that $\alpha(\omega) \geq n-1$, for all $\omega \leq \frac{3+\sqrt{3n^2-14n+9}}{2}$. Thus, if $\omega \geq \frac{3+\sqrt{3n^2-14n+9}}{2}$, we have proved that Conjecture 1 holds for all $k \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. □

Let Ω_n be a family of those connected graphs $G \in \Gamma_1$ for which the vertex covering number $\tau \in \{\omega-1, \omega, \omega+1\}$, that is,

$$\Omega_n = \{G \in \Gamma_1 : \tau = \omega - 1 \text{ or } \omega \text{ or } \omega + 1\}.$$

For the family of graphs Ω_n, we have the following observation.

Theorem 6. *If $G \in \Omega_n$, then*

$$S_k(G) \leq m + \frac{k(k+1)}{2}$$

holds for all k, if $\tau = \omega - 1$; holds for all k except for $k = \omega-2, \omega-1$ if $\tau = \omega$; holds for all k, $k \leq \frac{2\omega-1-\sqrt{8\omega+1}}{2}$ and $k \geq \frac{2\omega-1+\sqrt{8\omega+1}}{2}$, if $\tau = \omega + 1$.

Proof. Let $G \in \Omega_n$. Then $\tau \in \{\omega-1, \omega, \omega+1\}$. If $\tau = \omega - 1$, from inequality (3), we have

$$S_k^+(G) \leq k(\omega-2) + m - \frac{\omega(\omega-3)}{2} \leq m + \frac{k(k+1)}{2},$$

if $2k(\omega-2) \leq k^2 + k + \omega^2 - 3\omega$, that is,

$$k^2 - (2\omega-5)k + \omega^2 - 3\omega \geq 0. \tag{9}$$

For the polynomial $f(k) = k^2 - (2\omega-5)k + \omega^2 - 3\omega$, the discriminant $D = (2\omega-5)^2 - 4(\omega^2 - 3\omega) = 25 - 8\omega < 0$, if $\omega \geq 4$. This shows that (9) holds for all $\omega \geq 4$. By direct calculations, it can be seen that (9) holds for $\omega \leq 3$. Thus, it follows that (9) is true for all k.

If $\tau = \omega$, from inequality (3), we have

$$S_k^+(G) \leq k(\omega-1) + m - \frac{\omega(\omega-3)}{2} \leq m + \frac{k(k+1)}{2}$$

if $2k(\omega-1) \leq k^2 + k + \omega^2 - 3\omega$, that is,

$$k^2 - (2\omega-3)k + \omega^2 - 3\omega \geq 0. \tag{10}$$

For the polynomial $f(k) = k^2 - (2\omega-3)k + \omega^2 - 3\omega$, the roots are $\omega - 3$ and ω. It follows that $f(k) < 0$, for all $k \in (\omega-3, \omega)$. Since k and ω are integers and the only integers in $(\omega-3, \omega)$ are $\omega-2$, $\omega-1$, it follows that $f(k) \geq 0$ for all k except $k = \omega-2$, $\omega-1$. Thus, it follows that (10) holds for all $k \notin \{\omega-2, \ \omega-1\}$.

If $\tau = \omega + 1$, proceeding similarly as above, it can be seen that the conjecture holds for all k, $k \leq \frac{2\omega-1-\sqrt{8\omega+1}}{2}$ and $k \geq \frac{2\omega-1+\sqrt{8\omega+1}}{2}$. □

Theorem 7. *Let $G \in \Gamma_2$ be a connected graph of order $n \geq 2$ with m edges having vertex covering number τ. Let K_{s_1,s_1} be the maximal complete bipartite subgraph of G. Then Conjecture 1 holds for all k, if $\tau \leq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$ holds for all $k \leq \frac{2\tau-1-\sqrt{(2\tau-1)^2-8s_1(s_1-1)}}{2}$ and $k \geq \frac{2\tau-1+\sqrt{(2\tau-1)^2-8s_1(s_1-1)}}{2}$, if $\tau \geq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$.*

Proof. Using $s_1 = s_2$ in inequality (8), we have

$$S_k^+(G) \leq k\tau + m - s_1(s_1-1) \leq m + \frac{k(k+1)}{2},$$

if

$$k^2 - (2\tau-1)k + 2s_1(s_1-1) \geq 0. \tag{11}$$

The roots of the polynomial $f(k) = k^2 - (2\tau-1)k + 2s_1(s_1-1)$ are $\alpha = \frac{2\tau-1+\sqrt{\theta}}{2}$ and $\beta = \frac{2\tau-1-\sqrt{\theta}}{2}$, where $\theta = (2\tau-1)^2 - 8s_1(s_1-1)$. We have $(2\tau-1)^2 - 8s_1(s_1-1) \leq 0$, which implies that $4\tau^2 - 4\tau - (8s_1^2 - 8s_1 - 1) \leq 0$, which gives $\tau \leq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$. This shows that the discriminant of the polynomial $f(k)$ is non-positive for all $\tau \leq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$. That is, (11) holds for all $\tau \leq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$. On the other hand if the discriminant of the polynomial $f(k)$ is non-negative, then (11) holds for all $k \geq \alpha$ and for all $k \leq \beta$, completing the proof. □

Let G be a connected bipartite graph of order n having the vertex covering number τ. For bipartite graphs, it is well known that $\tau \leq \frac{n}{2}$. With this in mind, we have the following observation for bipartite graphs.

Theorem 8. *Let $G \in \Gamma_3$ be a connected bipartite graph of order $n \geq 4$ with m edges having the vertex covering number τ. If K_{s_1,s_1}, with $s_1 \geq \frac{n}{4}$, is the maximal complete bipartite subgraph of the graph G, then*

$$S_k(G) \leq m + \frac{k(k+1)}{2}$$

for all $k \leq \frac{n}{7} - 1$ and $k \geq \frac{6n}{7}$.

Proof. Using $s_1 = s_2$ in (8) and the fact that $\tau \leq \frac{n}{2}$, for bipartite graphs we have

$$S_k^+(G) \leq k\tau + m - s_1(s_1-1) \leq k(\frac{n}{2}) + m - s_1(s_1-1) \leq m + \frac{k(k+1)}{2}$$

if

$$kn \leq k(k+1) + 2s_1(s_1-1). \tag{12}$$

The right hand side of (10) is an increasing function of s_1. Therefore, to prove the assertion, it suffices to consider $s_1 = \frac{n}{4}$. With this value of s_1, from (12), we have

$$k^2 - (n-1)k + \frac{n(n-4)}{8} \geq 0.$$

The roots of the polynomial $f(k) = k^2 - (n-1)k + \dfrac{n(n-4)}{8}$ are

$$\alpha = \frac{n-1+\sqrt{0.5n^2+1}}{2}, \quad \beta = \frac{n-1-\sqrt{0.5n^2+1}}{2}.$$

This shows that $f(k) \geq 0$, for all $k \geq \alpha$; and $f(k) \geq 0$, for all $k \leq \beta$. By using elementary algebra it can be seen that $\alpha < 0.8535n$ and $\beta > 0.1464n - 1$. Hence the result follows. □

For graphs with girth $g \geq 5$, Rocha et al. [15] showed that Brouwer's conjecture holds for all $k \leq \lfloor \frac{g}{5} \rfloor$. Using similar analysis, we have the following observation.

Theorem 9. *For connected graphs with girth $g \geq 5$, Conjecture 1 holds for all k, $1 \leq k \leq \lfloor \frac{g}{5} \rfloor$.*

Using Theorem 3, the fact that

$$\cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})} \leq 2k+1$$

and proceeding similarly as in above theorems, we arrive at the following observation.

Theorem 10. *Let $G \in \Gamma_2$ be a connected graph of order $n \geq 3$ with m edges having diameter $d-1$ and vertex covering number τ. Then for $1 \leq k \leq n$, Conjecture 1 holds for all k, if $\tau \leq \frac{2\lfloor \frac{d}{2} \rfloor - 7 + \sqrt{8(d-1)}}{2}$; holds for all k,*

$$k \leq \frac{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - \sqrt{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - 8(d-1)}}{2}$$

and

$$k \geq \frac{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 + \sqrt{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - 8(d-1)}}{2},$$

if $\tau \geq \frac{2\lfloor \frac{d}{2} \rfloor - 7 + \sqrt{8(d-1)}}{2}$.

3 CONCLUDING REMARKS

The aim of this paper is twofold. Firstly, in Section 2, we obtained some upper bounds for the graph invariant $S_k^+(G)$, in terms of clique number ω, the vertex covering number τ and the diameter of the graph G. These bounds can be used to obtain the upper bounds for the signless Laplacian energy of the graph G and so can be helpful to obtain the extremal graphs among various families of the graphs. Secondly, in Section 3, we have used the results of Section 2 to verify the truth of the Conjecture 1 for some more families of graphs. Although, in Sections 2 and 3, we have restricted ourselves to graphs $G \in \{\Gamma_1, \Gamma_2, \Gamma_3\}$, the importance of these results can be realized from the fact that not many families of graphs are known for which Conjecture 1 holds.

Acknowledgements. The authors are grateful to the anonymous referee for his valuable comments and suggestions. The research of S. Pirzada (first author) is supported by SERB-DST, New Delhi under the research project number MTR/2017/000084.

REFERENCES

[1] Ashraf F., Omidi G.R., Tayfeh-Rezaie B. *On the sum of signless Laplacian eigenvalues of a graph*. Linear Algebra Appl. 2013, **438**, 4539–4546. doi:10.1016/j.laa.2013.01.023

[2] Bai H. *The Grone-Merris conjecture*. Trans. Amer. Math. Soc. 2011, **363** 4463–4474. doi:10.1.1.360.9876

[3] Brouwer A.E., Haemers W.H. Spectra of graphs. http://homepages.cwi.nl/aeb/math/ipm.pdf

[4] Cvetkovic D., Doob M., Sachs H. Spectra of Graphs-Theory and Application. Academic Press, New York, 1980.

[5] Das K.C., Mojallal S.A., Gutman I. *On Laplacian energy in terms of graph invariants*. Applied Mathematics and Computation 2015, **268**, 83–92. doi:10.1016/j.amc.2015.06.064

[6] Fulton W. *Eigenvalues, invariant factors, highest weights and Schubert calculus*. Bull. Amer. Math. Soc. (NS) 2000, **37**, 209–249. doi:10.1090/S0273-0979-00-00865-X

[7] Ganie H.A., Pirzada S. *On the bounds for signless Laplacian energy of a graph*. Discrete Appl. Math. 2017, **228**, 3–13. doi:10.1016/j.dam.2016.09.030

[8] Ganie H.A., Alghamdi A.M., Pirzada S. *On the sum of the Laplacian eigenvalues of a graph and Brouwer's conjecture*, Linear Algebra Appl. 2016, **501**, 376–389. doi:10.1016/j.laa.2016.03.034

[9] Ganie H.A., Pirzada S. *Corrigendum to "On the sum of the Laplacian eigenvalues of a graph and Brouwer's conjecture"*. Linear Algebra Appl. 2018, **538**, 228–230. doi:10.1016/j.laa.2017.10.020

[10] Ganie H.A., Pirzada S., Rezwan Ul Shaban, Li X. *Upper bounds for the sum of Laplacian eigenvalues of a graph and Brower's conjecture*. Discrete Math. Algorithms Appl. 2019, **11** (2), (15 pages). doi:10.1142/S1793830919500289

[11] Ganie H.A., Pirzada S., Rather B.A., Trevisan V. *Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs*. Linear Algebra Appl. 2020, **588**, 1–18. doi:10.1016/j.laa.2019.11.020

[12] Grone R., Merris R. *The Laplacian spectrum of a graph II*. SIAM J. Discrete Math. 1994, **7**, 221–229.

[13] Pirzada S. An Introduction to Graph Theory. Universities Press, OrientBlackSwan, Hyderabad, 2012.

[14] Pirzada S., Ganie H. A. *On the Laplacian eigenvalues of a graph and Laplacian energy*. Linear Algebra Appl. 2015, **486**, 454–468. doi:10.1016/j.laa.2015.08.032

[15] Rocha I., Trevisan V. *Bounding the sum of the largest Laplacian eigenvalues of graphs*. Discrete Appl. Math. 2014, **170**, 95–103. doi:10.1016/j.dam.2014.01.023

[16] Yang J., You L. *On a conjecture for the signless Laplacian eigenvalues*. Linear Algebra Appl. 2014, **446**, 115–132. doi:10.1016/j.laa.2013.12.032

Received 14.10.2018

Пірзада С., Гані Х.А., Альгамді А.М. *Про суму беззнакових лапласіанівських спектрів графів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 407–417.

Для деякого простого графа $G(V,E)$ з n вершинами і m ребрами, множиною вершин $V(G)=\{v_1,v_2,\ldots,v_n\}$ і множиною ребер $E(G)=\{e_1,e_2,\ldots,e_m\}$, матриця суміжності $A=(a_{ij})$ графа G — це $(0,1)$-квадратна матриця порядку n, для якої елементи з індексом (i,j) дорівнюють 1, якщо v_i суміжна з v_j і 0 у протилежному випадку. Нехай $D(G)=diag(d_1,d_2,\ldots,d_n)$ — діагональна матриця, асоційована з G, де $d_i=\deg(v_i)$, для всіх $i\in\{1,2,\ldots,n\}$. Матриці $L(G)=D(G)-A(G)$ і $Q(G)=D(G)+A(G)$ називаються лапласіанівські і беззнакові лапласіанівські матриці, відповідно, а їх спектри (власні значення), відповідно — лапласіанівським спектром (L-спектром) та беззнаковим лапласіанівським спектром (Q-спектром) графа G. Якщо $0=\mu_n\leq\mu_{n-1}\leq\cdots\leq\mu_1$ є лапласіанівські власні значення G, Броувер припустив, що сума k найбільших лапласіанівських значень $S_k(G)$ задовольняє $S_k(G)=\sum\limits_{i=1}^{k}\mu_i\leq m+\binom{k+1}{2}$ і це припущення є все ще відкритим. Якщо $q_1,q_2,\ldots,q_n$ — беззнакові лапласіанівські власні значення графа G для $1\leq k\leq n$, і нехай $S_k^+(G)=\sum_{i=1}^{k}q_i$ — сума k найбільших беззнакових лапласіанівських власних значень G. Аналогічно до припущення Броувера, Асхраф та ін. припустили, що $S_k^+(G)\leq m+\binom{k+1}{2}$ для всіх $1\leq k\leq n$. Це припущення було підтверджено для деяких класів графів. Ми отримали верхнє обмеження для $S_k^+(G)$ в термінах клікових чисел ω, чисел покриття вершин τ і діаметра графа G. Зрештою, ми показали, що припущення виконується для широкої сім'ї графів.

Ключові слова і фрази: беззнакові лапласіанівські спектри, припущення Броувера, клікові числа, числа покриття вершин, діаметр.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 418–421
doi:10.15330/cmp.11.2.418-421

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.418–421

PONOMARCHUK B.S.

METRIC DIMENSION OF METRIC TRANSFORM AND WREATH PRODUCT

Let (X, d) be a metric space. A non-empty subset A of the set X is called *resolving* set of the metric space (X, d) if for two arbitrary not equal points u, v from X there exists an element a from A, such that $d(u, a) \neq d(v, a)$. The smallest of cardinalities of resolving subsets of the set X is called *the metric dimension* $md(X)$ of the metric space (X, d).

In general, finding the metric dimension is an NP-hard problem. In this paper, metric dimension for metric transform and wreath product of metric spaces are provided. It is shown that the metric dimension of an arbitrary metric space is equal to the metric dimension of its metric transform.

Key words and phrases: metric dimension, metric transform, wreath product.

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INTRODUCTION

Let (X, d) be a metric space. It is said that a set A is the *resolving* set of the metric space (X, d) if A is a non-empty subset of X and for an arbitrary different points u, v from X there exists an element a in A, such that distances $d(u, a)$ and $d(v, a)$ are not equal. The smallest of cardinalities of resolving subsets of the set X is called *the metric dimension* $md(X)$ of the metric space (X, d).

Definition of the metric dimension for metric spaces was firstly introduced by Blumenthal in 1953 [4]. 20 years later Harari and Melter in [7] applied it to the graphs. After that the metric dimension concept found range of applications, like in combinatorial analysis, robotics, for finding its location, biology, chemistry etc. [14], [9], [13].

In 2013 S. Bau and F. Beardon [2] got the Blumenthal's ideas and proceeded research of the metric spaces metric dimension. They has managed to calculate the metric dimension of the sphere in a k-dimensional Euclidean space. Later, M. Heydarpour and S. Maghsoudi in [8] calculated the metric dimensions of geometric spaces.

As well as metric dimension, Blumenthal has also described metric transforms [3], which was studied further by other researchers, like by Schoenberg and von Neumann in scope of Euclidian subspace metric transforms into Hilbert space subsets [12], [15].

In general, finding of the metric dimension of a finite graph is an NP-hard problem [6]. Following that, metric dimension characterization for a finite metric space is also NP-hard. This is why there are several ways of conducting metric dimension research. One of those is researching metric dimension of constructions of two graphs, if we know metric dimensions of both of them. For example, metric dimensions of wreath products and cartesian products of two finite graphs characterized in [5], [1].

УДК 519.1
2010 *Mathematics Subject Classification:* 05C12, 51K05, 54E35.

In this paper we characterize metric dimensions of the wreath products of metric spaces which were considered in [11], [10]. This construction of metric spaces was called a wreath product because the isometry group of the wreath product of metric spaces is isomorphic to the wreath product of theirs isometry groups. In particular, we will also show that metric dimension of the metric transform of an arbitrary metric space is equal to the metric dimension of this space.

1 METRIC TRANSFORM

Denote by $\mathbb{R}^+$ the set of all non-negative real numbers. Let s be a continuous monotone increasing function and $s(0) = 0$. Such functions are called *scales*. *Transformation* of metric space (X, d_X) is the space $(X, s(d_X))$, where function $s(d_X)$ might not follow triangle inequality [3]. Transformation is called *metric*, if $s(d_X)$ is metric.

Definition 1 ([3]). *If for metric spaces (X, d_X) and (Y, d_Y) there is a bijection $g : X \to Y$, and scale s that for arbitrary $u, v \in X$ holds:*

$$d_X(u, v) = s(d_Y(g(u), g(v))),$$

then such metric spaces are called isomorphic.

Proposition 1. *Let (X, d) be a metric space and let $s : \mathbb{R}^+ \to \mathbb{R}^+$ be a metric transform. Then metric basis of X is also the metric basis of the metric transform $(X, s(d))$.*

Proof. Let $V = \{v_i, i \in I\}$ be a metric basis of the space (X, d). As follows from the definition of a metric basis, for every $u, w \in X$ there is $v_j \in V$, such that

$$d(u, v_j) \neq d(w, v_j),$$

i. e. v_j resolves points u and w. The function s is monotone increasing, so, we have

$$s(d(u, v_j)) \neq s(d(w, v_j)).$$

Hence, v_j resolves u and w in $(X, s(d))$. Therefore, V is resolving set of $(X, s(d))$.

We need to show, that V is minimal cardinality. Assume, that there is v_l, such that $V \setminus \{v_l\}$ also is a resolving set of $(X, s(d))$. But V is a minimal resolving set of (X, d). Hence, there are points $u, w \in X$ such that for any $v_j \in V \setminus \{v_l\}$ the following condition holds:

$$d(u, v_j) = d(w, v_j).$$

But it means, that $s(d(u, v_j)) = s(d(w, v_j))$. Hence, $V \setminus \{v_l\}$ is not a metric basis of $(X, s(d))$. □

Corollary 1. *Metric dimension of a metric space (X, d) is equal to the metric dimension of its metric transform $(X, s(d))$ for any scale s.*

2 WREATH PRODUCT

First, we recall the construction of a wreath product of metric spaces.

Definition 2. *A metric space (X, d) is called uniformly discrete if for an arbitrary $u, v \in X$ either $u = v$ or there exists a radius $r > 0$ such that $d(u, v) > r$.*

Let (X, d_X) be a uniformly discrete metric space, and (Y, d_Y) be a bounded metric space. Since space (X, d_X) is uniformly discrete, then there exists r such that for two different arbitrary points x_1, x_2 from set X inequality $d_X(x_1, x_2) \geq r$ holds. Let $s(x)$ be the scale such that

$$diam(s(Y)) < r. \tag{1}$$

Let us define a function ρ_s on the Cartesian product $X \times Y$ by:

$$\rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2. \end{cases}$$

Such a metric space is called *wreath product* of metric spaces (X, d_X) and (Y, d_Y) and denoted as Xwr_sY [11]. For different scales s_1 and s_2 metric spaces $Xwr_{s_1}Y$ and $Xwr_{s_2}Y$ are isomorphic.

Theorem 1. *Let X be a finite metric space and Y be a bounded metric space, $md(Y) < \infty$. Then, the dimension of wreath product of metric spaces (X, d_X) and (Y, d_Y) is equal to*

$$md(Xwr_sY) = |X| * md(Y).$$

If $md(Y) = \infty$, then $md(Xwr_sY) = \infty$.

Proof. Let $v_1, \dots, v_l$ be a metric basis of (Y, d_Y). We assume that $X = \{x_1, \dots, x_m\}$. Define a set

$$B = \{(x_j, v_i) | 1 \leq j \leq m, 1 \leq i \leq n\}.$$

We need to show that the set B is a basis of Xwr_sY.

Let (x_1, y_1) and (x_2, y_2) be two different points of Xwr_sY. From the definition of the wreath product of metric spaces follows, that if $x_1 \neq x_2$, then points (x_1, y_1) and (x_2, y_2) are resolved by point (x_2, v_2). Indeed, we will have:

$$\rho((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2), \quad \rho((x_2, y_2), (x_2, v_2)) = s(d_Y(y_2, v_2)).$$

From inequality (1) follows, that

$$\rho((x_1, y_1), (x_2, y_2)) < \rho((x_2, y_2), (x_2, v_2))$$

and therefore points (x_1, y_1) and (x_2, y_2) are resolved by (x_2, v_2).

Let $x_1 = x_2$. In this case, since $v_1, \dots, v_n$ is the metric basis of Y, exists v_j that resolves y_1 and y_2. Then

$$\rho((x_1, y_1), (x_1, v_j)) = s(d_Y(y_1, v_j)), \quad \rho((x_2, y_2), (x_1, v_j)) = s(d_Y(y_2, v_j)).$$

Since v_j resolves y_1 and y_2, $s(d_Y(y_1, v_j)) \neq s(d_Y(y_2, v_j))$. In this case all elements from the set X are supposed to be included into a basis of the cartesian product.

And now let us show that B is a basis. Assume that $B' = B/\{x_1, v_1\}$ is a basis. Since $v_1, \dots, v_n$ is the basis of the metric space Y, then there exists $y_1, y_2 \in Y$ which are not resolved by $v_2, v_3, \dots, v_n$ but are resolved by v_1 only. Then points (x, y_1) and (x, y_2) are not resolved by points from B'. This means that B is the minimal set, therefore B is a metric basis of the space Xwr_sY.

As a result we have that $md(Xwr_sY) = |X|md(Y)$. □

Theorem 1 implies the next statement.

Corollary 2. *If the space X is infinite, then $md(Xwr_sY) = \infty$.*

REFERENCES

[1] Assiyatun H., Bača M., Baskoro E.T., Salmana A.N.M., Saputro S.W., Simanjuntak R., Uttunggadewaa S. *The metric dimension of the lexicographic product of graphs*. Discr. Math. 2013, **313**, 1045–1051. doi: 10.1016/j.disc.2013.01.021

[2] Bau S., Beardon A.F. *The Metric Dimension of Metric Spaces*. Comput. Methods Funct. Theory 2013, **13**, 295–305. doi:10.1007/s40315-013-0024-0

[3] Blumenthal L.M. *Remarks concerning the euclidean four-point property*. Ergebnisse Math. Kolloq. 1936, **7**, 7–10.

[4] Blumenthal L.M. Theory and applications of distance geometry. Clarendon Press, Oxford, 1953.

[5] Chau K., Gosselin S. *The metric dimension of circulant graphs and their cartesian products*. Opuscula Math. 2017, **37** (4), 509–534. doi:10.7494/OpMath.2017.37.4.509

[6] Garey M.R., Johnson D.S. Computers and intractability. A guide to the theory of NP-completeness. W. H. Freeman and Company, New York, 1979.

[7] Harary F., Melter R.A. *On the metric dimension of a graph*. Ars Combin. 1976, **2**, 191–195.

[8] Heydarpour M., Maghsoudi S. *The metric dimension of geometric spaces*. Topology Appl. 2014, **178**, 230–235. doi:10.1016/j.topol.2014.09.012

[9] Khuller S., Raghavachari B., Rosenfeld A. *Landmarks in graphs*. Discrete Appl. Math. 1996, **70**, 217–219. doi:10.1016/0166-218X(95)00106-2

[10] Oliynyk B. *Infinitely iterated wreath products of metric spaces*. Algebra Discrete Math. 2013, **15** (1), 48–62.

[11] Oliynyk B. *Wreath product of metric spaces*. Algebra Discrete Math. 2007, **4**, 123–130.

[12] Schoenberg I.J. *Metric spaces and completely monotone functions*. Ann. of Math. 1938, **39**, 811–841. doi: 10.2307/1968466

[13] Sebo A., Tannier E. *On Metric Generators of Graphs*. Math. Oper. Res. 2004, **29**, 383–393. doi:10.1287/moor.1030.0070

[14] Slater P.J. *Leaves of trees*. Congr. Numer. 1975, **14**, 549–559.

[15] Von Neumann J., Schoenberg I.J. *Fourier integrals and metric geometry*. Trans. Amer. Math. Soc. 1941, **50**, 226–251. doi:10.2307/1989957

Received 28.08.2019

Пономарчук Б.С. *Метрична розмірність метричної трансформації та вінцевого добутку* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 418–421.

Для довільного метричного простору (X, d) множина $A \subset X$ називається розділяючою, якщо для довільних різних елементів u, v, що належать множині X існує такий елемент $a \in A$, що відстані $d(a, u)$ та $d(a, v)$ є різними. Метричною розмірністю $md(X)$ простору (X, d) називається розділяюча множина найменшої потужності.

В загальному випадку пошук метричної розмірності є NP–важкою задачею. В роботі охарактеризовано метричну розмірінсть метричної трансформації та вінцевого добутку метричних просторів. Також показано, що метрична розмірність довільного метричного простору співпадає з метричною розмірністю його метричної трансформації.

Ключові слова і фрази: метрична розмірність, метрична трансформація, вінцевий добуток.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 422–430
doi:10.15330/cmp.11.2.422-430

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.422–430

RAKDI M.A., MIDOUNE N.

WEIGHTS OF THE $\mathbb{F}_q$-FORMS OF 2-STEP SPLITTING TRIVECTORS OF RANK 8 OVER A FINITE FIELD

Grassmann codes are linear codes associated with the Grassmann variety $G(\ell, m)$ of ℓ-dimensional subspaces of an m dimensional vector space $\mathbb{F}_q^m$. They were studied by Nogin for general q. These codes are conveniently described using the correspondence between non-degenerate $[n, k]_q$ linear codes on one hand and non-degenerate $[n, k]$ projective systems on the other hand. A non-degenerate $[n, k]$ projective system is simply a collection of n points in projective space $\mathbb{P}^{k-1}$ satisfying the condition that no hyperplane of $\mathbb{P}^{k-1}$ contains all the n points under consideration. In this paper we will determine the weight of linear codes $C(3, 8)$ associated with Grassmann varieties $G(3, 8)$ over an arbitrary finite field $\mathbb{F}_q$. We use a formula for the weight of a codeword of $C(3, 8)$, in terms of the cardinalities certain varieties associated with alternating trilinear forms on $\mathbb{F}_q^8$. For $m = 6$ and 7, the weight spectrum of $C(3, m)$ associated with $G(3, m)$, have been fully determined by Kaipa K.V, Pillai H.K and Nogin Y. A classification of trivectors depends essentially on the dimension n of the base space. For $n \leq 8$ there exist only finitely many trivector classes under the action of the general linear group $GL(n)$. The methods of Galois cohomology can be used to determine the classes of nondegenerate trivectors which split into multiple classes when going from $\bar{\mathbb{F}}$ to $\mathbb{F}$. This program is partially determined by Noui L and Midoune N and the classification of trilinear alternating forms on a vector space of dimension 8 over a finite field $\mathbb{F}_q$ of characteristic other than 2 and 3 was solved by Noui L and Midoune N. We describe the $\mathbb{F}_q$-forms of 2-step splitting trivectors of rank 8, where char $\mathbb{F}_q \neq 3$. This fact we use to determine the weight of the $\mathbb{F}_q$-forms.

Key words and phrases: trivector, Grassmannian, weight.

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INTRODUCTION

Let V be an 8-dimensional vector space over a field K and let $\wedge^3 V$ denote the exterior power of degree 3 over V, the classification of trivectors is the study of the action of general linear group $GL(V)$ on the space $\wedge^3 V$ defined by $f.\omega = (\wedge^3 f)(\omega)$. The equivalence classes are the $GL(V)$-orbits under this action. As $\wedge^3 V^* \simeq (\wedge^3 V)^*$, there is no difference between trilinear alternating forms and trivectors. The support of the trivector ω is the least subspace F of V such that $\omega \in \wedge^3 F$, its dimension is the rank of ω. Let ω be a trilinear alternating form on V. The set $\{u \in V,\ \ \omega(u, \cdot, \cdot) = 0\}$ is called the radical of ω and will be denoted by $Rad\omega$. If $Rad\omega = \{0\}$, then ω is called nondegenerate (full rank).

This classification is motivated by many important applications, especially in the theory of codes. See [2, 4, 5, 7]. Let $C(3, 8)$ be a grassman code (linear code) associated with the

УДК 512.647
2010 *Mathematics Subject Classification:* 15A69, 20B40.

Grassmann variety $G(3,8)$ of 3-dimensional subspaces of an 8-dimensional vector space $\mathbb{F}_q^8$, where $\mathbb{F}_q$ is a finite field with q elements. The parameters n and k of the code $C(3,8)$ are

$$n = |C(3,8)| = \frac{(q^8-1)(q^{8-1}-1)(q^{8-3+1}-1)}{(q^3-1)(q^{3-1}-1)(q-1)},$$
$$k = \binom{8}{3}.$$

The minimum distance of grassmann codes $C(3,8)$ equals $d = q^{3(8-3)} = q^{15}$. The weight of $C(3,7)$, $C(3,6)$ and $C(2,m)$ is determined by [2], [5] and [4] respectively. In this paper,we are interested in the classification of $\mathbb{F}_q$-forms of the 2-step splitting trivectors of rank 8, and in determining the weights of $\mathbb{F}_q$-forms where $\mathbb{F}_q$ is a finite field of characteristic other than 3. Some undefined terms can be found in references [2,3,6] and [5].

1 $\mathbb{F}_q$-FORMS OF 2-STEP SPLITTING TRIVECTORS OF RANK ≤ 8

If ω is a trivector defined over the field K, a K-form of ω is another trivector of the same type as that of ω, defined over K which is isomorphic to ω over $\overline{K}$, the algebraic closure of K. The element ω of $\wedge^3 V$ is called splitting if there exists a decomposition $V = V_1 \oplus V_2$ such that $\omega \in V_1 \otimes \wedge^2 V_2$. If $dim V_1 = 2$, ω is called 2-step splitting.

Preliminary result

1.1 Degenerate forms

Theorem 1 ([1]). *Let V be a vector space of dimension 7 over a finite field $\mathbb{F}_q$. Then any trivector of rank ≤ 7 in $\wedge^3 V$ is equivalent to one of the trivectors in Table 1.*

Table1. Trivectors of rank ≤ 7 over $\mathbb{F}_q$ (degenerate forms).

Name	*Trivector*
ω_3	$e_1e_2e_3$
ω_5	$e_1(e_2e_3+e_4e_5)$
$\omega_{6,1}$	$e_1e_2e_3+e_4e_5e_6$
$\omega_{6,1,d_1}$	$e_1(e_3e_4+e_5e_6)+e_2(e_3e_6-d_1e_4e_5)$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{6,1,d_2}$	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_4-d_2e_3e_5+e_4e_5)$ *if char* $\mathbb{F}_q = 2$
$\omega_{6,2}$	$e_1e_2e_4+e_2e_3e_5+e_1e_3e_6$
$\omega_{7,1}$	$e_1(e_2e_3+e_4e_5+e_6e_7)$
$\omega_{7,2}$	$\omega_{7.1}+e_2e_4e_6$
$\omega_{7,3}$	$e_1e_2e_3+e_3e_4e_5+e_5e_6e_7$
$\omega_{7,3,d_1}$	$e_1(e_2e_5+e_3e_7)+e_4(e_2e_3+d_1e_5e_7)+e_6e_5e_3$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{7,3,d_2}$	$e_1(e_2e_3+e_4e_5)+e_6(e_2e_4-d_2e_3e_5+e_4e_5)+e_1e_6e_7$ *if char* $\mathbb{F}_q = 2$
$\omega_{7,4}$	$e_1(e_2e_3+e_4e_5)+e_2e_4e_6+e_3e_5e_7$
$\omega_{7,5}$	$\omega_{7.2}+e_3e_5e_7$

where $d_1 \notin (\mathbb{F}_{q^*})^2$, $d_2 \in (\mathbb{F}_{q^*})^2$.

Main results

1.2 Nondegenerate forms (full rank)

Theorem 2. *Let V be a vector space of dimension 8 over a finite field $\mathbb{F}_q$. Then any $\mathbb{F}_q$-form of 2-step splitting trivector of rank 8 in $\wedge^3 V$ is equivalent to one of the Table 2.*

Table2. Trivectors of rank 8 over $\mathbb{F}_q$ (nondegenerate forms).

Name	*Trivector*
$\omega_{8,1}$	$e_1(e_2e_3 + e_4e_5) + e_6e_7e_8$
$\omega_{8,2}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$
$\omega_{8,3}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_5 + e_7e_8)$
$\omega_{8,4}$	$e_1(e_2e_5 + e_3e_6) + e_4(e_7e_2 + e_8e_3)$
$\omega_{8,4,d_1}$	$e_5(e_1e_2 + e_3e_4) + e_6(e_1e_3 + d_1e_2e_4) + e_7(e_1e_4) + e_8(e_2e_3)$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{8,4,d_2}$	$e_8(e_1e_4 + e_3e_2) + e_7(e_1e_4 + e_4e_2 + d_2e_1e_3) + e_6e_1e_2 + e_5e_3e_4$ *if char* $\mathbb{F}_q = 2$
$\omega_{8,5}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_3 + e_7e_8)$
$\omega_{8,5,d_1}$	$e_7(e_1e_2 + e_3e_4 + e_5e_6) + e_8[e_1(e_4 + d_1e_5) + e_2e_6 + \frac{1}{d_1}e_3e_5]$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{8,5,d_2}$	$e_3(e_1e_2 + e_4e_7 + e_6e_8) + e_5(e_1e_4 + e_8e_2 + d_2e_6e_7)$ *if char* $\mathbb{F}_q = 2$
$\omega_{8,5,d_3}$	$e_1(d_3e_3e_4 + d_3e_5e_6 + e_7e_8) + e_2(e_3e_5 + e_4e_7 + e_6e_8)$ *if char* $\mathbb{F}_q \neq 3$
$\omega_{8,6}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_8(e_4e_3 + e_5e_6)$

where $d_1 \notin (\mathbb{F}_{q^*})^2, d_2 \in (\mathbb{F}_{q^*})^2, d_3 \notin (\mathbb{F}_{q^*})^3$.

Proof. The $\mathbb{F}_q$-forms of 2-step splitting trivectors of rank 8 where $\mathbb{F}_q$ is a field of characteristic other than 2 and 3 has been done in [6], hence, in characteristic 2, it is sufficient to study the case of orbits of type $\omega_{8,i}$, for $i = 4, 5$.

In characteristic 2, the trivectors of type $\omega_{8,i}$, for $i = 4, 5$ are written
$\omega_{8,4,d_2} = e_8(e_1e_4 + e_3e_2) + e_7(e_1e_4 + e_4e_2 + d_2e_1e_3) + e_6e_1e_2 + e_5e_3e_4$
$\omega_{8,5,d_2} = e_3(e_1e_2 + e_4e_7 + e_6e_8) + e_5(e_1e_4 + e_8e_2 + d_2e_6e_7)$.

If L is the quadratic extension of K, there exists a trivector $\omega_L \in \wedge^3 V$ such that $\omega_L \not\simeq \omega_{8,4}$ and $\omega_L \otimes L \in \wedge^3(V \otimes_K L)$ is L-isomorphic to $\omega_{8,4}$. We construct ω_L as follows: $\omega_{8,4} = e_1(e_2e_5 + e_3e_6) + e_4(e_7e_2 + e_8e_3)$ is a 4-step splitting because $\omega_{8,4} = e_5u_1 + e_6u_2 + e_7u_3 + e_8u_4$ where $u_1 = e_1e_2$, $u_2 = e_1e_3$, $u_3 = e_2e_4$ and $u_4 = e_3e_4$, thus $E = vect\{u_1, u_2, u_3, u_4\}$ is a subspace of dimension 4 of $\wedge^4 K^4$. We put $\omega_L = \omega_{8,4,d_2} = e_5v_1 + e_6v_2 + e_7v_3 + e_8v_4$, with $v_1 = e_3e_4$, $v_2 = e_1e_2$, $v_3 = e_1e_4 + e_4e_2 + d_2e_1e_3$, and $v_4 = e_1e_4 + e_3e_2$, where $K' = K(\alpha)$, $\alpha^2 + \alpha = d_2$, $\alpha \in K$. To each of the forms $\omega_{8,4}$, $\omega_{8,4,d_2}$, we associate a quadratic form on E [6]: $\gamma_2(xu_1 + yu_2 + zu_3 + tu_4)$, then we get $\gamma_2(xu_1 + yu_2 + zu_3 + tu_4) = (xt - yz)$, $\gamma_2(xv_1 + yv_2 + zv_3 + tv_4) = (y^2d_2 - x^2 + zt)$ respectively. The two forms are not equivalent over K but they may become equivalent over the algebraic closure $\overline{K}$. We can also prove that $\omega_{8,4}$ is not equivalent to $\omega_{8,4,d_2}$ by using the arithmetical invariant $d_1(\omega)$ [6].

Similar arguments apply to the case for $\omega_{8,5}$. □

2 FORMULA FOR THE WEIGHT OF A TRIVECTOR

The correspondence between equivalence classes of nondegenerate forms and equivalence classes of nondegenerate linear $[n, k]$-codes, is one-to-one. In what follows, we speak by abuse of language not only of a weight of a codeword, but also of a weight of hyperplane and a weight of a form $\omega \in \wedge^3 V$. Therefore, the problem on the spectrum of a Grassmann code (at least, on the weights of the codewords) is closely related to that on the classification of the elements of $\wedge^3 V$.

The cardinality of the general linear group $GL(8, \mathbb{F}_q)$ will be denoted by $[8]_q$

$$[8]_q = q^{8(8-1)/2}(q^8 - 1)(q^{8-1} - 1)\cdots(q-1).$$

Given a codeword of $C(3,8)$, let ω be the corresponding trivector on $\mathbb{F}_q^8$, and let $\mathcal{H}$ be the corresponding hyperplane of $\mathbb{P}(\wedge^3\mathbb{F}_q^8)$. The weight of the codeword ω

$$\text{wt}(\omega) = |\{P_i \ : \ 1 \le i \le n, \ \ P_i \notin \mathcal{H}\}|.$$

We have

$$[3]_q \cdot \text{wt}(\omega) = |\{[v_1, v_2, v_3] \ : \ \langle \omega, v_1 \wedge v_2 \wedge v_3 \rangle \neq 0\}|.$$

2.1 Weight of a degenerate trivector

If ω is degenerate, let $Rad\omega$ be r-dimensional. We pick a basis $\{e_1, \ldots, e_8\}$ of V such that $\{e_{8-r+1}, \ldots, e_8\}$ is a basis for $Rad\omega$. Let W denote the span of $\{e_1, \ldots, e_{8-r}\}$.

Let $\tilde{\omega}$ denote the restriction of the form ω to W. Since $W \cap Rad\omega = \{0\}$, it is clear that $\tilde{\omega}$ is a nondegenerate trivector on W. Thus, $\tilde{\omega}$ can be thought of as codeword in $C(3, 8-r)$.

Proposition 1 ([2]). *The weight of a degenerate trivector ω in $\mathbb{F}_q^8$ is given by*

$$\text{w}t(\omega) = q^{3r}\text{w}t(\tilde{\omega}).$$

The proposition shows that in order to calculate the weights of codewords of $C(3,8)$, it is enough to know only the weights of nondegenerate codewords of $C(3, m)$ for $m \le 8$.

Lemma 1. *The weights of degenerate trivectors are*

$$\begin{aligned}
\text{w}t(\omega_3) &= q^{15} \\
\text{w}t(\omega_5) &= q^{15} + q^{13} \\
\text{w}t(\omega_{6,1}) &= q^{15} + q^{13} + q^{12} - q^{10} \\
\text{w}t(\omega_{6,1,d}) &= q^{15} + q^{13} + q^{12} + q^{10} \\
\text{w}t(\omega_{6,2}) &= q^{15} + q^{13} + q^{12} \\
\text{w}t(\omega_{7,1}) &= q^{15} + q^{13} + q^{11} \\
\text{w}t(\omega_{7,2}) &= q^{15} + q^{13} + q^{12} + q^{11} \\
\text{w}t(\omega_{7,3}) &= q^{15} + q^{13} + q^{12} + q^{11} - q^{10} \\
\text{w}t(\omega_{7,3,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} \\
\text{w}t(\omega_{7,4}) &= q^{15} + q^{13} + q^{12} + q^{11} \\
\text{w}t(\omega_{7,5}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{9}.
\end{aligned}$$

Proof. According to Proposition 1, the weight of a degenerate form ω is q^3 times the weight of ω viewed as a trivector on $\mathbb{F}_q^7$ the span of $\{e_1, \ldots, e_7\}$. The latter weights were determined in [2]. We multiply them by q^3; we get the weights of ω. □

2.2 Weight varieties of a nondegenerate trivector

Let V be an 8-dimensional vector space over an arbitrary field F.

We consider the map $\varphi_w : V \longrightarrow \wedge^2 V^*$ sending $v \longmapsto \iota_v \omega$ where ι_v is the operation of the interior multiplication defined by

$$\langle \iota_v \omega, \beta \rangle = \langle \omega, v \wedge \beta \rangle, \quad \text{for all } \beta \in \wedge^2 V.$$

Here, $\langle , \rangle$ is the pairing between $\wedge^j V^*$ and $\wedge^j V$ for each j.

Given a two-from $\lambda \in \wedge^2 V^*$, we define certain quantities $\mathrm{Pf}_k(\lambda) \in \wedge^{2k} V^*$, for each $k \geq 1$ which we call the k-th Pfaffian of λ. Let $\mathrm{Pf}_0(\lambda) = 1$. We define $\mathrm{Pf}_k(\lambda) \in \wedge^{2k} V^*$ inductively by requiring

$$\iota_v \lambda \wedge \mathrm{Pf}_{k-1}(\lambda) = \iota_v \mathrm{Pf}_k(\lambda), \quad \text{for all } v \in V.$$

This $\mathrm{Pf}_k(\lambda)$ generalizes the forms $\frac{\lambda^k}{k!} = \frac{1}{k!}(\lambda \wedge \cdots \wedge \lambda)$.

Definition 1 ([2]). *Given a nondegenerate trivector ω on $\mathbb{F}_q^8$, the k-th weight variety of ω is the subvariety of $\mathbb{P}^7$ given by*

$$X_k(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_{k+1}(\iota_x \omega) = 0\}.$$

We have

$$\varnothing = X_0(\omega) \subset X_1(\omega) \subset X_2(\omega) \subset X_{\lfloor \frac{8-1}{2} \rfloor = 3}(\omega) = \mathbb{P}^7.$$

Lemma 2. *Given a nondegenerate trivector ω on $\mathbb{F}_q^8$.*

Let

$$n_i := |X_i(\omega)| - |X_{i-1}(\omega)|.$$

The weight $\mathrm{w}t(\omega)$ is given by

$$\mathrm{w}t(\omega) = q^6 \left[(q^9 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + 1) - \frac{n_2 + n_1(1+q^2)}{1+q+q^2} \right]. \tag{1}$$

Proof. We use Theorem 7 in [2], we get

$$\mathrm{w}t(\omega) = \frac{q^{2m-4}}{(q^2-1)(1+q+q^2)} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} n_i (1 - q^{-2i}),$$

for the case $m = 8$, we use $n_1 + n_2 + n_3 = |\mathbb{P}^7|$, we get this result in (1). □

3 WEIGHT CLASSIFICATION OF TRIVECTORS ON $\mathbb{F}_q^8$

The weights of the nondegenerate forms $\omega_{8,i}$, $1 \leq i \leq 6$ can be determined from formula (1) once the cardinalities of the varieties $X_1(\omega_{8,i})$ and $X_2(\omega_{8,i})$ are known. We recall that

$$X_1(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_2(\iota_x \omega) = 0\}$$

$$X_2(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_3(\iota_x \omega) = 0\}.$$

Proposition 2. *The varieties $X_1(\omega_{8,i})$ and their cardinalities for $1 \le i \le 6$ are*

$\omega_{8,i}$	$X_1(\omega_{8,i})$	$n_1(\omega_{8,i})$
$\omega_{8,1}$	$\mathbb{P}^2 \cup \mathbb{P}^3$	q^3+2q^2+2q+2
$\omega_{8,2}$	$\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^3$	q^3+2q^2+q+1
$\omega_{8,3}$	$\mathbb{P}^1 \cup \mathbb{P}^1$	$2q+2$
$\omega_{8,4}$	$\mathbb{P}^1 \times \mathbb{P}^1$	q^2+2q+1
$\omega_{8,4,d}$	$\mathbb{P}^1(\mathbb{F}_{q^2})$	q^2+1
$\omega_{8,5}$	$\mathbb{P}^1 \cup_{\mathbb{P}^0} \mathbb{P}^1$	$2q+1$
$\omega_{8,5,d}$	$\varnothing$	0
$\omega_{8,6}$	$\mathbb{P}^1$	$q+1$

Proof. Let $x = \sum_{j=1}^{8} x_j e_j$. We have

$$\mathrm{Pf}_2(\iota_x\omega) = \sum_{j=1}^{8} x_j^2 \mathrm{Pf}_2(\iota_{e_j}\omega) + \sum_{i<j} x_i x_j (\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega). \tag{2}$$

We calculate $\mathrm{Pf}_2(\iota_x\omega_i)$ using the above formula (2) and set it equal to zero to determine the varieties $X_1(\omega_i)$. We begin with $\omega_{8,1}$. The forms $\iota_{e_j}\omega_{8,1}$ for $j = 1\cdots 8$ are $e_2e_3+e_4e_5$, e_3e_1, e_1e_2, e_5e_1, e_1e_4, e_7e_8, e_8e_6, e_6e_7, respectively. For $j \ge 2$; the forms $\iota_{e_j}\omega_{8,1}$ are decomposable and hence $\mathrm{Pf}_2(\iota_{e_j}\omega_{8,1}) = 0$, whereas $\mathrm{Pf}_2(\iota_{e_1}\omega_{8,1}) = e_2e_3e_4e_5$.

We also note that $\iota_{e_2}\omega_{8,1} \wedge \iota_{e_j}\omega_{8,1} = 0$ for all $j = 3,4,5$, and $\iota_{e_3}\omega_{8,1} \wedge \iota_{e_4}\omega_{8,1} = \iota_{e_3}\omega_{8,1} \wedge \iota_{e_5}\omega_{8,1} = \iota_{e_4}\omega_{8,1} \wedge \iota_{e_5}\omega_{8,1} = \iota_{e_6}\omega_{8,1} \wedge \iota_{e_7}\omega_{8,1} = \iota_{e_6}\omega_{8,1} \wedge \iota_{e_8}\omega_{8,1} = \iota_{e_7}\omega_{8,1} \wedge \iota_{e_8}\omega_{8,1} = 0$. Using these relations, we get

$$\begin{aligned}
&\mathrm{Pf}_2(\iota_x\omega_{8,1}) = x_1^2 e_2e_3e_4e_5 + x_1[x_2e_4e_5e_3e_1 + x_3e_4e_5e_1e_2 + x_4e_2e_3e_5e_1 + x_5e_2e_3e_1e_4 \\
&+ x_6(e_2e_3e_7e_8 + e_4e_5e_7e_8) + x_7(e_2e_3e_8e_6 + e_4e_5e_8e_6 + x_8(e_2e_3e_6e_7 + e_4e_5e_6e_7))] \\
&+ x_2(x_6e_3e_1e_7e_8 + x_7e_3e_1e_8e_6 + x_8e_3e_1e_6e_7) + x_3(x_6e_1e_2e_7e_8 + x_7e_1e_2e_8e_6 + x_8e_1e_2e_6e_7) \\
&+ x_4(x_6e_5e_1e_7e_8 + x_7e_5e_1e_8e_6 + x_8e_5e_1e_6e_7) + x_5(x_6e_1e_4e_7e_8 + x_7e_1e_4e_8e_6 + x_8e_1e_4e_6e_7) = 0
\end{aligned}$$

Since the coefficient of $e_2e_3e_4e_5$ above is x_1^2, $x_1 = 0$ is necessary for $\mathrm{Pf}_2(\iota_x\omega_{8,1}) = 0$. Setting $x_1 = 0$ in the above equation, we get

$$\begin{aligned}
&\mathrm{Pf}_2(\iota_x\omega_{8,1})_{x_1=0} = x_2(x_6e_3e_1e_7e_8 + x_7e_3e_1e_8e_6 + x_8e_3e_1e_6e_7) + x_3(x_6e_1e_2e_7e_8 \\
&+ x_7e_1e_2e_8e_6 + x_8e_1e_2e_6e_7) + x_4(x_6e_5e_1e_7e_8 + x_7e_5e_1e_8e_6 + x_8e_5e_1e_6e_7) \\
&+ x_5(x_6e_1e_4e_7e_8 + x_7e_1e_4e_8e_6 + x_8e_1e_4e_6e_7) = e_1 \wedge (x_3e_2 - x_2e_3 \\
&+ x_5e_4 - x_4e_5) \wedge (x_6e_7e_8 + x_7e_8e_6 + x_8e_6e_7).
\end{aligned}$$

Therefore,

$$\begin{aligned}
X_1(\omega_{8,1}) &= \{x_1 = 0\} \cap [\{x_2 = x_3 = x_4 = x_5 = 0\} \cup \{x_6 = x_7 = x_8 = 0\}] \\
&= \mathbb{P}\{e_6, e_7, e_8\} \cup \mathbb{P}\{e_2, e_3, e_4, e_5\} \simeq \mathbb{P}^2 \cup \mathbb{P}^3.
\end{aligned}$$

Next, we consider $\mathrm{Pf}_2(\iota_x\omega_{8,2})$. The coefficients of $e_2e_3e_4e_5 + e_2e_3e_6e_7 + e_4e_5e_6e_7$, $e_1e_4e_6e_8$, $e_7e_1e_8e_5$ are x_1^2 and x_5^2 and x_6^2 respectively, $x_1 = x_5 = x_6 = 0$ is necessary for $\mathrm{Pf}_2(\iota_x\omega_{8,2}) = 0$. By Setting x_1, x_5 and x_6 to zero, in the equation, we get

$\mathrm{Pf}_2(\iota_x\omega_{8,2})_{x_1=x_5=x_6=0} = e_1 \wedge (x_3e_2 - x_2e_3) \wedge x_8e_5e_6$.
Therefore,

$$\begin{aligned} X_1(\omega_{8,2}) &= \{x_1 = x_5 = x_6 = 0\} \cap [\{x_2 = x_3 = 0\} \cup \{x_8 = 0\}] \\ &= \mathbb{P}\{e_4, e_7, e_8\} \cup_{\mathbb{P}\{e_4,e_7\}} \mathbb{P}\{e_2, e_3, e_4, e_7\} \simeq \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^3. \end{aligned}$$

In $\mathrm{Pf}_2(\iota_x\omega_{8,3})$, the coefficients of $e_3e_4e_5e_6$, $e_3e_5e_7e_8$, $e_4e_1e_5e_2$ and $e_6e_1e_2e_3$ are x_1^2, x_2^2, x_3^2 and x_5^2, respectively. By setting x_1, x_2, x_3 and x_5 to zero, $\mathrm{Pf}_2(\iota_x\omega_{8,3})$ is reduced to $e_1e_2 \wedge (x_4e_3 + x_6e_5) \wedge (x_8e_7 - x_7e_8)$.

Therefore, $X_1(\omega_{8,3}) = \{x_1 = x_2 = x_3 = x_5 = 0\} \cap [\{x_4 = x_6 = 0\} \cup \{x_7 = x_8 = 0\}] \simeq \mathbb{P}^1 \cup \mathbb{P}^1$. Similar arguments apply to the case for $X_1(\omega_{8,i})$ for $i = 4, \ldots, 6$. □

We now compute the varieties $X_2(\omega)$ and their cardinalities.

Proposition 3. *The varieties $X_2(\omega_{8,i})$ and their cardinalities for $1 \le i \le 6$ are*

$\omega_{8,i}$	$X_2(\omega_{8,i})$	$\lvert X_2(\omega_{8,i})\rvert$
$\omega_{8,1}$	$\mathbb{P}^6 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^6\rvert + \lvert\mathbb{P}^4\rvert - \lvert\mathbb{P}^3\rvert$
$\omega_{8,2}$	$\mathbb{P}^6$	$\lvert\mathbb{P}^6\rvert$
$\omega_{8,3}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^5\rvert + 2\lvert\mathbb{P}^4\rvert - 2\lvert\mathbb{P}^3\rvert$
$\omega_{8,4}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^5$	$2\lvert\mathbb{P}^5\rvert - \lvert\mathbb{P}^3\rvert$
$\omega_{8,4,d}$	$\mathbb{P}^3$	$\lvert\mathbb{P}^3\rvert$
$\omega_{8,5}$	$(\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4) \cup (\mathbb{F}_q)^2$	$\lvert\mathbb{P}^5\rvert + 2\lvert\mathbb{P}^4\rvert - 2\lvert\mathbb{P}^3\rvert + q^2$
$\omega_{8,5,d}$	$\mathbb{P}^5$	$\lvert\mathbb{P}^5\rvert$
$\omega_{8,6}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^5\rvert + \lvert\mathbb{P}^4\rvert - \lvert\mathbb{P}^3\rvert$

Proof. Let $x = \sum_{j=1}^{8} x_je_j$. We have

$$\mathrm{Pf}_3(\iota_x\omega) = \sum_{j=1}^{8} x_j^3\mathrm{Pf}_3(\iota_{e_j}\omega) + \sum_{i<j}[x_i^2x_j\mathrm{Pf}_2(\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega) + x_ix_j^2(\iota_{e_i}\omega) \wedge \mathrm{Pf}_2(\iota_{e_j}\omega)]. \tag{3}$$

We calculate $\mathrm{Pf}_3(\iota_x\omega_{8,i})$ using the above formula (3), and set it equal to zero to determine the varieties $X_2(\omega_{8,i})$. We begin with $\omega_{8,1}$.

For $j \ge 1$, $\mathrm{Pf}_3(\iota_{e_j}\omega_{8,1}) = 0$ and $\mathrm{Pf}_2(\iota_{e_1}\omega_{8,1}) = e_2e_3e_4e_5$, we get

$$\mathrm{Pf}_3(\iota_x\omega_{8,1}) = x_1^2x_6e_2e_3e_4e_5e_7e_8 + x_1^2x_7e_2e_3e_4e_5e_8e_6 + x_1^2x_8e_2e_3e_4e_5e_6e_7.$$

Since the coefficients of $e_2e_3e_4e_5e_7e_8$ and $e_2e_3e_4e_5e_8e_6$ and $e_2e_3e_4e_5e_6e_7$ above are $x_1^2x_6$ and $x_1^2x_7$ and $x_1^2x_8$, respectively, $x_1 = 0$ or $x_6 = x_7 = x_8 = 0$ is nececessary for $\mathrm{Pf}_3(\iota_x\omega_{8,1}) = 0$.

Therefore,

$$\begin{aligned} X_2(\omega_{8,1}) &= \{x_1 = 0\} \cup \{x_6 = x_7 = x_8 = 0\} \\ &= \mathbb{P}\{e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \cup_{\mathbb{P}\{e_2,e_3,e_4,e_5\}} \mathbb{P}\{e_1, e_2, e_3, e_4, e_5\} \simeq \mathbb{P}^6 \cup_{\mathbb{P}^3} \mathbb{P}^4. \end{aligned}$$

Next, we consider $\mathrm{Pf}_3(\iota_x\omega_{8,2})$. The coefficient of $e_2e_3e_4e_5e_6e_7$ is x_1^3; moreover, x_1 divides $\mathrm{Pf}_3(\iota_x\omega_{8,2})$. Therefore,

$$X_2(\omega_{8,2}) = \{x_1 = 0\} \simeq \mathbb{P}^6.$$

For $\mathrm{Pf}_3(\iota_x\omega_{8,3})$, the coefficients of $e_3e_4e_5e_6e_7e_8$, $e_3e_4e_5e_6e_8e_2$, $e_3e_4e_5e_6e_2e_7$, $e_3e_5e_7e_8e_4e_1$, $e_3e_5e_7e_8e_6e_1$, $e_7e_8e_4e_1e_5e_2$ and $e_7e_8e_6e_1e_2e_3$ are $x_1^2x_2$, $x_1^2x_7$, $x_1^2x_8$, $x_2^2x_3$, $x_2^2x_5$, $x_2x_3^2$ and $x_2x_5^2$ respectively. Reducing x_1x_2, x_1x_7, x_1x_8, x_2x_3 and x_2x_5 to zero is necessary for $\mathrm{Pf}_3(\iota_x\omega_{8,3}) = 0$.

Therefore,

$$\begin{aligned}
X_2(\omega_{8,3}) &= \{x_1 = x_2 = 0\} \cup \{x_2 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_5 = 0\} \\
&= \mathbb{P}\{e_3, e_4, e_5, e_6, e_7, e_8\} \cup_{\mathbb{P}\{e_3,e_4,e_5,e_6\}} \mathbb{P}\{e_1, e_3, e_4, e_5, e_6\} \cup_{\mathbb{P}\{e_4,e_6,e_7,e_8\}} \mathbb{P}\{e_2, e_4, e_6, e_7, e_8\} \\
&\simeq \mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4.
\end{aligned}$$

Similar arguments apply to the case for $X_2(\omega_{8,i})$ for $i = 4, \ldots, 6$. □

Theorem 3. *The weights of the nondegenerate forms* $\omega_{8,1}, \ldots, \omega_{8,6}$ *are*

$$\begin{aligned}
\mathrm{w}t(\omega_{8,1}) &= q^{15} + q^{13} + q^{12} + q^{11} - q^8 \\
\mathrm{w}t(\omega_{8,2}) &= q^{15} + q^{13} + q^{12} + q^{11} \\
\mathrm{w}t(\omega_{8,3}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^8 \\
\mathrm{w}t(\omega_{8,4}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^9 \\
\mathrm{w}t(\omega_{8,4,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 \\
\mathrm{w}t(\omega_{8,5}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^8 \\
\mathrm{w}t(\omega_{8,5,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^8 \\
\mathrm{w}t(\omega_{8,6}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10}.
\end{aligned}$$

Proof. We use the formula (1) with $n_2(\omega) + n_1(\omega) = |X_2(\omega)|$, we get

$$\mathrm{w}t(\omega_{8,i}) = q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^6 - q^6 \left(\frac{|X_2(\omega_{8,i})| + q^2|X_1(\omega_{8,i})|}{1 + q + q^2} \right),$$

the quantities $|X_1(\omega_{8,i})|$ and $|X_2(\omega_{8,i})|$ have been computed in Proposition 2 and 3.

For $\mathrm{w}t(\omega_{8,1})$,

we have $|X_1(\omega_{8,1})| = q^3 + 2q^2 + 2q + 2$ and $|X_2(\omega_{8,1})| = |\mathbb{P}^6| + |\mathbb{P}^4| - |\mathbb{P}^3| = q^6 + q^5 + 2q^4 + q^3 + q^2 + q + 1$, substituting these in the above equation we find

$$\begin{aligned}
\mathrm{w}t(\omega_{8,1}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^6 \\
&- q^6 \left(\frac{|\mathbb{P}^6| + |\mathbb{P}^4| - |\mathbb{P}^3| + q^2(q^3 + 2q^2 + 2q + 2)}{1 + q + q^2} \right) = q^{15} + q^{13} + q^{12} + q^{11} - q^8.
\end{aligned}$$

Similarly for the weights $\mathrm{w}t(\omega_{8,2}), \ldots, \mathrm{w}t(\omega_{8,6})$. □

REFERENCES

[1] Cohen A.M., Helminck A.G. *Trilinear alternating forms on a vector space of dimension 7*. Comm. Algebra 1988, **16** (1), 1–25. doi:10.1080/00927878808823558

[2] Kaipa K.V., Pillai H.K. *Weight Spectrum of Codes Associated With the Grassmannian* $G(3,7)$. IEEE Transactions on Information Theory 2013, **59** (2), 986–993. doi:10.1109/TIT.2012.2219497

[3] Midoune N., Noui L. *Trilinear alternating forms on a vector space of dimension 8 over a finite field*. Linear Multilinear Algebra 2013, **61** (1), 15–21. doi:10.1080/03081087.2012.661424

[4] Nogin Y. *Codes associated to Grassmannians*. In: Arithmetic, geometry and coding theory (Luminy, 1993). de Gruyter, 1996, 145–154.

[5] Nogin Y. *Spectrum of Codes Associated with the Grassmannian G (3, 6)*. Probl. Inf. Transm. 1997, **33** (2), 114–123.

[6] Noui L., Midoune N. *K-forms of 2-step splitting trivectors*. Int. J. Algebra 2008, **2** (5-8), 369–382.

[7] O'Brien E., Vojtěchovský P. *Code loops in dimension at most 8*. J. Algebra 2017, **473**, 607–626.

Received 10.02.2019

Ракді М.А., Мідуне Н. *Ваги $\mathbb{F}_q$-форм 2-ступінчастих тривекторів розщеплення рангу 8 над скінченним полем* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 422–430.

Коди Грассмана — це лінійні коди, пов'язані з многовидом Грассмана $G(\ell, m)$ ℓ-вимірного підпростору у m-вимірному векторному просторі $\mathbb{F}_q^m$. Їх вивчав Й. Ногін для довільних q. Ці коди зручно описати за допомогою відповідності між невиродженими $[n,k]_q$ лінійними кодами з одного боку, і невиродженими $[n,k]$ проективними системами з іншого боку. Невироджена $[n,k]$ проективна система — це просто набір n точок у проективному просторі $\mathbb{P}^{k-1}$, який задовольняє умови, що жодна гіперплощина $\mathbb{P}^{k-1}$ не містить n точок, що розглядаються. У цій роботі ми визначимо вагу лінійних кодів $C(3,8)$, асоційованих із многовидом Грассмана $G(3,8)$ над довільним скінченним полем $\mathbb{F}_q$. Ми використовуємо формулу для ваги кодового слова $C(3,8)$ у сенсі потужності певних многовидів, пов'язаних з чергуванням трилінійних форм на $\mathbb{F}_q^8$. Для $m = 6$ і 7, звужений спектр $C(3,m)$ асоційований з $G(3,m)$, був повністю визначений в роботах К.В. Кайпа, Х.К. Пілаі і Й. Ногіна. Класифікація тривекторів істотно залежить від розмірності n базового простору. Для $n \leq 8$ існує тільки скінченна кількість класів тривекторів під дією загальної лінійної групи $GL(n)$. Методи когомології Галуа можуть бути використані для визначення класів невироджених тривекторів, які поділяються на кілька класів при переході від $\bar{\mathbb{F}}$ до $\mathbb{F}$. Ця програма частково визначена Л. Ноуі і Н. Мідуне. Класифікація трилінійних змінних форм на векторному просторі розмірності 8 над скінченним полем $\mathbb{F}_q$ характеристик, відмінних від 2 і 3, була зроблена у роботах Л. Ноуі і Н. Мідуне. Ми описали $\mathbb{F}_q$-форми 2-ступінчастих тривекторів розщеплення рангу 8, де char $\mathbb{F}_q \neq 3$. Цей факт ми використовуємо для визначення ваги $\mathbb{F}_q$-форм.

Ключові слова і фрази: тривектор, грасманіан, вага.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 431–441
doi:10.15330/cmp.11.2.431-441

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.431–441

RAMSKYI A., SAMARUK N., POPLAVSKA O.

THE DERIVATIVE CONNECTING PROBLEMS FOR SOME CLASSICAL POLYNOMIALS

Given two polynomial sets $\{P_n(x)\}_{n\geq 0}$, and $\{Q_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = n, \deg(Q_n(x)) = n.$$

The so-called the connecting problem between them asks to find the coefficients $\alpha_{n,k}$ in the expression $Q_n(x) = \sum\limits_{k=0}^{n} \alpha_{n,k} P_k(x)$. Let $\{S_n(x)\}_{n\geq 0}$ be another polynomial set with $\deg(S_n(x)) = n$. The general connection problem between them consists in finding the coefficients $\alpha_{i,j}^{(n)}$ in the expansion

$$Q_n(x) = \sum_{i,j=0}^{n} \alpha_{i,j}^{(n)} P_i(x) S_j(x).$$

The connection problem for different types of polynomials has a long history, and it is still of interest. The connection coefficients play an important role in many problems in pure and applied mathematics, especially in combinatorics, mathematical physics and quantum chemical applications. For the particular case $Q_n(x) = P'_{n+1}(x)$ the connection problem is called the derivative connecting problem and the general derivative connecting problem associated to $\{P_n(x)\}_{n\geq 0}$.

In this paper, we give a closed-form expression of the derivative connecting problems for well-known systems of polynomials.

Key words and phrases: connection problem, inversion problem, derivative connecting problem, connecting coefficients, orthogonal polynomials.

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INTRODUCTION

Given the two polynomial sets $\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = \deg(Q_n(x)) = n,$$

for all n. The connection problem between them consists in finding the coefficients $\alpha_{n,k}$ in the expansion

$$Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x).$$

УДК 519.1
2010 *Mathematics Subject Classification:* 33C05, 05A19.

Let $\{S_n(x)\}_{n\geq 0}$ be another polynomial sets with $\deg(S_n(x)) = n$. The general connection problem between them consists in finding the coefficients $\alpha_{i,j}^{(n)}$ in the expansion

$$Q_n(x) = \sum_{i,j=0}^{n} \alpha_{i,j}^{(n)} P_i(x) S_j(x).$$

For the particular case $Q_n(x) = P'_{n+1}(x)$ the connection problem is called the derivative connecting problem and the general derivative connecting problem for the polynomial family $\{P_n(x)\}_{n\geq 0}$.

The study of such a problem has attracted a lot of interest in the last few years. For instance, the representations of parametric derivatives have been obtained by Froehlich [6] for Jacobi polynomials, by Koepf [7] for generalized Laguerre polynomials and Gegenbauer polynomials, by Koepf and Schmersau [8] for all the continuous and discrete classical orthogonal polynomials, in [5,9,11,13] for classic orthogonal polynomials.

The derivative connecting problem is considered for Chebyshev polynomials of the first and the second types [10], for some Koornwinder polynomials in [1]. In [2,3] the derivation connection problem was solved for the Fibonacci, Lucas and Kravchuk polynomials and the authors use the solutions to produce new combinatorial identities for these polynomials. Also, the derivative connecting problem is solved in [4] for some hypergeometrical polynomials.

As an example let us consider the sequence of Appel polynomials $\{A_n(x)\}_{n\geq 0}$ with exponential generating function

$$\mathcal{G}(A_n(x), z) = \mathcal{A}(z)e^{xz} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!},$$

where $\mathcal{A}(z)$ is an arbitrary formal power series, $\mathcal{A}(0) \neq 0$.

Then

$$\frac{d}{dx}\mathcal{G}(A_n(x), z) = \mathcal{A}(z)e^{xz}\, z = \mathcal{G}(A_n(x), z)z = \sum_{n=0}^{\infty} A_n(x)\frac{z^{n+1}}{n!}.$$

On the other side

$$\frac{d}{dx}\mathcal{G}(A_n(x), z) = \frac{d}{dx}\sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} A'_n(x)\frac{z^n}{n!}.$$

Equating the coefficients near z^n we will find

$$\frac{1}{n!}A_n(x)' = \frac{1}{(n-1)!}A_{n-1}(x),$$

and will obtain the solution of derivative connecting problem for Appel polynomials:

$$A_n(x)' = n\, A_{n-1}(x).$$

In the paper we solve these derivative connecting problems for many well-known classes of polynomials $P_n(x)$.

In Section 2, a general appearance of the decomposition of the derivative of the polynomial $P'_n(x)$ is established, depending on the appearance of the logarithmic derivative of the generating function. In Section 3, the derivative connecting problem is solved for Lagguerre, Kravchuk, Charlier, Stirling, Bell, Bernoulii, Euler and Hermite polynomials. In Section 4, the general derivative connecting problem is solved for Chebyshev, Gegenbauer and Legendre polynomials.

1 THE MAIN THEOREM

We propose a method for solving the derivative connecting problem based on the use of the generating functions of polynomial families. The generation function of the family of polynomials $\{P_n(x)\}_{n\geq 0}$ is the formal functional series

$$\mathcal{G}(P_n(x), z) = \sum_{n=0}^{\infty} c_n P_n(x) z^n,$$

where c_n is a certain numerical sequence. For case of $c_n = 1$ the generating function is called as ordinary generating function, and when $c_n = \frac{1}{n!}$ we obtain an exponential generating function.

Theorem 1. *Let the logarithmic derivative of the ordinary generating function $\mathcal{G}(P_n(x), z)$ of the polynomials family $\{P_n(x)\}$ can be represented by the following series with rational coefficients*

$$\frac{d}{dx} \ln \mathcal{G}(P_n(x), z) = \sum_{i=1}^{\infty} a_i z^i.$$

Then

$$P_n(x)' = \sum_{i=1}^{n} a_i P_{n-i}(x).$$

Let the logarithmic derivative of the exponential generating function $\mathcal{G}(x, z)$ of the polynomials family $\{P_n(x)\}$ is written as formal series with rational coefficients

$$\frac{d\mathcal{G}(x)}{dx} = \sum_{i=1}^{\infty} a_i \frac{z^i}{n!}.$$

Then

$$P_n(x)' = \sum_{i=1}^{n} a_i \frac{n!}{(n-i)!} P_{n-i}(x).$$

Proof. Assume that the generating function $\mathcal{G}(P_n(x), z)$ and its particular derivative $\mathcal{G}(P_n(x), z)'_x$ are connected

$$\mathcal{G}(P_n(x), z)'_x = \mathcal{G}(P_n(x), z) R(z),$$

where $R(z) = a_1 z + a_2 z^2 + \cdots$ is a formal power series. Then

$$\begin{aligned}\mathcal{G}(P_n(x), z)'_x = \sum_{n=0}^{\infty} c_n P'_n(x) z^n &= \left(\sum_{n=0}^{\infty} c_n P_n(x) z^n\right)\left(a_1 z + a_2 z^2 + \cdots\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=1}^{n} a_i c_{n-i} P_{n-i}(x)\right) z^n.\end{aligned}$$

Equating the coefficients at the same powers of z, we obtain that

$$c_n P'_n(x) = \sum_{i=1}^{n} a_i c_{n-i} P_{n-i}(x) = a_1 c_{n-1} P_{n-1} + a_2 c_{n-2} P_{n-2} + \cdots + a_n c_0 P_0(x),$$

which will be a solution of the derivative connection problem for the polynomial family $P_n(x)$.

For the case of the ordinary generating function, we have $c_n = 1$ and so

$$P_n'(x) = \sum_{i=1}^{n} a_i P_{n-i}(x).$$

Similarly, for the exponential generating function for $c_n = \frac{1}{n!}$ we obtain that

$$P_n(x)' = \sum_{i=1}^{n} a_i \frac{n!}{(n-i)!} P_{n-i}(x).$$

□

Proved theorem sets strict requirements for the generating function $\mathcal{G}(P_n(x), z)$-its logarithmic derivative must be a function of the one variable, although the generating function depends upon of two variables.

Suppose that the logarithmic derivative of the generating function is not a function of the variable and it has the following expansion

$$\frac{d}{dx} \ln \mathcal{G}(P_n(x), z) = \sum_{i=1}^{\infty} a_i(x) z^i,$$

where $a_i(x)$ – some polynomial. In this case for polynomials $S_n(x)$ their degree is equal n so they form the basis of the vector space of all polynomials from the variable x. Therefore the polynomials $a_i(x)$ can be expanded on this basis:

$$a_i(x) = \sum_{j=0}^{i} \alpha_{i,j} S_j(x).$$

The following Theorem 1 may be proved similarly

Theorem 2. *Let the logarithmic derivative of the generating function $\mathcal{G}(P_n(x), z)$ of the polynomials family $\{P_n(x)\}$ can be written as formal series*

$$\frac{d}{dx} \ln \mathcal{G}(P_n(x), z) = \sum_{i=1}^{\infty} a_i(x) z^i,$$

and

$$a_i(x) = \sum_{j=0}^{i} \alpha_{i,j} S_j(x),$$

for some coefficients $\alpha_{i,j}$. Then

$$P_n(x)' = \sum_{i=1}^{n} \sum_{j=0}^{i} \alpha_{i,j} Q_j(x) P_{n-i}(x).$$

2 THE DERIVATIVE CONNECTING PROBLEM

Let apply the proved theorems for solving of the derivative connecting problems for some types of the classical polynomials.

2.1 The Laguerre polynomials $L_n^\lambda(x)$

The Laguerre polynomials are defined by the following formula

$$L_n^{(\lambda)}(x) = \sum_{i=0}^{n} (-1)^i \binom{n+\lambda}{n-i} \frac{x^i}{i!}$$

with ordinary generating function

$$\mathcal{G}(L_n^\lambda(x), z) = (1-z)^{-\lambda-1} \mathrm{e}^{-\dfrac{xz}{1-z}}$$

We find the derivatives by parameters

$$\frac{d}{dx}\mathcal{G}(L_n^\lambda(x), z) = -(1-z)^{-2-\lambda} z\mathrm{e}^{-\dfrac{xz}{1-z}} = \frac{z\,\mathcal{G}(L_n^\lambda(x), z)}{z-1},$$

and

$$\frac{d}{d\lambda}\mathcal{G}(L_n^\lambda(x), z) = -(1-z)^{-\lambda-1} \mathrm{e}^{-\dfrac{xz}{1-z}} \ln(1-z).$$

Therefore, the logarithmic derivative has following form

$$\frac{d}{dx}\ln\mathcal{G}(L_n^\lambda(x), z) = \frac{z}{z-1} = -\sum_{i=1}^{\infty} z^i,$$
$$\frac{d}{d\lambda}\ln\mathcal{G}(L_n^\lambda(x), z) = -\ln(1-z) = \sum_{i=1}^{\infty} \frac{1}{i} z^i.$$

so we proved the theorem:

Theorem 3.

$$\frac{d}{dx} L_n^\lambda(x) = -\sum_{i=0}^{n-1} L_i^\lambda(x),$$
$$\frac{d}{d\lambda} L_n^\lambda(x) = \sum_{i=1}^{n} \frac{1}{i} L_{n-i}^\lambda(x).$$

This coincides with results [8] and [13] obtained by other methods.

2.2 The Kravchuk polynomials

The Kravchuk polynomials are defined such formula

$$K_n^{(p)}(x, N) = \sum_{j=0}^{n} (-1)^j (p-1)^{k-j} \binom{x}{j} \binom{N-x}{n-j},$$

and have following generating function

$$\mathcal{G}(K_n^{(p)}(x, N), z) = (1 + (p-1)z)^{N-x} (1-z)^x.$$

Theorem 4.

$$\frac{d}{dx} K_n^{(p)}(x,N) = \sum_{i=1}^{n} \frac{(-1)^i(p-1)^i - 1}{i} K_{n-i}^{(p)}(x,N),$$
$$\frac{d}{dN} K_n^{(p)}(x,N) = \sum_{i=1}^{n} \frac{(-1)^{i+1}(p-1)^i}{i} K_{n-i}^{(p)}(x,N),$$
$$\frac{d}{dp} K_n^{(p)}(x,N) = (N-x)\sum_{i=1}^{n} (-1)^{i-1}(p-1)^{i-1} K_{n-i}^{(p)}(x,N).$$

Proof. We find derivatives of the generating function for Kravchuk polynomials with respect to parameters x, N, p :

$$\frac{d}{dx}\mathcal{G}(K_n^{(p)}(x,N),z) = (1+(p-1)z)^{N-x}(1-z)^x(\ln(1-z) - \ln(1+(p-1)z))$$
$$= \mathcal{G}(K_n^{(p)}(x,N),z)\ln\left(\frac{1-z}{1+(p-1)z}\right),$$
$$\frac{d}{dN}\mathcal{G}(K_n^{(p)}(x,N),z) = (1+(p-1)z)^{N-x}\ln(1+(p-1)z)(1-z)^x,$$
$$\frac{d}{dp}\mathcal{G}(K_n^{(p)}(x,N),z) = \frac{(1+(p-1)z)^{N-x}(N-x)z(1-z)^x}{1+(p-1)z}.$$

So

$$\frac{d}{dx}\mathcal{G}(K_n^{(p)}(x,N),z) = \mathcal{G}(K_n^{(p)}(x,N),z)\ln\left(\frac{1-z}{1+(p-1)z}\right),$$
$$\frac{d}{dN}\mathcal{G}(K_n^{(p)}(x,N),z) = \mathcal{G}(K_n^{(p)}(x,N),z)\ln(1+(p-1)z),$$
$$\frac{d}{dp}\mathcal{G}(K_n^{(p)}(x,N),z) = \mathcal{G}(K_n^{(p)}(x,N),z)\frac{(N-x)z}{1+(p-1)z}.$$

From here we find expansion of a logarithmic derivative in a formal series

$$\frac{d}{dx}\ln\mathcal{G}(K_n^{(p)}(x,N),z) = \ln\left(\frac{1-z}{1+(p-1)z}\right) = \sum_{i=1}^{\infty}\frac{(-1)^i(p-1)^i - 1}{i}z^i,$$
$$\frac{d}{dN}\ln\mathcal{G}(K_n^{(p)}(x,N),z) = \ln(1+(p-1)z) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}(p-1)^i}{i}z^i,$$
$$\frac{d}{dp}\ln\mathcal{G}(K_n^{(p)}(x,N),z) = \frac{(N-x)z}{1+(p-1)z} = (N-x)\sum_{i=1}^{\infty}(-1)^{i-1}(p-1)^{i-1}z^i.$$

Applying the Theorem 1 we get the required result. □

For a particular case $p = 2$ the problem is solved in [3].

2.3 The Charlier polynomials $c_n^{(a)}(x)$

The Charlier polynomials $c_n^{(a)}(x)$have such an exponential generating function

$$\mathcal{G}(c_n^{(a)}(x),z) = e^z\left(1-\frac{z}{a}\right)^x.$$

From here it's easy to get that

$$\frac{d}{dx}\ln\mathcal{G}(c_n^{(a)}(x),z) = \ln\left(1-\frac{z}{a}\right) = -\sum_{i=1}^{\infty}\frac{z^i}{i\,a^i},$$
$$\frac{d}{da}\ln\mathcal{G}(c_n^{(a)}(x),z) = \frac{xz}{a^2}\left(1-\frac{z}{a}\right)^{-1} = \sum_{i=1}^{\infty}\frac{x}{a^{i+1}}z^i.$$

Therefore, the following theorem holds.

Theorem 5.

$$\frac{d}{dx}c_n^{(a)}(x) = -\sum_{i=1}^{n}\frac{n!}{(n-i)!\,i\,a^i}c_{n-i}^{(a)}(x),$$
$$\frac{d}{da}c_n^{(a)}(x) = a(c_1^{(a)}(x)-1)\sum_{i=1}^{n}\frac{n!}{(n-i)!\,a^{i+1}}c_{n-i}^{(a)}(x).$$

2.4 The Stirling and Bell polynomials.

The Stirling and Bell polynomials $S_n(x)$ (see [12]) are defined by the exponential generating function

$$\left(\frac{z}{1-\mathrm{e}^{-z}}\right)^{x+1} = \sum_{i=0}^{\infty}S_i(x)\frac{z^i}{i!}.$$

We have that the logarithmic derivatives is equal to

$$\frac{d}{dx}\ln\left(\frac{z}{1-\mathrm{e}^{-z}}\right)^{x+1} = \ln\left(\frac{z}{1-\mathrm{e}^{-z}}\right).$$

Let's expand to series the function

$$h(z) = \ln\left(\frac{z}{1-\mathrm{e}^{-z}}\right),$$

preliminary differentiating it.

We have

$$\frac{d}{dz}(\ln(h(z))) = \frac{h'(z)}{h(z)} = \frac{1}{z}-\frac{e^{-z}}{1-e^{-z}} = \frac{e^z-1-z}{z^2}\cdot\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}\cdot\sum_{n=0}^{\infty}\frac{B_n z^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_k}{(n-i+2)!i!}\right)z^n = \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}\sum_{k=0}^{n}\binom{n+2}{k}B_k = \frac{1}{2}-\sum_{n=1}^{\infty}\frac{B_{n+1}}{(n+1)!}z^n.$$

Here we used the known identity

$$\sum_{i=0}^{n-1}\binom{n}{i}B_i = 0,$$

and the fact that the generating function for the Bernoulli numbers B_i is equal to

$$\frac{z}{\mathrm{e}^z-1} = \sum_{i=1}^{n}B_i\frac{z^i}{i!}.$$

Note that the function $h(z)$ has a removable gap point at $z=0$ and

$$h(0) = \lim_{z\to 0}h(z) = \lim_{z\to 0}\frac{1}{(1-\mathrm{e}^{-z})'} = 1.$$

Therefore, by integrating, taking into account that $h(0) = 1$, we get

$$h(z) = \int \left(\frac{1}{2} - \sum_{n=1}^{\infty} \frac{B_{n+1}}{(n+1)!} z^n \right) dz = \frac{z}{2} - \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)!} z^{n+1}.$$

Consequently, we have proved the theorem.

Theorem 6.

$$\frac{d}{dx} S_n(x) = \sum_{i=1}^{n} \binom{n}{i} \frac{B_i}{i} S_{n-i}(x).$$

The Bell polynomials $\varphi_n(x)$ are determined through the Stirling numbers of second type

$$\varphi_n(x) = \sum_{i=0}^{n} S(n,i) x^n$$

and have the generating function

$$e^{x(e^z - 1)}.$$

In the same way as in the case of Stirling polynomials the following statement is proved.

Theorem 7.

$$\frac{d}{dx} \varphi_n(x) = \sum_{i=1}^{n} \binom{n}{i} \varphi_{n-i}(x).$$

2.5 Generalized Bernoulli, Euler and Hermite polynomials

Generalized Bernoulli $B_n^{(a)}(x)$, Euler $E_n^{(a)}(x)$ and Hermite $H_n^{(a)}(x)$ polynomials are defined by the following exponential generating function

$$e^{xz} \left(\frac{z}{e^z - 1} \right)^a = \sum_{i=0}^{\infty} B_n^{(a)}(x) \frac{z^n}{n!},$$

$$e^{xz} \left(\frac{2}{e^z + 1} \right)^a = \sum_{i=0}^{\infty} E_n^{(a)}(x) \frac{z^n}{n!},$$

$$e^{xz} e^{-at^2} = \sum_{i=0}^{\infty} H_n^{(a)}(x) \frac{z^n}{n!}.$$

With respect to the variable x these polynomials are the Appel polynomials, see [12], therefore for all three types of polynomials the following is performed

$$\frac{d}{dx} B_n^{(a)}(x) = n B_{n-1}^{(a)}(x),$$

$$\frac{d}{dx} E_n^{(a)}(x) = n E_{n-1}^{(a)}(x),$$

$$\frac{d}{dx} H_n^{(a)}(x) = n H_{n-1}^{(a)}(x).$$

Let's find the logarithmic derivatives by parameter a:

$$\frac{d}{da} \ln B_n^{(a)}(x) = \ln \left(\frac{z}{e^z - 1} \right) = -\frac{z}{2} + (-1)^{n+1} \sum_{i=2}^{\infty} \frac{B_i}{i \cdot i!} z^i,$$

$$\frac{d}{da} \ln E_n^{(a)}(x) = \ln \left(\frac{2}{e^z + 1} \right) = \frac{1}{2} \sum_{i=1}^{n} \frac{E_{i-1}(1)}{i!} z^i,$$

$$\frac{d}{da} \ln H_n^{(a)}(x) = -z^2,$$

here B_i, $E_i(1)$ – are the Bernoulli numbers and Euler numbers respectively. Expansion

$$\ln\left(\frac{z}{e^z-1}\right) = -\frac{z}{2} + \sum_{i=2}^{\infty} \frac{(-1)^{n+1}B_i}{i \cdot i!} z^i$$

is obtained in the same way as expansion in subsection 2.4.

So, the following statement takes place.

Theorem 8.

$$\frac{d}{da} B_n^{(a)}(x) = -\frac{n}{2} B_{n-1}^{(a)}(x) + \sum_{i=2}^{n} \frac{(-1)^{n+1}B_i}{i} \binom{n}{i} B_{n-i}^{(a)}(x),$$
$$\frac{d}{da} E_n^{(a)}(x) = \frac{1}{2}\sum_{i=1}^{n} \binom{n}{i} E_{i-1}(1) E_{n-i}^{(a)}(x),$$
$$\frac{d}{da} H_n^{(a)}(x) = -n(n-1) H_{n-2}^{(a)}(x).$$

3 A GENERALIZED DERIVATIVE CONNECTING PROBLEM

3.1 The Chebyshev polynomials

The Chebyshev polynomials $T_n(x)$ of the first kind and the Chebyshev polynomials $U_n(x)$ of the second kind are determined by such ordinary generating function

$$\mathcal{G}(T_n(x), z) = \frac{1-xz}{1-2xz+z^2}, \quad \mathcal{G}(U_n(x), z) = \frac{1}{1-2xz+z^2}.$$

The following theorem take place.

Theorem 9.

$$\frac{d}{dx} T_n(x) = T_0(x)T_{n-1}(x) + 3T_1(x)T_{n-2}(x)$$
$$+ \sum_{i=3}^{\infty} \left(T_0(x)T_1(x)^{i-1} + 2\sum_{k=1}^{i-1} T_k(x)T_1(x)^{i-1-k} \right) T_{n-i}(x),$$
$$\frac{d}{dx} U_n(x) = 2\sum_{i=1}^{n} U_{i-1}(x) U_{n-1-i}(x).$$

Proof. We have

$$\frac{d}{dx} \ln \mathcal{G}(T_n(x), z) = \frac{z\left(z^2-1\right)}{(1-xz)\left(1-2\,xz+z^2\right)}$$
$$= T_0(x)z + 3T_1(x)z^2 + \sum_{i=3}^{\infty} \left(T_0(x)T_1(x)^{i-1} + 2\sum_{k=1}^{i-1} T_k(x)T_1(x)^{i-1-k} \right) z^i.$$

□

For the Chebyshev polynomials $U_n(x)$ of the second kind we have

$$\frac{d}{dx} \ln \mathcal{G}(U_n(x), z) = \frac{2z}{1-2xz+z^2} = 2\sum_{i=1}^{\infty} U_{i-1}(x) z^i.$$

Therefore

$$U_n(x)' = 2\sum_{i=1}^{n} U_{i-1}(x) U_{n-1-i}(x).$$

3.2 The Gegenbauer and Legendre polynomials

The Gegenbauer polynomials $C_n^\lambda(x)$ are determined by ordinary generating function

$$\mathcal{G}(C_n^\lambda(x), z) = \frac{1}{(1-2xz+z^2)^\lambda}.$$

It is logarithmic derivative is expressed through Chebyshev polynomials

$$\frac{d}{dx}\ln \mathcal{G}(C_n^\lambda(x), z) = \frac{2\lambda z}{1-2xz+z^2} = 2\lambda \sum_{i=1}^{\infty} U_{i-1}(x) z^i,$$
$$\frac{d}{d\lambda}\ln \mathcal{G}(C_n^\lambda(x), z) = -\ln\left(1-2\,xz+z^2\right) = 2\sum_{i=1}^{\infty}\frac{1}{i}T_i(x)z^i.$$

Theorem 10.

$$\frac{d}{dx}C_n^\lambda(x) = 2\lambda \sum_{i=1}^{\infty} U_{i-1}(x) C_{n-i}^\lambda(x),$$
$$\frac{d}{d\lambda}C_n^\lambda(x) = 2\sum_{i=1}^{n}\frac{1}{i}T_i(x)C_{n-i}^{(\lambda)}(x).$$

In [13] another expressions for the Gegenbauer polynomials were obtained. The Legendre polynomials $P_n(x)$ are determined by generating function

$$\mathcal{G}(P_n(x), z) = \frac{1}{\sqrt{1-2xz+z^2}}.$$

We have

$$\frac{d}{dx}\ln \mathcal{G}(P_n(x), z) = \frac{z}{1-2\,xz+z^2} = \sum_{i=1}^{\infty} U_{i-1}(x) z^i.$$

Therefore there is the following assertion.

Theorem 11.

$$\frac{d}{dx}P_n(x) = \sum_{i=1}^{\infty} U_{i-1}(x) P_{n-i}(x).$$

REFERENCES

[1] Aktas R. *Representations for parameter derivatives of some Koornwinder polynomials in two variables*. J. Egyptian Math. Soc. 2016, **24**, 555–561

[2] Bedratyuk L. *Derivations and Identitites for Fibonacci and Lucas Polynomials*. Fibonacci Quart. 2013, **51** (4), 351–366.

[3] Bedratyuk L. *Derivations and Identities for Kravchuk Polynomials*. Ukrainian Math. J. 2014, **65** (12), 1755–1773. doi:10.1007/s11253-014-0896-9

[4] Bedratyuk L., Bedratuyk A. *The inverse and derivative connecting problems for some Hypergeometric polynomials*. Carpathian Math. Publ. 2018, **10** (2), 235–247. doi:10.15330/cmp.10.2.235-247

[5] Bonan S., Lubinsky D., Nevai P. *Orthogonal polynomials and their derivatives II*. SLAM J. Math. Anal. 1987, **18**, 1163–1176.

[6] Fröhlich J. *Parameter derivatives of the Jacobi polynomials and the Gaussian hypergeometric function*. Integral Transforms Spec. Funct. 1994, **2**, 252–266. doi:10.1080/10652469408819056

[7] Koepf W. *Identities for families of orthogonal polynomials and special functions*. Integral Transforms Spec. Funct. 1997, **5** (1-2), 69–102.

[8] Koepf W., Schmersau D. *Representations of orthogonal polynomials*. J. Comput. Appl. Math. 1998, **90**, 57–94. doi:10.1016/S0377-0427(98)00023-5

[9] Lewanowicz S. *Representations for the parameter derivatives of the classical orthogonal polynomials* Rend. Circ. Mat. Palermo, Ser. II, Suppl. 2002 **68**, 599–613.

[10] Prodinger H. *Representing derivatives of Chebyshev polynomials by Chebyshev polynomials and related questions*. Open Math. 2017, **15**, 1156–1160.

[11] Ronveaux A., Zarzo A., Area I., Godoy E. *Classical orthogonal polynomials: dependence on parameters* J.Comput. Appl. Math. 2000, **121**, 95–112

[12] Steven Roman. The umbral calculus. Academic Press, 1983.

[13] Szmytkowski R. *A note on parameter derivatives of classical orthogonal polynomials*. arXiv:0901.2639v3.

Received 28.02.2019

Рамський А.О., Самарук Н.М., Поплавська О.А. *Задачі диференціальної зв'язності для деяких класичних многочленів* // Карпатські матем. публ. — 2019. — Т.11, №2. — C. 431–441.

Нехай дано дві множини многочленів $\{P_n(x)\}_{n\geq 0}$ та $\{Q_n(x)\}_{n\geq 0}$ таких, що

$$\deg(P_n(x)) = n, \deg(Q_n(x)) = n.$$

Так звана задача диференціальної зв'язності між ними полягає у знаходженні коефіцієнтів $\alpha_{n,k}$ у виразі $Q_n(x) = \sum\limits_{k=0}^{n} \alpha_{n,k} P_k(x)$.

Нехай $\{S_n(x)\}_{n\geq 0}$ – це інша множина порядку $\deg(S_n(x)) = n$. Узагальнена задача зв'язності між ними полягає у знаходженні коефіцієнтів $\alpha_{i,j}^{(n)}$ у виразі

$$Q_n(x) = \sum_{i,j=0}^{n} \alpha_{i,j}^{(n)} P_i(x) S_j(x).$$

Задача зв'язності для різних типів многочленів має довгу історію, проте залишається цікавою і тепер. Коефіцієнти зв'язності грають важливу роль у багатьох задачах класичної та прикладної математики, особливо в комбінаториці, а також у математичній фізиці та прикладних застосуваннях квантової хімії. Для часткового випадку, коли $Q_n(x) = P'_{n+1}(x)$, задачу зв'язності називають диференціальною задачею зв'язності і відносять її до множини $\{P_n(x)\}_{n\geq 0}$.

У статті наведено вирази у замкнутій формі задач диференціальної зв'язності для відомих систем многочленів.

Ключові слова і фрази: задача зв'язності, обернена задача, задача диференціальної зв'язності, коефіцієнти звя'зності, гіпергеометричні функції, гіпергеометричні многочлени.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 442–452
doi:10.15330/cmp.11.2.442-452

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.442–452

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A NOTE ON COMPACT-LIKE SEMITOPOLOGICAL GROUPS

We present a few results related to separation axioms and automatic continuity of operations in compact-like semitopological groups. In particular, is provided a semiregular semitopological group G which is not T_3. We show that each weakly semiregular compact semitopological group is a topological group. On the other hand, constructed examples of quasiregular T_1 compact and T_2 sequentially compact quasitopological groups, which are not paratopological groups. Also we prove that a semitopological group (G, τ) is a topological group provided there exists a Hausdorff topology $\sigma \supset \tau$ on G such that (G, σ) is a precompact topological group and (G, τ) is weakly semiregular or (G, σ) is a feebly compact paratopological group and (G, τ) is T_3.

Key words and phrases: semitopological group, paratopological group, compact-like semitopological group, compact-like paratopological group, continuity of the inverse, joint continuity, separation axioms, countably compact paratopological group, feebly compact topological group, countably compact topological group.

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1 PRELIMINARIES

In this paper the word "space" means "topological space".

1.1 Topologized groups

A topologized group (G, τ) is a group G endowed with a topology τ. It is called a *semitopological group* provided the multiplication map $G \times G \to G$, $(x, y) \mapsto xy$ is separately continuous. Moreover, if the multiplication is continuous then G is called a *paratopological group*. A semitopological group with the continuous inversion map $G \to G$, $x \mapsto x^{-1}$ is called a *quasitopological group*. A topologized group which is both paratopological and quasitopological is called a *topological group*.

Whereas investigation of topological groups already is one of fundamental branches of topological algebra (see, for instance, [11, 29] and [5]), other topologized groups are not so well-investigated and have more variable structure.

Basic properties of semitopological or paratopological groups are described in book [5] by Arhangel'skii and Tkachenko, in author's PhD thesis [32] and papers [30,31]. New Tkachenko's survey [40] presents recent advances in this area.

УДК 512.546.82, 512.546.8, 512.546
2010 *Mathematics Subject Classification:* 22A15, 54H99, 54H11.
This work was supported by the budget program of Ukraine "Support for the development of priority research areas" (CPCEC 6451230).

1.2 Separation axioms

These axioms describe specific structural properties of a space. Basic separation axioms and relations between them are considered in [16, Section 1.5]. For more specific cases and topics, also related to semitopological and paratopological groups, see [7, 31], [40, Section 2], [22, 41].

All spaces considered in the present paper are *not* supposed to satisfy any of the separation axioms, if otherwise is not stated. We recall separation axioms which we use in our paper. A space X is

- T_0, if for any distinct points $x, y \in X$ there exists an open set $U \subset X$, which contains exactly one of the points x, y,
- T_1, if for any distinct points $x, y \in X$ there exists an open set $x \in U \subset X \setminus \{y\}$,
- T_2 or *Hausdorff*, if any distinct points $x, y \in X$ have disjoint neighborhoods,
- T_3, if any closed set $F \subset X$ and any point $x \in X \backslash F$ have disjoint neighborhoods,
- *regular*, if it is T_1 and T_3,
- *quasiregular*, if any nonempty open subset A of X contains the closure of some nonempty open subset B of X,
- *weakly semiregular*, if X has a base consisting of *regular open* sets, that is such sets U that $U = \operatorname{int} \overline{U}$,
- *semiregular*, if it is weakly semiregular and T_2,
- *functionally* T_2 of *functionally Hausdorff*, if for any distinct points $x, y \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$,
- $T_{3\frac{1}{2}}$ or *completely regular*, if it is T_1 and for any closed set $F \subset X$ and any point $x \in X \setminus F$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(F) \subset \{1\}$.

Remark that each T_3 space is quasiregular and weakly semiregular, so each regular space is semiregular.

1.3 Separation axioms in semitopological groups

It is easy to show that each topological group is T_3. Near 1936 Pontrjagin showed that each T_0 topological group is completely regular and T_1.

On the other hand, simple examples shows that for paratopological groups neither of the implications $T_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow T_3$ is necessary (see [30, Examples 1.6-1.8] and page 5 in any of papers [31] or [40]) and there are only a few backwards implications between different separation axioms, see [31, Section 1] or [40, Section 2]. Moreover, in 2014 Banakh and the author of the present paper similarly to Pontrjagin's proof showed that each T_1 weakly semiregular paratopological group is $T_{3\frac{1}{2}}$ and each T_2 paratopological group is functionally T_2 [7]. On the other hand, Banakh's announcement for a seminar for 28 November 2016 (see [39]) claims on an example of a regular quasitopological group which is not functionally Hausdorff.

It is easy to show that each weakly semiregular paratopological group is T_3 [31, Proposition 1.5], but there exists a semiregular semitopological group G which is not T_3, see Example 1. On the other hand, in Proposition 1 we shall prove that each T_0 weakly semiregular semitopological group is semiregular.

Given a topological space (X, τ) Stone [38] and Katětov [18] considered the topology τ_{sr} on X generated by the base consisting of all regular open sets of the space (X, τ). This topology is called the *semiregularization* of the topology τ. If (X, τ) is a semitopological group then (X, τ_{sr}) is a weakly semiregular semitopological group (see [31, p. 96]). If (X, τ) is a paratopological group then (X, τ_{sr}) is a T_3 paratopological group [31, Ex. 1.9], [32, p. 31], and [32, p. 28].

1.4 Compact-like spaces

Different classes of compact-like spaces and relations between them provide a well-known investigation topic of general topology, see, for instance, basic [16, Chap. 3] and general [13, 23, 25, 37, 42] works. The including relations between the classes are often visually represented by arrow diagrams, see, [25, Diag. 3 at p.17], [12, Diag. 1 at p. 58] (for completely regular spaces), [37, Diag. 3.6 at p. 611], and [17, Diag. at p. 3].

We recall the definitions of compact-like spaces with which we shall deal in the paper. A space X is called

- *sequentially compact*, if each sequence of X contains a convergent subsequence,
- *countably compact*, if each countable open over of X has a finite subcover,
- *feebly compact*, if each locally finite family of nonempty open subsets of the space X is finite,
- *pseudocompact*, if X is T_1 completely regular and each continuous real-valued function on X is bounded.

It is well-known and easy to show that each (sequentially) compact space is countable compact and each countable compact space is feebly compact. Moreover, by [16, Theorem 3.10.22] a T_1 completely regular space is feebly compact iff it is pseudocompact.

1.5 Automatic continuity of operations in semitopological groups

It turned out that if a space of a semitopological (resp. paratopological) group satisfies some conditions (sometimes with some conditions imposed on the group) then the multiplication (resp. inversion) in the group is continuous, that is the group is topological (resp. paratopological). Investigation of these conditions is one of main branches of the theory of paratopological groups, and, as far as the author knows, the firstly developed that. It turned out that automatic continuity essentially depends on compact-like properties and separation axioms of the space of a semitopological group. An interested reader can find known results and references on this subject in the survey Section 5.1 of [32] and in Section 3 of the survey [40] (both for semitopological and paratopological groups), and in Introduction of [1], [8, Section 1.6](for paratopological groups).

We briefly recall the history of the topic. In 1936 Montgomery [26] showed that every completely metrizable paratopological group is a topological group. In 1953 Wallace [43] asked whether every locally compact regular semitopological group a topological group. In 1957 Ellis obtained a positive answer of the Wallace question (see [14,15]) (remark that later the author of the present paper showed that regularity condition can be relaxed, see Proposition 5.5 in [32] or its counterpart in English in [33]). In 1960 Zelazko used Montgomery's result and showed that each completely metrizable semitopological group is a topological group. Since both locally compact and completely metrizable topological spaces are Čech-complete (recall that Čech-complete spaces are G_δ-subspaces of Hausdorff compact spaces), this suggested Pfister [28] in 1985 to ask whether each Čech-complete semitopological group a topological group. In 1996 Bouziad [9] and Reznichenko [36], as far as the author knows, independently answered affirmatively to the Pfister's question. To do this, it was sufficient to show that each Čech-complete semitopological group is a paratopological group since earlier, Brand [10] had proved that every Čech-complete paratopological group is a topological group. Brand's proof was later improved and simplified in [28]. For recent advances in this topic see Moors' paper [27] and references there.

If G is a paratopological group which is a T_1 space and $G \times G$ is countably compact (in particular, if G is sequentially compact) then G is a topological group, see [34]. On the other hand, we cannot weaken T_1 to T_0 here, because there exists a sequentially compact T_0 paratopological group which is not a topological group, see Example 5.27 from [8]. Also we cannot weaken countable compactness of $G \times G$ to that of G because under additional axiomatic assumptions there exists a countably compact (free abelian) paratopological group which is not a topological group, see [8, Example 3.22]. Also there exists a functionally Hausdorff second countable feebly compact paratopological group G which is not a topological group, see [8, Example 3.30]. On the other hand, by Proposition 3.15 from [8] each feebly compact quasiregular paratopological group is a topological group. In particular, each pseudocompact paratopological group is a topological group, see also [4, Theorem 1.7] and [2, Theorem 2.1].

According to [24, Corollary 6.3], a subgroup of a compact Hausdorff semitopological semigroup is a topological group. On the other hand, The group of integers $(\mathbb{Z}, +)$ endowed with the cofinite topology is a T_1 compact semitopological group which is not a paratopological group. On the other hand, it is easy to check that each T_1 regular countably compact space is strongly Baire (see, [19, p.158] for definition), so by [19, Theorem 2], each T_1 regular countably compact semitopological group G is a topological group. Nevertheless, there exists a pseudocompact quasitopological group G of period 2, which is not a paratopological group, (see [20, 21] and also [5, p.124-127]). On the other hand, Reznichenko in [35, Theorem 2.5] showed that each semitopological group $G \in \mathcal{N}$ is a topological group, where $\mathcal{N}$ is a family of all pseudocompact spaces X such that (X, X) is a *Grothendieck pair*, that is if each continuous image of X in $C_p(Y)$ has the compact closure in $C_p(Y)$. In particular, a pseudocompact space X belongs to $\mathcal{N}$ provided X has one of the following properties: countable compactness, countable tightness, separability, X is a k-space, see [35]. Also is known that every pseudocompact semitopological group of countable π-character is a compact metrizable topological group, see [5, Corollary 5.7.27]. Arhangel'skii, Choban, and Kenderov proved in [3, Proposition 8.5] that a T_2 locally countably compact semitopological group containing a compact of countable character is a paracompact locally compact topological group.

In the present paper we show that each weakly semiregular compact semitopological group G is a topological group, see Theorem 1. On the other hand, we construct examples of quasiregular T_1 compact and T_2 sequentially compact quasitopological groups, which are not paratopological groups, see Examples 2 and 3, respectively.

2 RESULTS

Example 1. *There exists a semiregular semitopological group G which is not T_3. Put $G = (\mathbb{R}^2, +)$ and $\mathcal{B} = \{U_n : 0 < n \in \mathbb{N}\}$, where $U_n = \{0\} \cup \{(x, y) \in \mathbb{R}^2 : |y| < |x| < 1/n\}$ for each n. Put $\tau = \{V \subset G : (\forall x \in V)(\exists U \in \mathcal{B}) : x + U \subset V\}$. It is easy to check that (G, τ) is a semitopological semigroup and $\mathcal{B}$ is its base at the unit. Let σ be the standard topology of $\mathbb{R}^2$. Since $\tau \supset \sigma$, the group (G, τ) is T_2. Since $\operatorname{int}_\varnothing \overline{U}^\sigma = U$ for each $U \in \mathcal{B}$, the group (G, τ) has a base $\{x + U : x \in G, U \in \mathcal{B}\}$, consisting of regular open sets. But the group (G, τ) is not T_3, because $U_1 \not\supset \overline{U_n}^\tau$ for each n.*

Let G be a semitopological group and $H \subset G$ be a normal subgroup of G. It is easy to check that the quotient group G/H endowed with the quotient topology with respect to the quotient map $\pi : G \to G/H$ is a semitopological group.

Lemma 1. (*see, [41, Theorem 3.1 and Corollary 3.2]*) *Let (G, τ) be a semitopological group, $N = \bigcap\{U : e \in U \in \tau\}$ and $K = N \cap N^{-1}$. Then K is a normal subgroup of the group G and $T_0 G = G/K$ is a T_0 semitopological group. Moreover, let $\pi : G \to G/K$ be the quotient homomorphism. Then $U = \pi^{-1}\pi(U)$ for each open set $U \subset G$ and hence the map π is clopen.*

Lemma 2. *A semitopological group G is a paratopological group iff $T_0 G$ is a paratopological group.*

Proof. The sufficiency is evident. The necessity follows from Lemma 1. □

Lemma 3. *Let (X, τ) be a weakly semiregular space, (Y, σ) be a space and $\pi : X \to Y$ be a continuous clopen surjection. Then Y is a weakly semiregular space.*

Proof. Let $y \in Y$ be any point and $V \in \sigma$ be any open neighborhood of y. Pick a point $x \in \pi^{-1}(y)$. Since $\pi^{-1}(V)$ is a neighborhood of x and X is a weakly semiregular space, there exists a regular open neighborhood U of the point x, contained in a set $\pi^{-1}(V)$. Then $y = \pi(x) \in \pi(U) \subset \overline{\pi(U)} \subset \pi(\overline{U}) \subset \pi\pi^{-1}(V) = V$ (the third inclusion here holds because the map π is closed). Therefore a canonical open set $V' = \operatorname{int} \overline{\pi(U)}$ is closed and $y \in V' \subset \overline{\pi(U)} \subset V$. □

Lemma 4. *Let (G, τ) be a weakly semiregular semitopological group. Put $N = \bigcap\{U : e \in U \in \tau\}$. Then N is a closed normal subgroup of the group G and*

$$N = \bigcap\{\overline{U} : e \in U \in \tau\} = \bigcap\{UU^{-1} : e \in U \in \tau\} = \bigcap\{U^{-1} : e \in U \in \tau\}.$$

Proof. Put $N' = \bigcap\{\overline{U} : e \in U \in \tau\}$ and $N'' = \bigcap\{UU^{-1} : e \in U \in \tau\}$. The set N' is a closed subset of the group G. Since for any $V \subset G$, $\overline{V} = \bigcap\{VU^{-1} : e \in U \in \tau\}$, we have $N' = N''$. Moreover, it is easy to see that $N^{-1} = \bigcap\{U^{-1} : e \in U \in \tau\}$, $N \subset N'$, $N \subset N''$ and $N^{-1} \subset N''$. Let $U \in \tau$ be any open neighborhood of the unit of the group G and x be any element of the set U. There exists an open neighborhood $V \in \tau$ the of unit of the group G such that $xV \subset U$. Then $xN' \subset x\overline{V} \subset \overline{U}$. Since this inclusion holds for an arbitrary element x of the set U, we see that $UN' \subset \overline{U}$. But UN' is an open subset of a group G and hence $N' \subset UN' \subset \operatorname{int} \overline{U}$. Then $N' \subset \bigcap\{\operatorname{int} \overline{U} : e \in U \in \tau\} = \bigcap\{U : e \in U \in \tau\} = N$ (the first equality holds because G is a weakly semiregular space). At last, since $N^{-1} \subset N'' = N' \subset N$, we have the inclusion $N \subset N^{-1}$.

Let x, y be arbitrary elements of N and $U \in \tau$ be an arbitrary open neighborhood of the unit of the group G. Then $x \in N \subset U$. There exists an open neighborhood $V \in \tau$ of the unit of the group G such that $xV \subset U$. Then $y \in N \subset V$. Hence $xy \in xV \subset U$. Since this holds for an arbitrary open neighborhood $U \in \tau$ of the unit of the group G, $xy \in \bigcap\{U : e \in U \in \tau\} = N$. So N is a subsemigroup of the group G. Since $N = N^{-1}$, N is a group.

Let g be an arbitrary element of the group G, and $U \in \tau$ be an arbitrary open neighborhood of the unit of the group G. There exists an open neighborhood $V \in \tau$ of the unit of the group G such that $g^{-1}Vg \subset U$. Then $g^{-1}Ng \subset g^{-1}Vg \subset U$. Since this holds for an arbitrary open neighborhood $U \in \tau$ of the unit of the group G, $g^{-1}Ng \subset \bigcap\{U : e \in U \in \tau\} = N$. So N is a normal subsemigroup of the group G. □

Proposition 1. *Each T_0 weakly semiregular semitopological group (G, τ) is semiregular.*

Proof. Put $N = \bigcap\{U : e \in U \in \tau\}$. Since G is a T_0 space, $N \cap N^{-1} = \{e\}$. But by Lemma 4, $N^{-1} = N = \bigcap\{UU^{-1} : e \in U \in \tau\} = N''$. Therefore $N'' = \{e\}$ and the group G is T_2. □

Lemma 1, Lemma 3 and Proposition 1 imply the following

Proposition 2. *If G is a weakly semiregular semitopological group then T_0G is a semiregular semitopological group.*

We remark that Proposition 2 cannot be generalized for arbitrary quotient groups even of regular paratopological groups, because in [6] Taras Banakh and the author constructed a countable regular abelian paratopological group G containing a closed discrete subgroup H such that the quotient G/H is T_2 but not T_3. The group G/H is even not weakly semiregular, because by [31, Proposition 1.5] each weakly semiregular paratopological group is T_3.

Lemma 5. *[35, Theorem 0.5] A T_2 compact semigroup with separately continouous multiplication and two-sides cancellations is a topological group.*

Lemma 6. (*see [32, Lemma 5.4], [41, Proposition 3.2], or [8, Proposition 3.2]*) *Each compact paratopological group is a topological group.*

Theorem 1. *Each weakly semiregular compact semitopological group G is a topological group.*

Proof. By Proposition 2, T_0G is a semiregular compact semitopological group. By Lemma 5, T_0G is a topological group. By Lemma 2, G is a paratopological group. By Lemma 6, G is a topological group. □

Let us illustrate the topic by the following simple

Proposition 3. *Let G be a group endowed with the cofinite topology, that is a set $U \subset G$ is open in G iff $U = \varnothing$ or a set $G \setminus U$ is finite. Then G is a T_1 semitopological group and the following conditions are equivalent.*

1. *The group G is a paratopological group.*

2.1. *The group G is T_2.*

2.2. *The group G is weakly semiregular.*

2.3. The group G is quasiregular.

3. The group G is finite.

Proof. The continuity of shifts on the group G and implications $3 \Rightarrow *$ are obvious, implications $2.* \Rightarrow 3$ follows from the fact that if the group G is infinite then each nonempty open subset of G is dense in G. It remains to show an implication $1 \Rightarrow 3$. Suppose to the contrary that G is an infinite paratopological group. Pick an element $x \in G \setminus \{e\}$. Since the multiplication at the unit of G is continuous, there exists a finite set $F \subset G \setminus \{e\}$ such that $(G \setminus F)^2 \subset G \setminus \{x\}$. Since the group G is infinite, there exists a point $y \in G \setminus (F \cup xF^{-1})$. Then $y(G \setminus F) \ni x$, a contradiction. □

Example 2. *There exists a T_1 quasiregular compact quasitopological group G, which is not a paratopological group. Let $G = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. We define an open base $\mathcal{B}$ at the unit of a topology of a semitopological group on G by putting $\mathcal{B} = \{U_n : 0 < n \in \mathbb{Z}\}$, where $U_n = \{z \in \mathbb{C} \setminus \{(-1,0)\} : \arg z \in (-1/n, 1/n) \cup (\pi - 1/n, \pi + 1/n)\}$.*□

Example 3. *There exists a T_2 quasiregular sequentially compact quasitopological group G, which is not a paratopological group. Let*

$$G = \Sigma_{\omega_1}\mathbb{Z}_2 = \{x \in \mathbb{Z}_2^{\omega_1} : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$

Put $\mathcal{B} = \{U_A \setminus S : A$ is a finite subset of $\omega_1\}$, where

$$U_A = \{x \in G : x_\alpha = x_\beta \text{ for each } \alpha, \beta \in A\}$$

and

$$S = \{x \in G : x_0 = 1 \text{ and } x_\gamma \geq x_\delta \text{ for each } \gamma < \delta < \omega_1\}.$$

We claim that the family $\mathcal{B}$ satisfies Pontrjagin conditions (see [30, Proposition 1]). Indeed, the one non-evident of these conditions for the family $\mathcal{B}$ is: for each $U \in \mathcal{B}$ and for each point $x \in U$ there exists $U' \in \mathcal{B}$ such that $x + U' \subset U$. Let's check this. Let $\mathcal{B} \ni U = U_A \setminus S$, where A is a finite subset of ω_1 and $x \in U$. If $x = 0$ then it suffices to put $U' = U$. If $x \neq 0$ then there exists an index $\gamma' \in \omega_1$ such that $x_{\gamma'} = 1$. Since $x \in \Sigma_{\omega_1}\mathbb{Z}_2$, there exists an index $\gamma' < \delta' < \omega_1$ such that $x_{\delta'} = 0$. Since $x \notin S$, there exist indexes $\gamma'', \delta'' \in \omega_1$, $\gamma'' < \delta''$ such that $x_{\gamma''} = 0$ and $x_{\delta''} = 1$. Put $A' = A \cup \{\gamma', \gamma'', \delta', \delta''\}$ and $U' = U_{A'}$. Then $x + U' \subset U$. Hence the family $\mathcal{B}$ is an open base at the unit of a topology of a semitopological group on G. Denote this topology as τ. Since $U_{A'} \supset \overline{U_{A'} \setminus S}$, the group (G, τ) is quasiregular. Since the set U_A is a group for any subset A of ω_1 and $\bigcap\{U_A : A$ is a finite subset of $\omega_1\} = \{0\}$, the group (G, τ) is T_2. Since the topology τ is weaker than the sequentially compact topology on the set $\Sigma_{\omega_1}\mathbb{Z}_2$, induced from the Tychonoff product, the group (G, τ) is sequentially compact too. At last, to show that (G, τ) is not a paratopological group, it suffices to show that for any finite set $A \subset \omega_1$ there exist points $x, y \in U_A \setminus S$ such that $x + y \in S$. Fix arbitrary two indexes $\alpha, \beta \in \omega_1$ such that $\sup A < \alpha < \beta$. For each $\gamma \in \omega_1$ put $x_\gamma = 1$ if $\gamma \in \{\alpha, \beta\}$ and $x_\gamma = 0$ otherwise. For each $\gamma \in \omega_1$ put $y_\gamma = 1$ if $\alpha \neq \gamma \leq \beta$ and $y_\gamma = 0$ otherwise.

Recall that a topological group G is *precompact* if for each neighborhood U of the unit of G there exists a finite subset F of G such that $FU = G$ (or, equivalently $UF = G$).

Theorem 2. *Let (G,σ) be a T_2 precompact topological group, (G,τ) be a weakly semiregular semitopological group and $\tau \subset \sigma$. Then (G,τ) is a topological group.*

Proof. Let $(\hat{G},\hat{\sigma})$ be a Raĭkov completion of the group (G,σ). Since the group G is a dense precompact subset of the group $(\hat{G},\hat{\sigma})$, by Corollary 3.7.6 from [5], the group $(\hat{G},\hat{\sigma})$ is precompact. Since the group $(\hat{G},\hat{\sigma})$ is Raĭkov complete, by Theorem 3.7.15 from [5] it is compact.

In this proof as $\bar{\cdot}$ we denote the closure with respect to the topology $\hat{\sigma}$.

Put $N = \bigcap\{\overline{U} : e \in U \in \tau\}$. We claim that N is a normal subgroup of the group $(\hat{G},\hat{\sigma})$. Indeed, let x, y be any elements of N, $U \in \tau$ be an any open neighborhood of the unit of the group G, and $\hat{W} = (\hat{W})^{-1} \in \hat{\sigma}$ be any symmetric open neighborhood of the unit of the group $\hat{G}$. Then there exists an element $u \in U \cap \hat{W}x$. There exists an open neighborhood $V \in \tau$ of the unit of the group G such that $uV \subset U$. Then there exists an element $v \in V \cap y\hat{W}$. Then $xy \in \hat{W}uv\hat{W} \subset \hat{W}U\hat{W}$. Since this holds for any symmetric open neighborhood $\hat{W} = (\hat{W})^{-1} \in \hat{\sigma}$ of the unit of the group $\hat{G}$, $xy \in \overline{U}$. Since this holds for any open neighborhood $U \in \tau$ of the unit of the group G, $xy \in \bigcap\{\overline{U} : e \in U \in \tau\} = N$. So N is a closed subsemigroup of a T_2 compact topological group $(\hat{G},\hat{\sigma})$. By Lemma 5, N is a group. Let g be any element of the group G and $U \in \tau$ be any open neighborhood of the unit of the group G. Since (G,τ) is a semitopological group and $g^{-1}eg = e$ there exists an open neighborhood $V \in \tau$ of the unit of the group G such that $g^{-1}Vg \subset U$. By continuity of multiplication on the group $(\hat{G},\hat{\sigma})$, $g^{-1}Ng \subset g^{-1}\overline{V}g \subset \overline{U}$. Since this holds for any open neighborhood $U \in \tau$ of the unit of the group G, $g^{-1}Ng \subset \bigcap\{\overline{U} : e \in U \in \tau\} = N$. Now suppose that there exists an element $\hat{g}$ of the group $\hat{G}$ such that $(\hat{g})^{-1}N\hat{g} \not\subset N$. Then there exists an element $x \in N$ such that $(\hat{g})^{-1}x\hat{g} \notin N$. Since N is a closed subset of the group $(\hat{G},\hat{\sigma})$ and the multiplication on the group $(\hat{G},\hat{\sigma})$ is continuous, there exists a symmetric open neighborhood $\hat{W} = (\hat{W})^{-1} \in \hat{\sigma}$ of the unit of the group $\hat{G}$ such that $\hat{W}(\hat{g})^{-1}x\hat{g}\hat{W} \cap N = \varnothing$. Since the group (G,σ) is dense in its completion $(\hat{G},\hat{\sigma})$, there exists an element $g \in G \cap \hat{g}\hat{W}$. But then $g^{-1}xg \in \hat{W}(\hat{g})^{-1}x\hat{g}\hat{W} \notin N$, a contradiction. Therefore $(\hat{g})^{-1}N\hat{g} \subset N$ for each element $(\hat{g}) \in \hat{G}$. Thus N is a normal subgroup of the group $\hat{G}$.

Define a topology $\hat{\sigma}_N$ on the group $\hat{G}$ by putting $\hat{\sigma}_N = \{\hat{W}N : \hat{W} \in \hat{\sigma}\}$. It is easy to check that $(\hat{G},\hat{\sigma}_N)$ is a topological group. We claim that $\hat{\sigma}_N|G = \tau$. Let's check this.

($\hat{\sigma}_N|G \subset \tau$) Let $\hat{W} \in \hat{\sigma}$ be any non-empty set and $x \in \hat{W}N \cap G$ be any point. Then $e \in x^{-1}\hat{W}N$, so $\bigcap\{\overline{U} : e \in U \in \tau\} = N \subset x^{-1}\hat{W}N$. Since $x^{-1}\hat{W}N$ is an open subset of the compact group $(\hat{G},\hat{\sigma}_N)$, there exists a set $e \in U \in \tau$ such that $\overline{U} \subset x^{-1}\hat{W}N$. Then xU is a neighborhood of the point x in the topology τ and $xU \subset \hat{W}N \cap G$.

($\hat{\sigma}_N|G \supset \tau$) Let $U \in \tau$ be any open neighborhood of the unit of the group G. We claim that $\overline{U}N \subset \overline{U}$. Indeed, let x be any element of the set U. There exists an open neighborhood of $V \in \tau$ the unit of the group G such that $xV \subset U$. Then $xN \subset x\overline{V} \subset \overline{U}$. Since this inclusion holds for any element x of the set U, we see that $UN \subset \overline{U}$. Let y be any element of the set N. Then $Uy \subset \overline{U}$ and $U \subset \overline{U}y^{-1}$. Since the set $\overline{U}y^{-1}$ is closed in the group $(\hat{G},\hat{\sigma}_N)$, we see that $\overline{U} \subset \overline{U}y^{-1}$. At last, since this inclusion holds for any element y of the set N, we see that $\overline{U}N \subset \overline{U}$. Since $\hat{\sigma}|G \supset \tau$, there exists an open neighborhood $\hat{W} \in \hat{\sigma}$ of the unit of the group G such that $\hat{W} \cap G \subset U$. Since the set G is dense in the space $(\hat{G},\hat{\sigma})$, $\hat{W} \subset \overline{\hat{W} \cap G} \subset \overline{U}$. Then $\hat{W}N \subset \overline{U}N \subset \overline{U}$. But $\hat{W}N \cap G \in \tau$, because $\hat{\sigma}_N|G \subset \tau$. Therefore $\hat{W}N \cap G \subset \operatorname{int}_\tau(\overline{U} \cap G) \subset \operatorname{int}_\tau \overline{U}^\tau$ (we have $\overline{U} \cap G \subset \overline{U}^\tau$, because $\hat{\sigma}|G \supset \tau$). At last, since $U \in \tau$ is any open neighborhood of the unit of the weakly semiregular group G, we have that $(\hat{\sigma}_N|G \supset \tau)$.

Thus, since $\hat{\sigma}_N|G = \tau$, (G, τ) is a topological group. □

Theorem 3. *Let (G, σ) be a T_2 feebly compact paratopological group, (G, τ) be a T_3 semitopological group and $\tau \subset \sigma$. Then (G, τ) is a topological group.*

Proof. The group G endowed with the topology σ_{sr} is a feebly compact T_2 and T_3 paratopological group. By [8, Proposition 3.15], (G, σ_{sr}) is a feebly compact topological group. Hence the group (G, σ_{sr}) is precompact. Let $U \in \tau$ be an arbitrary set and $x \in U$ be an arbitrary point. Since topology τ is T_3, there exists an open neighborhood $V \in \tau$ of the point x such that $\overline{V}^{\tau} \subset U$. Since $\tau \subset \sigma$, $V \in \sigma$. Then $x \in V = \operatorname{int}_\tau V \subset \operatorname{int}_\tau \overline{V}^{\sigma} \subset \operatorname{int}_\sigma \overline{V}^{\sigma} \subset \operatorname{int}_\sigma \overline{V}^{\tau} \subset \overline{V}^{\tau} \subset U$. Since $\operatorname{int}_\sigma \overline{V}^{\sigma} \in \sigma_{sr}$, $\tau \subset \sigma_{sr}$, and (G, σ_{sr}) is a weakly semiregular space, by Theorem 2, (G, τ) is a topological group. □

3 ACKNOWLEDGEMENTS

The author is grateful to Warren Moors for providing him the article [27].

REFERENCES

[1] Alas O.T., Sanchis M. *Countably Compact Paratopological Groups*. Semigroup Forum 2007, **74**, 423–438. doi:10.1007/s00233-006-0637-y

[2] Arhangel'skiĭ A.V., Choban M.M. *On paratopological groups and pseudocompactness*. Preprint.

[3] Arhangel'skiĭ A.V., Choban M.M., Kenderov P.S. *Topological games and topologies on groups*. Math. Maced. 2010, **8**, 1–19.

[4] Arhangel'skiĭ A.V., Reznichenko E.A. *Paratopological and semitopological groups versus topological groups*. Topology Appl. 2005, **151**, 107–119. doi:10.1016/j.topol.2003.08.035

[5] Arhangel'skiĭ A.V., Tkachenko M. Topological groups and related structures. Atlantis Press, Paris; World World Sci. Publ., Hackensack, NJ, 2008.

[6] Banakh T.O., Ravsky A.V. *The regularity of quotient paratopological groups*. Mat. Stud. 2018, **49** (2), 144–149. doi:10.15330/ms.49.2.144-149

[7] Banakh T.O., Ravsky A.V. *Each regular paratopological group is completely regular*. Proc. Amer. Math. Soc. 2017, **145** (3), 1373–1382. doi:10.1090/proc/13318

[8] Banakh T.O., Ravsky A.V. *Feebly compact paratopological groups* (version 7). -arXiv:1003.5343v7

[9] Bouziad A. *Every Čech-analytic Baire semitopological group is a topological group*. Proc. Amer. Math. Soc. 1996, **124** (3), 953–959.

[10] Brand N. *Another note on the continuity of the inverse*. Arch. Math. (Basel) 1982, **39**, 241–245. doi:10.1007/BF01899530

[11] Dikranjan D.N., Prodanov I.R., Stoyanov L.N. Topological Groups: Characters Dualities and Minimal Group Topologies, (2nd edn.), In: Kabayashi S., Hewitt E. (Eds.) Monographs and Textbooks in Pure and Applied Mathematics, **130**. Marcel Dekker, New York 1989.

[12] Dorantes-Aldama A., Shakhmatov D., *Selective sequential pseudocompactness*. Topology Appl. 2017, **222**, 53–69. doi:10.1016/j.topol.2017.02.016

[13] van Douwen E.K., Reed G.M., Roscoe A.W., Tree I.J. *Star covering properties*. Topology Appl. 1991, **39** (1), 71–103. doi:10.1016/0166-8641(91)90077-Y

[14] Ellis R. *Locally compact transformation groups*. Duke Math. J. 1957, **24**, 119–125. doi:10.1215/S0012-7094-57-02417-1

[15] Ellis R. *A note on the continuity of the inverse*. Proc. Amer. Math. Soc. 1957, **8**, 372–373. doi:10.1090/S0002-9939-1957-0083681-9

[16] Engelking R. General topology. Heldermann, Berlin, 1989.

[17] Gutik O.V., Ravsky A.V. *On old and new classes of feebly compact spaces*. Visnyk of the Lviv Univ. Ser. Mech. Math. 2018, **85**, 48–59. doi:10.30970/vmm.2018.85.048-059

[18] Katětov M. *On H-closed extensions of topological spaces*. Časopis Pěst. Mat. Fys. 1947, **72**, 17–32.

[19] Kenderov P.S., Kortezov I.S., Moors W.B. *Topological games and topological groups*. Topology Appl. 2001, **109**, 157–165. doi:10.1016/S0166-8641(99)00152-2

[20] Korovin A. Continuous actions of Abelian groups and topological properties in C_p-theory. Ph.D. Thesis, Moscow State University, Moscow, 1990. (in Russian)

[21] Korovin A. *Continuous actions of pseudocompact groups and the topological group axioms*. Deposited in VINITI 1990, #3734-D, Moscow. (in Russian)

[22] Li P., Tu J.-J., Xie L.-H. *Notes on (regular) T_3-reflections in the category of semitopological groups*. Topology Appl. 2014, **178**, 46–55. doi:10.1016/j.topol.2014.09.001

[23] Lipparini P. *A very general covering property*. -arXiv:1105.4342

[24] Lawson J.D. *Joint continuity in semitopological semigroups*. Illinois J. Math. 1974, **18** (2), 275–285.

[25] Matveev M. *A survey on star covering properties*. Topology Atlas. Preprint #330. http://at.yorku.ca/v/a/a/a/19.htm

[26] Montgomery D. *Continuity in topological groups*. Bull. Amer. Math. Soc. 1936, **42**, 879–882. doi:10.1090/S0002-9904-1936-06456-6

[27] Moors W.B. *Some Baire semitopological groups that are topological groups*. Topology Appl. 2017, **230**, 381–392. doi:10.1016/j.topol.2017.08.042

[28] Pfister H. *Continuity of the inverse*. Proc. Amer. Math. Soc 1985, **95**, 312–314. doi:10.1090/S0002-9939-1985-0801345-5

[29] Pontrjagin L.S. Continuous groups. Nauka, Moscow, 1973. (in Russian)

[30] Ravsky A.V. *Paratopological groups I*. Mat. Stud. 2001, **16** (1), 37–48.

[31] Ravsky A.V. *Paratopological groups II*. Mat. Stud. 2002, **17** (1), 93–101.

[32] Ravsky A.V. The topological and algebraical properties of paratopological groups. Ph.D. Thesis Lviv University, Lviv, 2002. (in Ukrainian)

[33] Ravsky A.V. *Post #209491 at MathOverflow*,

[34] Ravsky A.V., Reznichenko E.A. *The continuity of inverse in groups*. In: Zagorodnuyk A.V., Hryniv R.O. Book of Abstracts of International Conference on Functional Analysis and its Applications Dedicated to the 110th anniversary of Stefan Banach, Lviv, Ukraine, May 28-31, 2002, Publ. Cent. of Ivan Franko National University of Lviv, Lviv, 2002, 170–172.

[35] Reznichenko E.A. *Extension of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups*. Topology Appl. 1994, **59**, 33–44. doi:10.1016/0166-8641(94)90021-3

[36] Reznichenko E.A. *Čech complete semitopological group are topological groups*. (Preprint).

[37] Stephenson Jr. R.M. *Initially κ-compact and related compact spaces*. In K. Kunen, J. E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, 603–632. doi:10.1016/B978-0-444-86580-9.50016-1

[38] Stone M.H. *Applications of the theory of Boolean rings to general topology*. Trans. Amer. Math. Soc. 1937, **41**, 375–481. doi:10.1090/S0002-9947-1937-1501905-7

[39] Seminar "Topology & its Applications" at Chair of Geometry and Topology, Mechanics and Mathematics Faculty, Ivan Franko National University of Lviv. 28 November 2016. Not published.

[40] Tkachenko M. *Semitopological and paratopological groups vs topological groups*. In: Hart K.P., van Mill J., Simon P. (Eds.), Recent Progress in General Topology III, Springer Science & Business Media, 2013, 803–859.

[41] Tkachenko M. *Axioms of separation in semitopological groups and related functors*. Topology Appl. 2014, **161**, 364–376. doi:10.1016/j.topol.2013.10.037

[42] Vaughan J.E. *Countably compact and sequentially compact spaces*. In K. Kunen, J. E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., Amsterdam, 1984, 569–602. doi:10.1016/B978-0-444-86580-9.50015-X

[43] Wallace A.D. *The structure of topological semigroups*. Bull. Amer. Math. Soc. **61** (1955), 95–112.

Received 17.08.2019

Равський О.В. *Дещо про компактоподібні напівтопологічні групи* // Карпатські матем. публ. — 2019. — Т.11, №2. — C. 442–452.

Отримано деякі результати пов'язані з аксіомами відокремлення та автоматичною неперервністю у компактоподібних напівтопологічних групах. Зокрема, наведена напіврегулярна напівтопологічна група G, котра не є T_3. Показано, що кожна слабко напіврегулярна компактна напівтопологічна група є топологічною групою. З іншого боку, побудовані приклади квазірегулярних T_1 компактної та T_2 секвенціально компактної квазітопологічних груп, котрі не є паратопологічними групами. Також показано, що напівтопологічна група (G,τ) є топологічною групою за умови існування такої гаусдорфової топології $\sigma \supset \tau$ на G, що (G,σ) є прекомпактною топологічною групою і (G,τ) є слабко напіврегулярною або (G,σ) є слабко компактною паратопологічною групою і (G,τ) є T_3.

Ключові слова і фрази: напівтопологічна група, паратопологічна група, компактоподібна напівтопологічна група, компактоподібна паратопологічна група, неперервність оберненого, сукупна неперервність, аксіоми відокремлення, зліченно-компактна паратопологічна група, слабко компактна паратопологічна група, зліченно-компактна топологічна група.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 453–462
doi:10.15330/cmp.11.2.453-462

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.453–462

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ON ADDITIVITY OF DERIVATIONS

Let R be a ring and M be an R-bimodule. A mapping $d : R \to M$ (not necessarily additive) is called multiplicative derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In this paper, we intend to establish the additivity of d under some suitable restrictions. Moreover, we introduce multiplicative semi-derivations of rings and discuss their additivity.

Key words and phrases: derivation, multiplicative derivation, multiplicative semi-derivation, additivity, Peirce decomposition.

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INTRODUCTION

All through this paper, R denotes an associative ring (not necessarily with unity). A mapping $d : R \to R$ is called a derivation of R if for any $x, y \in R$

$$d(x+y) = d(x) + d(y) \tag{1}$$

$$\text{and } \ d(xy) = d(x)y + xd(y). \tag{2}$$

If d satisfies (2) but not necessarily (1), then d is called a multiplicative derivation of R (see [3]). In [2] Bergen extended the notion of a derivation by introducing *semi derivation* of a ring. Accordingly, a semi derivation (d, g) of a ring R is an additive mapping $d : R \to R$ associated with a ring endomorphism g of R such that $d(xy) = d(x)y + g(x)d(y) = d(x)g(y) + xd(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in R$. Clearly, every derivation is a semi derivation but the converse is not true always. We denote the Lie commutator $xy - yx$ by the symbol $[x, y]$. A non-zero element $e \in R$ is said to be *idempotent* if $e^2 = e$ and by a non-trivial idempotent we mean an idempotent element e different from the multiplicative identity of R. Let M be an $R-$bimodule and $e_1 \in R$ be a non-trivial idempotent element. For any $x \in M \cup R$ we shall write $x(1 - e_1)$ instead of $x - xe_1$, $(1 - e_1)x$ instead of $x - e_1 x$ and e_2 instead of $(1 - e_1)$. Then we set $R_{ij} = e_i R e_j$ and $M_{ij} = e_i M e_j$, where $i, j \in \{1, 2\}$. Therefore, R and M can be factorized as follows: $R = R_{11} \bigoplus R_{12} \bigoplus R_{21} \bigoplus R_{22}$ and $M = M_{11} \bigoplus M_{12} \bigoplus M_{21} \bigoplus M_{22}$. This representation of R and M is called *Peirce decomposition relative to* e_1 (see [[5], pg. 48]). Further, the following are some well-known facts related to this decomposition of R:

(i) $R_{ij} R_{jk} \subseteq R_{ik}$, where $i, j, k \in \{1, 2\}$.

(ii) $R_{ij} R_{kl} = 0$, where $j \neq k$, and $i, j, k, l \in \{1, 2\}$.

УДК 512.554.7
2010 *Mathematics Subject Classification:* 17C27, 13N15.

(iii) $x_{ij}^2 = 0$ for all $x_{ij} \in R_{ij}$, where $i \neq j$ and $i, j \in \{1, 2\}$.

The structure of rings is tightly connected with the additive mapping like isomorphisms, derivations, centralizers etc. Therefore, the problem of exploring the conditions under which these mappings become additive on rings (or algebras) has naturally grown as a fascinating area of research and has been attracted many algebraists for the last six decades. In this direction, Martindale [8] considered the so called problem "When a multiplicative mapping is additive?" He gave a remarkable technique and established a set of conditions on a ring that forces a multiplicative isomorphism to be additive. In particular, every multiplicative isomorphism from a prime ring containing a non-trivial idempotent onto any ring is additive. Inspired by this, Daif [3] obtained the additivity of multiplicative derivations of rings and consequently introduced the notion of multiplicative derivations. After that a number of results has been obtained in associative as well as alternative rings and algebras (see [4, 6, 7, 9–11]) and references therein). Recently, Wang [11] explored the additivity of $n-$multiplicative isomorphisms and $n-$multiplicative derivations of rings. As a consequence, one may deduce the theorem of Martindale and theorem of Daif from corollary 3.1 and 3.3 of [11] respectively. In this paper, we will continue the study of analogue problems for some derivable mappings on associative rings.

1 MAIN RESULTS

1.1 Additivity of multiplicative derivations

In view of *Peirce decomposition*, we see that any mapping $\delta : R \to M$ can be expressed as

$$\delta(x) = \delta_{11}(x) + \delta_{12}(x) + \delta_{21}(x) + \delta_{22}(x)$$

for all $x \in R$, where $\delta_{ij} : R \to M_{ij}$ be a mapping defined as $x \mapsto e_i x e_j$ for all $i, j \in \{1, 2\}$. For any $x, y \in R$, we have $x = x_{11} + x_{12} + x_{21} + x_{22}$ and $y = y_{11} + y_{12} + y_{21} + y_{22}$. Further,

$$xy = (x_{11}y_{11} + x_{12}y_{21}) + (x_{11}y_{12} + x_{12}y_{22}) + (x_{21}y_{11} + x_{22}y_{21}) + (x_{21}y_{12} + x_{22}y_{22}).$$

Now, we extend the notion of multiplicative derivation of a ring R as follows:

Definition 1. *Let R be a ring (not necessarily with unity) and M be a bimodule over R. A mapping $d : R \to M$ (not necessarily additive) is said to be a multiplicative derivation of R into M if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.*

Since $d(e_1) \in M_{11} \bigoplus M_{21} \bigoplus M_{12} \bigoplus M_{22}$ i.e., $d(e_1) = m_{11} + m_{12} + m_{21} + m_{22}$, where $m_{ij} \in M_{ij}$ for all $i, j \in \{1, 2\}$. Also $d(e_1) = d(e_1^2) = d(e_1)e_1 + e_1 d(e_1)$. By using the value of $d(e_1)$ we obtain that $m_{11} = 0 = m_{22}$ and hence $d(e_1) \in M_{12} \bigoplus M_{21}$. For some fixed $x \in M$ and $z \in R$, we define a function $f : R \to M$ by $a \mapsto [z, x]a + a[x, z]$. Clearly, f is a derivation. Fix $x = m_{12} + m_{21}$ and $z = e_1$. Re-defining f as $a \mapsto [e_1, m_{12} + m_{21}]a + a[m_{12} + m_{21}, e_1]$. Thus, we have

$$\begin{aligned} f(e_1) &= [e_1, m_{12} + m_{21}]e_1 + e_1[m_{12} + m_{21}, e_1] \\ &= (m_{12} - m_{21})e_1 + e_1(m_{12} - m_{21}) = -m_{12} - m_{21} = -d(e_1). \end{aligned}$$

Hence, $(f+d)(e_1)=0$. We set $f+d=D$. That means $D(e_1)=0$. Now, we have the following relations:

$$D_{11}(xy) = D_{11}(x)y_{11} + x_{11}D_{11}(y) + D_{12}(x)y_{21} + x_{12}D_{21}(y), \tag{3}$$

$$D_{12}(xy) = D_{11}(x)y_{12} + D_{12}(x)y_{22} + x_{11}D_{12}(y) + x_{12}D_{22}(y), \tag{4}$$

$$D_{21}(xy) = x_{21}D_{11}(y) + D_{21}(x)y_{11} + x_{22}D_{21}(y) + D_{22}(x)y_{21},$$

$$D_{22}(xy) = x_{21}D_{12}(y) + D_{21}(x)y_{12} + D_{22}(x)y_{22} + x_{22}D_{22}(y).$$

Further, it is easy to check that $D_{ij}(e_1)=0$ and $D_{ij}(xy)=D_{ij}(x)y+xD_{ij}(y)$ for all $i,j\in\{1,2\}$.

Lemma 1. *Let R be a ring (not necessary with unity) and M be a bimodule over R. Suppose that R contains a non-trivial idempotent e_1 such that for any $m\in M$, the following are satisfied:*

(H1) $e_1me_1R_{12}=(0)$ implies $e_1me_1=0$,

(H2) $e_1me_2R_{22}=(0)$ implies $e_1me_2=0$,

(H3) $e_1me_2R_{21}=(0)$ implies $e_1me_2=0$.

Then D_{11} and D_{12} are additive.

Proof. Firstly, we shall show that D_{11} is additive on $R_{11}\bigoplus R_{12}\bigoplus R_{22}$ and that D_{12} is additive on $R_{11}\bigoplus R_{12}\bigoplus R_{21}$. We begin with

$$\begin{aligned} D_{11}(x_{11}+x_{12}+x_{21}+x_{22}) &= e_1D(x_{11}+x_{12}+x_{21}+x_{22})e_1 = e_1D((x_{11}+x_{12}+x_{21}+x_{22})e_1)e_1 \\ &= e_1D(x_{11}+x_{21})e_1 = D_{11}(x_{11}+x_{21}). \end{aligned}$$

That is

$$D_{11}(x_{11}+x_{12}+x_{21}+x_{22}) = D_{11}(x_{11}+x_{21}). \tag{5}$$

In particular, we have

$$D_{11}(x_{11}+x_{12}+x_{22}) = D_{11}(x_{11}). \tag{6}$$

For any $y_{12}\in R_{12}$, we have $D_{11}(x_{12})y_{12} = D_{11}(x_{12}y_{12}) - x_{12}D_{11}(y_{12}) = 0$. That means $D_{11}(x_{12})R_{12}=(0)$. By (H1), we obtain $D_{11}(x_{12})=0$ for all $x_{12}\in R_{12}$. Likewise $D_{11}(x_{22})R_{12} = (0)$ for all $x_{22}\in R_{22}$. Again by (H1), we find $D_{11}(x_{22})=0$ for all $x_{22}\in R_{22}$. Now, we can rewrite (6) as

$$D_{11}(x_{11}+x_{12}+x_{22}) = D_{11}(x_{11}) + D_{11}(x_{12}) + D_{11}(x_{22}).$$

It means that D_{11} is additive on $R_{11}\bigoplus R_{12}\bigoplus R_{22}$. On the other hand, for any $r\in R$, we find that

$$\begin{aligned} &(D_{12}(x_{11}+x_{12}+x_{21}+x_{22}) - D_{12}(x_{12}+x_{22}))r \\ &= D_{12}(x_{11}+x_{12}+x_{21}+x_{22})r - D_{12}(x_{12}+x_{22})r \\ &= D_{12}(x_{11}+x_{12}+x_{21}+x_{22})(r_{21}+r_{22}) - D_{12}(x_{12}+x_{22})(r_{21}+r_{22}) \\ &= D_{12}((x_{11}+x_{12}+x_{21}+x_{22})(r_{21}+r_{22})) - (x_{11}+x_{12}+x_{21}+x_{22}) \\ &\quad D_{12}(r_{21}+r_{22}) - D_{12}((x_{12}+x_{22})(r_{21}+r_{22})) + (x_{12}+x_{22})D_{12}(r_{21}+r_{22}) \\ &= -(x_{11}+x_{21})D_{12}(r_{21}+r_{22}) = -(x_{11}+x_{21})e_1D_{12}(r_{21}+r_{22}) \\ &= -(x_{11}+x_{21})D_{12}(e_1(r_{21}+r_{22})) + (x_{11}+x_{21})D_{12}(e_1)(r_{21}+r_{22}) = 0. \end{aligned}$$

Hence $(D_{12}(x_{11}+x_{12}+x_{21}+x_{22})-D_{12}(x_{12}+x_{22}))R=(0)$. In particular, $(D_{12}(x_{11}+x_{12}+x_{21}+x_{22})-D_{12}(x_{12}+x_{22}))R_{22}=(0)$. By (H2), we find

$$D_{12}(x_{11}+x_{12}+x_{21}+x_{22})=D_{12}(x_{12}+x_{22}).$$

Consequently

$$D_{12}(x_{11}+x_{12}+x_{21})=D_{12}(x_{12}). \tag{7}$$

Now, for any $z_{22}\in R_{22}$, we get $D_{12}(x_{11})z_{22}=D_{12}(x_{11}z_{22})-x_{11}D_{12}(z_{22})=-x_{11}e_1D_{12}(z_{22})=-x_{11}D_{12}(e_1z_{22})+x_{11}D_{12}(e_1)z_{22}=0$. That is $D_{12}(x_{11})R_{22}=(0)$ for all $x_{11}\in R_{11}$. Thus we may apply hypothesis (H2), which forces that $D_{12}(x_{11})=0$ for all $x_{11}\in R_{11}$. In the similar manner, we find that $D_{12}(x_{21})R_{22}=(0)$ for all $x_{21}\in R_{21}$. Again applying (H2), we get $D_{12}(x_{21})=0$ for all $x_{21}\in R_{21}$. Thus expression (7) assures that D_{12} is additive on $R_{11}\bigoplus R_{12}\bigoplus R_{21}$.

We now proceed to show that D_{11} is additive on R_{21} and D_{12} is additive on R_{22}. For any $x,y\in R$, we have

$$\begin{aligned}&D_{11}(xy)=D_{11}((x_{11}+x_{12}+x_{21}+x_{22})(y_{11}+y_{12}+y_{21}+y_{22}))\\&=D_{11}((x_{11}y_{11}+x_{12}y_{21})+(x_{21}y_{11}+x_{22}y_{21})+(x_{11}y_{12}+x_{12}y_{22})\\&+(x_{21}y_{12}+x_{22}y_{22}))=D_{11}((x_{11}y_{11}+x_{12}y_{21})+(x_{21}y_{11}+x_{22}y_{21}))(\ using\ (5)\ ).\end{aligned}$$

and

$$\begin{aligned}D_{11}(x)y_{11}+x_{11}D_{11}(y)+D_{12}(x)y_{21}+x_{12}D_{21}(y)&=D_{11}(x_{11}+x_{21})y_{11}\\+x_{11}D_{11}(y_{11}+y_{21})+D_{12}(x_{12}+x_{22})y_{21}&+x_{12}D_{21}(y_{11}+y_{21}).\end{aligned}$$

Now, relation (3) can be expressed as

$$\begin{aligned}D_{11}((x_{11}y_{11}+x_{12}y_{21})+(x_{21}y_{11}+x_{22}y_{21}))&=D_{11}(x_{11}+x_{21})y_{11}\\+x_{11}D_{11}(y_{11}+y_{21})+D_{12}(x_{12}+x_{22})y_{21}&+x_{12}D_{21}(y_{11}+y_{21}).\end{aligned} \tag{8}$$

In particular, putting $x_{11}=0=x_{12}$ in (8), we obtain

$$D_{11}(x_{21}y_{11}+x_{22}y_{21})=D_{11}(x_{21})y_{11}+D_{12}(x_{22})y_{21}. \tag{9}$$

It follows that

$$D_{11}(x_{21}y_{11})=D_{11}(x_{21})y_{11},\quad D_{11}(x_{22}y_{21})=D_{12}(x_{22})y_{21}. \tag{10}$$

Thus, (9) can be written as

$$D_{11}(x_{21}y_{11}+x_{22}y_{21})=D_{11}(x_{21}y_{11})+D_{11}(x_{22}y_{21}). \tag{11}$$

Replacing y_{11} by $x_{12}y_{21}$ and x_{22} by $z_{21}x_{12}$ in (11), we get

$$\begin{aligned}D_{11}(x_{21}x_{12}y_{21}+z_{21}x_{12}y_{21})&=D_{11}(x_{21}x_{12}y_{21})+D_{11}(z_{21}x_{12}y_{21}),\\D_{11}((x_{21}+z_{21})x_{12}y_{21})&=D_{11}((x_{21})(x_{12}y_{21}))+D_{11}((z_{21})(x_{12}y_{21})).\end{aligned}$$

Application of (10) yields that

$$D_{11}(x_{21}+z_{21})x_{12}y_{21}=D_{11}(x_{21})(x_{12}y_{21})+D_{11}(z_{21})(x_{12}y_{21}).$$

That is,

$$(D_{11}(x_{21}+z_{21})-D_{11}(x_{21})-D_{11}(z_{21}))R_{12}R_{21}=(0).$$

Application of (H3) and (H1) respectively yields

$$D_{11}(x_{21}+z_{21})=D_{11}(x_{21})+D_{11}(z_{21}) \text{ for all } x_{21},z_{21}\in R_{21}.$$

From (10), we have $D_{12}(x_{22})y_{21}=D_{11}(x_{22}y_{21})$. Therefore

$$\begin{aligned} D_{12}(x_{22}+z_{22})y_{21} &= D_{11}((x_{22}+z_{22})y_{21})=D_{11}(x_{22}y_{21}+z_{22}y_{21}) \\ &= D_{11}(x_{22}y_{21})+D_{11}(z_{22}y_{21})=D_{12}(x_{22})y_{21}+D_{12}(z_{22})y_{21}. \end{aligned}$$

It implies that

$$(D_{12}(x_{22}+z_{22})-D_{12}(x_{22})-D_{12}(z_{22}))R_{21}=(0).$$

We may apply (H3) in order to obtain $D_{12}(x_{22}+z_{22})=D_{12}(x_{22})+D_{12}(z_{22})$. Hence, D_{12} is additive on R_{22}.

Next, we shall show that D_{11} is additive on R_{11} and D_{12} is additive on R_{11}. It is straight forward to check that, for any $x_{12},y_{12}\in R_{12}$

$$(D_{11}(x_{12}+y_{12})-D_{11}(x_{12})-D_{11}(y_{12}))R_{12}=(0).$$

Thus, hypothesis (H1) forces $D_{11}(x_{12}+y_{12})=D_{11}(x_{12})+D_{11}(y_{12})$. Let $r_{12}\in R_{12}$. Then

$$\begin{aligned} &D_{11}(x_{11}+y_{11})r_{12}=D_{11}((x_{11}+y_{11})r_{12})-(x_{11}+y_{11})D_{11}(r_{12})=D_{11}(x_{11}r_{12}+y_{11}r_{12}) \\ &-x_{11}D_{11}(r_{12})-y_{11}D_{11}(r_{12})=D_{11}(x_{11}r_{12})+D_{11}(y_{11}r_{12})-x_{11}D_{11}(r_{12})-y_{11}D_{11}(r_{12}) \\ &=D_{11}(x_{11})r_{12}+D_{11}(y_{11})r_{12}. \end{aligned}$$

That is $(D_{11}(x_{11}+y_{11})-D_{11}(x_{11})-D_{11}(y_{11}))r_{12}=0$ for all $r_{12}\in R_{12}$. Again we apply (H1) in order to obtain

$$D_{11}(x_{11}+y_{11})=D_{11}(x_{11})+D_{11}(y_{11}) \text{ for all } x_{11},y_{11}\in R_{11}.$$

In like manner, for any $r_{21}\in R_{21}$, we see $(D_{12}(x_{11}+y_{11})-D_{12}(x_{11})-D_{12}(y_{11}))r_{21}=0$. Thus $(D_{12}(x_{11}+y_{11})-D_{12}(x_{11})-D_{12}(y_{11}))R_{21}=(0)$. On utilizing (H3), D_{12} is additive on R_{11}. Further, we consider

$$\begin{aligned} &(D_{12}(x_{12}+y_{12})-D_{12}(x_{12})-D_{12}(y_{12}))r_{21}=D_{12}(x_{12}+y_{12})r_{21}-D_{12}(x_{12})r_{21}-D_{12}(y_{12})r_{21} \\ &=D_{12}(x_{12}r_{21}+y_{12}r_{21})-D_{12}(x_{12}r_{21})-D_{12}(y_{12}r_{21})=0. \end{aligned}$$

Therefore, we obtain $(D_{12}(x_{12}+y_{12})-D_{12}(x_{12})-D_{12}(y_{12}))R_{21}=(0)$. Hypothesis (H3) yields

$$D_{12}(x_{12}+y_{12})=D_{12}(x_{12})+D_{12}(y_{12}).$$

Now, we are well occupied to prove that D_{11} and D_{12} are additive on R. Observe that, as per the results derived above it is enough to show that $D_{11}(x_{11}+x_{21})=D_{11}(x_{11})+D_{11}(x_{21})$ and $D_{12}(x_{12}+x_{22})=D_{12}(x_{12})+D_{12}(x_{22})$.

Firstly, note that

$$\begin{aligned} &D_{21}(y)=D_{21}(y_{11}+y_{12}+y_{21}+y_{22})=e_2D(y_{11}+y_{12}+y_{21}+y_{22})e_1 \\ &=e_2D((y_{11}+y_{12}+y_{21}+y_{22})e_1)e_1=e_2D(y_{11}+y_{21})e_1=D_{21}(y_{11}+y_{21}). \end{aligned}$$

and

$$
\begin{aligned}
&(D_{22}(x_{11}+x_{12}+x_{21}+x_{22})-D_{22}(x_{12}+x_{22}))r\\
&=D_{22}(x_{11}+x_{12}+x_{21}+x_{22})(r_{21}+r_{22})-D_{22}(x_{12}+x_{22})(r_{21}+r_{22})\\
&=D_{22}((x_{11}+x_{12}+x_{21}+x_{22})(r_{21}+r_{22}))-(x_{11}+x_{12}+x_{21}+x_{22})D_{22}(r_{21}\\
&+r_{22})-D_{22}((x_{12}+x_{22})(r_{21}+r_{22}))+(x_{12}+x_{22})D_{22}(r_{21}+r_{22})=0.
\end{aligned}
$$

Let us rewrite expression (4) as

$$
\begin{aligned}
&D_{12}((x_{11}y_{12}+x_{12}y_{22})+(x_{21}y_{12}+x_{22}y_{22}))=D_{11}(x_{11}+x_{21})y_{12}\\
&+D_{12}(x_{12}+x_{22})y_{22}+x_{11}D_{12}(y_{12}+y_{22})+x_{12}D_{22}(y_{12}+y_{22}).
\end{aligned}
\tag{12}
$$

In particular, we put $x_{12}=0=x_{21}$ in (12), we find

$$D_{12}(x_{11}y_{12}+x_{22}y_{22})=D_{11}(x_{11})y_{12}+D_{12}(x_{22})y_{22}+x_{11}D_{12}(y_{12}+y_{22}). \tag{13}$$

On substituting $x_{11}=e_1, y_{12}=z_{12}y_{22}$ in (13), we get

$$
\begin{aligned}
&D_{12}((z_{12}+x_{22})y_{22})=D_{11}(e_1)z_{12}y_{22}+D_{12}(x_{22})y_{22}+e_1D_{12}(z_{12}y_{22}+y_{22})\\
&=D_{12}(x_{22})y_{22}+D_{12}(e_1(z_{12}y_{22}+y_{22}))-D_{12}(e_1)(z_{12}y_{22}+y_{22})\\
&=D_{12}(x_{22})y_{22}+D_{12}(z_{12}y_{22})=D_{12}(x_{22})y_{22}+D_{12}(z_{12})y_{22}.
\end{aligned}
$$

That gives

$$D_{12}((z_{12}+x_{22})y_{22})=D_{12}(z_{12})y_{22}+D_{12}(x_{22})y_{22}. \tag{14}$$

We next put $y_{12}=0=x_{11}$ in (12), we get

$$D_{12}((x_{12}+x_{22})y_{22})=D_{12}(x_{12}+x_{22})y_{22}+x_{12}D_{22}(y_{22}). \tag{15}$$

On combining (14) and (15), it follows that

$$D_{12}(x_{12}+x_{22})y_{22}+x_{12}D_{22}(y_{22})=(D_{12}(z_{12})+D_{12}(x_{22}))y_{22}.$$

On substituting $y_{22}=y_{21}t_{12}$ in the above expression in order to obtain

$$
\begin{aligned}
&(D_{12}(z_{12})+D_{12}(x_{22}))y_{21}t_{12}=D_{12}(x_{12}+x_{22})y_{21}t_{12}+x_{12}D_{22}(y_{21}t_{12})\\
&=D_{12}(x_{12}+x_{22})y_{21}t_{12}+x_{12}D_{22}(y_{21})t_{12}+x_{12}y_{21}D_{22}(t_{12})=D_{12}(x_{12}+x_{22})y_{21}t_{12}.
\end{aligned}
$$

That is $(D_{12}(x_{12}+x_{22})-D_{12}(z_{12})-D_{12}(x_{22}))y_{21}t_{12}=0$ for all $y_{21}\in R_{21}$ and $t_{12}\in R_{12}$. Thus $(D_{12}(x_{12}+x_{22})-D_{12}(z_{12})-D_{12}(x_{22}))R_{21}R_{12}=(0)$. An application of (H1) and (H3) successively yields $D_{12}(z_{12}+x_{22})=D_{12}(z_{12})+D_{12}(x_{22})$. Moreover, we put $x_{12}=0=y_{22}$ in (14) in order to obtain

$$
\begin{aligned}
&D_{11}(x_{11}+x_{21})y_{12}+x_{11}D_{12}(y_{12})=D_{12}(x_{11}y_{12}+x_{21}y_{12})\\
&=D_{12}(x_{11}y_{12})+D_{12}(x_{21}y_{12}).
\end{aligned}
\tag{16}
$$

It follows that

$$D_{12}(x_{11}y_{12})=D_{11}(x_{11})y_{12}+x_{11}D_{12}(y_{12}),\quad D_{12}(x_{21}y_{12})=D_{11}(x_{21})y_{12}. \tag{17}$$

By utilizing (17) in (16), we find $(D_{11}(x_{11}+x_{21})-D_{11}(x_{11})-D_{11}(x_{21}))y_{12}=0$ for all $y_{12}\in R_{12}$. That means $(D_{11}(x_{11}+x_{21})-D_{11}(x_{11})-D_{11}(x_{21}))R_{12}=(0)$. By (H1), we get $D_{11}(x_{11}+x_{21})=D_{11}(x_{11})+D_{11}(x_{21})$. $\square$

Analogously, we can prove the following lemma:

Lemma 2. *Let R be a ring (not necessary with unity) and M be a bimodule over R. Suppose that R contains a non-trivial idempotent e_1 such that for any $m \in M$, the following are satisfied:*

(H4) $e_2me_2R_{21} = (0)$ *implies* $e_2me_2 = 0$,

(H5) $e_2me_1R_{11} = (0)$ *implies* $e_2me_1 = 0$,

(H6) $e_2me_1R_{12} = (0)$ *implies* $e_2me_2 = 0$.

Then D_{21} and D_{22} are additive.

Since $D = D_{11} + D_{12} + D_{21} + D_{22}$, Lemma 1 and Lemma 2 proves our main result:

Theorem 1. *Let R be a ring and M be a bimodule over R. If e_1 is a non-trivial idempotent in R such that for all $m \in M$ the conditions (H1)-(H6) hold. Then every multiplicative-derivation $d : R \to M$ is additive.*

Recall that R is said to be a prime ring if $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ for all $a \in R$. Let R be a semiprime ring and Q be the two sided Martindale quotient ring of R. The maximal left ring of quotients (also called left Utumi quotient ring) of R is denotes by Q_{ml}. The center C of Q is called the extended centroid of R. If R happens to be prime, then C is a field. Moreover, the extended centroid C of R coincides with the center of Q_{ml} and is reduced in the sense that C does not have nonzero nilpotent elements. For more information of these objects, we refer the reader to [1]. As an application of Theorem 1, we obtain the following consequent results:

Corollary 1. *Let R be a semiprime ring containing a non-trivial idempotent e. Suppose that for any $a \in Q_{ml}$ the following holds:*

(I) $e_1ae_1Re_2 = (0)$ *implies* $e_1ae_1 = 0$,

(II) $e_2ae_2Re_1 = (0)$ *implies* $e_2ae_2 = 0$.

Then any multiplicative-derivation $d : R \to Q_{ml}$ is additive.

Proof. Let $a \in Q_{ml}$ be an element such that $e_iae_jRe_k = (0)$ for all $i, j, k \in \{1, 2\}$. We have the following possible cases:
Case 1. If $i = k$, then we have $(e_iae_jRe_i)ae_j = 0$. It yields that $e_iae_j = 0$ for all $i, j \in \{1, 2\}$.
Case 2. Suppose that $j = k$. In the view of proposition 2.1.7 (ii) of [1], there exist a dense left ideal D of R such that $De_ia \subseteq R$. It implies that $(De_iae_j)R(De_iae_j) \subseteq (De_iae_j)Re_j = (0)$. It follows that $De_iae_j = (0)$ for all $i, j \in \{1, 2\}$. With the aid of proposition 2.1.7 (iii) of [1], we obtain $e_iae_j = 0$ for all $i, j \in \{1, 2\}$.
Case 3. In latter case $i = j$. By our hypothesis $e_iae_iRe_k = (0)$ implies $e_iae_i = 0$ for all $i \in \{1, 2\}$. Now, we see that the condition (H1)-(H6) hold here. Therefore, d is additive by Theorem 1. □

In case R is a prime ring, every derivation $d : R \to Q_{ml}$ is additive automatically, since if for any $q_1, q_2 \in Q_{ml}$, $q_1Rq_2 = (0)$ implies $q_1 = 0$ or $q_0 = 0$. Thus, we obtain

Corollary 2. *Let R be a prime ring containing a non-trivial idempotent e. Then every multiplicative-derivation $d : R \to Q_{ml}$ is additive.*

1.2 Additivity of multiplicative semi-derivations

In [8] Martindale give a set of conditions that are sufficient for the additivity of ring isomorphisms. Precisely, he proved that "Let R be a ring containing a family $\{e_\lambda : \lambda \in \Lambda\}$ of idempotents satisfying (Martindale's conditions)

(I) $xR = (0)$ implies $x = 0$,

(II) If for each $\lambda \in \Lambda$, $e_\lambda Rx = (0)$, then $x = 0$ (hence $Rx = (0)$ implies $x = 0$),

(III) If $e_\lambda x e_\lambda R(1 - e_\lambda) = (0)$ for each $\lambda \in \Lambda$, then $e_\lambda x e_\lambda = 0$.

Then any multiplicative isomorphism of R onto an arbitrary ring S is additive". It is natural to think of a unified notion of multiplicative derivation and a semi derivation. In view of this idea, we now give the notion of multiplicative semi-derivation, as follows:

Definition 2. *Let R be a ring. A mapping $g : R \to R$ (not necessarily additive) defined by $g(xy) = g(x)g(y)$ for all $x, y \in R$ is called a multiplicative homomorphism of R. Then the mapping $\delta : R \to R$ (not necessarily additive) together with g is called multiplicative semi-derivation of R if*

$$\delta(xy) = \delta(x)g(y) + x\delta(y) = \delta(x)y + g(x)\delta(y).$$

holds for all $x, y \in R$.

Example 1. *Let $R = \{\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} : u, v, w \in \mathbb{R}\}$, where $\mathbb{R}$ denotes the field of real numbers. Define a mapping $g : R \to R$ by $g\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & det\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \end{pmatrix}$, which is clearly a ring endomorphism of R. Now, it can be easily verified that $\delta = g - I$ is the multiplicative semi-derivation of R.*

In this section, our aim is to obtain the additivity of multiplicative semi-derivations of rings under certain conditions. Precisely, we obtain the following result:

Theorem 2. *Let R be a ring satisfying Martindale's conditions (I)-(III). If $d : R \to R$ is a multiplicative semi-derivation of R associated with a multiplicative isomorphism $g : R \to R$, then d is additive.*

Let us define a function $\varphi : R \times R \to R$ that $\varphi(x, y) = d(x + y) - d(x) - d(y)$, where d is a multiplicative semi-derivation of R. Clearly, φ is a well-define mapping and $\varphi(x, 0) = 0 = \varphi(0, x)$ for all $x \in R$. Now, it is clear that d is additive if and only if $\varphi = 0$. This observation motivated the technique opted in this paper. We prove Theorem 2 through a sequence of lemmas.

Lemma 3. *For any $x, y, k \in R$, $k\varphi(x, y) = \varphi(kx, ky)$ and $\varphi(x, y)k = \varphi(xk, yk)$.*

Proof. In the view of [[8], Theorem], g must be additive on R. For any $x, y, k \in R$, we have $\varphi(kx, ky) = d(k(x + y)) - d(kx) - d(ky) = d(k)g(x + y) + kd(x + y) - d(k)g(x) - kd(x) - d(k)g(y) - kd(y) = k(d(x + y) - d(x) - d(y)) = k\varphi(x, y)$. On the other hand, let us consider $\varphi(xk, yk) = d((x + y)k) - d(xk) - d(yk) = d(x + y)k + g(x + y)d(k) - d(x)k - g(x)d(k) - d(y)k - g(y)d(k) = (d(x + y) - d(x) - d(y))k = \varphi(x, y)k$. ◻

Lemma 4. *$\varphi(x_{ii}, x_{jk}) = 0 = \varphi(x_{jk}, x_{ii})$; $j \neq k$, where $i, j, k \in \{1, 2\}$.*

Proof. In case $i = j$. For any $r_{il} \in R_{il}$, we find $\varphi(x_{ii}, x_{jk})r_{il} = \varphi(x_{ii}r_{il}, x_{jk}r_{il}) = \varphi(z_{il}, 0) = 0$ for all $i, j, , k, l \in \{1, 2\}$, by Lemma 3. For any $r_{kl} \in R_{kl}$, we have $\varphi(x_{ii}, x_{jk})r_{kl} = \varphi(x_{ii}r_{kl}, x_{jk}r_{kl}) = \varphi(0, w_{jl}) = 0$ for all $i, j, k, l \in \{1, 2\}$. Since $i = j \neq k$, it implies $\varphi(x_{ii}, x_{jk}) \quad R = (0)$. By hypothesis (I), we obtain $\varphi(x_{ii}, x_{jk}) = 0$. In the latter case, we assume $i \neq j$. For any $r_{mi} \in R_{mi}$, we have $r_{mi}\varphi(x_{ii}, x_{jk}) = \varphi(r_{mi}x_{ii}, r_{mi}x_{jk}) = \varphi(z_{mi}, 0) = 0$ for all $i, j, k, m \in \{1, 2\}$. Similarly, we may infer that $r_{mj}\varphi(x_{ii}, x_{jk}) = 0$ for all $r_{mj} \in R_{mj}$ and $i, j, k, m \in \{1, 2\}$. Combining these relation, we get $R\varphi(x_{ii}, x_{jk}) = (0)$. By hypothesis (II), we get $\varphi(x_{ii}, x_{jk}) = 0$. Hence, we conclude that $\varphi(x_{ii}, x_{jk}) = 0$ for all $j \neq k$ and $i, j, k \in \{1, 2\}$. Analogously, we obtain $\varphi(x_{jk}, x_{ii}) = 0$ for all $j \neq k$ and $i, j, k \in \{1, 2\}$. $\square$

Lemma 5. $\varphi(x_{12}, y_{12}) = 0$.

Proof. Clearly, $e_1\varphi(x_{12}, y_{12}) = \varphi(e_1x_{12}, e_1y_{12}) = \varphi(x_{12}, y_{12})$ and $\varphi(x_{12}, y_{12})e_1 = \varphi(x_{12}e_1, y_{12}\, e_1) = \varphi(0, 0) = 0$. It implies that $\varphi(x_{12}, y_{12}) \in R_{12}$. Therefore, $\varphi(x_{12}, y_{12})a_{11} = 0$ and $\varphi(x_{12}, y_{12})a_{12} = 0$ for all $a_{11} \in R_{11}, a_{12} \in R_{12}$. Now for any $a_{21} \in R_{21}$, we have

$$\begin{aligned}\varphi(x_{12}, y_{12})a_{21} &= \varphi(x_{12}a_{21}, y_{12}a_{21}) = \varphi(x_{12}(a_{21} + y_{12}a_{21}), e_1(a_{21} + y_{12}a_{21})) \\ &= \varphi(x_{12}, e_1)(a_{21} + y_{12}a_{21}) = 0 \quad (\textit{using Lemma } 4).\end{aligned}$$

In the similar way, we can show that $\varphi(x_{12}, y_{12})a_{22} = 0$ for all $a_{22} \in R_{22}$. Combining all these relations, we get $\varphi(x_{12}, y_{12})R = (0)$. Hence, $\varphi(x_{12}, y_{12}) = 0$ by condition (I). $\square$

Lemma 6. $\varphi(x_{11}, y_{11}) = 0$.

Proof. Under the influence of Lemma 3, it is easy to see that $\varphi(x_{11}, y_{11}) \in R_{11}$. For any $a_{12} \in R_{12}$, we have $\varphi(x_{11}, y_{11})a_{12} = \varphi(x_{11}a_{12}, y_{11}a_{12}) = \varphi(y_{12}, z_{12}) = 0$ by Lemma 5. That means

$$\varphi(x_{12}, y_{12})R_{12} = (0). \tag{18}$$

Since $\varphi(x_{11}, y_{11}) \in R_{11}$, so $\varphi(x_{11}, y_{11}) = e_1\varphi(x_{11}, y_{11})e_1$. From Eq. (18), we get $\varphi(x_{11}, y_{11})R_{12} = e_1\varphi(x_{11}, y_{11})e_1R(1 - e_1) = (0)$. By condition (III), we obtain $e_1\varphi(x_{11}, y_{11})e_1 = 0$ and hence $\varphi(x_{11}, y_{11}) = 0$. $\square$

Lemma 7. $\varphi(x_{11} + x_{12}, y_{11} + y_{12}) = 0$.

Proof. For any $a_{11} \in R_{11}$ and $a_{12} \in R_{12}$ we see that $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{11} = \varphi(x_{11}a_{11}, y_{11}\, a_{11}) = 0$ by Lemma 6, and $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{12} = \varphi(x_{11}a_{12}, y_{11}a_{12}) = 0$ by Lemma 5. By repeating same arguments and utilization of Lemma 5, 6 we get $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{21} = 0$ for all $a_{21} \in R_{21}$ and $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{22} = 0$ for all $a_{22} \in R_{22}$. Add up all these equations in order to find $\varphi(x_{11} + x_{12}, y_{11} + y_{12})R = (0)$. Hence, $\varphi(x_{11} + x_{12}, y_{11} + y_{12}) = 0$ by hypothesis (I). $\square$

Proof of Theorem 2: By Lemma 7, $\varphi(u, v) = 0$ for all $u, v \in e_1R$. For any $x, y, r \in R$, we have $e_1r\varphi(x, y) = \varphi(e_1rx, e_1ry) = 0$. Since e_1 was arbitrary member chosen from the family $\{e_\lambda : \lambda \in \Lambda\}$, so we must have $e_\lambda R\varphi(x, y) = (0)$ for all $\lambda \in \Lambda$. By our hypothesis (II), we find that $\varphi(x, y) = 0$ for all $x, y \in R$. $\square$

REFERENCES

[1] Beidar K. I., Martindale III W. S., Mikhalev A. V. Rings with generalized identities. Marcel Dekker Inc., New York, 1996.

[2] Bergen J. *Derivations in prime rings*. Canad. Math. Bull. 1983, **26**(3), 267–270. doi:10.4153/CMB-1983-042-2

[3] Daif M. N. *When is a multiplicative derivation additive?* Internat. J. Math. Math. Sci. 1991, **14**(3), 615–618. doi:10.1155/S0161171291000844

[4] Ferreira B. L. M., Guzzo Jr. H., Ferreira J. C. M. *Additivity of Jordan elementary maps on standard rings*. Alg. Disc. Math. 2014, **18**(2), 203–233.

[5] Jacobson N. Structure of rings. Colloq. Pub. (Amer. Math. Soc.), Rhode Island, Vol. 37, 1964.

[6] Ling Z., Lu F. *Jordan maps of nest algebras*. Linear Algebra Appl. 2004, **387** (1), 361–368. doi:10.1016/j.laa.2004.02.031

[7] Lu F. *Jordan derivable maps of prime rings*. Commun. Algebra 2010, **38** (12), 4430–4440. doi:10.1080/00927870903366884

[8] Martindale III W. S. *When are multiplicative mappings additive?* Proc. Amer. Math. Soc. 1969, **21**(3), 695–698. doi:10.2307/2036449

[9] Peisheng J. *Jordan maps on triangular algebras*. Linear Algebra Appl. 2007, **426** (1), 190–198. doi:10.1016/j.laa.2007.04.009

[10] Rodrigues R. L., Guzzo Jr. H., Ferreira J. C. M. *When is a multiplicative derivation additive in alternative rings?* Comm. Algebra 2016, **44** (6), 2561–2566. doi:10.1080/00927872.2015.1065835

[11] Wang Y. *The additivity of multiplicative maps on rings*. Comm. Algebra 2009, **37** (7), 2351–2356. doi: 10.1080/00927870802623369

Received 11.01.2019

Сандху Г.С., Кумар Д. *Про адитивність диференціювань* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 453–462.

Нехай R — деяке кільце і M — деякий R-бімодуль. Відображення $d : R \to M$ (не обов'язково адитивне) називається мультиплікативним диференціюванням кільця R, якщо $d(xy) = d(x)y + xd(y)$ для всіх $x, y \in R$. У цій статті ми намагаємось встановити адитивність d при деяких додаткових обмеженнях. Крім того ми вводимо мультиплікативне напівдиференціювання кільця і обговорюємо його адитивність.

Ключові слова і фрази: диференціювання, мультиплікативне диференціювання, мультиплікативне напівдиференціювання кільця, адитивність, розклад Пірса.

ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 463–474
doi:10.15330/cmp.11.2.463-474

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.463–474

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THE NONLOCAL BOUNDARY VALUE PROBLEM FOR ONE-DIMENSIONAL BACKWARD KOLMOGOROV EQUATION AND ASSOCIATED SEMIGROUP

This paper is devoted to a partial differential equation approach to the problem of construction of Feller semigroups associated with one-dimensional diffusion processes with boundary conditions in theory of stochastic processes. In this paper we investigate the boundary-value problem for a one-dimensional linear parabolic equation of the second order (backward Kolmogorov equation) in curvilinear bounded domain with one of the variants of nonlocal Feller-Wentzell boundary condition. We restrict our attention to the case when the boundary condition has only one term and it is of the integral type. The classical solution of the last problem is obtained by the boundary integral equation method with the use of the fundamental solution of backward Kolmogorov equation and the associated parabolic potentials. This solution is used to construct the Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle leaves the boundary of the domain by jumps.

Key words and phrases: parabolic potential, boundary integral equation method, Feller semigroup, nonlocal boundary condition.

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INTRODUCTION

Let $\Pi[0,T] = \{(s,x):\ 0 \le s \le T,\ x \in \mathbb{R}\}$ and let $S_t \subset \Pi[0,T]$ be the curvilinear domain

$$S_t = \{(s,x):\ 0 \le s < t \le T,\ r_1(s) < x < r_2(s)\},$$

where T is a fixed positive number, and r_1, r_2 are given functions defined on $[0,T]$. Denote by D_s the interval $(r_1(s), r_2(s))$ and by $\overline{S}_t$ and $\overline{D}_s$ the closure of S_t and D_s respectively. Denote also by $\mathcal{C}_i$ the curves $\{(s, r_i(s)):\ s \in [0,T]\}$ $(i = 1,2)$ and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

In $\Pi[0,T]$ we consider the parabolic operator of the second order with bounded continuous coefficients

$$\frac{\partial}{\partial s} + L_s \equiv \frac{\partial}{\partial s} + \frac{1}{2} b(s,x) \frac{\partial^2}{\partial x^2} + a(s,x) \frac{\partial}{\partial x}.$$

The main problem is to find a classical solution $u(s,x,t)$ of equation

$$\frac{\partial u}{\partial s} + L_s u = 0, \quad (s,x) \in S_t, \tag{1}$$

УДК 519.21
2010 *Mathematics Subject Classification:* 60J60, 35K20.

which satisfies the "initial" condition

$$\lim_{s\uparrow t} u(s,x,t) = \varphi(x), \quad x \in \overline{D}_t, \tag{2}$$

and two boundary conditions

$$\int_{D_s} [u(s,r_i(s),t) - u(s,y,t)]\mu_i(s,dy) = 0, \quad 0 \le s \le t \le T, \quad i = 1,2, \tag{3}$$

where φ is the given function and $\mu_i(s,\cdot)$ $(s \in [0,T],\ i = 1,2)$ are given finite nonnegative measures on D_s, $s \in [0,T]$.

The problem (1)-(3) appears, in particular, in the theory of stochastic processes while studying the diffusion processes with boundary conditions. Recall that the most general form of boundary conditions for one-dimensional diffusion processes was established in works of W. Feller [2] and A. D. Wentzell [12] (see also [13], where the multidimensional case is considered). From the assertions proved there, it follows that if the ordinary differential operator of the second order is a generator of the Feller semigroup in $C[r_1,r_2]$ (r_1, r_2 are fixed, $-\infty < r_1 < r_2 < \infty$), then its domain of definition consists of functions satisfying nonlocal boundary conditions. In the general case, these boundary conditions contain the values of the function and its first-order derivatives with respect to the time variable and with respect to the spatial variable at points r_i, $i = 1,2$, and the nonlocal component of the integral type that correspond, respectively, to such properties of process after it reaches the boundary point r_i as its termination, delay, reflection and jump out of r_i.

In the present paper we shall establish the classical solvability of problem (1)-(3) by the boundary integral equation method with the use of the fundamental solution of the equation (1) and the associated parabolic potentials, and prove that its solution $u(s,x,t) \equiv T_{st}\varphi(x)$ can be treated as the two parameter semigroup of operators describing an inhomogeneous Feller process in $\mathbb{R}$ which trajectories are located in curvilinear domain $\overline{S}_T$. It is easy to understand that the trajectories of this process in $\overline{S}_T \setminus \mathcal{C}$ can be treated as the trajectories of the diffusion process generated by the operator L_s and at the points of curves $\mathcal{C}_i$ $(i = 1,2)$ their behavior is determined by Feller-Wentzell boundary conditions in (3). The conditions in (3) correspond to jump discontinuity of trajectories of process which is caused by inward jump of a Markovian particle from the boundary.

It is necessary to note that the scheme we shall use to solve the problem (1)-(3) is partially presented in work [6], where the similar problem was investigated in the case when the backward Kolmogotov equation is given in $\cup_{i=1}^2 S_t^{(i)} = \cup_{i=1}^2 \{(s,x) : 0 \le s < t \le T,\ (-1)^i(x - r(s)) > 0\}$ and, at the common boundary $x = r(s)$ of domains $S_t^{(1)}$ and $S_t^{(2)}$, the Feller-Wentzell conjugation condition, which, in addition to the integral term, contains also the local term corresponding to the termination of process, is imposed. We should also mention works [8], [11], which give the results concerning the construction of diffusion processes with nonlocal boundary conditions of the integral type by the methods of stochastics [8] and functional analysis [11].

We need the following conditions:

I. The operator $\partial/\partial s + L_s$ is uniformly parabolic in $\Pi[0,T]$, i.e., there exist constants b and B such that $0 < b \le b(s,x) \le B < \infty$ for all $(s,x) \in \Pi[0,T]$.

II. The coefficients of L_s are bounded and continuous functions in $\Pi[0,T]$ which belong to Hölder class $H^{\frac{\alpha}{2},\alpha}(\Pi[0,T])$, $0<\alpha<1$ (to recall the definitions of Hölder classes see [7, p.16]).

III. The function φ in (2) is assumed to be defined on $\mathbb{R}$ and belongs to the space of bounded continuous functions on $\mathbb{R}$, which we will denote by $C_b(\mathbb{R})$. The norm in this space is defined by the equality $\|\varphi\| = \sup_{s\in\mathbb{R}} |\varphi(x)|$. Furthermore, two fitting conditions

$$\int\limits_{D_t} [\varphi(r_i(t)) - \varphi(y)]\mu_i(t,dy) = 0, \quad i=1,2, \quad \text{hold.}$$

IV. The nonnegative measures μ_i in (3) are such that $\mu_i(s,D_s)=1$, $s\in[0,T]$ and for all $f\in C_b(\mathbb{R})$ the integrals

$$\int\limits_{D_s} f(y)\mu_i(s,dy), \quad i=1,2,$$

belong to $H^{\frac{1+\alpha}{2}}([0,T])$ as functions of s.

V. The functions $r_i(s)$, $i=1,2$, are continuous and belong to $H^{\frac{1+\alpha}{2}}([0,T])$.

Conditions I, II ensure the existence of the fundamental solution of the parabolic operator $\partial/\partial s + L_s$ in $\Pi[0,T]$ (see [7, Ch.IV, §15], [9, Ch.II, §3]), i.e., a function $G(s,x,t,y)$ defined for all (s,x) and (t,y) in $\Pi[0,T]$, $s<t$, satisfying the following condition:

for any $\varphi\in C_b(\mathbb{R})$, the function

$$u_0(s,x,t) = \int\limits_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy \tag{4}$$

satisfies the equation (1) if $0\le s<t\le T$, $x\in\mathbb{R}$ and the condition (2) if $t\in(0,T]$, $x\in\mathbb{R}$.

Note that the function G admits the representation

$$G(s,x,t,y) = Z_0(s,x,t,y) + Z_1(s,x,t,y), \quad i=1,2,$$

where

$$Z_0(s,x,t,y) = [2\pi b(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b(t,y)(t-s)}\right\},$$

$$Z_1(s,x,t,y) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} Z_0(s,x,\tau,z)Q(\tau,z,t,y)dz,$$

and the function $Q(s,x,t,y)$ is the solution of some singular Volterra integral equation of the second kind. Note also that

$$\left|D_s^r D_x^p Z_0(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{5}$$

$$\left|D_s^r D_x^p Z_1(s,x,t,y)\right| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y-x)^2}{t-s}\right\} \tag{6}$$

$(0 \le s < t \le T,\ x, y \in \mathbb{R})$, and that for the function u_0 defined by (4) $(\varphi \in C_b(\mathbb{R}))$ which is called the Poisson potential in the theory of parabolic equations, the inequality

$$\left|D_s^r D_x^p u_0(s,x,t)\right| \le C\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \quad 0 \le s < t \le T,\ x \in \mathbb{R}, \tag{7}$$

holds. Here C and c are positive constants (we shall subsequently denote various positive constants by symbols C or c without specifying their values), r and p are the nonnegative integers for which $2r + p \le 2$, D_s^r is the partial derivative with respect to s of order r, D_x^p is the partial derivative with respect to x of order p.

In addition to the integral $u_0(s,x,t)$ we need to consider two more integrals

$$u_{i1}(s,x,t) = \int_s^t G(s,x,\tau,r_i(\tau))V_i(\tau,t)d\tau, \quad i = 1,2,$$

where $0 \le s < t \le T$, $x \in \mathbb{R}$ and V_1, V_2 are some functions. The function u_{i1} is called the parabolic simple-layer potential. If we assume that the density $V_i(\tau,t)$ is continuous for $\tau \in [s,t)$ and admits a weak singularity with an exponent of not less than $-\frac{1}{2}$ when $\tau = t$, then the function $u_{i1}(s,x,t)$, $i = 1,2$, is bounded continuous in $0 \le s \le t \le T$, $x \in \mathbb{R}$ and satisfies the equation (1) in $(s,x) \in [0,t) \times (\mathbb{R} \setminus r_i(s))$ with the initial condition: $u_{i1}(s,x,t) \to 0$ if $s \uparrow t$ $(x \in \mathbb{R},\ i = 1,2)$.

The important property of the function u_{i1} is reflected in the so-called theorem on the jump of conormal derivative of parabolic simple-layer potential (see, e.g. [3, Ch.V, §2], [7, Ch.IV, §15]). In the present paper this assertion is not used, and therefore we do not formulate it.

1 SOLVING THE PARABOLIC BOUNDARY VALUE PROBLEM

We shall find a solution u of problem (1)-(3) as a sum of Poisson potential u_0 and two simple-layer potentials u_{11} and u_{21}, namely:

$$u(s,x,t) = \int_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy + \sum_{j=1}^{2} \int_s^t G(s,x,\tau,r_j(\tau))V_j(\tau,t)d\tau, \quad (s,x) \in \overline{S}_t. \tag{8}$$

Here φ is the function in (2) and V_i, $i = 1,2$, are the unknown densities to be determined.

Note that since $\mu_i(s,D_s) = 1$ for every $s \in [0,T]$ (see the condition IV), the conditions (3) and the fitting conditions in III can be reduced to

$$u(s,r_i(s),t) - \int_{D_s} u(s,y,t)\mu_i(s,dy) = 0, \quad 0 \le s \le t \le T,\ i = 1,2, \tag{9}$$

and

$$\varphi(r_i(t)) - \int_{D_t} \varphi(y)\mu_i(t,dy) = 0, \quad i = 1,2, \tag{10}$$

respectively.

Substituting (8) into (9), we get the system of two Volterra integral equations of the first kind for the unknowns V_i, $i = 1,2$, namely

$$\sum_{j=1}^{2} \int_s^t K_{ij}(s,\tau)V_j(\tau,t)d\tau = \Phi_i(s,t), \quad 0 \le s < t \le T,\, i = 1,2, \tag{11}$$

where

$$K_{ij}(s,\tau) = G(s, r_i(s), \tau, r_j(\tau)) - \int_{D_s} G(s,y,\tau,r_j(\tau))\mu_i(s,dy),$$

$$\Phi_i(s,t) = \int_{D_s} u_0(s,y,t)\mu_i(s,dy) - u_0(s,r_i(s),t).$$

Using Holmgren's method [4] (see also [5]) we shall reduce (11) to an equivalent system of Volterra integral equations of the second kind. For this purpose we consider the integro-differential operator

$$\mathcal{E}(s,t)f = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho - s)^{-\frac{1}{2}} f(\rho,t)d\rho, \quad 0 \le s < t \le T$$

and apply it to the both sides of each equation in (11).

The application of the operator $\mathcal{E}$ to the left-hand side of (11) gives the expression which after interchanging the order of integration takes on the form

$$I_i(s,t) \equiv \sum_{j=1}^{2} \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t V_j(\tau,t)d\tau \int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}(\rho,\tau)d\rho.$$

Write K_{ij} as $K_{ij}(\rho,\tau) = K_{ij}^{(1)}(\rho,\tau) + K_{ij}^{(2)}(\rho,\tau) - K_{ij}^{(3)}(\rho,\tau)$, where

$$\begin{aligned}
K_{ij}^{(1)}(\rho,\tau) &= Z_0(\rho, r_i(\tau), \tau, r_j(\tau)),\\
K_{ij}^{(2)}(\rho,\tau) &= Z_1(\rho, r_i(\tau), \tau, r_j(\tau)) + [G(\rho, r_i(\rho), \tau, r_j(\tau)) - G(\rho, r_i(\tau), \tau, r_j(\tau))],\\
K_{ij}^{(3)}(\rho,\tau) &= \int_{D_\rho} Z_0(\rho,y,\tau,r_j(\tau))\mu_i(\rho,dy) + \int_{D_\rho} Z_1(\rho,y,\tau,r_j(\tau))\mu_i(\rho,dy),
\end{aligned}$$

and denote by $J_{ij}(s,\tau)$ the integral $\int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}(\rho,\tau)d\rho$, and by $J_{ij}^{(k)}(s,\tau)$ the integral $\int_s^\tau (\rho - s)^{-\frac{1}{2}} K_{ij}^{(k)}(\rho,\tau)d\rho$, $k = 1,2,3$.

Note that $J_{ij}^{(1)}(s,\tau)$ is equal to

$$\frac{1}{\sqrt{2\pi b(\tau, r_i(\tau))}} \int_s^\tau (\tau - \rho)^{-\frac{1}{2}}(\rho - s)^{-\frac{1}{2}} d\rho = \sqrt{\frac{\pi}{2b(\tau, r_i(\tau))}},$$

when $i = j$, and tends to zero as $s \uparrow \tau$ when $i \ne j$. Note also that application of the mean value theorem to difference $G(\rho, r_i(\rho), \tau, r_j(\tau)) - G(\rho, r_i(\tau), \tau, r_j(\tau))$ together with the condition V and the estimates (5), (6) lead to the estimate

$$|K_{ij}^{(2)}(\rho,\tau)| \le |Z_1(\rho, r_i(\tau), \tau, r_j(\tau))| + |D_x^1 G(\rho, x_0, \tau, r_j(\tau))| \cdot |r_i(\tau) - r_i(\rho)| \le C(\tau - \rho)^{-\frac{1}{2}+\frac{\alpha}{2}}$$

(x_0 is a point in the open interval with endpoints $r_i(\tau)$ and $r_i(\rho)$) from which it follows that $J_{ij}^{(2)}(s,\tau) \to 0$ as $s \uparrow \tau$.

Hence,

$$I_{ij}^{(k)}(s,t) \equiv \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t V_j(\tau,t) J_{ij}^{(k)}(s,\tau) d\tau = \sqrt{\frac{2}{\pi}} \int_s^t V_j(\tau,t) \frac{\partial}{\partial s} J_{ij}^{(k)}(s,\tau) d\tau \tag{12}$$

if $k=1$, $i \neq j$ or if $k=2$. If $k=1$ and $i=j$, then $I_{ij}^{(k)}(s,t) = -\frac{V_i(s,t)}{\sqrt{b(s,r_i(s))}}$.

Let us show that the relation (12) is true also for $k=3$. For this it suffices to prove that

$$\lim_{s \uparrow \tau} J_{ij}^{(3)}(s,\tau) = 0. \tag{13}$$

Let us denote by $K_{ij}^{(31)}$ the first term in the expression for $K_{ij}^{(3)}$ and by $J_{ij}^{(31)}$ the integral $J_{ij}^{(3)}$ with $K_{ij}^{(3)}$ replaced by $K_{ij}^{(31)}$. In view of (5) and (6), we may verify (13) only for $J_{ij}^{(31)}$.

Write $J_{ij}^{(31)}$ in the form $J_{ij}^{(31)}(s,\tau) = L_{ij}^{(1)}(s,\tau) + L_{ij}^{(2)}(s,\tau) + L_{ij}^{(3)}(s,\tau)$, $i=1,2$, $j=1,2$, where

$$\begin{aligned} L_{ij}^{(1)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} d\rho \Bigg[\int_{D_\rho} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-\rho)}\right\} \mu_i(\rho,dy) \\ - \int_{D_s} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-\rho)}\right\} \mu_i(s,dy) \Bigg], \end{aligned}$$

$$\begin{aligned} L_{ij}^{(2)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_{D_s} \Bigg[\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\} \\ - \exp\left\{-\frac{(y-r_j(s))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\} \Bigg] R_j(s,\tau,y)\mu_i(s,dy), \end{aligned}$$

$$L_{ij}^{(3)}(s,\tau) = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}} \int_{D_s} \exp\left\{-\frac{(y-r_j(s))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\} R_j(s,\tau,y)\mu_i(s,dy),$$

and $R_j(s,\tau,y)$ denotes the integral

$$R_j(s,\tau,y) = \int_s^\tau (\rho-s)^{-\frac{1}{2}}(\tau-\rho)^{-\frac{1}{2}} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)} \cdot \frac{\rho-s}{\tau-\rho}\right\} d\rho,$$

which after the change of variables $z = \frac{\rho-s}{\tau-\rho}$ takes on the form

$$R_j(s,\tau,y) = \int_0^\infty z^{-\frac{1}{2}}(z+1)^{-1} \exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)} \cdot z\right\} dz,$$

and so

$$|R_j(s,\tau,y)| \leq C. \tag{14}$$

From this and IV it follows immediately that

$$|L_{ij}^{(1)}(s,\tau)| \le C(\tau - s)^{\frac{1+\alpha}{2}}, \tag{15}$$

$$|L_{ij}^{(3)}(s,\tau)| \le C\left(\mu_i(s, U_\delta(r_j(s))) + \exp\left\{-\frac{\delta^2}{2B(\tau - s)}\right\}\right), \tag{16}$$

where $U_\delta(r_j(s)) = \{y \in D_s : |y - r_j(s)| < \delta\}$, δ is any positive constant, B is the constant from I. Applying the mean value theorem to the difference of exponents within the braces in the expression for $L_{ij}^{(2)}$, we get, after using the condition V as well as the estimate (14) and the inequality $\sigma^\nu \exp\{-c\sigma\} \le C$ $(0 \le \sigma < \infty, 0 \le \nu < \infty)$,

$$|L_{ij}^{(2)}(s,\tau)| \le C(\tau - s)^{\frac{\alpha}{2}}. \tag{17}$$

The estimates (15)–(17) imply that $J_{ij}^{(31)}(s,\tau) \to 0$ as $s \uparrow \tau$. This completes the proof of (13). Thus, the relation (12) holds also for $k = 3$.

Let us apply the operator $\mathcal{E}$ to the right-hand side of (11). In order to simplify the expression for $\Upsilon_i(s,t) \equiv \mathcal{E}(s,t)\Phi_i(s,t)$ we need to prove the following two relations:

$$\Phi_i(s,t) \to 0 \text{ as } s \uparrow t, \tag{18}$$

$$|\Phi_i(s,t) - \Phi_i(\widetilde{s},t)| \le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s - \widetilde{s})^{\frac{1+\alpha}{2}}, \quad 0 \le \widetilde{s} < s < t \le T. \tag{19}$$

Passing to the limit as $s \uparrow t$ in the expression for Φ_i $(i = 1, 2)$, and recalling that the Poisson potential u_0 satisfies the condition (2), we get the expression which equals the left side of (10) taken with the opposite sign and which therefore vanishes. Thus (18) holds.

We proceed to prove (19). Write the difference $\Phi_i(s,t) - \Phi_i(\widetilde{s},t)$ in the form

$$\begin{aligned}\Phi_i(s,t) - \Phi_i(\widetilde{s},t) = &\int_{D_s} [u_0(s,y,t) - u_0(\widetilde{s},y,t)]\mu_i(s,dy) \\ &+ \left[\int_{D_s} u_0(\widetilde{s},y,t)\mu_i(s,dy) - \int_{D_{\widetilde{s}}} u_0(\widetilde{s},y,t)\mu_i(\widetilde{s},dy)\right] \\ &+ [u_0(\widetilde{s}, r_i(\widetilde{s}), t) - u_0(s, r_i(\widetilde{s}), t)] + [u_0(s, r_i(\widetilde{s}), t) - u_0(s, r_i(s), t)]\end{aligned} \tag{20}$$

and note that for $\widetilde{s} < s$

$$\begin{aligned}|u_0(s,y,t) - u_0(\widetilde{s},y,t)| &= |u_0(s,y,t) - u_0(\widetilde{s},y,t)|^{\frac{1+\alpha}{2}}|u_0(s,y,t) - u_0(\widetilde{s},y,t)|^{\frac{1-\alpha}{2}} \\ &\le \left|\left.\frac{\partial u_0(\hat{s},y,t)}{\partial \hat{s}}\right|_{\hat{s}=\widetilde{s}+\theta(s-\widetilde{s})} \cdot (s-\widetilde{s})\right|^{\frac{1+\alpha}{2}} (|u_0(s,y,t)| + |u_0(\widetilde{s},y,t)|)^{\frac{1-\alpha}{2}} \\ &\le C\|\varphi\|\left[(t - \widetilde{s} - \theta(s-\widetilde{s}))^{-1}(s-\widetilde{s})\right]^{\frac{1+\alpha}{2}} \\ &\le C\|\varphi\|\left[((t-s)+(s-\widetilde{s})(1-\theta))^{-1}(s-\widetilde{s})\right]^{\frac{1+\alpha}{2}} \\ &\le C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}}, \quad 0 < \theta < 1.\end{aligned}$$

Using this inequality for differences $u_0(s,y,t) - u_0(\widetilde{s},y,t)$, $u_0(\widetilde{s}, r_i(\widetilde{s}), t) - u_0(s, r_i(\widetilde{s}), t)$ and the condition IV to estimate the difference of integrals in the second line of the expression (20) as well as the Lagrange formula together with the condition V and the inequality (7) (with $r = 0$, $p = 1$) to estimate the last term $u_0(s, r_i(\widetilde{s}), t) - u_0(s, r_i(s), t)$ in (20), we arrive at (19).

Taking into account (18) and (19) we see thus that the application of the operator $\mathcal{E}$ to the function Φ_i gives

$$\Upsilon_i(s,t) = \frac{1}{\sqrt{2\pi}} \int_s^t (\rho - s)^{-\frac{3}{2}} [\Phi_i(\rho, t) - \Phi_i(s,t)] d\rho - \sqrt{\frac{2}{\pi}} (t-s)^{-\frac{1}{2}} \Phi_i(s,t). \tag{21}$$

Having considered the action of the operator $\mathcal{E}$ on both sides of (11), we can now write the system of Volterra integral equations of the second kind for the unknowns V_i, $i = 1, 2$, which is equivalent to (11) and has the form

$$V_i(s,t) = \sum_{j=1}^{2} N_{ij}(s,\tau) V_j(\tau, t) d\tau + \Psi_i(s,t), \quad 0 \le s < t \le T, \, i = 1, 2, \tag{22}$$

where

$$\begin{aligned}
\Psi_i(s,t) &= -\sqrt{b(s, r_i(s))}\, \Upsilon_i(s,t), \\
N_{ii}(s,\tau) &= \sqrt{\frac{2b(s, r_i(s))}{\pi}} \frac{\partial}{\partial s} \big(J_{ii}^{(2)}(s,\tau) - J_{ii}^{(3)}(s,\tau) \big), \quad i = j, \\
N_{ij}(s,\tau) &= \sqrt{\frac{2b(s, r_i(s))}{\pi}} \frac{\partial}{\partial s} J_{ij}(s,\tau), \quad i \ne j.
\end{aligned}$$

Note that from (21), (19) and (7) (with $r = p = 0$), it follows that

$$|\Psi_i(s,t)| \le C \|\varphi\| (t-s)^{-\frac{1}{2}}.$$

Unfortunately, the kernels N_{ij} do not have a weak singularity. We can not find the estimate for $N_{ij}(s,\tau)$ better than $C(\tau - s)^{-1}$. However this difficulty arises due to only one term

$$\int_{U_\delta(r_j(s))} \frac{\partial}{\partial y} Z_0(s, y, \tau, r_j(\tau)) \mu_i(s, dy)$$

which appears after writing $\frac{\partial}{\partial s} J_{ij}^{(31)}(s,\tau)$ in the form

$$\begin{aligned}
\frac{\partial}{\partial s} J_{ij}^{(31)}(s,\tau) &= \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} \Big(\int_{D_\rho} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(\rho, dy) \\
&- \int_{D_{s_0}} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(s_0, dy) \Big) \Big|_{s_0 = s} + \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int_{D_{s_0}} Z_0(\rho, y, \tau, r_j(\tau)) \mu_i(s_0, dy) \Big|_{s_0 = s}
\end{aligned}$$

and then taking the derivative of the last term in this expression. Namely,

$$\frac{\partial}{\partial s}\int_s^\tau (\rho-s)^{-\frac{1}{2}}d\rho \int_{D_{s_0}} Z_0(\rho,y,\tau,r_j(\tau))\mu_i(s_0,dy)\Big|_{s_0=s} = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}}$$
$$\times\frac{\partial}{\partial s}\int_{D_{s_0}}\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\right\}R_j(s,\tau,y)\mu_i(s_0,dy)\Big|_{s_0=s} = \frac{1}{\sqrt{2\pi b(\tau,r_j(\tau))}}$$
$$\times\frac{\partial}{\partial s}\int_{D_{s_0}}\mu_i(s_0,dy)\int_0^\infty z^{-\frac{1}{2}}(z+1)^{-1}\exp\left\{-\frac{(y-r_j(\tau))^2}{2b(\tau,r_j(\tau))(\tau-s)}\cdot(z+1)\right\}dz\Big|_{s_0=s}$$
$$=\sqrt{\frac{\pi b(\tau,r_j(\tau))}{2}}\int_{D_s}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy)=\sqrt{\frac{\pi b(\tau,r_j(\tau))}{2}}$$
$$\times\Bigg(\int_{U_\delta(r_j(s))}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy)+\int_{D_s\setminus U_\delta(r_j(s))}\frac{\partial}{\partial y}Z_0(s,y,\tau,r_j(\tau))\mu_i(s,dy)\Bigg).$$

All other terms in the expression for N_{ij} can be estimated by $C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}$, where $C(\delta)$ is the positive constant depending on δ.

Despite the strong singularity of kernels N_{ij}, the system of equations (22) has a solution and this solution can be found by the method of successive approximations:

$$V_i(s,t)=\sum_{n=0}^\infty V_i^{(n)}(s,t),\quad 0\le s<t\le T,\ i=1,2, \tag{23}$$

where

$$V_i^{(0)}(s,t)=\Psi_i(s,t),\qquad V_i^{(n)}(s,t)=\sum_{j=1}^2\int_s^t N_{ij}(s,\tau)V_j^{(n-1)}(\tau,t)d\tau,\quad n=1,2,\dots.$$

The convergence of series (23) and so the existence of the function V_i follows from the next inequality

$$|V_i^{(n)}(s,t)|\le C\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^n C_n^k a^{(n-k)}m^k,\quad 0\le s<t\le T,\ i=1,2, \tag{24}$$

where

$$a^{(k)}=\frac{\left(2C(\delta_0)T^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})\right)^k\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1+k\alpha}{2}\right)},\quad k=0,1,\dots,n,$$
$$m=\max_{s\in[0,T]}\left\{\sum_{j=1}^2\mu_i(s,U_{\delta_0}(r_j(s))),\ i=1,2\right\}$$

and the constant $\delta=\delta_0$ is chosen to be sufficiently small so that $m<1$. One can prove the estimate (24) by induction and by using the scheme analogous to those used in the proofs of (15), (16) and (17). Note also that the similar scheme was used in [10] in the study of the system of Volterra integral equations of the second kind with strong singularity in the kernels.

From (24) it also follows that the function $V_i(s,t)$, $i=1,2$, satisfies the inequality

$$|V_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}. \tag{25}$$

Thus, the formula (23) represents the unique solution of (22), which is continuous in the domain $0 \le s < t \le T$ and satisfies the inequality (25).

From estimates (5) (with $r=p=0$) and (25) it follows that there exist the simple-layer potentials $u_{i1}(s,x,t)$, $i=1,2$, in (8), and for them the condition $u_{i1}(s,x,t) \to 0$ if $s \uparrow t$ and the inequality

$$|u_{i1}(s,x,t)| \le C\|\varphi\|, \quad (s,x) \in \overline{S}_t, \tag{26}$$

hold. It is obvious (see (7)) that the same inequality is also true for the Poisson potential $u_0(s,x,t)$ in (8) and thus for the function $u(s,x,t)$ as well. Recalling that $u_0(s,x,t) \to \varphi(x)$ if $s \uparrow t$ and that the functions $u_0(s,x,t)$ and $u_{i1}(s,x,t)$ satisfy equation (1) in the domain $(s,x) \in S_t$ we conclude that $u(s,x,t)$ is the desired classical solution of problem (1)-(3).

Let us prove the uniqueness of the solution of the problem (1)-(3). Suppose that the problem (1)-(3) has two solutions $u_1(s,x,t)$ and $u_2(s,x,t)$ which are continuous in $\overline{S}_t$. Then the function $\overline{u} \equiv u_1 - u_2$ satisfies equation (1), the initial condition (2) with $\varphi \equiv 0$ and two boundary conditions

$$u(s,r_i(s),t) = g_i(s,t), \quad 0 \le s < t \le T,\ i=1,2,$$

where

$$g_i(s,t) = \int\limits_{D_s} \overline{u}(s,y,t)\mu_i(s,dy).$$

The above problem is the first boundary value problem and since the function g_i is continuous in s, it has a unique classical solution, continuous in $\overline{S}_t$, which can be represented in the form (8) with $\varphi \equiv 0$. Thus, the function $\overline{u}$ can be expressed in the form (8) where there are no Poisson potential and V_i are the unknown functions, continuous in $s \in [0,t)$, which are determined by $g_i(s,t)$. Further, if we repeat the considerations of this section concerning the construction of solution of the problem (1)-(3), we obtain the system (22) with $\Psi_i \equiv 0$ for the unknowns V_i. Then $V_i \equiv 0$ and hence $\overline{u} \equiv 0$. This completes the proof of the uniqueness.

Thus we have proved the following theorem:

Theorem 1. *Let conditions I-V hold. Then problem* (1)-(3) *has a unique classical solution, continuous in $\overline{S}_t$ for all $t \in (0,T]$. Furthermore, this solution has the form* (8) *and satisfies the inequality* (26).

2 FELLER SEMIGROUP

Suppose that the conditions I-V hold and consider the two-parameter family of linear operators T_{st}, $0 \le s < t \le T$, acting on the function $\varphi \in C_b(\mathbb{R})$ by the rule:

$$T_{st}\varphi(x) = \int\limits_{\mathbb{R}} G(s,x,t,y)\varphi(y)dy + \sum_{j=1}^{2} \int\limits_{s}^{t} G(s,x,\tau,r_j(\tau))V_j(\tau,t)d\tau, \tag{27}$$

where the pair of functions (V_1,V_2) is the solution of (22). Recall that the function V_i $(i=1,2)$ has the form (23) and satisfy the inequality (25).

We introduce the subspace $C_0(\mathbb{R})$ of $C_b(\mathbb{R})$ which consists of all functions $\varphi \in C_b(\mathbb{R})$ for which the fitting conditions in III holds. Since the subspace $C_0(\mathbb{R})$ is closed in $C_b(\mathbb{R})$, it is a Banach space. Furthermore, it is invariant under the operators T_{st}, i.e.,

$$\varphi \in C_0(\mathbb{R}) \Longrightarrow T_{st}\varphi \in C_0(\mathbb{R}).$$

Let us study properties of the family of operators T_{st} in $C_0(\mathbb{R})$.

First we note that if the sequence $\varphi_n \in C_b(\mathbb{R})$ is such that $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$ and, in addition, $\sup\limits_n \|\varphi_n\| < \infty$, then $\lim\limits_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $0 \le s < t \le T$, $x \in \overline{D}_s$. The proof of this property is based on well known assertions of calculus on passage of the limit under the summation and integral signs (here this concerns series (23) and integrals on the right-hand side of (8)). This property allows us to prove the following properties of the operator family T_{st} without loss of generality, under the assumption that the function φ has a compact support.

Now we prove that the operators T_{st}, $0 \le s < t \le T$, remain the cone of nonnegative functions invariant.

Lemma 1. *If $\varphi \in C_0(\mathbb{R})$ and $\varphi(x) \ge 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \ge 0$ for all $x \in \overline{D}_s$, $0 \le s < t \le T$.*

Proof. Let φ be any nonnegative function in $C_0(\mathbb{R})$ with a compact support. Denote by γ the minimum of $T_{st}\varphi(x)$ in $\overline{S}_t$ and assume that $\gamma < 0$. From the minimum principle [3, Ch.II] it follows that the value γ may be attained only when $s \in (0,t)$ and $x = r_i(s)$, $i = 1,2$. Fix $s_0 \in (0,t)$ and $i_0 \in \{1,2\}$ for which $T_{s_0t}\varphi(r_{i_0}(s_0)) = \gamma$. But then

$$\int\limits_{D_{s_0}} [T_{s_0t}\varphi(r_{i_0}(s_0)) - T_{s_0t}\varphi(y)]\mu_{i_0}(s_0, dy) < 0$$

which contradicts (3). Therefore $\gamma \ge 0$ and the assertion of the lemma follows. □

Note also that $T_{st}\varphi_0(x) = 1$ for all $0 \le s < t \le T$, $x \in \overline{D}_s$ if $\varphi_0 \equiv 1$. This property together with the assertion of lemma 1 allow us to assert that operators T_{st} are contractive, i.e.,

$$\|T_{st}\varphi\| \le \|\varphi\|$$

for all $0 \le s < t \le T$.

Finally, we show that the operator family T_{st} has the semigroup property

$$T_{st} = T_{s\tau}T_{\tau t}, \quad 0 \le s < \tau < t \le T.$$

This property is a consequence of the assertion of uniqueness of the solution of the problem (1)-(3). Indeed, to find $u(s,x,t) = T_{st}\varphi(x)$, when it is given that $u(s,x,t) \to \varphi(x)$ as $s \uparrow t$, one can solve the problem first in time interval $[\tau, t]$ and then solve it in the time interval $[s, \tau]$ with that initial function $u(\tau, x, t) = T_{\tau t}\varphi(x)$ which was obtained; in other words, $T_{st}\varphi(x) = T_{s\tau}(T_{\tau t}\varphi)(x)$, $\varphi \in C_0(\mathbb{R})$ or $T_{st} = T_{s\tau}T_{\tau t}$.

The above properties of operators T_{st} imply the following assertion (see [1, Ch.II, §1]).

Theorem 2. *Let conditions I-V hold. Then the two-parameter family of operators T_{st}, $0 \le s < t \le T$, defined by formula* (27) *describes the inhomogeneous Feller process in $\mathbb{R}$ which trajectories are located in curvilinear domain $\overline{S}_T$. In $\overline{S}_T \setminus \mathcal{C}$, the trajectories of this process can be treated as the trajectories of the diffusion process generated by the operator L_s and at the points of curves $\mathcal{C}_i$ $(i = 1, 2)$ they behave according to boundary conditions in* (3).

REFERENCES

[1] Dynkin E.B. Markov processes. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. (in Russian)

[2] Feller W. *The parabolic differential equations and associated semi-groups of transformations*. Ann. of Math. 1952, **55**, 468–519. doi:10.2307/1969644

[3] Friedman A. Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[4] Holmgren E. *Sur une application de l'équation de M. Volterra*. Arkiv för Matematik, Astronomi och Fysik 1907, **3** (12), 1–4.

[5] Kamynin L. I. *A boundary value problem in the theory of heat conduction with a nonclassical boundary condition*. Zh. Vychisl. Mat. Mat. Fiz. 1964, **4** (6), 1006–1024.

[6] Kopytko B.I., Shevchuk R.V. *On Feller semigroup generated by solution of nonlocal parabolic conjugation problem*. Carpathian Math. Publ. 2018, **10** (2), 333–345. doi:10.15330/cmp.10.2.333-345

[7] Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N. Linear and Quasilinear Equations of Parabolic Type. Nauka, Moscow, 1967. (in Russian)

[8] Pilipenko A.Yu. *On the Skorokhod mapping for equations with reflection and possible jumplike exit from a boundary*. Ukrainian Math. J. 2012, **63** (9), 1415–1432. (in Russian)

[9] Portenko M.I. Diffusion Processes in Media with Membranes. Proceedings of the Institute of Mathematics of the National Academy of Sciences of the Ukraine, Kyiv, 1995. (in Ukrainian)

[10] Shevchuk R.V. *Pasting of two one-dimensional diffusion processes*. Annales Mathematicae et Informaticae 2012, **39**, 225–239.

[11] Skubachevskii A.L. *On Feller semigroups for multidimensional diffusion processes*. Dokl. AN SSSR 1995, **341** (2), 173–176. (in Russian)

[12] Wentzell A.D. *Semigroups of operators that correspond to a generalized differential operator of second order*. Dokl. AN SSSR 1956, **111** (2), 269–272. (in Russian)

[13] Wentzell A.D. *On boundary conditions for multi-dimensional diffusion processes*. Theory Probab. Appl. 1959, **4** (2), 164–177. doi:10.1137/1104014

Received 21.06.2019

Шевчук Р.В., Савка І.Я., Нитребич З.М. *Нелокальна крайова задача для одновимірного оберненого рівняння Колмогорова і пов'язана з нею напівгрупа операторів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 463–474.

Стаття присвячена вивченню методами теорії диференціальних рівнянь в частинних похідних проблеми побудови напівгруп Феллера, які описують одновимірні дифузійні процеси в областях із заданими крайовими умовами. У цій статті ми досліджуємо крайову задачу для одновимірного лінійного параболічного рівняння другого порядку (оберненого рівняння Колмогорова) у криволінійній обмеженій області з одним із варіантів нелокальної крайової умови типу Феллера-Вентцеля. Ми зосереджуємо увагу на випадку, коли крайова умова Феллера-Вентцеля містить лише компоненту інтегрального типу. Класичну розв'язність останньої задачі одержано нами методом граничних інтегральних рівнянь з використанням фундаментального розв'язку оберненого рівняння Колмогорова і породжених ним параболічних потенціалів. Цей розв'язок використано для побудови напівгрупи Феллера, яка описує явище дифузії в обмеженій області з властивістю повернення дифундуючої частинки в середину області стрибками.

Ключові слова і фрази: параболічний потенціал, метод граничних інтегральних рівнянь, напівгрупа Феллера, нелокальна крайова умова.

ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 475–492
doi:10.15330/cmp.11.2.475-492

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.475–492

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A NEW GENERALIZATION OF α-TYPE ALMOST-F-CONTRACTIONS AND α-TYPE F-SUZUKI CONTRACTIONS IN METRIC SPACES AND THEIR FIXED POINT THEOREMS

In this paper a new generalization of α-type almost-F-contractions and an extension of α-type F-Suzuki contractions are given. Moreover, some new fixed point theorems of them are discussed. Some examples and applications in order to illustrate the main results are presented. The results of this article can be considered as improvements of some well-known results appeared in the literature.

Key words and phrases: α-type almost-F-contraction, α-type F-Suzuki contraction.

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1 INTRODUCTION

After innovation of the Banach contraction principle [2], fixed point theory, which was one of the most celebrated tool in nonlinear analysis, acquires a distinguished role in research activity. Due to its applications in the nonlinear integro-differential equations, nonlinear Volterra integral equations, game theory etc, existence of a fixed point for contraction type mappings in metric spaces have been considered by many authors. see, for instance, [4,12,13,17,19,22,23] and the references therein.

During the past decades, scholars extend this principle towards different contractions. Specially, in 2012, Wardowski [24] generalized it interestingly by introducing a new type of contractions called F-contractions. After presentation of F-contractions, many authors extended them in various forms. Some extensions and generalizations are obtained in [1,6–11,14–21,25]. Wardowski and Van Dung [25] (also independently Minak et al. [14]) with using Ćirić-type generalized contraction [5] in definition of F-contractions, introduced the notion of F-weak contractions and utilize the same to generalize the main result of [24].

Very recently (in 2016) Gopal et al. [7] generalized it by introducing the concept of α-type F-contraction. On the other hand, In 2014 Piri and Kumam [16] extended the results of Wardowski [24] by introducing the concept of an F-Suzuki contraction. Also, in the same year, Minak et al. [14] introduced a new concept of an almost-F-contraction. Most recently (in 2016) Budhia et al. [3] introduced the new concepts of an α-type almost-F-contraction and an α-type F-Suzuki contraction and proved some fixed point theorems concerning such contractions. In this research, we extended the results of [7] and [3], by introducing a new type of contractions that is called α-type almost-F-weak contraction and an α-type F-weak Suzuki contraction.

УДК 517.988.523
2010 *Mathematics Subject Classification:* 47H10, 54H25, 37C25.

2 PRELIMINARIES

Here, we express a series of definitions of some fundamental notions.

First, let us, following [24], denote with $\mathcal{F}$ the family of all functions $F : (0, +\infty) \to \mathbb{R}$ that satisfy the following conditions:

(F1) F is strictly increasing,

(F2) for every sequence $\{\alpha_n\}$ in $(0, +\infty)$, we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\to\infty} \alpha_n = 0$,

(F3) there exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = -\infty$.

And following [20], denote by $\mathcal{G}$ the collection of all functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

(G1) F is strictly increasing,

(G2) there exists a sequence $\{\alpha_n\}$ in $(0, +\infty)$ such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$, or $\inf F = -\infty$,

(G3) F is a continuous map.

Example 1 ([3]). *The following functions belong to* $\mathcal{F}$*:*

$$F(\alpha) = \ln \alpha, \ \ F(\alpha) = \ln \alpha + \alpha, \ \ F(\alpha) = -\frac{1}{\sqrt{\alpha}},$$

and the following functions $F : (0, +\infty) \to \mathbb{R}$ *belongs to* $\mathcal{G}$*:*

$$F(\alpha) = \ln \alpha, \ \ F(\alpha) = -\frac{1}{\alpha} + \alpha, \ \ F(\alpha) = -\frac{1}{\alpha}.$$

Definition 1 ([24]). *Let (X, d) be a metric space. The mapping $T : X \to X$ is called an F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Example 2 ([24], Example 2.1). *It is easy to verify that every Banach contraction is an F-contraction with $F(t) = \ln t$ and $\tau = \ln r$. For more details and examples see [24].*

Definition 2 ([25]). *Let (X, d) be a metric space. The mapping $T : X \to X$ is called an F-weak contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have*

$$\tau + F(d(Tx, Ty)) \leq F(m(x, y)),$$

where

$$m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Remark 1. *Every F-contraction is an F-weak contraction but converse is not necessarily true [25].*

Definition 3 ([25]). *Let (X, d) be a metric space and $\alpha : X \times X \to (0, +\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is called an α-type F-contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have*

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 4 ([25]). *Let (X,d) be a metric space and $\alpha : X \times X \to (0,+\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is called an α-type F-weak contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx,Ty) > 0$, we have*

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y)),$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Remark 2. *Every α-type F-contraction is an α-type F-weak contraction but the converse is not necessarily true.*

Remark 3. *It is clear that every F-weak contraction is an α-type F-weak contraction with $\alpha(x,y) = 1$, for all $x, y \in X$. But every α-type F-weak contraction is not necessarily an F-weak contraction. For example, see ([25], Example 3.4).*

Definition 5 ([14]). *Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be an almost-F-contraction, if there exist $F \in \mathcal{F}$, $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \;\Rightarrow\; \tau + F(d(Tx,Ty)) \leq F(d(x,y) + Ld(y,Tx))$$

and

$$d(Tx,Ty) > 0 \;\Rightarrow\; \tau + F(d(Tx,Ty)) \leq F(d(x,y) + Ld(x,Ty)).$$

Remark 4. *Every F-contraction is an almost-F-contraction with $L = 0$, but the converse is not necessarily true [14]. Also, it is obvious that every F-weak contraction is an α-type F-weak contraction with $\alpha(x,y) = 1$, for all $x, y \in X$, but the converse is not necessarily true. For examples, see [14].*

Definition 6 ([3]). *Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be an α-type almost-F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ and $L \geq 0$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \;\Rightarrow\; \tau + F(d(Tx,Ty)) \leq F(d(x,y) + Ld(y,Tx))$$

and

$$d(Tx,Ty) > 0 \;\Rightarrow\; \tau + F(d(Tx,Ty)) \leq F(d(x,y) + Ld(x,Ty)).$$

Remark 5. *Every almost-F-contraction is an α-type almost-F-contraction with $\alpha(x,y) = 1$, for all $x, y \in X$. But the converse is not necessarily true. For some examples, see [3, Example 3.1].*

Definition 7 ([16]). *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-Suzuki contraction if there exist $F \in \mathcal{G}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \textit{implies that} \quad \tau + F(d(Tx,Ty)) \leq F(d(x,y)).$$

Definition 8 ([3]). *Let (X,d) be a metric space and $\alpha : X \times X \to (0,+\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is said to be an α-type F-Suzuki contraction if there exist $F \in \mathcal{G}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \textit{implies that} \quad \tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(d(x,y)).$$

Remark 6. *Every α-type F-Suzuki contraction is an F-Suzuki contraction with $\alpha(x,y)=1$, for all $x,y \in X$. But the converse is not necessarily true. For example, see [3, Example 3.2].*

Definition 9 ([19])**.** *Let $\alpha : X \times X \to (0,+\infty)$ be a given mapping. The mapping $T : X \to X$ is said to be an α-admissible, whenever $\alpha(Tx,Ty) \geq 1$ provided $\alpha(x,y) \geq 1$ and $x,y \in X$.*

Definition 10. *An α-admissible map T is said to have the K-property, if for each sequence $\{x_n\} \subseteq X$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, there exists a natural number k such that $\alpha(Tx_n, Tx_m) \geq 1$, for all $m > n \geq k$.*

We state the following lemmas which are useful in proving our main results.

Lemma 1 ([16])**.** *Let $F : (0,+\infty) \to \mathbb{R}$ be an increasing function and $\{\alpha_n\}$ be a sequence of positive real numbers. Then, the following holds:*

(a) if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$,

(b) if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Lemma 2 ([4])**.** *Let (X,d) be a metric space, and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence then there exists $\varepsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ and*

(1) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$,

(2) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$,

(3) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon$,

(4) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \varepsilon$.

3 MAIN RESULTS

In this section, two new contractions are introduced. In the first part of this section, the concept of an α-type almost-F-weak contraction is defined in metric spaces. And in the second part the concept of an α-type F-weak Suzuki contraction is introduced. Some fixed point theorems for these contractions are proved and suitable examples are furnished to demonstrate the validity of the hypotheses of our results and reality of our generalizations.

We commence our main result with the following definition.

Definition 11. *Let (X,d) be a metric space and $\alpha : X \times X \to (0,+\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is said to be an α-type almost-F-weak contraction (for simplicity we write almost-α F-weak contraction), if there exist $F \in \mathcal{F}$, $\tau > 0$ and $L \geq 0$ such that $d(Tx,Ty) > 0$ implies that*

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y) + LN_1(x,y)),$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\},$$

and

$$N_1(x,y) = \min\{d(x,Ty), d(y,Tx)\}.$$

Example 3. *Let $X = \{(0,0),(0,4),(5,0),(4,5)\}$ be endowed with the metric d defined by*

$$d\Big((x_1,x_2),(y_1,y_2)\Big) = |x_1 - y_1| + |x_2 - y_2|.$$

It is easy to see that (X,d) is a complete metric space.

Suppose that $T : X \to X$ is defined as follows :

$$T(0,0) = T(5,0) = T(0,4) = (0,0), T(4,5) = (5,0).$$

Furthermore, suppose $\alpha((x_1,x_2),(y_1,y_2)) = 1$, for all $(x_1,x_2),(y_1,y_2) \in X$. It is easily verified that, for each $F \in \mathcal{F}$, the mapping T is not an α-type almost-F-contraction. Indeed, for any $\tau > 0$ and $F \in \mathcal{F}$, we have

$$\tau + \alpha\Big((0,4),(4,5)\Big)F\Big(d\Big(T(0,4),T(4,5)\Big)\Big) = \tau + F\Big(d\Big((0,0),(5,0)\Big)\Big) = \tau + F(5).$$

On the other hand, we have

$$F\Big(d\Big((0,4),(4,5)\Big) + Ld\Big((4,5),T(0,4)\Big)\Big) = F(5).$$

And $\tau + F(5) > F(5)$. So, T is not an α-type almost-F-contraction. But, one can easily see that, for $0 < \tau < \ln\frac{6}{5}$ and $F(t) = \ln t$, if $d\Big(T(x_1,x_2),T(y_1,y_2)\Big) > 0$ then

$$\begin{aligned}\tau + \alpha\Big((x_1,x_2),(y_1,y_2)\Big)F\Big(d\Big(T(x_1,x_2),T(y_1,y_2)\Big)\Big) \leq F\Big(m\Big((x_1,x_2),(y_1,y_2)\Big) +\\ LN_1\Big((x_1,x_2),(y_1,y_2)\Big)\Big),\end{aligned} \tag{1}$$

where

$$\begin{aligned} m\Big((x_1,x_2),(y_1,y_2)\Big) &= \max\Big\{d\Big((x_1,x_2),(y_1,y_2)\Big), d\Big((x_1,x_2),T(x_1,x_2)\Big),\\ &\quad d\Big((y_1,y_2),T(y_1,y_2)\Big), \frac{d\Big((x_1,x_2),T(y_1,y_2)\Big)+d\Big((y_1,y_2),T(x_1,x_2)\Big)}{2}\Big\},\end{aligned}$$

and

$$N_1\Big((x_1,x_2),(y_1,y_2)\Big) = \min\Big\{d\Big((x_1,x_2),T(y_1,y_2)\Big), d\Big((y_1,y_2),T(x_1,x_2)\Big)\Big\}.$$

For example $d\Big(T(0,4),T(4,5)\Big) = d\Big((0,0),(5,0)\Big) = 5 > 0$ and

$$\begin{aligned} m\Big((0,4),(4,5)\Big) &= \max\Big\{d\Big((0,4),(4,5)\Big), d\Big((0,4),T(0,4)\Big), d\Big((4,5),T(4,5)\Big),\\ &\quad \frac{d\Big((0,4),T(4,5)\Big)+d\Big((4,5),T(0,4)\Big)}{2}\Big\} = \max\{5,4,6,\tfrac{9+9}{2}\} = 9,\end{aligned}$$

and we have

$$\tau+\alpha\Big((0,4),(4,5)\Big)F\Big(d\Big(T(0,4),T(4,5)\Big)\Big)=\tau+F(5)<\ln\frac{6}{5}+\ln 5=\ln 6.$$

On the other hand, we have

$$F\Big(m\Big((0,4),(4,5)\Big)+LN_1\Big((0,4),(4,5)\Big)\Big)=F(9)=\ln 9.$$

Hence,

$$\tau+\alpha\Big((0,4),(4,5)\Big)F\Big(d\Big(T(0,4),T(4,5)\Big)\Big)<F\Big(m\ \Big(\ (0,4),(4,5)\Big)+LN_1\Big((0,4),(4,5)\Big)\Big).$$

Or for $(5,0)$ *and* $(4,5)$*, we have* $d\Big(\ T(5,0),T(4,5)\Big)=d\Big((0,0),(5,0)\Big)=5>0$ *and*

$$\begin{aligned} m\Big((5,0),(4,5)\Big) &= \max\Big\{d\Big((5,0),(4,5)\Big),d\Big((5,0),T(5,0)\Big),d\Big((4,5),T(4,5)\Big),\\ &\qquad \frac{d\Big(\ (5,0),T(4,5)\Big)+d\Big(\ (4,5),T(5,0)\Big)}{2}\Big\}=\max\{6,5,6,\tfrac{0+9}{2}\}=6, \end{aligned}$$

and we have

$$\tau+\alpha\Big((5,0),(4,5)\Big)F\Big(d\Big(T(5,0),T(4,5)\Big)\Big)=\tau+F(5)<\ln\frac{6}{5}+\ln 5=\ln 6.$$

On the other hand, we have

$$F\Big(m((5,0),(4,5))+LN_1((5,0),(4,5))\Big)=F(6)=\ln 6.$$

Hence,

$$\tau+\alpha\Big((5,0),(4,5)\Big)F\Big(d\Big(T(5,0),T(4,5)\Big)\Big)\le F\Big(m\Big((5,0),(4,5)\Big)+LN_1\Big((5,0),(4,5)\Big)\Big).$$

In the same manner, we can easily check that (1) is satisfied for $(0,0)$ *and* $(4,5)$*. Therefore,* T *is an almost-*α F*-weak contraction.*

Now, we present our first result.

Theorem 1. *Let* (X,d) *be a complete metric space,* $\alpha: X\times X\to(0,+\infty)\cup\{-\infty\}$ *be a symmetric function,* $F\in\mathcal{F}$ *and* $T: X\to X$ *be an almost-*α F*-weak contraction satisfying the following conditions:*

(i) T *is* α*-admissible,*

(ii) *there exists* $x_0\in X$ *such that* $\alpha(x_0,Tx_0)\ge 1$*,*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n\to x$ *as* $n\to\infty$ *and* $\alpha(x_n,x_{n+1})\ge 1$*, for all* $n\in\mathbb{N}$*, then* $\alpha(x_n,x)\ge 1$*, for all* $n\in\mathbb{N}$*.*

Then, if T or F is continuous then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. For any $n \in \mathbb{N}$, define:

$$x_{n+1} = T(x_n).$$

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$ then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$, for each $n \in \mathbb{N}$. Since T is α-admissible, one can easily obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \ \ \forall n \in \mathbb{N}. \tag{2}$$

Now since T is an almost-$\alpha F - weak\ contraction$, there exist $\tau > 0$ and $L \geq 0$ such that if $d(Tx, Ty) > 0$, then

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y) + LN_1(x,y)). \tag{3}$$

Therefore, by (2) and (3)

$$\begin{aligned}\tau + F(d(Tx_n, Tx_{n+1})) &\leq \tau + \alpha(x_n, x_{n+1})F(d(Tx_n, Tx_{n+1}))\\ &\leq F(m(x_n, x_{n+1}) + LN_1(x_n, x_{n+1})) \leq F(m(x_n, x_{n+1}) + Ld(x_{n+1}, Tx_n))\\ &= F(m(x_n, x_{n+1}) + 0) = F(m(x_n, x_{n+1})).\end{aligned}$$

Hence, we have

$$\tau + F(d(x_{n+1}, x_{n+2})) \leq F(m(x_n, x_{n+1})). \tag{4}$$

But

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \tfrac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\}\\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2})}{2}\right\}\\ &\leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right\}\\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.\end{aligned}$$

Now, if $d(x_{n_0+1}, x_{n_0+2}) \geq d(x_{n_0}, x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then

$$m(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0+1}, x_{n_0+2}),$$

and since F is strictly increasing,

$$F(m(x_{n_0}, x_{n_0+1})) \leq F(d(x_{n_0+1}, x_{n_0+2})),$$

so, it follow from (4)

$$\tau + F(d(x_{n_0+1}, x_{n_0+2})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

So, $\tau \leq 0$ is a contradiction. Consequently,

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \ \forall n \in \mathbb{N}. \tag{5}$$

Hence, from (4) and (5), we have

$$\tau + F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})),$$

or

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

In general, one can get

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_0, x_1)) - n\tau. \tag{6}$$

Hence $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. So, from (F_2) we have,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Therefore, with notice to (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) = 0.$$

Now, (6) implies that

$$(d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) \leq (d(x_n, x_{n+1}))^k (F(d(x_0, x_1)) - n\tau).$$

Then, it can be easily seen that

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^k = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \geq n_0.$$

Consequently, if $m > n > n_0$, then

$$d(x_n, x_m) \quad \leq \quad \sum_{i=n}^{m} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n_0}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since $k \in (0, 1)$, the series $\sum_{i=n_0}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. Therefore, $\{x_n\}$ is a Cauchy sequence, and since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. We claim that u is a fixed point of T.

To prove the claim, at first suppose that T is continuous, then we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(u),$$

and so u is a fixed point of T. Now, suppose that F is continuous and in contrary, suppose that $Tu \neq u$. Without lose of generality, one can assume that there exists $n_0 \in \mathbb{N}$ such that $Tx_n \neq Tu$, for all $n \geq n_0$. (Indeed, if $x_{n+1} = Tx_n = Tu$ for infinite values of n, then uniqueness of the limit concludes that $Tu = u$).

From (iii) and (4), we have

$$\begin{aligned} \tau + F(d(Tx_n, Tu)) \quad &\leq \quad \tau + \alpha(x_n, u)F(d(Tx_n, Tu)) \leq F(m(x_n, u) + LN_1(x_n, u)) \\ &\leq \quad F(m(x_n, u) + Ld(Tx_n, u)) = F(m(x_n, u) + Ld(x_{n+1}, u)) \end{aligned}$$

And since F is continuous, as $n \to \infty$ we get

$$\tau + F(d(u, Tu)) \leq F(\lim_{n\to\infty}(m(x_n, u) + Ld(x_{n+1}, u))), \tag{7}$$

where

$$m(x_n, u) = \max\left\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), \frac{d(x_n, Tu) + d(u, x_{n+1})}{2}\right\},$$

so,

$$\lim_{n\to\infty} m(x_n, u) = \max\left\{0, 0, d(u, Tu), \frac{d(u, Tu) + 0}{2}\right\} = d(u, Tu).$$

Also, we have

$$\lim_{n\to\infty} Ld(x_{n+1}, u) = 0.$$

Therefore, from (7) we have

$$\tau + F(d(u, Tu)) \leq F(d(u, Tu)),$$

which is contradicted by positivity of τ . So, $d(u, Tu) = 0$ i.e. $Tu = u$. □

The next result establishes a sufficient condition for uniqueness of fixed point.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist $F \in \mathcal{F}$ and $\tau > 0$ and $L \geq 0$ such that $d(Tx, Ty) > 0$ implies that*

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(m(x, y) + LN_2(x, y)), \tag{8}$$

where $m(x, y)$ is defined as in Definition 11 and

$$N_2(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}.$$

We further assume that $\alpha(x, y) \geq 1$, for each $x, y \in Fix(T)$. Then if T is satisfies the conditions (i), (ii) and (iii) of Theorem 1 and T or F is continuous, then T has a unique fixed point.

Proof. It is clear that T is an almost-α F-weak contraction. So, by Theorem 1, T has a fixed point.

Now, suppose that u and v are two fixed point of T. If $u \neq v$, then $d(Tu, Tv) > 0$. Also $\alpha(u, v) \geq 1$, because $u, v \in Fix(T)$, hence (8) implies that

$$\begin{aligned}
\tau + F(d(u, v)) &= \tau + F(d(Tu, Tv)) \leq \tau + \alpha(u, v)F(d(Tu, Tv)) \\
&\leq F(m(u, v) + LN_2(u, v)) \leq F(m(u, v) + Ld(u, Tu)) \\
&= F(m(u, v) + 0) = F(m(u, v)),
\end{aligned}$$

where

$$\begin{aligned}
m(u, v) &= \max\left\{d(u, v), d(u, Tu), d(v, Tv), \tfrac{d(u,Tv)+d(v,Tu)}{2}\right\} \\
&= \max\{d(u, v), 0, 0, \tfrac{d(u,v)+d(v,u)}{2}\} = d(u, v).
\end{aligned}$$

So, we have

$$\tau + F(d(u, v)) \leq F(d(u, v)),$$

which is contradicted by positivity of τ. So, $u = v$. □

Corollary 1 ([3], Theorem 3.1)**.** *Let (X, d) be a complete metric space and $T : X \to X$ be an α-type almost-F-contraction, where $F \in \mathcal{F}$, satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. It is enough to notice that T is an almost-α F-weak contraction in which $m(x, y) = d(x, y)$. One can prove this corollary by applying the proof of Theorem 1, without needing to continuity of T or F. □

The following corollaries are some obvious results of Theorem 1.

Corollary 2. *Let (X, d) be a complete metric space and $T : X \to X$ be an almost F-contraction. Then, T has a fixed point.*

Proof. In Theorem 1, put $\alpha(x, y) = 1$, for each $x, y \in X$. □

Corollary 3. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then, T has a unique fixed point.*

Proof. In the Theorem 1, put $\alpha(x, y) = 1$, for each $x, y \in X$, and $L = 0$. □

The following example shows that Theorem 1 is a generalization of the Theorem 3.1 in [3].

Example 4. *In the Example 3, we observed that the mapping T is not an α-type almost-F-contraction. So, T does not satisfy to Theorem 3.1 in [3]. But T is an almost-α F-weak contraction, and we can easily see that T satisfies all conditions of Theorem 1 and $(0, 0)$ is a fixed point of T. Also, all conditions of the Theorem 2 are satisfied and $(0, 0)$ is the unique fixed point of the map T.*

Here, to obtain our next results, we first introduce the following definition.

Definition 12. *Let (X, d) be a metric space and $\alpha : X \times X \to (0, +\infty) \cup \{-\infty\}$ be a symmetric function. The mapping $T : X \to X$ is said to be an α-type F-weak Suzuki contraction (for simplicity we write α F-weak Suzuki contraction) if there exists $F \in \mathcal{G}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \textit{implies that} \quad \tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(m(x, y)),$$

where $m(x, y)$ is defined as in Definition 11.

Example 5. *Let $X = \{0, 1, 2\}$ be endowed with the metric d defined by*

$$d(x, y) = |x - y|.$$

And $T : X \to X$ is defined as follows

$$T(1) = T(2) = 1 \quad \textit{and} \quad T(0) = 2.$$

Furthermore, suppose that $\alpha(x,y) = 1$, for all $x,y \in X$. It is easily verified that, for each $F \in \mathcal{F}$, the mapping T is not an α-type F-Suzuki contraction. Indeed, for any $\tau > 0$ and $F \in \mathcal{F}$, we have

$$\frac{1}{2}d(0,T0) = \frac{1}{2}d(0,2) = 1 = d(0,1),$$

and

$$\tau + \alpha(0,1)F(d(T0,T1)) = \tau + F(d(2,1)) = \tau + F(1).$$

On the other hand, we have

$$F(d(0,1)) = F(1).$$

And $\tau + F(1) > F(1)$. So, T is not an α-type F-Suzuki contraction. But one can easily see that, for $0 < \tau \le \ln 2$ and $F(t) = \ln t$, if $d(Tx,Ty) \neq 0$ then

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \textit{implies that} \quad \tau + \alpha(x,y)F(d(Tx,Ty)) \le F(m(x,y)), \tag{9}$$

where $m(x,y)$ is defined as in Definition 11. For example, $d(T(0),T(1)) = d(2,1) = 1$ and

$$m(0,1) = \max\left\{d(0,1), d(0,T0), d(1,T1), \frac{d(0,T1)+d(1,T0)}{2}\right\} = 2,$$

and we have

$$\tau + \alpha(0,1)F(d(T0,T1)) = \tau + F(1) \le \ln 2 + \ln 1 = \ln 2.$$

On the other hand, we have

$$F(m(0,1)) = F(2) = \ln 2.$$

Hence,

$$\tau + \alpha(0,1)F(d(T0,T1)) \le F(m(0,1)).$$

In the same manner, we can easily check that (9) is satisfied for $x = 0, y = 2$. Therefore, (9) is satisfied for any $x,y \in X$ which $d(Tx,Ty) \neq 0$. So, T is an α F-weak Suzuki contraction.

Theorem 3. *Let (X,d) be a complete metric space and $T : X \to X$ be an α F-weak Suzuki contraction, satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(iii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$,*

(iv) *T has the K-property.*

Then, T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. For any $n \in \mathbb{N} \cup \{0\}$, define:

$$x_{n+1} = T(x_n).$$

Since T is α-admissible, one can easily obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{10}$$

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N} \cup \{0\}$, i.e. $d(x_n, x_{n+1}) > 0$ and so

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x_{n+1}). \tag{11}$$

Now, since T is an α F-weak Suzuki contraction, there exist $F \in \mathcal{G}$ and $\tau > 0$ such that if $d(Tx, Ty) > 0$, then

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{implies that} \quad \tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(m(x, y)), \tag{12}$$

where $m(x, y)$ is defined as in Definition 11.
Therefore, by (11) and (12)

$$\begin{aligned} \tau + F(d(Tx_n, Tx_{n+1})) &\leq \tau + \alpha(x_n, x_{n+1})F(d(Tx_n, Tx_{n+1})) \\ &\leq F(m(x_n, x_{n+1})), \end{aligned} \tag{13}$$

in which

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \tfrac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Now, if $d(x_{n_0+1}, x_{n_0+2}) \geq d(x_{n_0}, x_{n_0+1})$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then

$$m(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0+1}, x_{n_0+2}),$$

and since F is strictly increasing,

$$F(m(x_{n_0}, x_{n_0+1})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

Therefore by (13)

$$\tau + F(d(x_{n_0+1}, x_{n_0+2})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

So, $\tau \leq 0$ a contradiction. Consequently,

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \ \forall n \in \mathbb{N}. \tag{14}$$

Therefore,

$$m(x_n, x_{n+1}) \leq d(x_n, x_{n+1}), \; \forall n \in \mathbb{N} \cup \{0\}. \tag{15}$$

So, from (13) and (14) one can obtain that

$$\tau + F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})),$$

or

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

In general, one can get

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_0, x_1)) - n\tau.$$

Hence,

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty,$$

which together with (G2) and Lemma 1, gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. If it is not true, then by Lemma 2, there exists $\varepsilon_0 > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \varepsilon_0$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon_0$ and

$(L1)$ $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon_0$,

$(L2)$ $\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon_0$,

$(L3)$ $\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k}) = \varepsilon_0$,

$(L4)$ $\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k-1}) = \varepsilon_0$.

Therefore, with notice to definition of $m(x, y)$ we have:

$$\begin{aligned}\lim_{k\to\infty} m(x_{n_k}, x_{m_k-1}) &= \lim_{k\to\infty} \max\Big\{ d(x_{n_k}, x_{m_k-1}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k-1}, x_{m_k}), \\ &\qquad \tfrac{d(x_{n_k}, x_{m_k} + d(x_{m_k-1}, x_{n_k+1}))}{2} \Big\} = \max\{\varepsilon_0, 0, 0, \tfrac{\varepsilon_0+\varepsilon_0}{2}\} = \varepsilon_0.\end{aligned}$$

So

$$\lim_{k\to\infty} m(x_{n_k}, x_{m_k-1}) = \varepsilon_0. \tag{16}$$

On the other hand, since $\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon_0 > 0$, and $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$, by considering a subsequence if necessary, one can assume that, there exists $k_1 \in \mathbb{N}$ such that for any $k > k_1$ and $n_k > m_k > k$

$$d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, x_{m_k-1}).$$

So, it is clear that

$$\frac{1}{2} d(x_{n_k}, Tx_{n_k}) = \frac{1}{2} d(x_{n_k}, x_{n_k+1}) < d(x_{n_k}, x_{m_k-1}), \;\; \forall k > k_1 \;\; and \;\; n_k > m_k > k. \tag{17}$$

Also, using the K-property, there exists $k_2 \in \mathbb{N}$ such that

$$\alpha(x_{n_k}, x_{m_k-1}) \geq 1, \ \ \forall k > k_2. \tag{18}$$

Let $k \geq \max\{k_1, k_2\}$, then from (18), (17) and (12) we have

$$\begin{aligned} \tau + F(d(Tx_{n_k}, x_{m_k-1})) &\leq \tau + \alpha(x_{n_k}, x_{m_k-1})F(d(Tx_{n_k}, Tx_{m_k-1})) \\ &\leq F(m(x_{n_k}, x_{m_k-1})). \end{aligned}$$

Letting $n \to \infty$, since F is continuous, by (L1) and (16) we have

$$\tau + F(\varepsilon_0) \leq F(\varepsilon_0),$$

which is a contradiction, as $\tau > 0$. Consequently, $\{x_n\}$ is a Cauchy sequence in the complete metric space X. So, there exists $u \in X$ such that $x_n \to u$, as $n \to \infty$. To complete the proof, we show that u is a fixed point of T. At first, we claim that, for all $n \geq 0$

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, u) \quad \text{or} \quad \frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, u). \tag{19}$$

In fact, if for some $n_0 \geq 0$, both of them are false then we will have

$$\frac{1}{2}d(x_{n_0}, x_{n_0+1}) > d(x_{n_0}, u) \quad \text{and} \quad \frac{1}{2}d(x_{n_0+1}, x_{n_0+2}) > d(x_{n_0+1}, u).$$

So, with notice of (14) we have

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\leq d(x_{n_0}, u) + d(u, x_{n_0+1}) < \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) + \tfrac{1}{2}d(x_{n_0+1}, x_{n_0+2}) \\ &\leq \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) + \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) = d(x_{n_0}, x_{n_0+1}). \end{aligned}$$

Which is a contradiction and the claim is proved.

Well, let us begin with the first part of (19), i.e. suppose that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, u),$$

and in contrary, assume that $Tu \neq u$. Without lose of generality, one can assume that $Tx_n \neq Tu$, for all $n \in \mathbb{N}$. (Indeed, if $x_{n+1} = Tx_n = Tu$ for infinite values of n, then uniqueness of the limit concludes that $Tu = u$). Then, from (14) and (iii) we get

$$\begin{aligned} \tau + F(d(x_{n+1}, Tu)) &= \tau + F(d(Tx_n, Tu)) \\ &\leq \tau + \alpha(x_n, u)F(d(Tx_n, Tu)) \leq F(m(x_n, u)), \end{aligned}$$

and since F is continuous on $(0, +\infty)$ and $d(u, Tu) > 0$ as $n \to \infty$, we get

$$\tau + F(d(u, Tu)) \leq F(\lim_{n\to\infty} m(x_n, u)). \tag{20}$$

But

$$m(x_n, u) = \max\left\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), \frac{d(x_n, Tu) + d(u, x_{n+1})}{2}\right\}.$$

So, we have

$$\lim_{n\to\infty} m(x_n, u) = \max\left\{0, 0, d(u, Tu), \frac{d(u, Tu) + 0}{2}\right\} = d(u, Tu).$$

Therefore, if $d(u, Tu) \neq 0$ then from (20) we have

$$\tau + F(d(u, Tu)) \leq F(d(u, Tu)),$$

which is contradicted by positivity of τ. So, $d(u, Tu) = 0$, i.e. $Tu = u$. Finally, if we assume that the second part of (19) is true, i.e.

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, u).$$

Then, as the same manner, we can prove that $d(u, Tu) = 0$, i.e. $Tu = u$. □

The next result establishes a sufficient condition for uniqueness of fixed point of an α F-weak Suzuki contraction.

Theorem 4. *Suppose that all the conditions of Theorem 3 are satisfied. In addition, assume that $\alpha(x, y) \geq 1$, for all $x, y \in Fix(T)$. Then, T has a unique fixed point.*

Proof. Suppose that u and v are two fixed point of T. If $u \neq v$, then $d(Tu, Tv) > 0$. Also $\alpha(u, v) \geq 1$, because $u, v \in Fix(T)$. Also, it is clear that $\frac{1}{2}d(u, Tu) = 0 < d(u, v)$. Hence, (12) implies that

$$\tau + F(d(u, v)) \quad = \quad \tau + F(d(Tu, Tv)) \leq \tau + \alpha(u, v)F(d(Tu, Tv)) \leq F(m(u, v)),$$

where

$$\begin{aligned} m(u, v) &= \max\left\{d(u, v), d(u, Tu), d(v, Tv), \tfrac{d(u,Tv)+d(v,Tu)}{2}\right\} \\ &= \max\{d(u, v), 0, 0, \tfrac{d(u,v)+d(v,u)}{2}\} = d(u, v). \end{aligned}$$

So, we have

$$\tau + F(d(u, v)) \leq F(d(u, v)),$$

which is a contradiction, as $\tau > 0$. So, $u = v$. □

Since each α-type F-Suzuki contraction is obviously an α F-weak Suzuki contraction, the following two corollaries are elementary results of Theorems 3 and 4 respectively.

Corollary 4 ([3], Theorem 3.3)**.** *Let (X, d) be a complete metric space and $T : X \to X$ be an α-type F-Suzuki contraction, satisfying the conditions (i)–(iv) of Theorem 3. Then, T has a fixed point.*

Corollary 5 ([3], Theorem 3.4)**.** *If in the Corollary 4, we further assume that $\alpha(x, y) \geq 1$, for all $x, y \in Fix(T)$, then T has a unique fixed point.*

The following example shows that Theorem 3 is a generalization of Theorem 3.3 in [3].

Example 6. *In the Example 5, we saw that the mapping T is not an α-type F-Suzuki contraction. So, T does not satisfy to Theorem 3.3 in [3]. But T is an α F-weak Suzuki contraction, and we can easily see that T satisfies all conditions of Theorem 3. And $u = 1$ is a fixed point of T. Also, all conditions of Theorem 4 are satisfied and $u = 1$ is the unique fixed point of T.*

4 CONSEQUENCES

In this section, one of the consequences of our research in metric spaces with graph is introduced. First, we remind a series of definitions and notions in graph theory.

Let (X, d) be a metric space and $\Delta = \{(x, x), x \in X\}$. Suppose that G is a graph, $V(G)$ is the set of all its vertices and $E(G)$ is the set of all edges of G. We say that G has no parallel edge, if $(x, y), (y, x) \in E(G)$ implies that $x = y$. Also G is directed if the edges have a direction associated with them. We denoted by $\mathcal{G}(X)$ the set of all directed graph G with no parallel edge in which $V(G) = X$ and $\Delta \subseteq E(G)$.

Definition 13 ([9]). *The mapping $T : X \to X$ is called G-continuous, if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X that $(x_n, x_{n+1}) \in E(G)$ $\forall n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ one can conclude that $Tx_n \to Tx$ as $n \to \infty$.*

Theorem 5. *Let (X, d) be a complete metric space endowed with a graph $G \in \mathcal{G}(X)$ and $T : X \to X$ be a mapping with the following conditions:*

(i) for all $x, y \in X, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$,

(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,

(iii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x \in X$ if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$, for all $n \in \mathbb{N}$,

(iv) there exist $F \in \mathcal{F}$, and $\tau > 0$ and $L \geq 0$ such that if $(x, y) \in E(G)$ and $d(Tx, Ty) > 0$ then

$$\tau + F(d(Tx, Ty)) \leq F(m(x, y) + LN_1(x, y)), \tag{21}$$

where $m(x, y)$ and $N_1(x, y)$ are defined as in Definition 11.

Then, if T is G-continuous or F is continuous, then T has a fixed point.

Proof. Define $\alpha : X \times X \to (0, +\infty) \cup \{-\infty\}$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ -\infty, & \text{otherwise.} \end{cases}$$

We show that all condition of Theorem 1 are satisfied. First, prove that T is α-admissible, it is enough to notice that if $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$ and it follows from (i) that $(Tx, Ty) \in E(G)$. Hence, $\alpha(Tx, Ty) \geq 1$. By (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ i.e. $\alpha(x_0, Tx_0) \geq 1$. Now, suppose that $\{x_n\}_{n=1}^{\infty} \subseteq X$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, Then, $(x_n, x_{n+1}) \in E(G)$ and it follows from (iv) that $(x_n, x) \in E(G)$, for all $n \in \mathbb{N}$, i.e. $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Finally, we show that T is an almost-α F-weak contraction on X. For this, suppose that $x, y \in X$ and $d(Tx, Ty) > 0$. If $(x, y) \notin E(G)$, then $\alpha(x, y) = -\infty$ and so we have

$$\tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(m(x, y) + LN_1(x, y)).$$

If $(x, y) \in E(G)$, then $\alpha(x, y) = 1$ and it follows from (21) that

$$\tau + \alpha(x, y)F(d(Tx, Ty)) = \tau + F(d(Tx, Ty)) \leq F(m(x, y) + LN_1(x, y)).$$

Thus, T is an almost-α F-weak contraction on X. It follow from all the conditions of Theorem 1 are satisfied and T has a fixed point in X. $\square$

The following result is immediately deduced from Theorem 5.

Corollary 6 ([6], Theorem 4.1)**.** *Let (X, d) be a complete metric space endowed with a graph $G \in \mathcal{G}(X)$ and $T : X \to X$ be a mapping with the following conditions:*

(i) for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$,

(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$,

(iii) for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x \in X$ if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$, for all $n \in \mathbb{N}$ or T is G-continuous.

(iv) there exist $F \in \mathcal{F}$, $\tau > 0$ and $L \geq 0$ such that if $(x, y) \in E(G)$ and $d(Tx, Ty) > 0$ then

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y) + LN_1(x, y)).$$

Then, T has a fixed point.

Acknowledgment. The authors would like to thank anonymous referees for valuable comments and careful reading.

References

[1] Batra R., Vashistha S. *Fixed points of an F-contraction on metric spaces with a graph*. Int. J. Comput. Math. 2014, **91**, 1–8.

[2] Banach S. *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*. Fund. Math. 1922, **3**, 133–181.

[3] Budhia L.B, Kumam P., Martinez-Moreno J., Gopal D. *Extensions of almost-F and F-Suzuki contractions with graph and some applications to fractional calculus*. Fixed Point Theory Appl. 2016. Article number 2 (2016). doi:10.1186/s13663-015-0480-5

[4] Choudhury B.S, Konar P., Rhoades B.E, Metiya N. *Fixed point theorems for generalized weakl contractive mappings*. Nonlinear Anal. 2011, **74** (6), 2116–2126.

[5] Ćirić L.B. *A generalization of Banach́s contraction principle*. Proc. Amer. Math. Soc. 1974, **45** (2), 267–273.

[6] Cosentino M., Vetroa P. *Fixed point results for F-contractive mappings of Hardy-Rogers-type*. Filomat 2014, **28** (4), 715–722. doi:10.2298/FIL1404715C

[7] Gopal D., Abbas M., Patel D.K., Vetro C. *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*. Acta Math. Sci. 2016, **36** (3), 957–970.

[8] Gopal D., Abbas M., Vetro C., Patel D.K. *Some coincidence and periodic points results in a metric space endowed with a graph and applications*. Banach J. Math. Anal. 2015, **9** (3), 128–139. doi:10.15352/bjma/09-3-9

[9] Jachymski J. *The contraction principle for mapping on a metric space with a graph*. Proc. Am. Math. Soc. 2008, **136**, 1359–1373.

[10] Karapinar E., Kutbi M.A., Piri H., ÓRegan D. *Fixed points of conditionally F-contractions incomplete metric-like spaces*. Fixed Point Theory Appl. 2015. Article number 126 (2015). doi:10.1186/s13663-015-0377-3

[11] Karapinar E., Samet B. *Generalized α-ψ contractive type mappings and related fixed point theorems with applications*. Abstr. Appl. Anal. 2012, **2012**. Article ID 793486. doi:10.1155/2012/793486

[12] Kirk W., Sims B. Handbook of metric fixed point theory. Springer, 2001.

[13] Meir A., Keeler E. *A theorem on contraction mappings*. J. Math. Anal. Appl. 1969, **28**, 326–329.

[14] Minak G., Helvaci A., Altun I. *Ćirić-type generalized F-contractions on complete metric spaces and fixed point results*. Filomat 2014, **28** (6), 1143–1151.

[15] Mohammadi B., Rezapour S., Shahzad N. *Some results on fixed points of α-ψ-Ciric generalized multifunctions*. Fixed Point Theory Appl. 2013. Article number 24 (2013). doi:10.1186/1687-1812-2013-24

[16] Piri H., Kumam P. *Some fixed point theorems concerning F-contraction in complete metric spaces*. Fixed Point Theory Appl. 2014. Article number 210 (2014). doi:10.1186/1687-1812-2014-210

[17] Rhoades B.E. *Some theorems on weakly contractive maps*. Nonlinear Anal. 2001, **47**, 2683–2693.

[18] Salimi P., Latif A., Hussain N. *Modified α-ψ-contractive mappings with applications*. Fixed Point Theory Appl. 2013. Article number 151 (2013). doi:10.1186/1687-1812-2013-151

[19] Samet B., Vetro C., Vetro P. *Fixed point theorems for α-ψ-contractive type mappings*. Nonlinear Anal. 2012, **75**, 2154–2165.

[20] Secelean N.A. *Iterated function systems consisting of F-contractions*. Fixed Point Theory Appl. 2013. Article number 277 (2013). doi:10.1186/1687-1812-2013-277

[21] Sgroi M., Vetro C. *Multi-valued F-contractions and the solution of certain functional and integral equations*. Filomat 2013, **27**, 1259 –1268.

[22] Singh S.L, Kamal R., De la Sen M., Chugh R. *A fixed point theorem for generalized weak contractions*. Filomat 2015, **29** (7), 1481–1490.

[23] Suzuki T. *A new type of fixed point theorem in metric spaces*. Nonlinear Anal. 2009, **71** (11), 5313–5317.

[24] Wardowski D. *Fixed points of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. 2012. Article number 94 (2012). doi:10.1186/1687-1812-2012-94

[25] Wardowski D., Van Dung N. *Fixed points of F-weak contractions on complete metric spaces*. Demonstr. Math. 2014, **47** (1), 146–155.

Received 20.01.2019

Тахері А., Фараджаде А.П. *Нова характеризація майже-F-стиску α-типу і F-Сузукі стиску α-типу в метричних просторах і теореми про фіксовану точку для них* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 475–492.

У цій статті запропоновано нове узагальнення майже-F-стиску α-типу і продовження F-Сузукі стиску α-типу. Крім того, доведено деякі нові теореми про фіксовану точку для цих випадків. Наведено приклади і застосування, які ілюструють основні результати. Результати цієї статті покращують результати, які добре відомі у літературі.

Ключові слова і фрази: майже-F-стиск α-типу, F-Сузукі стиск α-типу.

ISSN 2075-9827 e-ISSN 2313-0210 http://www.journals.pnu.edu.ua/index.php/cmp
Carpathian Math. Publ. 2019, **11** (2), 493–501 Карпатські матем. публ. 2019, Т.11, №2, С.493–501
doi:10.15330/cmp.11.2.493-501

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POINT-EVALUATION FUNCTIONALS ON ALGEBRAS OF SYMMETRIC FUNCTIONS ON $(L_\infty)^2$

It is known that every continuous symmetric (invariant under the composition of its argument with each Lebesgue measurable bijection of $[0,1]$ that preserve the Lebesgue measure of measurable sets) polynomial on the Cartesian power of the complex Banach space L_∞ of all Lebesgue measurable essentially bounded complex-valued functions on $[0,1]$ can be uniquely represented as an algebraic combination, i.e., a linear combination of products, of the so-called elementary symmetric polynomials. Consequently, every continuous complex-valued linear multiplicative functional (character) of an arbitrary topological algebra of the functions on the Cartesian power of L_∞, which contains the algebra of continuous symmetric polynomials on the Cartesian power of L_∞ as a dense subalgebra, is uniquely determined by its values on elementary symmetric polynomials. Therefore, the problem of the description of the spectrum (the set of all characters) of such an algebra is equivalent to the problem of the description of sets of the above-mentioned values of characters on elementary symmetric polynomials.

In this work the problem of the description of sets of values of characters, which are point-evaluation functionals, on elementary symmetric polynomials on the Cartesian square of L_∞ is completely solved. We show that sets of values of point-evaluation functionals on elementary symmetric polynomials satisfy some natural condition. Also we show that for any set c of complex numbers, which satisfies the above-mentioned condition, there exists the element x of the Cartesian square of L_∞ such that values of the point-evaluation functional at x on elementary symmetric polynomials coincide with the respective elements of the set c.

Key words and phrases: symmetric polynomial, point-evaluation functional.

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INTRODUCTION

In general, the problem of the description of the spectrum (the set of continuous complex-valued linear multiplicative functionals, or characters) of a topological algebra of analytic functions on a Banach space is unsolved. But if a topological algebra or its dense subalgebra has a countable algebraic basis (the subset B of the algebra A is called an algebraic basis of A, if every element of A can be uniquely represented as an algebraic combination (a linear combination of products) of elements of B), then the problem of the description of the spectrum simplifies, because in this case every character is uniquely determined by the sequence of its values on elements of the algebraic basis and, consequently, the problem of the description of the spectrum is equivalent to the problem of the description of the set of such sequences. For

УДК 517.98
2010 *Mathematics Subject Classification:* 46G25, 46G20.
The publication contains the results of studies conducted by President's of Ukraine grant for competitive projects 0119U103204.

example, in [2] it was constructed an algebraic basis of the algebra of all continuous symmetric (see definition below) polynomials on the complex Banach space L_∞ of all complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$. Also, using this result, in [2] it was described the spectrum of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on L_∞ and it was shown that every character of $H_{bs}(L_\infty)$ is a point-evaluation functional.

Firstly algebraic bases of algebras of symmetric continuous polynomials on real Banach spaces of Lebesgue measurable integrable in a power p functions on $[0,1]$ and $[0,+\infty)$, where $1 \leq p < +\infty$, were studied by Nemirovskii and Semenov in [7]. Some of their results were generalized to real separable rearrangement invariant Banach spaces of Lebesgue measurable functions on $[0,1]$ and $[0,+\infty)$ by González, Gonzalo and Jaramillo in [4]. Symmetric polynomials and symmetric analytic functions on the complex Banach spaces of all complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$ and $[0,+\infty)$ were studied in [2] and [3] respectively. Symmetric polynomials on Cartesian products of some Banach spaces were studied in [6,8–12]. In particular, in [10] it was constructed a countable algebraic basis of the algebra of continuous symmetric polynomials on the Cartesian power of L_∞.

In this work the problem of the description of sequences of values of point-evaluation functionals on the elements of the algebraic basis of the algebra of continuous symmetric polynomials on the Cartesian square of L_∞ is completely solved. We show that the above-mentioned sequences satisfy some natural condition. Also we show that for any sequence c of complex numbers, which satisfies this condition, there exists an element x of the Cartesian square of L_∞ such that the sequence of values of the point-evaluation functional at x coincides with c. We generalize the results of [11].

1 PRELIMINARIES

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

A mapping $P : X \to \mathbb{C}$, where X is a Banach space with norm $\|\cdot\|_X$, is called an *N-homogeneous polynomial*, where $N \in \mathbb{N}$, if there exists an N-linear mapping $A_P : X^N \to \mathbb{C}$ such that

$$P(x) = A_P(\underbrace{x, \ldots, x}_{N})$$

for every $x \in X$. It is known that an N-homogeneous polynomial $P : X \to \mathbb{C}$ is continuous if and only if

$$\|P\| = \sup_{\|x\|_X \leq 1} |P(x)| < +\infty.$$

Consequently, if P is a continuous N-homogeneous polynomial, then

$$|P(x)| \leq \|P\| \|x\|_X^N \tag{1}$$

for every $x \in X$.

A mapping $P = P_0 + P_1 + \ldots + P_N$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$, is called a *polynomial* of degree at most N.

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |x(t)|.$$

Let $(L_\infty)^2$ be the Cartesian square of L_∞ with norm

$$\|x\|_{\infty,2} = \max\{\|x_1\|_\infty, \|x_2\|_\infty\}$$

where $x = (x_1, x_2) \in (L_\infty)^2$.

Let Ξ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure. A function $f : (L_\infty)^2 \to \mathbb{C}$ is called *symmetric* if

$$f(x \circ \sigma) = f(x)$$

for every $x = (x_1, x_2) \in (L_\infty)^2$ and for every $\sigma \in \Xi$, where $x \circ \sigma = (x_1 \circ \sigma, x_2 \circ \sigma)$.

For every multi-index $k = (k_1, k_2) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ let us define a mapping $R_k : (L_\infty)^2 \to \mathbb{C}$ by

$$R_k(x) = \int_{[0,1]} \prod_{\substack{s=1\\ k_s>0}}^{2} (x_s(t))^{k_s}\, dt, \tag{2}$$

where $x = (x_1, x_2) \in (L_\infty)^2$. Note that R_k is a continuous symmetric $|k|$-homogeneous polynomial, where $|k| = k_1 + k_2$, and $\|R_k\| = 1$. By [10, Theorem 2], the set of polynomials $\{R_k : \ k \in \mathbb{Z}_+^2 \setminus \{(0,0)\}\}$ is an algebraic basis of the algebra $\mathcal{P}_s((L_\infty)^2)$ of all continuous symmetric polynomials on $(L_\infty)^2$.

Let A be an algebra of functions $f : D \to \mathbb{C}$, where the set D is such that $D \supset (L_\infty)^2$. For $x \in (L_\infty)^2$, let the mapping $\delta_x : A \to \mathbb{C}$ be defined by

$$\delta_x(f) = f(x),$$

where $f \in A$. The mapping δ_x is called a point-evaluation functional at the point x. Note that a point-evaluation functional is linear and multiplicative.

We shall use the following result.

Theorem 1. *(see [2, Section 3]) For every sequence $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that*

$$\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty,$$

there exists $v_\xi \in L_\infty$ such that

$$\int_{[0,1]} (v_\xi(t))^n\, dt = \xi_n$$

for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \le \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|}$, where

$$M = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{n+1}\right). \tag{3}$$

2 THE MAIN RESULT

Theorem 2. *For every mapping $c : \mathbb{Z}_+^2 \setminus \{(0,0)\} \to \mathbb{C}$ such that*

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|} < +\infty$$

there exists a function $x_c \in (L_\infty)^2$ such that $R_n(x_c) = c(n)$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ and

$$\|x_c\|_{\infty,2} \le \frac{24}{M^3} \sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|},$$

where M is defined by (3).

Proof. Let ε_k be the kth Rademacher function, that is, $\varepsilon_k(t) = \operatorname{sign}(\sin 2^k \pi t)$. It is well known (see, e.g., [1, p. 162] or [5, Chapter 3]) that the series $\sum_{k=1}^{\infty} \varepsilon_k(t) u_k$ is convergent almost everywhere on $[0,1]$ if and only if the series $\sum_{k=1}^{\infty} |u_k|^2$ converges. Consequently, the series $\sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k+1}$ converges almost everywhere on $[0,1]$.

For every $n = (n_1, n_2) \in \mathbb{N}^2$ let us define a function $p_n : [0,1] \to \mathbb{C}^2$ by

$$p_n(t) = \left(\exp\Big(\frac{\mathrm{i}\pi}{2n_1} \sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big), \exp\Big(\frac{\mathrm{i}\pi}{2n_2} \sum_{k=1}^{\infty} \frac{\varepsilon_{2k}(t)}{k+1} \Big) \right).$$

Note that the function p_n belongs to the space $(L_\infty[0,1])^2$ and $||p_n|| = 1$.

The sequence of the functions $\{p_n^{(l)}\}_{l=1}^{\infty}$, where

$$p_n^{(l)}(t) = \left(\exp\Big(\frac{\mathrm{i}\pi}{2n_1} \sum_{k=1}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big), \exp\Big(\frac{\mathrm{i}\pi}{2n_2} \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) \right),$$

converges pointwise to p_n. Therefore, for every $m = (m_1, m_2) \in \mathbb{N}^2$, according to the dominated convergence theorem,

$$R_m(p_n) = \lim_{l \to \infty} R_m(p_n^{(l)}).$$

Note that

$$\begin{aligned}
R_m(p_n^{(l)}) &= \int_{[0,1]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=1}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \frac{1}{2} \Big) \int_{[0,\frac{1}{2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&+ \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \frac{-1}{2} \Big) \int_{[\frac{1}{2},1]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \int_{[0,\frac{1}{2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= 4 \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{2} \Big) \\
&\times \int_{[0,\frac{1}{4}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=2}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= 4^2 \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{3} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{3} \Big) \\
&\times \int_{[0,\frac{1}{4^2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=3}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=3}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt = \ldots \\
&= 4^l \int_{[0,\frac{1}{4^l}]} dt \prod_{k=1}^{l} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big) \\
&= \prod_{k=1}^{l} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big).
\end{aligned}$$

Therefore,

$$R_m(p_n) = \prod_{k=1}^{\infty} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big).$$

For $k \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, let

$$a_{j,k} = \exp\Big(\frac{2\pi i j}{k}\Big).$$

For every $k \in \mathbb{N}$ let us define a function $S_k : [0,1] \to \mathbb{C}$ in the following way. For $t \in [\frac{j-1}{k}, \frac{j}{k}]$, where $j \in \{1, \ldots, k\}$, let

$$S_k(t) = a_{j,k}.$$

Let $\mathrm{frac}(t)$ be the fractional part of a real number t. For every $n = (n_1, n_2) \in \mathbb{N}^2$ let us define a function $y_n : [0,1] \to \mathbb{C}^2$ by a formula

$$y_n(t) = \Big(S_{n_1}(t)p_{n,1}(\mathrm{frac}(n_1 n_2 t)), S_{n_2}(\mathrm{frac}(n_1 t))p_{n,2}(\mathrm{frac}(n_1 n_2 t))\Big).$$

Note that $\|y_n\|_{\infty,2} = 1$. For every $m = (m_1, m_2) \in \mathbb{N}^2$, we have

$$\begin{aligned} R_m(y_n) &= \int_{[0,1]} S_{n_1}^{m_1}(t)p_{n,1}^{m_1}(\mathrm{frac}(n_1 n_2 t))S_{n_2}^{m_2}(\mathrm{frac}(n_1 t))p_{n,2}^{m_2}(\mathrm{frac}(n_1 n_2 t))dt = \\ &= \sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \int_{[\frac{j-1}{n_1}, \frac{j}{n_1}]} S_{n_2}^{m_2}(\mathrm{frac}(n_1 t))p_{n,1}^{m_1}(\mathrm{frac}(n_1 n_2 t))p_{n,2}^{m_2}(\mathrm{frac}(n_1 n_2 t))dt. \end{aligned}$$

Let us make the substitution $u = n_1 t - (j-1)$ in the jth integral. Then $n_1 t = u + j - 1$ and, consequently, $\mathrm{frac}(n_1 t) = \mathrm{frac}(u + j - 1) = \mathrm{frac}(u)$ and $\mathrm{frac}(n_1 n_2 t) = \mathrm{frac}(n_2 u + n_2(j-1)) = \mathrm{frac}(n_2 u)$. Therefore,

$$R_m(y_n) = \frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \int_{[0,1]} S_{n_2}^{m_2}(\mathrm{frac}(u))p_{n,1}^{m_1}(\mathrm{frac}(n_2 u))p_{n,2}^{m_2}(\mathrm{frac}(n_2 u))du.$$

Note that

$$\begin{aligned} &\int_{[0,1]} S_{n_2}^{m_2}(\mathrm{frac}(u))p_{n,1}^{m_1}(\mathrm{frac}(n_2 u))p_{n,2}^{m_2}(\mathrm{frac}(n_2 u))du \\ &\qquad = \sum_{r=1}^{n_2} a_{r,n_2}^{m_2} \int_{[\frac{r-1}{n_2}, \frac{r}{n_2}]} p_{n,1}^{m_1}(\mathrm{frac}(n_2 u))p_{n,2}^{m_2}(\mathrm{frac}(n_2 u))du. \end{aligned}$$

Let us make the substitution $v = n_2 u - (r-1)$ in the rth integral. Then $n_2 u = v + r - 1$ and, consequently, $\mathrm{frac}(n_2 u) = \mathrm{frac}(v + r - 1) = \mathrm{frac}(v) = v$. Therefore,

$$\begin{aligned} R_m(y_n) &= \frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \frac{1}{n_2}\sum_{r=1} n_2 a_{r,n_2}^{m_2} \int_{[0,1]} p_{n_1}^{m_1}(v)p_{n_2}^{m_2}(v)dv = \Big(\frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1}\Big)\Big(\frac{1}{n_2}\sum_{r=1}^{n_2} a_{r,n_2}^{m_2}\Big) \\ &\times R_m(p_n) = \Big(\frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1}\Big)\Big(\frac{1}{n_2}\sum_{r=1}^{n_2} a_{r,n_2}^{m_2}\Big)\prod_{k=1}^{\infty}\cos\Big(\frac{\pi m_1}{2n_1}\frac{1}{k+1}\Big)\cos\Big(\frac{\pi m_2}{2n_2}\frac{1}{k+1}\Big). \end{aligned}$$

If m_1 is not a multiple of n_1, then

$$\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} = 0.$$

Similarly, if m_2 is not a multiple of n_2, then

$$\sum_{r=1}^{n_2} a_{r,n_2}^{m_2} = 0.$$

Let $m_1 = k_1 n_1$ and $m_2 = k_2 n_2$, where $k_1, k_2 \in \mathbb{N}$. Then

$$\frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} = \frac{1}{n_2}\sum_{r=1}^{n_2} a_{r,n_2}^{m_2} = 1.$$

Therefore,

$$R_m(y_n) = \prod_{k=1}^{\infty} \cos\Big(\frac{\pi k_1}{2}\frac{1}{k+1}\Big)\cos\Big(\frac{\pi k_2}{2}\frac{1}{k+1}\Big).$$

If $k_1 > 1$ or $k_2 > 1$, then there is a multiplier $\cos\frac{\pi}{2} = 0$ in the given product. Thus $R_m(y_n) = 0$, if $m \neq n$. If $m = n$, then $R_m(y_n) = M^2$, where M is defined by (3).

For every $n = (n_1, n_2) \in \mathbb{N}^2$, let us define a function $z_n : [0,1] \to \mathbb{C}^2$ by

$$z_n = \frac{1}{\sqrt[|n|]{M^2}} y_n.$$

Note that

$$\|z_n\|_{\infty,2} = \frac{1}{\sqrt[|n|]{M^2}} \leq \frac{1}{M^2}, \tag{4}$$

since $0 < M < 1$. For every $m \in \mathbb{N}^2$,

$$R_m(z_n) = \begin{cases} 1, & \text{if } m = n,\\ 0, & \text{if } m \neq n. \end{cases} \tag{5}$$

Let us define sequences $\xi = \{\xi_l\}_{l=1}^{\infty}, \eta = \{\eta_l\}_{l=1}^{\infty} \subset \mathbb{C}$ by

$$\xi_l = 4c((l,0)) - 4\sum_{k=1}^{\infty}\frac{1}{k2^{k+1}}\sum_{j=1}^{k}\big(c((j,k-j))k2^{k+1}\big)^{l/k} R_{(l,0)}(z_{(j,k-j)}) \tag{6}$$

and

$$\eta_l = 4c((0,l)) - 4\sum_{k=1}^{\infty}\frac{1}{k2^{k+1}}\sum_{j=1}^{k}\big(c((j,k-j))k2^{k+1}\big)^{l/k} R_{(0,l)}(z_{(j,k-j)})$$

for $l \in \mathbb{N}$. Let us show that $\sup_{l\in\mathbb{N}} |\xi_l|^{1/l} < +\infty$ and $\sup_{l\in\mathbb{N}} |\eta_l|^{1/l} < +\infty$. Let

$$a = \sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|}.$$

Then $|c(n)| \leq a^{|n|}$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. By (1), $|R_{(l,0)}(z_{(j,k-j)})| \leq \|R_{(l,0)}\| \|z_{(j,k-j)}\|_{\infty,2}^{l}$. By (4), taking into account the equality $\|R_{(l,0)}\| = 1$,

$$|R_{(l,0)}(z_{(j,k-j)})| \leq \frac{1}{M^{2l}}.$$

Therefore,

$$|\xi_l| \leq 4a^l + \frac{4a^l}{M^{2l}}\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\big(k2^{k+1}\big)^{l/k}.$$

Note that $\sup_{k\in\mathbb{N}}(k2^{k+1})^{1/k} = 4$. Therefore, $k2^{k+1} \leq 4^k$ for every $k \in \mathbb{N}$. Consequently,

$$\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\big(k2^{k+1}\big)^{l/k} \leq 4^l\sum_{k=1}^{\infty}\frac{1}{2^{k+1}} = \frac{4^l}{2}.$$

Therefore,

$$|\xi_l| \le 4a^l + \frac{2(4a)^l}{M^{2l}}.$$

Taking into account the estimate $0 < M < 1$,

$$4a^l + \frac{2(4a)^l}{M^{2l}} \le \frac{4a^l + 2(4a)^l}{M^{2l}} \le \frac{3(4a)^l}{M^{2l}} \le \frac{(12a)^l}{M^{2l}}.$$

Thus,

$$|\xi_l| \le \frac{(12a)^l}{M^{2l}}.$$

Analogically,

$$|\eta_l| \le \frac{(12a)^l}{M^{2l}}.$$

Since $\sup_{l\in\mathbb{N}} |\xi_l|^{1/l} \le 12a/M^2$ and $\sup_{l\in\mathbb{N}} |\eta_l|^{1/l} \le 12a/M^2$, by Theorem 1, there exist $v_\xi, v_\eta \in L_\infty$ such that

$$\int_{[0,1]} (v_\xi(t))^l\,dt = \xi_l \quad \text{and} \quad \int_{[0,1]} (v_\eta(t))^l\,dt = \eta_l \tag{7}$$

for every $l \in \mathbb{N}$ and

$$\|v_\xi\|_\infty, \|v_\eta\|_\infty \le \frac{24a}{M^3}. \tag{8}$$

For $k \in \mathbb{N}$ and $j \in \{1,\dots,k\}$, let

$$\Delta_{j,k} = \Big(1 - \frac{1}{2^k} + \frac{j-1}{k2^{k+1}}, 1 - \frac{1}{2^k} + \frac{j}{k2^{k+1}}\Big)$$

and $h_{j,k} : \Delta_{j,k} \to (0,1)$ let be defined by

$$h_{j,k}(t) = \Big(t - \Big(1 - \frac{1}{2^k} + \frac{j-1}{k2^{k+1}}\Big)\Big)k2^{k+1}.$$

Note that $h_{j,k}$ is a bijection. Let us define a function $x_c : [0,1] \to \mathbb{C}^2$ by

$$x_c(t) = \begin{cases} (v_\xi(4t), 0), & \text{if } t \in (0, 1/4), \\ (0, v_\eta(4t-1)), & \text{if } t \in (1/4, 1/2), \\ \big(c((j,k-j))k2^{k+1}\big)^{1/k} z_{(j,k-j)}(h_{j,k}(t)), & \text{if } t \in \Delta_{j,k},\ k \in \mathbb{N},\ j \in \{1,\dots,k\}, \\ (0,0), & \text{otherwise.} \end{cases}$$

Note that $x_c \in (L_\infty)^2$ and, taking into account estimations (4), (8) and the inequality $\big(c((j,k-j))k2^{k+1}\big)^{1/k} \le 4a$, we obtain

$$\|x_c\|_{\infty,2} \le \max\Big\{\frac{24a}{M^3}, \frac{4a}{M^2}\Big\}.$$

Since $0 < M < 1$, it follows that $4a/M^2 \le 4a/M^3 \le 24a/M^3$. Therefore, $\|x_c\|_{\infty,2} \le 24a/M^3$. Let us show that $R_n(x_c) = c(n)$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. Consider the case $n = (n_1, n_2) \in \mathbb{N}^2$. In this case, taking into account (5),

$$\begin{aligned} R_n(x_c) = &\int_{(0,1/4)} (v_\xi(4t))^{n_1} 0^{n_2}\,dt + \int_{(1/4,1/2)} 0^{n_1} (v_\eta(4t-1))^{n_2}\,dt + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{|n|/k} \\ &\times \int_{\Delta_{j,k}} \big(z_{(j,k-j),1}(h_{j,k}(t))\big)^{n_1} \big(z_{(j,k-j),2}(h_{j,k}(t))\big)^{n_2}\,dt \\ = &\sum_{k=1}^{\infty}\sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{|n|/k} \frac{1}{k2^{k+1}} R_n(z_{(j,k-j)}) = \big(c((n_1,n_2))|n|2^{|n|+1}\big)^{|n|/|n|} \frac{1}{|n|2^{|n|+1}} \\ = &\, c((n_1,n_2)). \end{aligned}$$

Consider the case $n=(l,0)$, where $l\in\mathbb{N}$. In this case, taking into account (6) and (7),

$$\begin{aligned} R_n(x_c) &= \int_{(0,1/4)} (v_\xi(4t))^l\,dt + \int_{(1/4,1/2)} 0^l\,dt \\ &\qquad + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \left(c((j,k-j))k2^{k+1}\right)^{l/k} \int_{\Delta_{j,k}} \left(z_{(j,k-j),1}(h_{j,k}(t))\right)^l dt \\ &= \frac{1}{4}\int_{(0,1)} (v_\xi(t))^l\,dt + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \left(c((j,k-j))k2^{k+1}\right)^{l/k} \frac{1}{k2^{k+1}} R_{(l,0)}(z_{(j,k-j)}) \\ &= \frac{1}{4}\xi_l + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \left(c((j,k-j))k2^{k+1}\right)^{l/k} \frac{1}{k2^{k+1}} R_{(l,0)}(z_{(j,k-j)}) = c((l,0)). \end{aligned}$$

Analogically, in the case $n=(0,l)$, where $l\in\mathbb{N}$, we have $R_n(x_c)=c((0,l))$. This completes the proof. ☐

Corollary 1. *Let A be a topological algebra of complex-valued functions on $(L_\infty)^2$, which contains the algebra $\mathcal{P}_s((L_\infty)^2)$ as a dense subalgebra. Let A be such that for each $x\in L_\infty$ the point-evaluation functional δ_x is continuous on A. Let $\varphi: A\to\mathbb{C}$ be a continuous linear multiplicative functional. Then φ is a point-evaluation functional if and only if*

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} < +\infty.$$

Proof. Let $\varphi: A\to\mathbb{C}$ be a continuous linear multiplicative functional such that

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} < +\infty.$$

By Theorem 2, there exists $x\in(L_\infty)^2$ such that $R_n(x)=\varphi(R_n)$ for every $n\in\mathbb{Z}_+^2\setminus\{(0,0)\}$, that is, $\delta_x(R_n)=\varphi(R_n)$ for every $n\in\mathbb{Z}_+^2\setminus\{(0,0)\}$. Since both δ_x and φ are linear and multiplicative, it follows that $\delta_x(P)=\varphi(P)$ for every $P\in\mathcal{P}_s((L_\infty)^2)$. Since both δ_x and φ are continuous and $\mathcal{P}_s((L_\infty)^2)$ is dense in A, it follows that $\delta_x=\varphi$.

Let $\varphi=\delta_x$ for some $x=(x_1,x_2)\in(L_\infty)^2$. By (1), for every $n=(n_1,n_2)\in\mathbb{Z}_+^2\setminus\{(0,0)\}$, $|\varphi(R_n)|=|R_n(x)|\le\|x\|^{|n|}$. Consequently,

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} \le \|x\|.$$

☐

REFERENCES

[1] Albiac F., Kalton N. Topics in Banach space theory. Graduate Texts in Mathematics, Springer, 2006.

[2] Galindo P., Vasylyshyn T.V., Zagorodnyuk A.V. *The algebra of symmetric analytic functions on L_∞*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics 2017, **147** (4), 743–761. doi:10.1017/S0308210516000287

[3] Galindo P., Vasylyshyn T.V., Zagorodnyuk A.V. *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0,+\infty)$*. Math. Nachr. 2018, **291** (11-12), 1712–1726. doi:10.1002/mana.201700314

[4] González M., Gonzalo R., Jaramillo J.A. *Symmetric polynomials on rearrangement invariant function spaces*. J. London Math. Soc. 1999, **59** (2), 681–697. doi:10.1112/S0024610799007164

[5] Kahane J.P. Some random series of functions. Cambridge University Press, 1985.

[6] Kravtsiv V.V., Vasylyshyn T.V., Zagorodnyuk A.V. *On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$*. Journal of Function Spaces 2017, **2017**, 8 p. doi:10.1155/2017/4947925

[7] Nemirovskii A.S., Semenov S.M. *On polynomial approximation of functions on Hilbert space*. Mat. USSR Sbornik 1973, **21** (2), 255–277. doi:10.1070/SM1973v021n02ABEH002016

[8] Vasylyshyn T.V. *Symmetric polynomials on $(L_p)^n$*. European Journal of Math. doi:10.1007/s40879-018-0268-3

[9] Vasylyshyn T.V. *Symmetric polynomials on the Cartesian power of L_p on the semi-axis*. Mat. Stud. 2018, **50** (1), 93–104. doi:10.15330/ms.50.1.93-104

[10] Vasylyshyn T.V. *The algebra of symmetric polynomials on $(L_\infty)^n$*. Mat. Stud. 2019, **52** (1), 71–85. doi:10.30970/ms.52.1.71-85

[11] Vasylyshyn T.V. *Some properties of elementary symmetric polynomials on the Cartesian square of the complex Banach space $L_\infty[0,1]$*. Precarpathian Bull. Shevchenko Sci. Soc. 2018, **46**, 9–16. (in Ukrainian) doi:10.31471/2304-7399-2018-2(46)-9-16

[12] Vasylyshyn T.V., Zagorodnyuk A.V. *Continuous symmetric 3-homogeneous polynomials on spaces of Lebesgue measurable essentially bounded functions*. Methods Funct. Anal. Topology 2018, **24** (4), 381–398.

Received 10.10.2019

Василишин Т.В. *Функціонали обчислення значень в точках на алгебрах симетричних функцій на просторі* $(L_\infty)^2$ // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 493–501.

Відомо, що кожен неперервний симетричний (інваріантний відносно дії композиції аргумента з будь-якою вимірною за Лебегом бієкцією відрізка $[0,1]$, яка зберігає міру Лебега вимірних множин) поліном на декартовому степені комплексного банахового простору L_∞ всіх вимірних за Лебегом суттєво обмежених комплекснозначних функцій на відрізку $[0,1]$ може бути єдиним чином подано як алгебраїчну комбінацію, тобто лінійну комбінацію добутків, так званих елементарних симетричних поліномів. Як наслідок, кожен неперервний комплекснозначний лінійний мультиплікативний функціонал (характер) довільної топологічної алгебри функцій на декартовому степені простору L_∞, яка містить алгебру неперервних симетричних поліномів на декартовому степені простору L_∞ як щільну підалгебру, однозначно визначається своїми значеннннями на елементарних симетричних поліномах. Тому задача опису спектра (множини всіх характерів) такої алгебри еквівалентна до задачі опису множин вищезгаданих значень характерів на елементарних симетричних поліномах.

В даній роботі розв'язано задачу опису множин значень характерів, які є функціоналами обчислення значення в точках, на елементарних симетричних поліномах на декартовому квадраті простору L_∞. Показано, що множини значень функціоналів обчислення значення в точках на елементарних симетричних поліномах задовольняють деяку природну умову. Також показано, що для кожної множини c комплексних чисел, яка задовольняє вищезгадану умову, існує елемент x декартового квадрата простору L_∞ такий, що значення функціонала обчислення значення в точці x на елементарних симетричних поліномах збігаються з відповідними елементами множини c.

Ключові слова і фрази: симетричний поліном, функціонал обчислення значення в точці.