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COUPLED COINCIDENCE POINT RESULTS FOR CONTRACTION OF C-CLASS MAPPINGS IN ORDERED UNIFORM SPACES

In the literature there is a lot of works related to fixed point theory. The theory has many applications and some authors are interested in these applications in various spaces. In 2009, Altun I. and Imdad M. defined the order relation on uniform spaces and the concept of compatibility of mappings. Later Ansari A.H. defined the C-class function concept. In this paper, we take some ultra altering distance and C-class functions, then we prove some coupled coincidence point theorems for a mapping providing mixed g-monotonicity property in ordered uniform spaces. We also give the appropriate examples.

Key words and phrases: coupled coincidence point, C-class mapping, ordered uniform space.

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INTRODUCTION AND PRELIMINARIES

In the literature there is a lot of works related to fixed point theory. Some of them are fixed or common fixed point results in uniform space (e.g. [1–3, 12]). Lately, Aamri M. and El Moutawakil D. [1] have introduced the concept of E-distance function on uniform spaces and utilize it to improve some well known results of the existing literature involving both E-contractive or E-expansive mappings. Later, Altun I. and Imdad M. [3] have introduced a partial ordering on uniform spaces utilizing E-distance function and have used the same to prove a fixed point theorem for single-valued non-decreasing mappings on ordered uniform spaces.

In this paper, we use the C-class function defined by Ansari A.H. [4], the order relation on uniform spaces defined by Altun I. and Imdad M. [3] and the concept of compatibility of mappings, then we prove coupled coincidence point theorems in ordered uniform spaces. We also discuss an example.

Now, we mention some relevant definitions and properties from the foundation of uniform spaces. We call a pair (X, ϑ) to be a uniform space which consists of a non-empty set X together with a uniformity ϑ, wherein the latter begins with a special kind of filter on $X \times X$, whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$, then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for

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$m, n \geq N$. A uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to diagonal Δ of X, i.e. $(x, y) \in V$ for $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V = V^{-1} = \{(y, x) : (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close and then one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they are naturally interpreted with respect to the topological space $(X, \tau(\vartheta))$.

In the sequel we shall require the following definitions and lemmas.

Definition 1 ([1]). *Let (X, ϑ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if*

(p1) for any $V \in \vartheta$ there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, imply $(x, y) \in V$,

(p2) $p(x, y) \leq p(x, z) + p(z, y)$ for any $x, y, z \in X$.

The following lemma embodies some useful properties of E-distance.

Lemma 1 ([1,2]). *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds.*

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a p-Cauchy sequence in (X, ϑ).

Let (X, ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2 ([1,2]). *Let (X, ϑ) be a uniform space and p be an E-distance on X. Then*

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$,

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(fx_n, fx) = 0$,

(iv) $f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark 1 ([1]). *Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.*

Lemma 2 ([3]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows;*

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x, y) \leq \varphi(x) - \varphi(y).$$

Then "$\preceq$" is a (partial) order on X induced by φ.

Definition 3 ([6]). *We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$, $T(y, x) = y$.*

Definition 4 ([11]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = g(x)$, $T(y, x) = g(y)$.*

Definition 5 ([11]). *Let X be a non-empty set and $T : X \times X \to X$ and $g : X \to X$. We say T and g are commutative if $g(T(x, y)) = T(g(x), g(y))$ for any $x, y \in X$.*

Definition 6 ([7]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X. The mappings T and g, where $T : X \times X \to X$ and $g : X \to X$, are said to be compatible if*

$$\lim_{n\to\infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} p(g(T(y_n, x_n)), T(g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, for any $x, y \in X$ are satisfied.

In 2014, the concept of C-class functions (see Definition 7) was introduced by A.H. Ansari in [4] that is pivotal result in fixed point theory. Also see [5,8,9].

Definition 7. *A mapping $f : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

(1) $f(s, t) \leq s$;

(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Remark 2. *Note for some f we have that $f(0, 0) = 0$.*

We denote C-class functions as $\mathcal{C}$.

Example 1. *The following functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:*

(1) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;

(2) $f(s, t) = ms$, $0<m<1$, $f(s, t) = s \Rightarrow s = 0$;

(3) $f(s,t) = \frac{s}{(1+t)^r}; r \in (0,\infty), f(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $f(s,t) = \log(t + a^s)/(1+t), a > 1, f(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $f(s,t) = \ln(1 + a^s)/2, a > e, f(s,t) = s \Rightarrow s = 0$.

Definition 8 ([10]). *A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 9. *An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.*

We denote by Φ_u the set of ultra altering distance functions.

Definition 10 ([12]). *Let (X,ϑ) be a uniform space and let " $\preceq$ " be an order relation on X and let $T : X \times X \to X$ be an operator. We say that T has the mixed monotone property if $T(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).$$

Definition 11 ([12]). *Let (X,ϑ) be a uniform space and let " $\preceq$ " be an order relation on X and let $T : X \times X \to X$, $g : X \to X$ be operators. We say T has the mixed g-monotone property if T is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \text{ implies } T(x_1, y) \preceq T(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \text{ implies } T(x, y_1) \succeq T(x, y_2).$$

Remark 3. *If g is the identity mapping, then Definition 11 reduces to Definition 10.*

1 THE MAIN RESULTS

Theorem 1. *Let (X,ϑ) be a Hausdorff uniform space, " $\preceq$ " is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Assume there is a function $F \in C, \varphi \in \Phi_u$ and also suppose $T : X \times X \to X$ and $g : X \to X$ are such that T has the mixed g-monotone property and*

$$\begin{aligned} &p(T(x,y), T(u,v)) \\ &\leq F\left(\left(\frac{p(g(x),g(u)) + p(g(y),g(v))}{2}\right), \varphi\left(\frac{p(g(x),g(u)) + p(g(y),g(v))}{2}\right)\right) \end{aligned} \tag{1}$$

for all $x, y, u, v \in X$ for which $g(x)$, $g(u)$ are comparable and $g(y)$, $g(v)$ are comparable. Suppose $T(X \times X) \subseteq g(X)$, g is $\tau(\vartheta)$-continuous and monotone increasing and T and g be compatible mappings. Also suppose *(a) T is $\tau(\vartheta)$-continuous*

or

(b) X has the following property :

(i) if a non-decreasing sequence

$$\{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n, \tag{2}$$

(ii) if a non-increasing sequence

$$\{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n. \tag{3}$$

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = T(x, y)$ and $g(y) = T(y, x)$, that is, T and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$. Since $T(X \times X) \subseteq g(X)$, we can define $x_1, y_1 \in X$ such that $g(x_1) = T(x_0, y_0)$ and $g(y_1) = T(y_0, x_0)$.

In the same way we construct, $g(x_2) = T(x_1, y_1)$ and $g(y_2) = T(y_1, x_1)$. Continuing in this way we construct two sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that,

$$g(x_{n+1}) = T(x_n, y_n) \text{ and } g(y_{n+1}) = T(y_n, x_n) \text{ for all } n \geq 0. \tag{4}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{5}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{6}$$

Since $g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$, in view of $g(x_1) = T(x_0, y_0)$ and $g(y_1) = T(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (5) and (6) hold for $n = 0$.

We presume that (5) and (6) hold for some $n > 0$. As T has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (4), we get

$$g(x_{n+1}) = T(x_n, y_n) \preceq T(x_{n+1}, y_n) \text{ and } T(y_{n+1}, x_n) \preceq T(y_n, x_n) = g(y_{n+1}). \tag{7}$$

Also for the same reason we have

$$g(x_{n+2}) = T(x_{n+1}, y_{n+1}) \succeq T(x_{n+1}, y_n) \text{ and } T(y_{n+1}, x_n) \succeq T(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{8}$$

Then from (7) and (8), $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$. Then, by mathematical induction it follows that (5) and (6) hold for all $n \geq 0$.

Let

$$\delta_n = p(g(x_n), g(x_{n+1})) + p(g(y_n), g(y_{n+1}))$$

and

$$\delta'_n = p(g(x_{n+1}), g(x_n)) + p(g(y_{n+1}), g(y_n)).$$

Next we prove that

$$\delta_n \leq 2F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) \text{ and } \delta'_n \leq 2F\left(\frac{\delta'_{n-1}}{2}, \varphi\left(\frac{\delta'_{n-1}}{2}\right)\right). \tag{9}$$

Since for all $n \geq 0$, $g(x_{n-1}) \preceq g(x_n)$ and $g(y_{n-1}) \succeq g(y_n)$, we have from (1) and (4),

$$\begin{aligned}
&p(g(x_n), g(x_{n+1})) = p(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\
&\leq F\left(\left(\frac{p(g(x_{n-1}), g(x_n) + p(g(y_{n-1}), g(y_n))}{2}\right), \varphi\left(\frac{p(g(x_{n-1}), g(x_n)) + p(g(y_{n-1}), g(y_n))}{2}\right)\right) \\
&= F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
p(g(x_{n+1}), g(x_n)) &= p(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\
\leq & F\left(\left(\frac{p(g(x_n), g(x_{n-1})) + p(g(y_n), g(y_{n-1}))}{2}\right),\right. \\
&\left.\varphi\left(\frac{p(g(x_n), g(x_{n-1})) + p(g(y_n), g(y_{n-1}))}{2}\right)\right) = F\left(\frac{\delta^{\shortmid}_{n-1}}{2}, \varphi\left(\frac{\delta^{\shortmid}_{n-1}}{2}\right)\right).
\end{aligned} \tag{10}$$

Similarly from (1) and (4), we have for all $n \geq 0$,

$$\begin{aligned}
&p(g(y_n), g(y_{n+1})) = p(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\
&\leq \varphi\left(\frac{p(g(y_{n-1}), g(y_n)) + p(g(x_{n-1}), g(x_n))}{2}\right) = F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&p(g(y_{n+1}), g(y_n)) = p(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\
&\leq \varphi\left(\frac{p(g(y_n), g(y_{n-1})) + p(g(x_n), g(x_{n-1}))}{2}\right) = F\left(\frac{\delta^{\shortmid}_{n-1}}{2}, \varphi\left(\frac{\delta^{\shortmid}_{n-1}}{2}\right)\right).
\end{aligned} \tag{11}$$

Combining (10) and (11) we obtain (9). Since $\varphi(t) > 0$ for $t > 0$, it follows from (9) that the sequences $\{\delta_n\}$ and $\{\delta^{\shortmid}_n\}$ are monotone decreasing sequence of non-negative real numbers. Hence there exist $\delta \geq 0$ and $\delta^{\shortmid} \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$ and $\lim_{n\to\infty} \delta^{\shortmid}_n = \delta^{\shortmid}$. Taking the limit as $n \to \infty$ in (9),we obtain $\delta = \lim_{n\to\infty} \delta_n \leq 2 \lim_{n\to\infty} F\left(\frac{\delta_{n-1}}{2}, \varphi\left(\frac{\delta_{n-1}}{2}\right)\right) = 2F\left(\left(\frac{\delta}{2}\right), \varphi\left(\frac{\delta}{2}\right)\right)$. So, $\frac{\delta}{2} = 0$ or $\varphi\left(\frac{\delta}{2}\right) = 0$. Thus $\delta = 0$. Hence we have

$$\lim_{n\to\infty} [p(g(x_n), g(x_{n+1})) + p(g(y_n), g(y_{n+1}))] = \lim_{n\to\infty} \delta_n = 0$$

and similarly $\delta^{\shortmid} = 0$ that is

$$\lim_{n\to\infty} [p(g(x_{n+1}), g(x_n)) + p(g(y_{n+1}), g(y_n))] = \lim_{n\to\infty} \delta^{\shortmid}_n = 0. \tag{12}$$

Next we show that $\{g(x_n)\}$ and $\{g(y_n)\}$ are p-Cauchy sequences. Let at least one of $\{g(x_n)\}$ and $\{g(y_n)\}$ be not a p-Cauchy sequence. Then there exists $\varepsilon > 0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every natural number k, $m(k) > l(k) \geq k$ and

$$p_k = p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right) \geq \varepsilon. \tag{13}$$

Now corresponding to $l(k)$ we can choose $m(k)$ to be the smallest positive integer for which (13) holds. Then,

$$p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) < \varepsilon. \tag{14}$$

Further from (13) and (14), for all $k \geq 0$, we have

$$\begin{aligned}\varepsilon \leq p_k \leq\ & p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right)\\ &+ p\left(g\left(x_{m(k)-1}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) + p\left(g\left(y_{m(k)-1}\right), g\left(y_{m(k)}\right)\right)\\ &= p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)-1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right)\right) + \delta_{m(k)-1} < \varepsilon + \delta_{m(k)-1}.\end{aligned}$$

Taking the limit as $k \to \infty$, we have by (12),

$$\lim_{k\to\infty} p_k = \varepsilon. \tag{15}$$

Again, for all $k \geq 0$, we have,

$$\begin{aligned}p_k &= p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)\\ &\leq p\left(g\left(x_{l(k)}\right), g\left(x_{l(k)+1}\right)\right) + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) + p\left(g\left(x_{m(k)+1}\right), g\left(x_{m(k)}\right)\right)\\ &+ p\left(g\left(y_{l(k)}\right), g\left(y_{l(k)+1}\right)\right) + p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right) + p\left(g(y_{m(k)+1}), g\left(y_{m(k)}\right)\right)\\ &= p\left(g\left(x_{l(k)}\right), g\left(x_{l(k)+1}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{l(k)+1}\right)\right) + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right)\\ &+ p(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right) + p\left(g\left(x_{m(k)+1}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{m(k)+1}\right), g\left(y_{m(k)}\right)\right).\end{aligned}$$

Hence, for all $k \geq 0$

$$p_k \leq \delta_{l(k)} + \delta^{\shortmid}_{m(k)} + p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) + p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right). \tag{16}$$

From (1), (4), (5), (6) and (13), for all $k \geq 0$, we obtain

$$\begin{aligned}&p\left(g\left(x_{l(k)+1}\right), g\left(x_{m(k)+1}\right)\right) = p\left(T\left(x_{l(k)}, y_{l(k)}\right), T\left(x_{m(k)}, y_{m(k)}\right)\right)\\ &\leq F\Big(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2},\\ &\qquad \varphi\Big(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\Big)\Big) = F\Big(\frac{p_k}{2}, \varphi\Big(\frac{p_k}{2}\Big)\Big).\end{aligned} \tag{17}$$

Also by (1), (4), (5), (6) and (13), for all $k \geq 0$, we have,

$$\begin{aligned}&p\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right)\right) = p\left(T\left(y_{l(k)}, x_{l(k)}\right), T\left(y_{m(k)}, x_{m(k)}\right)\right)\\ &\leq F\left(\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\right),\right.\\ &\qquad \left.\varphi\left(\frac{p\left(g\left(x_{l(k)}\right), g\left(x_{m(k)}\right)\right) + p\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right)\right)}{2}\right)\right) = F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right).\end{aligned} \tag{18}$$

Putting (17) and (18) in (16), for all $k \geq 0$, we obtain, $p_k \leq \delta_{l(k)} + \delta^{\shortmid}_{m(k)} + 2F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right)$.

Letting $n \to \infty$ in the above inequality and using (12), (13) and (15) we obtain,

$$\varepsilon \leq 2 \lim_{k\to\infty} F\left(\frac{p_k}{2}, \varphi\left(\frac{p_k}{2}\right)\right) = 2F\left(\frac{\varepsilon}{2}, \varphi\left(\frac{\varepsilon}{2}\right)\right).$$

So, $\frac{\varepsilon}{2} = 0$, or $\varphi\left(\frac{\varepsilon}{2}\right) = 0$ which is a contradiction. Therefore, $\{g(x_n)\}$ and $\{g(y_n)\}$ are p-Cauchy sequences in X and hence they are convergent in the p-Cauchy complete uniform space (X, ϑ). Let

$$\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y. \tag{19}$$

Since T and g are compatible mappings, we have by (19),

$$\lim_{n\to\infty} p\left(g\left(T(x_n, y_n)\right), T\left(g(x_n), g(y_n)\right)\right) = 0 \tag{20}$$

and

$$\lim_{n\to\infty} p\left(g\left(T(y_n, x_n)\right), T\left(g(y_n), g(x_n)\right)\right) = 0. \tag{21}$$

Next we prove that $g(x) = T(x, y)$ and $g(y) = T(y, x)$. Let (a) hold. For all $n \geq 0$, we have,

$$p\left(g(x_n), T\left(g(x_n), g(y_n)\right)\right) \leq p\left(g(x_n), g\left(T(x_n, y_n)\right)\right) + p\left(g\left(T(x_n, y_n)\right), T\left(g(x_n), g(y_n)\right)\right)$$

Taking the limit as $n \to \infty$, using (4), (19), (20) and the fact that T and g are continuous, we have $p\left(g(x_n), T(x, y)\right) = 0$.

Similarly, from (4), (19), (21) and the continuities of T and g, we have $p\left(g(y_n), T(y, x)\right) = 0$.

Combining the above two results we get $g(x) = T(x, y)$ and $g(y) = T(y, x)$.

Next we suppose that (b) holds. By (5), (6) and (19) we have $\{g(x_n)\}\}$ is non-decreasing sequence, $g(x_n) \to x$ and $\{g(y_n)\}$ is non-increasing sequence, $g(y_n) \to y$ as $n \to \infty$. Then by (2) and (3) we have for all $n \geq 0$,

$$g(x_n) \preceq x \text{ and } g(y_n) \succeq y. \tag{22}$$

Since, T and g are compatible mappings and g is continuous, by (20) and (21) we have,

$$\lim_{n\to\infty} g\left(g(x_n)\right) = g(x) = \lim_{n\to\infty} g\left(T(x_n, y_n)\right) = \lim_{n\to\infty} T\left(g(x_n), g(y_n)\right) \tag{23}$$

and

$$\lim_{n\to\infty} g\left(g(y_n)\right) = g(y) = \lim_{n\to\infty} g\left(T(y_n, x_n)\right) = \lim_{n\to\infty} T\left(g(y_n), g(x_n)\right). \tag{24}$$

Now we have $p\left(g(x), T(x, y)\right) \leq p\left(g(x), g(gx_{n+1})\right) + p\left(g\left(g(x_{n+1})\right), T(x, y)\right)$. Taking the limit as $n \to \infty$ in the above inequality, using (4) and (23) we have,

$$\begin{aligned} p\left(g(x), T(x, y)\right) &\leq \lim_{n\to\infty} p\left(g(x), g\left(g(x_{n+1})\right)\right) + \lim_{n\to\infty} p\left(g\left(T(x_n, y_n)\right), T(x, y)\right) \\ &\leq \lim_{n\to\infty} p\left(T\left(g(x_n), g(y_n)\right), T(x, y)\right). \end{aligned}$$

Since the mapping g is monotone increasing, by (1), (22) and the above inequality, we have for all $n \geq 0$, Using (19)

$$\begin{aligned} &p\left(g(x), T(x, y)\right) \leq \lim_{n\to\infty} F\Big(\frac{p\left(g\left(g(x_n)\right), g(x)\right) + p\left(g\left(g(y_n)\right), g(y)\right)}{2}, \\ &\varphi\Big(\frac{p\left(g\left(g(x_n)\right), g(x)\right) + p\left(g\left(g(y_n)\right), g(y)\right)}{2}\Big)\Big) = F\big(p\left(g(x), T(x, y)\right), \varphi(p\left(g(x), T(x, y)\right))\big). \end{aligned}$$

So, $p(g(x), T(x,y)) = 0$, or , $\varphi(p(g(x), T(x,y))) = 0$.

That is $g(x) = T(x,y)$ and similarly, by virtue of (4), (19) and (24) we obtain $g(y) = T(y,x)$. Thus we have proved that T and g have coupled coincidence point in X. This completes the proof. □

Remark 4. *If we take $F(s,t) = \varphi(s)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$ in the above theorem then we obtain a corollary in [12].*

Corollary 1. *Let (X,ϑ) be a Hausdorff uniform space, " $\preceq$ " is an order on X and suppose there is an E-distance p on X such that (X,p) is a p-Cauchy complete uniform space. Assume there is a function $F \in C, \varphi \in \Phi_u$ and also suppose $T : X \times X \to X$ and $g : X \to X$ are such that T has the mixed g-monotone property and*

$$p(T(x,y),T(u,v)) \leq F\left(\left(\frac{p(g(x),g(u)) + p(g(y),g(v))}{2}\right), \varphi\left(\frac{p(g(x),g(u)) + p(g(y),g(v))}{2}\right)\right)$$

for all $x,y,u,v \in X$ for which comparable $g(x), g(u)$ and comparable $g(y), g(v)$. Suppose $T(X \times X) \subseteq g(X)$, g is $\tau(\vartheta)$-continuous and commutes with T and also suppose either (a) T is $\tau(\vartheta)$-continuous or (b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$,then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq T(x_0,y_0)$ and $g(y_0) \succeq T(y_0,x_0)$, then there exist $x,y \in X$ such that $g(x) = T(x,y)$ and $g(y) = T(y,x)$, that is, T and g have a coupled coincidence.

Example 2. *Let $X = [0,1]$, $p(x,y) = |x-y|$. Then for $x,y \in X$ and " $\preceq$ " is a partially ordered with the natural ordering of real numbers. Then $(X,\preceq)$ is an ordered uniform space and (X,p) is a p-Cauchy complete uniform space. Let $g : X \to X$ be defined as $g(x) = x$ for all $x \in X$.*

Let $T : X \times X \to X$ be defined as $T(x,y) = \begin{cases} \frac{x-y}{2}, & x,y \in X,\ x \succeq y \\ 0, & x \prec y \end{cases}$. T obeys the mixed g-monotone property.

Let $\varphi : [0,\infty) \to [0,\infty)$ be defined as $\varphi(s) = s$, for $s \in [0,\infty)$ and $F(s,\varphi(s)) = \varphi(s)$. Therefore $F(s,\varphi(s)) = \varphi(s) = s \leq s$ and $F(s,\varphi(s)) = s \Rightarrow s = 0$ or $\varphi(s) = 0$ and $\varphi(s) = 0 \Rightarrow s = 0$. So $F \in C, \varphi \in \Phi_u$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that, $\lim_{n\to\infty} T(x_n,y_n) = a$, $\lim_{n\to\infty} g(x_n) = a$, and $\lim_{n\to\infty} T(y_n,x_n) = b$, $\lim_{n\to\infty} g(y_n) = b$. Then obviously, $a = 0$ and $b = 0$.

Now, for all $n \geq 0$; $g(x_n) = x_n$, $x_n \in X$ and $g(y_n) = y_n$, $y_n \in X$,

$$T(x_n,y_n) = \begin{cases} \frac{x_n-y_n}{2}, & \text{if } x_n \succeq y_n, \\ 0, & \text{if } x_n \prec y_n, \end{cases} \quad \text{and} \quad T(y_n,x_n) = \begin{cases} \frac{y_n-x_n}{2}, & \text{if } y_n \succeq x_n, \\ 0, & \text{if } y_n \prec x_n. \end{cases}$$

Then, it follows that

$$\lim_{n\to\infty} p(g(T(x_n,y_n)), T(g(x_n), g(y_n))) \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\lim_{n\to\infty} p(g(T(y_n,x_n)), T(g(y_n), g(x_n))) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, the mappings T and g are compatible in X. Also, $x_0 = 0$ and for a positive number m, $y_0 = m$ are two points in X such that $g(x_0) = g(0) = 0 = T(0,m) = T(x_0,y_0)$ and $g(y_0) = g(m) = m \succeq \frac{m}{2} = T(m,0) = T(y_0,x_0)$. We next verify inequality (1) *of Theorem 1. We take $x,y,u,v \in X$, such that $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, that is, $x \preceq u$ and $y \succeq v$.*

We consider the following cases:

Case 1: *$x \succeq y$ and $u \succeq v$.*

Then

$$\begin{aligned} p(T(x,y),T(u,v)) &= p\left(\frac{x-y}{2},\frac{u-v}{2}\right) = \left|\frac{x-y}{2}-\frac{u-v}{2}\right| = \left|\frac{x-u}{2}-\frac{y-v}{2}\right| \\ &\preceq \left|\frac{x-u}{2}\right| + \left|\frac{y-v}{2}\right| = \varphi(\frac{|x-u|}{2}+\frac{|y-v|}{2}) = \varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2}) \\ &= F\left(\frac{p(g(x),g(u))+p(g(y),g(v))}{2},\varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2})\right). \end{aligned}$$

Case 2: *$x \succeq y$ and $u \prec v$.*

Then

$$\begin{aligned} p(T(x,y),T(u,v)) &= p\left(\frac{x-y}{2},0\right) = \left|\frac{x-y}{2}\right| = \frac{x-y}{2} = \frac{u+x-y-u}{2} \\ &= \frac{(u-y)-(u-x)}{2}\,(\textit{since } v \succ u) \preceq \frac{|u-x|}{2}+\frac{|v-y|}{2} \\ &= \varphi(\frac{|x-u|}{2}+\frac{|y-v|}{2}) = \varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2}) \\ &= F\left(\frac{p(g(x),g(u))+p(g(y),g(v))}{2},\varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2})\right). \end{aligned}$$

Case 3: *$x \prec y$ and $u \succeq v$.*

Then

$$\begin{aligned} p(T(x,y),T(u,v)) &= p\left(0,\frac{u-v}{2}\right) = \left|\frac{u-v}{2}\right| = \frac{u-v}{2} = \frac{u+x-v-x}{2} \\ &= \frac{(u-x)-(v-x)}{2}\,(\textit{since } y \succ x) \preceq \frac{|u-x|}{2}+\frac{|v-y|}{2} = \varphi(\frac{|x-u|}{2}+\frac{|y-v|}{2}) \\ &= \varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2}) \\ &= F\left(\frac{p(g(x),g(u))+p(g(y),g(v))}{2},\varphi(\frac{p(g(x),g(u))+p(g(y),g(v))}{2})\right). \end{aligned}$$

Case 4: *$x \prec y$ and $u \prec v$.*

Then $T(x,y) = 0$ and $T(u,v) = 0$, that is $p(T(x,y),T(u,v)) = 0$. Obviously (1) *is satisfied.*

Thus it is verified that the functions T,g,φ satisfy all the conditions of Theorem 1. Here $(0,0)$ is the coupled coincidence point of T and g in X.

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Ансарі А.Г., Бінбасіоглу Д., Туркоглу Д. *Результати про зв'язану точку збігу для стискуючих відображень класу C у впорядкованих рівномірних просторах* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 3–13.

У літературі існує багато робіт, пов'язаних з теорією нерухомої точки. Ця теорія має багато застосувань, тому деякі автори зацікавлені в цих застосуваннях в різних просторах. У 2009 р. Алтун І. та Імдад М. визначили відношення порядку на рівномірних просторах і поняття сумісності відображень. Ансарі А. ввів концепцію функцій C-класу. У цій статті ми вибираємо функції C-класу, що ультра змінюють відстань, та доводимо деякі теореми про зв'язану точку збігу для відображень, що задовольняють властивість змішаної g-монотонності у впорядкованих рівномірних просторах. Ми також наводимо відповідні приклади.

Ключові слова і фрази: зв'язана точка збігу, C-клас відображень, впорядкований рівномірний простір.

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SOME WEAKER SUFFICIENT CONDITIONS OF L-INDEX BOUNDEDNESS IN DIRECTION FOR FUNCTIONS ANALYTIC IN THE UNIT BALL

We partially reinforce some criteria of L-index boundedness in direction for functions analytic in the unit ball. These results describe local behavior of directional derivatives on the circle, estimates of maximum modulus, minimum modulus of analytic function, distribution of its zeros and modulus of directional logarithmic derivative of analytic function outside some exceptional set. Replacement of universal quantifier on existential quantifier gives new weaker sufficient conditions of L-index boundedness in direction for functions analytic in the unit ball. The results are also new for analytic functions in the unit disc. The logarithmic criterion has applications in analytic theory of differential equations. This is convenient to investigate index boundedness for entire solutions of linear differential equations. It is also apllicable to infinite products.

Auxiliary class of positive continuous functions in the unit ball (so-denoted $Q_{\mathbf{b}}(\mathbb{B}^n)$) is also considered. There are proved some characterizing properties of these functions. The properties describe local behavior of these functions in the polydisc neighborhood of every point from the unit ball.

Key words and phrases: bounded L-index in direction, analytic function, unit ball, maximum modulus, directional derivative, distribution of zero.

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INTRODUCTION

The paper is addendum to papers [4–6, 20]. There was introduced a concept of analytic functions in the unit ball of bounded L-index in a direction, where $L : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function, $R_+ = (0, +\infty)$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Besides, there were deduced necessary and sufficient conditions of belonging of analytic function in the unit ball to functions of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where $\mathbf{0} = (0, \ldots, 0)$. The conditions describe local behavior directional derivatives, maximum modulus and minimum modulus of the analytic function on the circle of arbitrary radii. There are also an estimate of logarithmic directional derivative outside some exceptional set by the function L and an estimate of distribution of zeros for the analytic functions. Moreover, we established connection [4] betweeen analytic functions in the unit ball of bounded L-index in direction and analytic function in the unit ball of bounded value L-distribution.

Of course, there are two big classes of functions analytic in bounded domains from $\mathbb{C}^n$. These domains are unit ball and unit polydisc. The domains are not biholomorpic equivalent. Nevertheless, they are importance domains in function theory of several complex variables. Many methods are firstly developing for these domains. Particularly, there are papers [8–10]

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on the concept of bounded $\mathbf{L}$-index in joint variables for functions analytic in the unit polydisc or in the unit ball. It was demonstrated application [17] of the concept to study properties of analytic solutions of some systems of partial differential equations.

Recently, for entire functions of bounded L-index in direction new weaker sufficient conditions are obtained [2, 16]. They require validity of some conditions for one value of radius instead each positive value. Moreover, there was presented class [7] of entire functions of unbounded index in any direction. The proof of this fact checks validity of some conditions for some radius. It is simpler than for any radius. Also this idea [11] was applied to investigate L-index boundedness in direction of entire solutions of linear directional differential equations.

Here we will consider similar problems for analytic functions in the unit ball.

1 AUXILIARY CLASS OF POSITIVE CONTINUOUS FUNCTIONS IN THE UNIT BALL

This section is devoted to auxiliary class of of positive continuous functions in the unit ball. Note that positivity and continuity are still weak restrictions to construct a deep theory of bounded index. Thus, we suppose that the functions satisfy additional assumptions on local behavior.

Let $\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a fixed direction, where $\mathbf{0} = (0, \ldots, 0)$. For $z \in \mathbb{B}^n$ we denote $D_z = \{t \in \mathbb{C} : |t| \leq \frac{1-|z|}{|\mathbf{b}|}\}$,

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1 - t_2| \leq \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{B}^n)$ stands for a class of positive continuous functions $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying

$$(\forall \eta \in [0, \beta]) : \ \lambda_{\mathbf{b}}(\eta) < +\infty \tag{1}$$

and

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}, \tag{2}$$

where $\beta > 0$ is some constant. It is easy to check that class $Q_{\mathbf{b}}(\mathbb{B}^n)$ can be defined as follows. For $\eta \in [0, \beta]$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L : \mathbb{B}^n \to \mathbb{R}_+$, satisfying (2), we define

$$\lambda_1^{\mathbf{b}}(\eta) = \inf_{z\in\mathbb{B}^n} \inf \{L(z+t\mathbf{b})/L(z) : |t| \leq \eta/L(z)\},$$

$$\lambda_2^{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{B}^n} \sup \{L(z+t\mathbf{b})/L(z) : |t| \leq \eta/L(z)\}.$$

Then the class $Q_{\mathbf{b}}(\mathbb{B}^n)$ consists from the functions L, providing inequality

$$(\forall \eta \in [0, \beta]) : \ 0 < \lambda_1^{\mathbf{b}}(\eta) \leq \lambda_2^{\mathbf{b}}(\eta) < +\infty, \tag{3}$$

i.e., conditions (3) and (1) equivalent. Actually it is enough to require validity of any inequality in (3) for one value $\eta \in (0, \beta]$ (for $\eta = 0$ the inequality is trivial). If $n = 1$ then $Q(\mathbb{D}) \equiv Q_{\mathbf{1}}(\mathbb{B}^1)$.

The reasoning leads us to the proposition.

Proposition 1. *Let* $L : \mathbb{B}^n \to \mathbb{R}_+$ *be a positive continuous functions such that* $(\forall z \in \mathbb{B}^n) : L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, *where* $\beta > 1$. *Then the following statements are equivalent:*

1. $(\forall \eta \in [0,\beta]) : \lambda_{\mathbf{b}}(\eta) < +\infty$;
2. $(\forall \eta \in [0,\beta]) : 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$;
3. $(\exists \eta \in (0,\beta]) : 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$.

The proof of this proposition is elementary and uses the definition of class $Q_{\mathbf{b}}(\mathbb{B}^n)$. Other propositions on class $Q_{\mathbf{b}}$ are in [1, 14, 20].

2 LOCAL BEHAVIOR OF DIRECTIONAL DERIVATIVE

Henceforth, we everywhere suppose that $\beta > 1$.

Analytic function $F : \mathbb{B}^n \to \mathbb{C}$ is called a function of *bounded* L*-index [4–6, 20] in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for each $z \in \mathbb{B}^n$

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max_{0 \le k \le m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}, \tag{4}$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}} \left(\partial_{\mathbf{b}}^{k-1} F(z) \right)$, $k \ge 2$. There is also papers about analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables [19]. A connection between these classes is established in [17].

Theory of entire functions of bounded L-index in direction is deeply considered in [13].

We need the following criterion of L-index boundedness in direction.

Theorem 1 ([5, 6])**.** *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n$ *if and only if for every* η, $0 < \eta \le \beta$, *there exist* $n_0 = n_0(\eta) \in \mathbb{Z}_+$ *and* $P_1 = P_1(\eta) \ge 1$ *such that for each* $z \in \mathbb{B}^n$ *there exists* $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, *and the following inequality*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \eta / L(z)\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|$$

holds.

Let us formulate some auxiliary propositions.

Lemma 1 ([5, 6])**.** *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta} < \theta_1 \le \theta_2 < +\infty$, $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* L^**-index in the direction* $\mathbf{b}$ *if and only if the function* F *has bounded* L*-index in the direction* $\mathbf{b}$.

Lemma 2 ([5, 6])**.** *Let* $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $m \in \mathbb{C}, m \ne 0$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *is of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n$ *if and only if the function* $F(z)$ *is of bounded* L*-index in the direction* $m\mathbf{b}$.

Theorem 2 ([5, 6])**.** *Let* $\beta > 1$, $L \in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. *Analytic function* $F(z)$ *in* $\mathbb{B}^n$ *has bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *if and only if for any* r_1 *and for any* r_2, $0 < r_1 < r_2 \le \beta$, *there exists* $P_1 = P_1(r_1, r_2) \ge 1$ *such that for each* $z^0 \in \mathbb{B}^n$

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_1 \max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_1}{L(z_0)}\right\}. \tag{5}$$

Theorem 2 is criterion of L-index boudnedness in direction providing maximum modulus estimate on the greater circle by maximum modulus estimate on the lesser circle. Also it is known some stronger proposition as sufficient conditions.

Theorem 3 ([5,6])**.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. Analytic function $F(z)$ in $\mathbb{B}^n$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exist r_1 and r_2, $0 < r_1 < 1 < r_2 \le \beta$, and $P_1 \ge 1$ such that for every $z^0 \in \mathbb{B}^n$ inequality* (5) *is true.*

The theorems distinguish universal and existential quantifiers for r_1 and r_2 such that $0 < r_1 < 1 < r_2 < +\infty$.

This leads to a natural question: *Is it possible to replace quantifiers in other criteria of L-index boundedness in direction?*

Using Fricke's idea [21], we deduce a modification of Theorem 1.

Theorem 4. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. If there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, and*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \eta / L(z)\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

then analytic function $F : \mathbb{B}^n \to \mathbb{C}$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

Proof. Besides mentioned paper of Fricke [21], our proof is similar to [3] (entire functions of bounded L-index in direction) and to [29] (entire functions of bounded l-index).

Assume that there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \ge 1$ such that for any $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \le \frac{\eta}{L(z)}\} \le P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{6}$$

If $\eta \in (1, \beta]$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \le \eta^{j_0}$. And for $\eta \in (0; 1]$ we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0! k_0!}{(j_0+k_0)!} P_1 < 1$. The j_0 is well-defined because

$$\frac{j_0! k_0!}{(j_0 + k_0)!} P_1 = \frac{k_0!}{(j_0 + 1)(j_0 + 2) \cdot \ldots \cdot (j_0 + k_0)} P_1 \to 0, \ j_0 \to \infty.$$

Applying integral Cauchy's formula to the function $F(z + t\mathbf{b})$ as analytic function of one complex variable t for $j \ge j_0$ we obtain that for every $z \in \mathbb{B}^n$ there exists $k_0 = k_0(z)$, $0 \le k_0 \le n_0$, and

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int\limits_{|t| = \frac{\eta}{L(z)}} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

Taking into account (6), we deduce

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \le \frac{L^j(z)}{\eta^j} \max\left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : \ |t| = \frac{\eta}{L(z)} \right\} \le P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{7}$$

In view of choice j_0 with $\eta \in (1, \beta]$, for all $j \ge j_0$ one has

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \le \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z + t_0\mathbf{b})} \le \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \le n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ do not depend of z, and $z \in \mathbb{B}^n$ is arbitrary, the last inequality is equivalent to the assertion that F has boudned L-index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F, L) \le n_0 + j_0$.

If $\eta \in (0, 1)$, then from (7) it follows that for all $j \ge j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)! L^{k_0+j}(z)} \le \frac{j! k_0! P_1}{(j+k_0)!} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)}$$

or in view of choice j_0

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)!} \frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!} \frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function F is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then by Lemma 1 the function F has bounded L-index in the direction $\mathbf{b}$, if $\eta\beta > 1$. When $\eta \le \frac{1}{\beta}$, we choose arbitrary $\gamma > \frac{1}{\eta\beta}$. By Lemma 1 the function F is of bounded L_1-index in the direction $\mathbf{b}$, where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then be Lemma 2 the function F has bounded L_1-index in the direction $\gamma\mathbf{b}$. Since $\partial_{\gamma\mathbf{b}}^k F = \gamma^k \partial_{\mathbf{b}}^k F$ and $L_1^k(z) = \gamma^k L^k(z)$, in inequality (4) with the definition of L-index boundedness in direction the corresponding multiplier γ is reduced. Hence, the function F is of bounded L-index in the direction $\mathbf{b}$. Theorem is proved. $\square$

The following propostion is easy directly deduced from the definition of L-index boundedness in direction.

Proposition 2. *Let $L : \mathbb{B}^n \to \mathbb{C}$ be a positive continuous function. An analytic function $F : \mathbb{B}^n \to \mathbb{C}$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if the function $G(z) = F(\mathbf{a}z + \mathbf{c})$ has bounded L_*-index in the direction $\frac{\mathbf{b}}{\mathbf{a}}$ for any $\mathbf{c} \in \mathbb{C}^n$ and $\mathbf{a} \in \mathbb{B}^n$ such that $|c| < 1 - |a|$, $a_j \ne 0$ $(\forall j)$, where $\mathbf{a}z + \mathbf{c} = (a_1 z_1 + c_1, \ldots, a_n z_n + c_n)$, $\frac{\mathbf{b}}{\mathbf{a}} = (\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$, $L_*(z) = L(\mathbf{a}z + \mathbf{c})$.*

The proof of the proposition is elementary and it is similar to proof in the case of entire functions (see [12]).

Analog of Proposition 2 for entire functions has generated the following still open problem.

Problem 1 ([12]). *Does exist numbers a_1, a_2, c_1, $c_2 \in \mathbb{C}$ and an entire function $F(z_1, z_2)$ such that $F(z_1, z_2)$ is of bounded L-index in a direction $\mathbf{b} = (b_1, b_2)$, but $F(a_1 z_1 + c_1, a_2 z_2 + c_2)$ is of unbounded L-index in the same direction $\mathbf{b} = (b_1, b_2)$?*

3 ESTIMATE MAXIMUM MODULUS BY MINIMUM MODULUS

Previously (see [5,6]) we proved few criteria of L-index boundedness in direction. They are analogs of one-dimensional criterion of l-index boundedness [29]. Moreover, we found that some assertions (Theorems 1 and 2) have modified stronger versions. In fact, their reinforcement is to replace universal quantifiers by existential quantifiers (see Theorems 3 and 4).

Also we can weaken sufficient conditions of Theorem 3, replacing the condition $0 < r_1 < 1 < r_2 < +\infty$ by $0 < r_1 < r_2 < +\infty$.

Theorem 5. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, F be a function analytic in $\mathbb{B}^n$. If there exist r_1 and r_2, $0 < r_1 < r_2 \le \beta$, and $P_1 \ge 1$ such that for all $z^0 \in \mathbb{B}^n$ inequality* (5) *is satisfied, then the function F is of bounded L-index in the direction* $\mathbf{b}$.

Proof. Our proof is based on idea of A. D. Kuzyk and M. M. Sheremeta [24]. They proposed this method to investigate the l-index boundedness of entire solutions of linear differential equations. Later their idea was applied for entire functions of bounded L-index in the direction and in joint variables [2, 15].

Inequality (5) for $0 < r_1 < r_2 < \beta$ implies

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_2}{r_1+r_2}\frac{r_1+r_2}{2L(z^0)}\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_1}{r_1+r_2}\frac{r_1+r_2}{2L(z_0)}\right\}.$$

Putting $L^*(z) = \frac{2L(z)}{r_1+r_2}$, we obtain

$$\begin{aligned} &\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_2}{(r_1+r_2)L^*(z^0)}\right\} \\ &\qquad \le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2r_1}{(r_1+r_2)L^*(z^0)}\right\}, \end{aligned} \tag{8}$$

where $0 < \frac{2r_1}{r_1+r_2} < 1 < \frac{2r_2}{r_1+r_2} < \frac{2\beta}{r_1+r_2}$. Clearly, $L^*(z) = \frac{2L(z)}{r_1+r_2} > \frac{2\beta|b|}{(r_1+r_2)(1-|z|)}$, i.e., L^* satisfies (2) and belongs to the class $Q_{\mathbf{b}}(\mathbb{B}^n)$ with $\frac{2\beta}{r_1+r_2}$ instead β. From validity of inequality (8) we get that by Theorem 3 the function F has bounded L^*-index in the direction $\mathbf{b}$. And by Lemma 1 the function F has bounded L-index in the direction $\mathbf{b}$. □

Theorem 6 ([5, 6])**.** *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. An analytic function $F(z)$ in $\mathbb{B}^n$ has bounded L-index in the direction $\mathbf{b}$ if and only if for every R, $0 < R \le \beta$, there exist $P_2(R) \ge 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in [\eta(R), R]$ the inequality*

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \le P_2 \min\left\{|F(z^0+t\mathbf{b})| : \ |t| = r/L(z^0)\right\} \tag{9}$$

is true.

Taking into account analogs of Theorems 4 and 5 for entire functions there was posed the following question in [12].

Problem 2 ([12, Problem 6])**.** *Is the following Conjecture 1 true?*

Conjecture 1 ([12, 1])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F : \mathbb{C}^n \to \mathbb{C}$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exist $R > 0$, $P_2(R) \ge 1$ and $\eta(R) \in (0, R)$ such that for all $z^0 \in \mathbb{C}^n$ and some $r = r(z^0) \in [\eta(R), R]$ inequality* (9) *is valid.*

The was fully proved for entire functions in [2, 16].

Now, we will try to deduce similar results for functions analytic in the unit ball.

Theorem 7. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If there exists $R \in (0, \beta/2)$ (or if there exists $R \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) and there exist $P_2 \ge 1$, $\eta \in (0, R)$ such that for all $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in [\eta, R]$ inequality* (9) *holds, thenthe function F has bounded L-index in the direction* $\mathbf{b}$.

Proof. In view of Theorem 5 we need to show existence P_1 such that for all $z^0 \in \mathbb{B}^n$

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = (\beta-R)/L(z^0)\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})| : |t| = R/L(z^0)\right\}. \tag{10}$$

Assume that there exist $R \in (0,\beta/2)$, $P_2 \ge 1$ and $\eta \in \big(0,R\big)$ such that for every $z^0 \in \mathbb{B}^n$ and some $r = r(z^0) \in \big[\eta, R\big]$ we have

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \le P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\}.$$

Denote $L^* = \max\left\{L(z^0+t\mathbf{b}) : |t| \le \beta/L(z^0)\right\}$, $\rho_0 = R/L(z^0)$, $\rho_k = \rho_0 + k\eta/L^*$, $k \in \mathbb{Z}_+$. We obtain

$$\frac{\eta}{L^*} < \frac{R}{L^*} \le \frac{R}{L(z^0)} = \frac{\beta}{L(z^0)} - \frac{\beta-R}{L(z^0)}.$$

Therefore, there exists $n^* \in \mathbb{N}$, independent of z^0 and such that

$$\rho_{p-1} < \frac{\beta-R}{L(z^0)} \le \rho_p \le \frac{\beta}{L(z^0)},$$

for some $p = p(z^0) \le n^*$. It is possible because $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. Ar first, one has

$$\begin{aligned}\left(\frac{\beta}{L(z^0)} - \rho_0\right)\Big/\left(\frac{\eta}{L^*}\right) =& \frac{(\beta-R)L^*}{\eta L(z^0)}\\ =& \frac{\beta-R}{\eta}\max\left\{\frac{L(z^0+t\mathbf{b})}{L(z^0)} : |t| \le \frac{\beta}{L(z^0)}\right\} \le \frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta).\end{aligned}$$

Therefore, $n^* = \left[\frac{\beta-R}{\eta}\lambda_{\mathbf{b}}(\beta)\right]$, where $[a]$ is an entire part of number $a \in \mathbb{R}$. Let $|F(z^0+t_k^{**}\mathbf{b})| = \max\{|F(z^0+t\mathbf{b})| : t \in c_k\}$, $c_k = \{t \in \mathbb{C} : |t| = \rho_k\}$, and t_k^* be the intersection point of the segment $[0, t_k^{**}]$ with the circle c_{k-1}. Hence, for every $r > \eta$ and for each $k \le n^*$ we get the inequality $|t_k^{**} - t_k^*| = \frac{\eta}{L^*} \le \frac{r}{L(z^0+t_k^*\mathbf{b})}$. Thus, for some $r = r(z^0+t_k^*\mathbf{b}) \in [\eta, R]$ we deduce

$$\begin{aligned}|F(z^0+t_k^{**}\mathbf{b})| &\le \max\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b})\right\}\\ &\le P_2\min\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b})\right\}\\ &\le P_2\min\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b}), |t-t_0| \le \rho_{k-1}\right\}\\ &\le P_2\max\{|F(z^0+t\mathbf{b})| : t \in c_{k-1}\}.\end{aligned}$$

Hence,

$$\begin{aligned}\max\left\{|F(z^0+t\mathbf{b})| : |t| = (\beta-R)/L(z^0)\right\} &\le \max\{|F(z^0+t\mathbf{b})| : t \in c_p\}\\ &\le P_2\max\{|F(z^0+t\mathbf{b})| : t \in c_{p-1}\}\\ &\le \ldots \le (P_2)^p\max\{|F(z^0+t\mathbf{b})| : t \in c_0\}\\ &\le (P_2)^{n^*}\max\left\{|F(z^0+t\mathbf{b})| : |t| = R/L(z^0)\right\}.\end{aligned}$$

We get (10) with $P_1 = (P_2)^{n^*}$. Thus, for $R \in (0,\beta/2)$ Theorem 7 is proved.

Now, suppose that $R \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$. Then inequality (9) can be rewritten as

$$\max\left\{|F(z^0 + \frac{t}{2} \cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + \frac{t}{2} \cdot 2\mathbf{b})| : |t/2| = \frac{r/2}{L(z^0)}\right\}.$$

Denoting $t' = t/2$, one has

$$\max\left\{|F(z^0 + t' \cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t' \cdot 2\mathbf{b})| : |t'| = \frac{r/2}{L(z^0)}\right\}.$$

Since $r \le R \in [\beta/2, \beta)$, we have $r/2 \le R \in [\beta/4, \beta/2) \subset (0, \beta/2)$. Therefore, as shown above the function F has bounded L-index in the direction $2\mathbf{b}$, but by Lemma 2 the function is also of bounded L-index in the direction $\mathbf{b}$. $\square$

4 ESTIMATE OF DIRECTIONAL LOGARITHMIC DERIVATIVE

Below we formulate another criterion of L-index boundedness in direction. It describes behavior of logarithmic derivative in direction and distribution of zeros. Firstly the criterion was obtained by Fricke [21,22] for entire function of bounded index.

We need additional notations.

Let $g_{z^0}(t) := F(z^0 + t\mathbf{b})$. If for given $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \neq 0$ for all $t \in D_{z^0}$, then $G_r^{\mathbf{b}}(F, z^0) := \varnothing$; if for given $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \equiv 0$, then $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} \colon t \in D_{z^0}\}$. And if for some $z^0 \in \mathbb{B}^n$ $g_{z^0}(t) \not\equiv 0$ and a_k^0 are zeros of the functions $g_{z^0}(t)$, i.e., $F(z^0 + a_k^0\mathbf{b}) = 0$, then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{z^0 + t\mathbf{b} \colon |t - a_k^0| \le \frac{r}{L(z^0 + a_k^0\mathbf{b})}\right\}, \quad r > 0.$$

Let

$$G_r^{\mathbf{b}}(F) = \bigcup_{z^0 \in \mathbb{B}^n} G_r^{\mathbf{b}}(F, z^0).$$

By $n\big(r, z^0, 1/F\big) = \sum_{|a_k^0| \le r} 1$ we denote counting functions of number of zeros a_k^0.

Theorem 8 ([5,6]). *Let F be an analytic function in $\mathbb{B}^n$, $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $\mathbb{B}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \varnothing$. The function $F(z)$ has bounded L-index in the direction $\mathbf{b}$ if and only if*

1) *for every $r \in (0, \beta]$ there exists $P = P(r) > 0$ such that for any $z \in \mathbb{B}^n \setminus G_r^{\mathbf{b}}(F)$*
$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le PL(z); \tag{11}$$

2) *for each $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{B}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$ one has*
$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r). \tag{12}$$

We weak sufficient conditions in Theorem 8. The one-dimensional analog of Theorem 8 for entire functions revealed its efficiency in the investigation of boundedness of the l-index of infinite products in the one-dimensional case [27,28]. Recently, in [18], it has also used this criterion to establish the sufficient conditions of boundedness of the L-index in joint variables in terms of the restrictions imposed on the partial logarithmic derivatives and the distribution of zeros. There was posed the following problem.

Problem 3 ([12, Problem 7])**.** *Is the following Conjecture 2 true?*

Conjecture 2 ([12, 2])**.** *Let $F(z)$ be an entire function in $\mathbb{C}^n$, $L \in Q^n_{\mathbf{b}}$. The function F has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if*

1) *there exist $r > 0$, $P > 0$ such that for every $z \in \mathbb{C}^n \setminus G_r$ inequality (11) holds;*

2) *there exist $r > 0$, $\widetilde{n} \in \mathbb{Z}_+$ such that for every $z \in \mathbb{C}^n$ inequality (12) is true.*

By some additional restriction there was proved the conjecture in [2,16].

Now we consider similar problem for analytic functions in the unit ball with $r \in (0, \beta]$ instead $r > 0$. Let us denote

$$G_r(F) := G^{\mathbf{b}}_r(F) = \bigcup_{z\colon F(z)=0} \{z + t\mathbf{b}\colon |t| < r/L(z)\},$$

a^0_k are zeros of the function $F(z^0 + t\mathbf{b})$ for fixed $z^0 \in \mathbb{B}^n$. By $n_{z^0}(r, F) = n_{\mathbf{b}}\big(r, z^0, 1/F\big) := \sum_{|a^0_k| \le r} 1$ we denote the counting function of zeros a^0_k for the slice function $F(z^0 + t\mathbf{b})$ in the disc $\{t \in \mathbb{C}\colon |t| \le r\}$. If for given $z^0 \in \mathbb{B}^n$ and for all $t \in D_z$ $F(z^0 + t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z \in \mathbb{B}^n} n_z(r/L(z))$.

Theorem 9. *Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\mathbb{B}^n \setminus G^{\mathbf{b}}_{\beta}(F) \neq \varnothing$, $F : \mathbb{B}^n \to \mathbb{C}$ be an analytic function. If the following conditions are satisfied*

1) *there exists $r_1 \in (0, \beta/2)$ (either there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{B}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1; \infty)$;*

2) *there exist $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{B}^n \setminus G_{r_2}(F)$ inequality (11) is true;*

then the function F has bounded L-index in the direction $\mathbf{b}$.

Proof. Suppose that conditions 1) and 2) are true.

At first, we consider the case $n(r_1) \in \{-1; 0\}$. Then in the best case the function F can only identically equals zero on the complex line $z^* + t\mathbf{b}$ for some $z^* \in \mathbb{B}^n$, i.e., $F(z^* + t\mathbf{b}) \equiv 0$. For all points lying on such complex lines inequality (9) is obvious.

Let $z^0 \in \mathbb{B}^n \setminus G_{r_2}$. For any points t_1 and t_2 such that $|t_j| = \frac{r_2}{L(z_0)}$, $j \in \{1, 2\}$, one has

$$\ln\left|\frac{F(z^0 + t_2\mathbf{b})}{F(z^0 + t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2} \left|\frac{\partial_{\mathbf{b}} F(z^0 + t\mathbf{b})}{F(z^0 + t\mathbf{b})}\right| |dt| \le P \int_{t_1}^{t_2} L(z^0 + t\mathbf{b})|dt| \\ \le P\lambda_{\mathbf{b}}(r_2)\, L(z^0) \frac{\pi r_2}{L(z^0)} \le \pi r_2 P \lambda_{\mathbf{b}}(r_2)$$

(we also use that $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$). Hence,

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t\mathbf{b})| :\ |t| = \frac{r_1}{L(z^0)}\right\},$$

where $P_2 = \exp\{\pi r_2 P \lambda_2(r_2)\}$. Therefore, by Theorem 7 the function F has bounded L-index in the direction $\mathbf{b}$.

Let $r_1 > 0$ be a such that $n(r_1) \in [1;\infty)$ and $2n(r_1)r_2 < r_1/\lambda_{\mathbf{b}}(r_1)$. Put $c = \frac{r_1}{2r_2\lambda_{\mathbf{b}}(r_1)} - n(r_1) > 0$. Clearly, $r_2 = r_1/(2(n(r_1)+c)\lambda_{\mathbf{b}}(r_1))$.

Under condition 1) each set $\overline{K} = \left\{z^0 + t\mathbf{b} : |t| \le \frac{r_1}{L(z^0)}\right\}$ has no more $n(r_1)$ zeros of the function F, where $F(z^0 + t\mathbf{b}) \not\equiv 0$.

Under condition 2) there exists $P > 0$ such that $|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}| \le PL(z)$ for every $z \in \mathbb{B}^n \backslash G_{r_2}$, i.e., for all $z \in \overline{K}$, lying outside the sets $\left\{z^0 + t\mathbf{b} : \ |t - a_k^0| < \frac{r_2}{L(z^0 + a_k^0\mathbf{b})}\right\}$, where $a_k^0 \in \overline{K}$ are zeros of the slice function $F(z^0 + t\mathbf{b}) \not\equiv 0$. By definition $\lambda_{\mathbf{b}}$ we obtain $L(z^0)/\lambda_{\mathbf{b}}(r_1) \le L(z^0 + a_k^0\mathbf{b})$. Then $|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}| \le PL(z)$ for every point $z \in \mathbb{B}^n$, lying outside union of the sets

$$c_k^0 = \left\{z^0 + t\mathbf{b} : |t - a_k^0| \le \frac{r_2\lambda_{\mathbf{b}}(r_1)}{L(z^0)} = \frac{r_1}{2(n(r_1) + c)L(z^0)}\right\}.$$

The total sum of diameters of the sets c_k^0 does not exceed the value $\frac{r_1 n(r_1)}{(n(r_1)+c)L(z^0)} < \frac{r_1}{L(z^0)}$. Hence, there exists a set $\tilde{c}^0 = \left\{z^0 + t\mathbf{b} : |t| = \frac{r}{L(z^0)}\right\}$, where $\frac{r_1 \min\{1,c\}}{2(n(r_1)+c)} = \eta < r < r_1$, such that for all $z \in \tilde{c}^0$ we have $\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le PL(z) \le P\lambda_{\mathbf{b}}(r)L(z^0) \le P\lambda_{\mathbf{b}}(r_1)\, L(z^0)$. For any points $z_1 = z^0 + t_1\mathbf{b}$ and $z_2 = z^0 + t_2\mathbf{b}$ with $\tilde{c}^0$ one has

$$\ln\left|\frac{F(z^0 + t_2\mathbf{b})}{F(z^0 + t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2} \left|\frac{\partial_{\mathbf{b}} F(z^0 + t\mathbf{b})}{F(z^0 + t\mathbf{b})}\right| |dt| \le P\lambda_2(r_1)\, L(z^0)\frac{\pi r}{L(z^0)} \le \pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1).$$

Therefore,

$$\max\left\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0 + t\mathbf{b})| : \ |t| = \frac{r}{L(z^0)}\right\}, \tag{13}$$

where $P_2 = \exp\{\pi r_1\, P(r_2)\lambda_{\mathbf{b}}(r_1)\}$. If $F(z^0 + t\mathbf{b}) \equiv 0$, then inequality (13) is obvious. By Theorem 7 the function $F(z)$ has bounded L-index in the direction $\mathbf{b}$. Theorem 9 is proved. □

Remark 1. We proved Hypothesis 2 for analytic function in the unit ball under the additional condition $2r_2 n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$. The same condition was firstly appeared for entire functions in [16]. At present, we do not know whether this condition is essential (see Problem 3 in [16]).

Note that Theorems 4, 5, 7 and 9 are new even for analytic functions in the unit disc (cf. [23, 25, 26]). Particularly, for $n = 1$ and analytic functions of bounded l-index Theorem 9 implies the following corollary.

Corollary 1. *Let $l \in Q(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{C}$ be an analytic function in the unit disc. If the function f satisfies the condition:*

1) *there exists $r_1 \in (0, \beta/2)$ (either there exists $r_1 \in [\beta/2, \beta)$ and $(\forall t \in \mathbb{D}) : l(t) > \frac{2\beta}{1-|t|}$) such that $n(r_1) \in [0;\infty)$;*

2) *there exists $r_2 \in (0, \beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$, $\mathbb{D}\backslash G_{r_2}(f) \ne \varnothing$ and for all $t \in \mathbb{D}\backslash G_{r_2}(f)$ $\frac{|f'(t)|}{|f(t)|} \le Pl(t)$;*

then the function f has bounded l-index.

As we have written that similar criteria (estimate of maximum modulus, minimum modulus, logarithmic derivative and distribution of zeros for arbitrary radii) are also known for function analytic in the unit disc and in arbitrary domain on the complex plane [23,25,26]. But they contain the universal quantifier in their assumptions.

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Бандура А.І. *Деякі слабші достатні умови обмеженості* L*-індексу за напрямком для аналітичних в одиничній кулі функцій* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 14–25.

Частково посилюються деякі критерії обмеженості L-індексу за напрямком для аналітичних в одиничній кулі функцій. Ці результати описують локальне поводження похідних за напрямком на колі, оцінки максимуму модуля, мінімуму модуля аналітичної функції, розподілу її нулів та модуля логарифмічної похідної за напрямком від аналітичної функції зовні деякої виняткової множини. Заміна квантора універсальності на квантор загальності дає нові слабші достатні умови обмеженості L-індексу за напрямком для аналітичних в одиничній кулі функцій. Ці результати також є новими для функцій, аналітичних в одиничному крузі. Отриманий логарифмічний критерій має застосування в аналітичній теорії диференційних рівнянь. Він зручний у дослідженні обмеженості індексу цілих розв'язків лінійних диференційних рівнянь. Також він застосовний до нескінченних добутків.

Досліджено допоміжний клас додатних неперервних функцій в одиничній кулі (так званий $Q_{\mathbf{b}}(\mathbb{B}^n)$). Для функцій з цього класу доведено деякі характеристизаційні властивості. Ці властивості описують локальне поводження таких функцій в полікругових околах кожної точки з одиничної кулі.

Ключові слова і фрази: обмежений L-індекс за напрямком, аналітична функція, одинична куля, максимум модуля, похідна за напрямком, розподіл нулів.

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ZABOLOTSKYI M.V., BASIUK YU.V.

ASYMPTOTICS OF THE ENTIRE FUNCTIONS WITH v-DENSITY OF ZEROS ALONG THE LOGARITHMIC SPIRALS

Let v be the growth function such that $rv'(r)/v(r) \to 0$ as $r \to +\infty$, $l_\varphi^c = \{z = te^{i(\varphi + c\ln t)}, 1 \leqslant t < +\infty\}$ be the logarithmic spiral, f be the entire function of zero order. The asymptotics of $\ln f(re^{i(\theta + c\ln r)})$ along ordinary logarithmic spirals l_θ^c of the function f with v-density of zeros along l_φ^c outside of the C_0-set is found. The inverse statement is true just in case zeros of f are placed on the finite logarithmic spirals system $\Gamma_m = \bigcup_{j=0}^m l_{\theta_j}^c$.

Key words and phrases: entire function, density of zeros, logarithmic spiral.

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INTRODUCTION

The issues related to the study of behavior of entire functions along the logarithmic spirals were considered in [1–4, 6]. In particular, Macintyre [6] introduced the notion of an indicator along the logarithmic spiral and generalized the concept of associated function. Kennedy [3] generalized the concept of Mittag - Leffler function on the curvilinear area. Valiron-type and Valiron-Titchmarsh-type theorems for entire functions of positive order with zeros on the logarithmic spiral were proved by Balašov [2] and Kheifits [4] correspondingly. The relation between regular behavior of logarithm of modulus of entire function f of positive order along the curves of regular rotation (in particular, the logarithmic spirals) and existance of density of zeros of f along these curves was investigated in [1]. The results of [1] generalize the well-known Levin and Pfluger research of entire functions of completely regular growth (see, for example, [5, p. 118-122; p. 199]).

In this paper we study issues that similar to ones considered in [1] for entire functions of zero order.

1 SECTION WITH RESULTS

For $c \in \mathbb{R}$, $\varphi \in [-\pi; \pi)$ we denote by $l_\varphi^c(a, r) = \{z : z = te^{i(\varphi + c\ln t)}, a \leqslant t < r\}$, $l_\varphi^c(1, +\infty) = l_\varphi^c$ the logarithmic spiral, $D^c(r; \alpha, \beta) = \bigcup\limits_{\alpha \leqslant \varphi < \beta} l_\varphi^c(1, r)$ the curvilinear sector, $-\pi \leqslant \alpha < \beta < \pi$.

Let L be the set of all growth functions v such that $rv'(r)/v(r) \to 0$ as $r \to +\infty$ where growth function $v : [0; +\infty) \to \mathbb{R}_+$ is a continuously differentiable increasing to $+\infty$ function. It is clear that a set L coincides with accuracy to equivalent functions with a set of slow growing

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functions in the sense of Karamata ([7, p. 15]). For $v \in L$ we denote by $H_0(v)$ the class of entire functions f of zero order that satisfy the condition $n(r) = O(v(r))$, $r \to +\infty$, where $n(r) = n(r, 0, f)$ is counting function of zeros $(a_n)_{n=1}^{+\infty}$ of function f.

We say that zeros of the function $f \in H_0(v)$ have v-density $\Delta^c(\alpha, \beta)$ along logarithmic spirals l^c_φ if the limit

$$\lim_{r\to\infty} \frac{n^c(r; \alpha, \beta)}{v(r)} = \Delta^c(\alpha, \beta)$$

exists for all $\alpha, \beta \in \mathbb{R}$, $0 < \beta - \alpha \leqslant 2\pi$ with the exception, perhaps, of α or β belongs to some countable set $\mathcal{N}$, where $n^c(r; \alpha, \beta)$ is a number of zeros of the function f in $D^c(r; \alpha, \beta)$.

The equality $\Delta^c(\varphi) = \Delta^c(\varphi_1, \varphi)$ for a fixed $\varphi_1 \notin \mathcal{N}$ defines on the segment $[\varphi_1, \varphi_1 + 2\pi]$ a non-decreasing function $\Delta^c(\varphi)$ which we extend on $\mathbb{R}$ by the rule $\Delta^c(\varphi + 2\pi) - \Delta^c(\varphi) = \Delta^c(\varphi_1 + 2\pi) - \Delta^c(\varphi_1)$.

The logarithmic spiral l^c_θ satisfying the condition

$$\lim_{h\to 0+} \overline{\lim_{r\to+\infty}} \frac{n^c(r; \theta - h, \theta + h)}{v(r)} = 0$$

is called *ordinary* for $f \in H_0(v)$. The other logarithmic spirals are called *exceptional*. It follows from monotonicity of the function $\Delta^c(\varphi)$ that the set of exceptional logarithmic spirals is no more than countable if zeros of $f \in H_0(v)$ have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ.

Denote by $\ln\left(1 - \dfrac{z}{a_n}\right)$, $a_n \in l^c_\theta$ the single-valued in the domain $D(l^c_\theta) = \mathbb{C} \setminus l^c_\theta(|a_n|, +\infty)$ branch of multi-valued function $Ln\left(1 - \dfrac{z}{a_n}\right)$ such that $\ln\left(1 - \dfrac{z}{a_n}\right)\Big|_{z=0} = 0$. Let

$$f(z) = \prod_{n=1}^{+\infty} \left(1 - \frac{z}{a_n}\right) \in H_0(v). \tag{1}$$

Then

$$\ln f(z) = \sum_{n=1}^{+\infty} \ln\left(1 - \frac{z}{a_n}\right),\ z \in \mathbb{C} \setminus \bigcup_{n=1}^{+\infty} l^c_{\varphi_i}(r_i, +\infty),$$

where r_i is the minimum module of zeros a_i of f that lie on the logarithmic spiral $l^c_{\varphi_i}$, $\varphi_i = \arg a_i \in [-\pi, \pi)$.

We call a set $E \in \mathbb{C}$ the C_0-set if it can be covered by a system of circles $\{z : |z - a_k| < r_k\}$, $k \in \mathbb{N}$ such that $\sum\limits_{|a_k|\leqslant r} r_k = o(r)$, $r \to +\infty$.

We write $\hat{h}(\theta; \psi)$ for the 2π-periodic extension of the function $h(\theta; \psi) = \theta - \psi - \pi$ from $(\psi; \psi + 2\pi)$ to $\mathbb{R}$, $-\pi \leqslant \psi < \pi$. Note $N(r) = N(r, 0, f) = \displaystyle\int_0^r \frac{n(t)}{t}\, dt$,

$$H^c_f(\theta) = \int_{\theta-2\pi}^{\theta} (\theta - \psi - \pi)\, d\Delta^c(\psi) = \int_{-\pi}^{\pi} \hat{h}(\theta; \psi) d\Delta^c(\psi). \tag{2}$$

Theorem 1. *Let $v \in L$, $f \in H_0(v)$, zeros of f have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ. Then there is a C_0-set E such that the following asymptotic relation holds ($|z| = r$):*

$$\ln f(z) = (1 + ic)N(r) + iH^c_f(\theta)v(r) + o(v(r)),\ z \in l^c_\theta,\ z \notin E, \tag{3}$$

where l^c_θ is ordinary logarithmic spiral.

Let $\Gamma_m = \bigcup\limits_{j=1}^{m} l^c_{\theta_j}$, $-\pi \leqslant \theta_1 < \ldots < \theta_m < \pi$ be a finite system of logarithmic spirals, $\theta_{m+1} = \theta_1 + 2\pi$.

Theorem 2. *Let $v \in L$, $f \in H_0(v)$, zeros of f lie on Γ_m, H be a piecewise continuous on $[-\pi, \pi)$ function. If for any $\delta > 0$ the following asymptotic relation*

$$\ln f\left(re^{i(\theta + c\ln r)}\right) = (1+ic)N(r) + iH(\theta)v(r) + o(v(r)),\ r \to \infty \tag{4}$$

holds uniformly with respect to $\theta \in [-\pi, \pi) \setminus \bigcup\limits_{j=1}^{m+1} (\theta_j - \delta; \theta_j + \delta)$, then zeros of f have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ.

Remark. *The condition that zeros of $f \in H_0(v)$ lie on a finite system of logarithmic spirals Γ_m is significant in Theorem 2. In the general case of zeros arrangement the statement of Theorem 2 is wrong (see [8] in case c = 0).*

2 THE PROOF OF RESULTS

At first we present the lemmas that will be used in the proof of the theorems.

Lemma 1 ([11])**.** *Let $\Delta > 0$, $v \in L$, $f \in H_0(v)$, zeros of f lie on the logarithmic spiral l^c_ψ, $\psi \in \mathbb{R}$,*

$$n(r) = (1+o(1))\Delta v(r),\ r \to +\infty.$$

Then for $\theta \in \mathbb{R}\setminus\{\psi + 2\pi k : k \in \mathbb{Z}\}$ the following asymptotic relation holds:

$$\ln f\left(re^{i(\theta + c\ln r)}\right) = (1+ic)N(r) + i\Delta\hat{h}(\theta;\psi)v(r) + o(v(r)),\ r \to \infty, \tag{5}$$

moreover, relation (5) *is uniform with respect to $\theta \in [\psi + \delta; \psi + 2\pi - \delta]$, $0 < \delta < 1$.*

Lemma 2. *Let f has the form defined in* (1)*, zeros of f have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ, $\varepsilon > 0$ is arbitrary number. Then there exist $\delta > 0$ and a C_0-set E such that for all ordinary logarithmic spirals l^c_θ of the function f the following inequality holds:*

$$\left|\ln f(z) - \ln f^\delta(z)\right| < \varepsilon v(r),\ z \in l^c_\theta,\ z \notin E,$$

where $f^\delta(z) = \prod\limits_{n=1}^{+\infty}\left(1 - \dfrac{z}{a'_n}\right)$, $|a'_n| = |a_n|$, $|\arg a_n - \arg a'_n| < \delta$.

The proof of the Lemma 2 follows from the considerations similar to [5, p. 132-133], [1, p. 352-353] and Theorem 1 from [10].

We say that a set $F \subset \mathbb{R}_+$ is E_0-set if F is a measurable and $mes(E \cap [0,r]) = o(r)$, $r \to +\infty$.

In view of Lemmas 4 and 5 from [9], we get

Lemma 3. *Let $\theta \in [-\pi, \pi)$, $v \in L$, $f \in H_0(v)$, $\delta > 0$. Then there exists a E_0-set F such that*

$$r\int\limits_{\theta-\delta}^{\theta+\delta}\left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right| d\varphi = O(v(r))\left(\delta + \delta\ln\left(1 + \frac{1}{\delta}\right)\right),\ r \to +\infty,\ r \notin F.$$

Proof of Theorem 1. Let $\varepsilon > 0$ is given arbitrary number, function $H_f^c(\theta)$ defined by formula (2). Choose $\delta > 0$ such that the integral sum

$$S_m(\theta) = \sum_{j=0}^{m-1} \hat{h}(\theta;\psi_j)(\Delta^c(\psi_{j+1}) - \Delta^c(\psi_j)),$$

where $-\pi = \psi_0 < \psi_1 < \ldots < \psi_{m-1} < \psi_m = \pi$, $\max\limits_{0\leqslant j\leqslant m-1} |\psi_{j+1} - \psi_j| < \delta$, satisfies the inequality

$$\left| H_f^c(\theta) - S_m(\theta) \right| < \frac{\varepsilon}{3}. \tag{6}$$

Then take numbers a_k' such that $|a_k'| = |a_k|$, $a_k' \in l_{\psi_j}^c$ if $a_k \in l_\psi^c$, $\psi_j \leqslant \psi < \psi_{j+1}$ $(j = 0, 1, \ldots, m-1)$ and build the function $f^\delta(z)$. Applying Lemma 2 we obtain that there exist $\delta > 0$ and C_0-set E_1 such that for all ordinary logarithmic spirals l_θ^c of f and f^δ the following inequality holds:

$$\left| \ln f(z) - \ln f^\delta(z) \right| < \frac{\varepsilon}{3} v(r),\ z \notin E_1,\ z \in l_\theta^c. \tag{7}$$

Zeros of $f^\delta(z)$ lie on a finite system of logarithmic spirals Γ_m so $f^\delta(z)$ can be depicted as a product of m entire functions such that zeros of each function lie on a single logarithmic spiral $l_{\psi_j}^c$. From Lemma 1 (see (5)) we get that inequality

$$\left| \frac{\ln f^\delta(z) - (1+ic)N(r)}{v(r)} - iS_m(\theta) \right| < \varepsilon,\ z \in l_\theta^c$$

holds uniformly with respect to $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j - \delta; \psi_j + \delta)$, where $\delta > 0$ is an arbitrary number.

Further taking into account (6), (7) we obtain that for $z \notin E_1$, $z \in l_\theta^c$, $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j - \delta; \psi_j + \delta)$ the following inequality holds:

$$\left| \frac{\ln f(z) - (1+ic)N(r)}{v(r)} - iH_f^c(\theta) \right| < \varepsilon. \tag{8}$$

Choosing another segmentation of $[-\pi;\pi]$ by points $(\psi_j')_{j=0}^m$, $|\psi_{j+1}' - \psi_j'| < \delta$ such that intervals $(\psi_j' - \delta; \psi_j' + \delta)$ do not have the mutual points with intervals $(\psi_j - \delta; \psi_j + \delta)$, we get that (8) holds for $z \notin E_2$, $z \in l_\theta^c$, $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j' - \delta; \psi_j' + \delta)$, where E_2 is some C_0-set.

This yields that (3) holds for all ordinary logarithmic spirals l_θ^c of function f. So Theorem 1 is proved. □

Proof of Theorem 2. Let $v \in L$, $\Omega = \{|a_n| : n \in \mathbb{N}\}$, a_n be zeros of $f \in H_0(v)$ that lie on a finite system of logarithmic spirals $\Gamma_m = \bigcup\limits_{j=1}^m l_{\theta_j}^c$, $-\pi \leqslant \theta_1 < \ldots < \theta_m < \pi$. Set

$$\partial D^c(r;\alpha,\beta) = l_\alpha^c(1,r) \cup \Gamma(r;\alpha,\beta) \cup \left(l_\beta^c(1,r) \right)^{-1} \cup (\Gamma(1;\alpha,\beta))^{-1},$$

where $r \notin \Omega$, $-\pi \leqslant \theta_{k_0-1} < \alpha < \theta_{k_0} < \ldots < \theta_{s_0} < \beta < \theta_{s_0+1} < \pi$,

$$\Gamma(\tau;\alpha,\beta) = \{z = \tau e^{i(\varphi + c\ln\tau)} : \alpha \leqslant \varphi \leqslant \beta\}.$$

Since $dz = (1+ic)e^{i(\varphi + c\ln t)}dt$ for $l^c_\theta(1,r)$ then with the notation

$$F(\tau,\varphi) = \tau e^{i(\varphi + c\ln\tau)} \frac{f'(\tau e^{i(\varphi+c\ln\tau)})}{f(\tau e^{i(\varphi+c\ln\tau)})}$$

using Residue theorem we have

$$\begin{aligned} 2\pi i\, n^c(r;\alpha,\beta) &= \int\limits_{\partial D^c(r;\alpha,\beta)} \frac{f'(z)}{f(z)}\,dz = \left(\int\limits_{l^c_\alpha(1,r)} + \int\limits_{\Gamma(r;\alpha,\beta)} - \int\limits_{l^c_\beta(1,r)} - \int\limits_{\Gamma(1;\alpha,\beta)} \right) \frac{f'(z)}{f(z)}\,dz \\ &= (1+ic)\int\limits_1^r \left(\frac{F(t,\alpha)}{t} - \frac{F(t,\beta)}{t} \right) dt + \int\limits_\alpha^\beta \left(F(r,\theta) - F(1,\theta)\right) id\theta \\ &= \ln f(re^{i(\alpha + c\ln r)}) - \ln f(re^{i(\beta+c\ln r)}) \\ &\qquad + \left(\int\limits_\alpha^{\theta_{k_0}-\delta} + \sum_{j=k_0}^{s_0-1} \int\limits_{\theta_j+\delta}^{\theta_{j+1}-\delta} + \int\limits_{\theta_{s_0}+\delta}^{\beta} + \sum_{j=k_0}^{s_0} \int\limits_{\theta_j-\delta}^{\theta_j+\delta} \right) F(r,\theta)id\theta + C, \end{aligned} \tag{9}$$

where $C = -\ln f(e^{i\alpha}) + \ln f(e^{i\beta}) - \int\limits_\alpha^\beta F(1,\theta)id\theta$, $0 < \delta < \min\left\{ \dfrac{\theta_{k_0}-\alpha}{2}, \dfrac{\beta - \theta_{s_0}}{2}, \dfrac{\theta_{j+1}-\theta_j}{2} \right\}$, $j = \overline{k_0, s_0-1}$.

Taking account of $\int\limits_{\theta_j+\delta}^{\theta_{j+1}-\delta} F(r,\theta)id\theta = \ln f(re^{i(\theta_{j+1}-\delta+c\ln r)}) - \ln f(re^{i(\theta_j+\delta+c\ln r)})$, from (9) we obtain

$$\begin{aligned} 2\pi i\, n^c(r;\alpha,\beta) &= \sum_{j=k_0}^{s_0} \left(\ln f(re^{i(\theta_j-\delta+c\ln r)}) - \ln f(re^{i(\theta_j+\delta+c\ln r)}) \right) \\ &\qquad + \sum_{j=k_0}^{s_0} \int\limits_{\theta_j-\delta}^{\theta_j+\delta} F(re^{i(\theta+c\ln r)})id\theta = \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned} \tag{10}$$

Applying (4) we get

$$\sum\nolimits_1 = i\sum_{j=k_0}^{s_0} \left(H(\theta_j - \delta) - H(\theta_j+\delta) \right) v(r) + o(v(r)),\ r\to\infty.$$

In view of Lemma 3, there exist E_0-sets F_j such that $(j = \overline{k_0, s_0})$

$$\begin{aligned} \left| \int\limits_{\theta_j-\delta}^{\theta_j+\delta} F(re^{i(\theta+c\ln r)})id\theta \right| &\leqslant r\int\limits_{\theta_j-\delta}^{\theta_j+\delta} \left| \frac{f'(re^{i(\theta+c\ln r)})}{f(re^{i(\theta+c\ln r)})} \right| d\theta = r\int\limits_{\theta_j-\delta}^{\theta_j+\delta} \left| \frac{f'(re^{i\varphi})}{f(re^{i\varphi})} \right| d\varphi \\ &= O(v(r)) \left(\delta + \delta\ln\left(1+\frac{1}{\delta}\right)\right),\ r\to+\infty,\ r\notin F_j. \end{aligned}$$

So,

$$\left|\sum\nolimits_2\right| \leqslant K_1(v(r))\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right),\ r\to+\infty,\ r\notin F,$$

where $F=\bigcup\limits_{j=k_0}^{s_0}F_j$ is a E_0-set, K_1 is some constant.

Combining the last inequalities and (10) yields

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{n^c(r;\alpha,\beta)}{v(r)}=\frac{1}{2\pi}\sum_{j=k_0}^{s_0}\big(H(\theta_j-\delta)-H(\theta_j+\delta)\big)+K_2\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right).$$

Directing δ to 0+ gives

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{n^c(r;\alpha,\beta)}{v(r)}=\frac{1}{2\pi}\sum_{j=k_0}^{s_0}\big(H(\theta_j-0)-H(\theta_j+0)\big):=\Delta(\alpha,\beta).$$

Whereas F is E_0-set, then any interval $(R,(1+\eta)R)$, $\eta>0$, includes points that are not in F. Due to the monotonicity of the function $n^c(r;\alpha,\beta)$ with respect to r for $r>R_0$ we can assert that

$$\frac{n^c(r_1;\alpha,\beta)}{v(r_1)}\frac{v(r_1)}{v(r)}\leqslant\frac{n^c(r;\alpha,\beta)}{v(r)}\leqslant\frac{n^c(r_2;\alpha,\beta)}{v(r_2)}\frac{v(r_2)}{v(r)},$$

where $r(1-\eta)<r_1<r<r_2<(1+\eta)r,\ r_1,r_2\notin F$.

Since $v(r_2)\sim v(r)\sim v(r_1), r\to\infty$, the last relation yields

$$\lim_{r\to\infty}\frac{n^c(r;\alpha,\beta)}{v(r)}=\Delta(\alpha,\beta).$$

Theorem 2 is proved. □

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Нехай функція зростання v така, що $rv'(r)/v(r) \to 0$ при $r \to +\infty$, $l^c_\varphi = \{z = te^{i(\varphi + c\ln t)}, 1 \leqslant t < +\infty\}$ — логарифмічна спіраль, f — ціла функція нульового порядку. За умови існування v-щільності нулів f вздовж l^c_φ знайдено асимптотику $\ln f(re^{i(\theta + c\ln r)})$ вздовж звичайних логарифмічних спіралей l^c_θ функції f зовні C_0-множини. Показано, що обернене до цього твердження правильне лише у випадку розташування нулів f на скінченній системі логарифмічних спіралей $\Gamma_m = \bigcup_{j=0}^m l^c_{\theta_j}$.

Ключові слова і фрази: ціла функція, щільність нулів, логарифмічна спіраль.

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REPRESENTATION OF A QUOTIENT OF SOLUTIONS OF A FOUR-TERM LINEAR RECURRENCE RELATION IN THE FORM OF A BRANCHED CONTINUED FRACTION

The quotient of two linearly independent solutions of a four-term linear recurrence relation is represented in the form of a branched continued fraction with two branches of branching by analogous with continued fractions. Formulas of partial numerators and partial denominators of this branched continued fraction are obtained. The solutions of the recurrence relation are canonic numerators and canonic denominators of $\mathcal{B}$-figured approximants. Two types of figured approximants $\mathcal{A}$-figured and $\mathcal{B}$-figured are often used. A nth $\mathcal{A}$-figured approximant of the branched continued fraction is obtained by adding a next partial quotient to the $(n-1)$th $\mathcal{A}$-figured approximant. A nth $\mathcal{B}$-figured approximant of the branched continued fraction is a branched continued fraction that is a part of it and contains all those elements that have a sum of indexes less than or equal to n. $\mathcal{A}$-figured approximants are widely used in proving of formulas of canonical numerators and canonical denominators in a form of a determinant, $\mathcal{B}$-figured approximants are used in solving the problem of corresponding between multiple power series and branched continued fractions. A branched continued fraction of the general form cannot be transformed into a constructed branched continued fraction. For calculating canonical numerators and canonical denominators of a branched continued fraction with N branches of branching, $N > 1$, the linear recurrent relations do not hold. $\mathcal{B}$-figured convergence of the constructed fraction in a case when coefficients of the recurrence relation are real positive numbers is investigated.

Key words and phrases: branched continued fraction, four-term recurrence relation.

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INTRODUCTION

It is well known that the general solution of a linear homogeneous recurrence relation of second order: $y_n = b_n y_{n-1} + a_n y_{n-2},\ n = 1, 2, \ldots,$ where the $a_n, b_n,\ n \geq 1$, are complex numbers, can be represented in a form of a linear combination of two linearly independent solutions

$$y^{(1)} = \left(1, 0, y_1^{(1)}, y_2^{(1)}, \ldots\right),\ y^{(2)} = \left(0, 1, y_1^{(2)}, y_2^{(2)}, \ldots\right).$$

These solutions are, respectively, canonical numerators and canonical denominators of approximants of the continued fractions [15, 18, 19]

$$\underset{k=1}{\overset{\infty}{\mathrm{D}}} \frac{a_k}{b_k}.$$

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In this paper, an analogous idea for a four-term linear recurrence relation

$$y_n = c_n y_{n-1} + b_n y_{n-2} + a_n y_{n-3}, \tag{1}$$

where the $a_n, b_n, c_n,\ n \geq 2$, are complex numbers, is considered.

Different constructions of multidimensional generalizations arise as a result of considering the N-term recurrent relation, $N > 1$, [8, 12, 18]. They are widely used for compatible approximations, for representation of solutions of algebraic equations, etc. The formulas of the elements of these fractions were not obtained, in general, except for the Furshtenau's two-dimensional generalization of continued fractions [14]. B. V. Krukowski has proved the theorem of convergence of these fractions [16].

This investigation leads to branched continued fractions (BCF) that are a multidimensional generalization of continued fractions. Thus, BCF of the general form are under consideration [7, 9, 11, 21]. Also, the different forms of BCF exist, in particular, BCF of the special form [1, 3–6, 10, 13], two-dimentional continued fractions [2, 17, 20], etc. The different constructions of their approximants [7] and, respectively, the different types of convergence appear in the considering of different mathematical problems.

Let

$$\mathcal{I} = \left\{ i(k) = (i_1, i_2, \ldots, i_k)\ : 1 \leq i_p \leq 2,\ p = \overline{1,k},\ k \geq 1 \right\}$$

be the set of multiindices. Let us introduse an order relation $\prec$ on the set $\mathcal{I}$ for $i(p) \in \mathcal{I}$ and $j(q), j(p) \in \mathcal{I}$, where $j(s) = (j_1, j_2, \ldots, j_s)$, $s \in \mathbb{N}$:

1) $i(p) \prec j(q)$, if $p < q$;

2) $i(p) \prec j(p)$, if $i_1 < j_1$;

3) $i(p) \prec j(p)$, if exists r, $1 \leq r < p$, such that $i_k = j_k$, $k = \overline{1,r}$, $i_{r+1} < j_{r+1}$.

Let we have sequenses of complex numbers $\left\{\xi_{i(k)}\right\}$, $\left\{\eta_{i(k)}\right\}$, where $i(k) \in \mathcal{I}$, then

$$\sum_{i_1=1}^{2} \cfrac{\xi_{i(1)}}{\eta_{i(1)} + \sum\limits_{i_2=1}^{2} \cfrac{\xi_{i(2)}}{\eta_{i(2)} + \ddots}} = \sum_{i_1=1}^{2} \frac{\xi_{i(1)}}{\eta_{i(1)}} + \sum_{i_2=1}^{2} \frac{\xi_{i(2)}}{\eta_{i(2)}} + \ldots = \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{2} \frac{\xi_{i(k)}}{\eta_{i(k)}} \tag{2}$$

be a general branched continued fraction with two branches of branching with complex elements.

A nth approximant of the BCF (2) is a finite BFC of the form

$$f_n = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi_{i(k)}}{\eta_{i(k)}},\ n \geq 1. \tag{3}$$

The continued fraction

$$\frac{\xi_{i(1)}}{\eta_{i(1)}} + \frac{\xi_{i(2)}}{\eta_{i(2)}} + \ldots + \frac{\xi_{i(k)}}{\eta_{i(k)}} + \ldots \tag{4}$$

is called a $(i_1, i_2, \ldots, i_k, \ldots)$ branch of the BCF (2). Let us fix $i(n) \in \mathcal{I}$, then a $(i_1, i_2, \ldots, i_n)$ branch be a finite branch of the BCF (3).

Length of a finite $(i_1, i_2, \ldots, i_n)$ branch of the BCF (3) is a number of partial quotient of the nth approximant of the continued fraction (4).

Each branch in the finite BFC (3) has length equal n. A figured approximant of the BFC (2) is a BFC that is a part of (2) and has at least two branches with nonequal length. Two types of figured approximants are often used. In particular, $\mathcal{A}$-figured approximants are widely used in proving of formulas of canonical numerators and canonical denominators in a form of a determinant [7], $\mathcal{B}$-figured approximants are used in solving the problem of corresponding betweeen a multiple power series and a BFC.

Let $\frac{a}{b} \equiv \frac{c}{d}$ denotes that $a = c, b = d$.

A nth $\mathcal{B}$-figured approximanth of (2) is a BCF

$$\widehat{f_n} = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}},\ n \geq 1, \tag{5}$$

where

$$\frac{\xi^*_{i(k)}}{\eta^*_{i(k)}} \equiv \begin{cases} \dfrac{\xi_{i(k)}}{\eta_{i(k)}}, & \text{if } i_1 + i_2 + \ldots + i_k \leq n; \\ \dfrac{0}{1}, & \text{if } i_1 + i_2 + \ldots + i_k > n. \end{cases}$$

The BCF (2) converges ($\mathcal{B}$-figured converges), if the finite limit of its sequence of approximants f_n ($\mathcal{B}$-figured approximants $\widehat{f_n}$) exists.

The canonical numerator A_n and the canonical denominator B_n of the $\mathcal{B}$-figured approximant $\widehat{f_n}$ are, respectively, the numerator and the denominator of a calculated BCF (5), $\widehat{f_n} = A_n/B_n$. In calculating we use the following algorithm [7]

$$\frac{A_n}{B_n} \equiv \sum_{i_1=1}^{2} \frac{\xi^*_{i(1)}\eta'_{i(1)}}{\eta^*_{i(1)}\eta'_{i(1)} + \xi'_{i(1)}}, n \geq 1, \tag{6}$$

and

$$\frac{\xi'_{i(m)}}{\eta'_{i(m)}} \equiv \sum_{i_{m+1}=1}^{2} \frac{\xi^*_{i(m+1)}\eta'_{i(m+1)}}{\eta^*_{i(m+1)}\eta'_{i(m+1)} + \xi'_{i(m+1)}},\ i(m) \in \mathcal{I},\ m = n-1, n-2, \ldots, 1;\ n \geq 2, \tag{7}$$

where

$$\xi'_{i(n)} = 0,\ \eta'_{i(n)} = 1,\ i_p = \overline{1,2},\ p = \overline{1,n}, n \geq 1. \tag{8}$$

The algorithm (6)–(8) is equivalent to the gradual algorithm of calculation of the BCF (5) without any reductions in the process.

1 SECTION WITH RESULTS

Let the $y^{(1)} = (1, 0, b_1, y_2^{(1)}, y_3^{(1)}, \ldots)$, $y^{(2)} = (0, 1, c_1, y_2^{(2)}, y_3^{(2)}, \ldots)$, be the two solutions of equation (1), where the b_1, c_1 are complex numbers. These solutions yield all three linear independent solutions of (1), for example,

$$y^{(1)} = (1, 0, 0, y_2^{(1)}, y_3^{(1)}, \ldots),\ y^{(2)} = (0, 1, 0, y_2^{(2)}, y_3^{(2)}, \ldots),\ y^{(3)} = (1, 0, 1, y_2^{(3)}, y_3^{(3)}, \ldots).$$

Put $A_k = y_k^{(1)}$, $B_k = y_k^{(2)}$, $k = -1, 0, 1, \ldots$, where

$$\begin{aligned} A_n &= c_n A_{n-1} + b_n A_{n-2} + a_n A_{n-3},\ n = 2, 3 \ldots, \\ B_n &= c_n B_{n-1} + b_n B_{n-2} + a_n B_{n-3},\ n = 2, 3 \ldots, \end{aligned} \tag{9}$$

and

$$A_{-1} = 1,\ A_0 = 0,\ A_1 = b_1,\ B_{-1} = 0,\ B_0 = 1,\ B_1 = c_1. \tag{10}$$

By analogous with a continued fraction let us construct the BCF such that each its nth $\mathcal{B}$-figured approximant equals A_n/B_n, $n \geq 1$.

If $n = 1$ then $A_1/B_1 = b_1/c_1$. For $n = 2$ we have $A_2/B_2 = b_1/(c_1 + b_2 c_2^{-1}) + a_1/(c_2 c_1 + b_2)$. If $n \geq 3$ we replace n by $n-1$ in (9) and put the obtained value A_{n-1} in (9), we get

$$A_n = \gamma_{n-1}^{(n)} A_{n-2} + \beta_{n-1}^{(n)} A_{n-3} + \alpha_{n-1}^{(n)} A_{n-4}, \tag{11}$$

where $\gamma_{n-1}^{(n)} = c_{n-1} c_n + b_n$, $\beta_{n-1}^{(n)} = b_{n-1} c_n + a_n$, $\alpha_{n-1}^{(n)} = a_{n-1} c_n$. Next, if $n \geq 4$, by substituting $n-2$ for n in (9) and putting obtained A_{n-2} in (11), we get a new formula for A_n, etc. If $n \geq r+2$, after $(n-r)$ steps we have

$$A_n = \gamma_r^{(n)} A_{r-1} + \beta_r^{(n)} A_{r-2} + \alpha_r^{(n)} A_{r-3}, \tag{12}$$

where

$$\gamma_r^{(n)} = c_r \gamma_{r+1}^{(n)} + \beta_{r+1}^{(n)},\ \beta_r^{(n)} = b_r \gamma_{r+1}^{(n)} + \alpha_{r+1}^{(n)},\ \alpha_r^{(n)} = a_r \gamma_{r+1}^{(n)}, \tag{13}$$

$r = n-1, n-2, \ldots, 2$, and $\gamma_n^{(n)} = c_n$, $\beta_n^{(n)} = b_n$, $\alpha_n^{(n)} = a_n$.

An analogous relation holds for B_n

$$B_n = \gamma_r^{(n)} B_{r-1} + \beta_r^{(n)} B_{r-2} + \alpha_r^{(n)} B_{r-3}, \tag{14}$$

where $\gamma_r^{(n)}, \beta_r^{(n)}, \alpha_r^{(n)}, r = n-1, n-2, \ldots, 2$, are defined by (13) and $\gamma_n^{(n)} = c_n$, $\beta_n^{(n)} = b_n$, $\alpha_n^{(n)} = a_n$, with initial conditions from (10).

Let us introduse the following notation

$$\begin{aligned} c'_k &= c_k c_{k-1} + b_k,\ k = \overline{2, n};\ n \geq 2; \\ b'_k &= b_k c_{k-2} + a_k,\ k = \overline{3, n};\ n \geq 3; \\ a'_k &= a_k c_{k-3},\ k = \overline{4, n};\ n \geq 4; \end{aligned} \tag{15}$$

and

$$w_j^{(n)} = \frac{\beta_j^{(n)}}{\gamma_j^{(n)}},\ j = \overline{1, n}, \qquad v_j^{(n)} = \frac{c_{j-2}\beta_j^{(n)} + \alpha_j^{(n)}}{\gamma_j^{(n)}},\ j = \overline{3, n},\ n \geq 3. \tag{16}$$

Combining this with the initial conditions (10) and relations (12)–(14), for $r = 2$, we obtain

$$\frac{A_n}{B_n} = \frac{\gamma_2^{(n)} A_1 + \beta_2^{(n)} A_0 + \alpha_2^{(n)} A_{-1}}{\gamma_2^{(n)} B_1 + \beta_2^{(n)} B_0 + \alpha_2^{(n)} B_{-1}} = \frac{\gamma_2^{(n)} b_1 + \alpha_2^{(n)}}{\gamma_2^{(n)} c_1 + \beta_2^{(n)}} = \frac{\beta_1^{(n)}}{\gamma_1^{(n)}} = w_1^{(n)},\ n \geq 3.$$

Using the denoting (15) and (16) we get

$$w_1^{(n)} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2 \gamma_3^{(n)}}{c_1(c_2 \gamma_3^{(n)} + \beta_3^{(n)}) + b_2 \gamma_3^{(n)} + \alpha_3^{(n)}} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2}{c'_2 + v_3^{(n)}}.$$

Let us prove the recurrent formulas for $w_k^{(n)}$, $k=\overline{2,n-2}$, $n\ge 4$, $v_k^{(n)}$, $k=\overline{3,n-2}$, $n\ge 5$. We obtain

$$w_k^{(n)}=\frac{\beta_k^{(n)}}{\gamma_k^{(n)}}=\frac{b_k\gamma_{k+1}^{(n)}+\alpha_{k+1}^{(n)}}{c_k\gamma_{k+1}^{(n)}+\beta_{k+1}^{(n)}}=\frac{b_k}{c_k+w_{k+1}^{(n)}}+\frac{a_{k+1}}{c'_{k+1}+v_{k+2}^{(n)}}. \tag{17}$$

Analogously

$$v_k^{(n)}=\frac{c_{k-2}\beta_k^{(n)}+\alpha_k^{(n)}}{\gamma_k^{(n)}}=\frac{c_{k-2}(b_k\gamma_{k+1}^{(n)}+\alpha_{k+1}^{(n)})+a_k\gamma_{k+1}^{(n)}}{c_k\gamma_{k+1}^{(n)}+\beta_{k+1}^{(n)}}=\frac{b'_k}{c_k+w_{k+1}^{(n)}}+\frac{a'_{k+1}}{c'_{k+1}+v_{k+2}^{(n)}}. \tag{18}$$

Let us now consider the case $k=n-1$

$$w_{n-1}^{(n)}=\frac{b_{n-1}}{c_{n-1}+\dfrac{b_n}{c_n}}+\frac{a_n}{c'_n},\ n\ge 2,\ v_{n-1}^{(n)}=\frac{b'_{n-1}}{c_{n-1}+\dfrac{b_n}{c_n}}+\frac{a'_n}{c'_n},\ n\ge 4. \tag{19}$$

If we put $w_n^{(n)}=\dfrac{b_n}{c_n}$, $v_n^{(n)}=\dfrac{b'_n}{c_n}$, $w_{n+1}^{(n)}=v_{n+1}^{(n)}=0$, $w_{n+2}^{(n)}=v_{n+2}^{(n)}=\infty$ we have that recurrent formulas (17), (18) hold for $k=n-1,n$, as well.

Consider the BCF (2), where

$$\xi_1=b_1,\ \xi_2=a_2, \tag{20}$$

and for all $i(k)\in\mathcal{I}, k\ge 2$

$$\xi_{i(k)}=\begin{cases} b_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=i_k=1;\\ b'_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=2, i_k=1;\\ a_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=1, i_k=2;\\ a'_{i_1+i_2+\ldots+i_k}, & \text{if } i_{k-1}=2, i_k=2,\end{cases} \tag{21}$$

and for all $i(k)\in\mathcal{I}, k\ge 1$

$$\eta_{i(k)}=\begin{cases} c_{i_1+i_2+\ldots+i_k}, & \text{if } i_k=1;\\ c'_{i_1+i_2+\ldots+i_k}, & \text{if } i_k=2,\end{cases} \tag{22}$$

where the a_i,b_i,c_i, $i\ge 1$, are coeficients of (1), the a'_{i+2},b'_{i+1},c'_i, $i\ge 2$, are obtained from (15).

Theorem 1. *Let $\{A_n\}$, $\{B_n\}$ be sequences of complex numbers such that*

$$A_{-1}=1,\ A_0=0,\ A_1=b_1,\ B_{-1}=0,\ B_0=1,\ B_1=c_1,$$

and

$$A_n=c_nA_{n-1}+b_nA_{n-2}+a_nA_{n-3},\ n=2,3\ldots,$$

$$B_n=c_nB_{n-1}+b_nB_{n-2}+a_nB_{n-3},\ n=2,3\ldots,$$

where the a_n, b_n, c_n, $n\ge 1$, are complex constants. Then the A_n, B_n are the canonical numerator and the canonical denominator of nth $\mathcal{B}$-figured approximant of the BCF (2), i.e. $\widehat{f}_n=A_n/B_n$.

Proof. Applying the equality $A_n/B_n = w_1^{(n)}$, $n \geq 1$, we use the recurrent relations (17)–(19) and step by step write the value A_n/B_n in a form of a finite BCF that is equal $\widehat{f}_n$. On the first step we have

$$\frac{A_n}{B_n} = \frac{b_1}{c_1 + w_2^{(n)}} + \frac{a_2}{c_2' + v_3^{(n)}} = \frac{\xi_1}{\eta_1 + w_2^{(n)}} + \frac{\xi_2}{\eta_2 + v_3^{(n)}}.$$

After the second step we get

$$\frac{A_n}{B_n} = \cfrac{\xi_1}{\eta_1 + \cfrac{\xi_{1,1}}{\eta_{1,1} + w_3^{(n)}} + \cfrac{\xi_{1,2}}{\eta_{1,2} + v_4^{(n)}}} + \frac{\xi_2}{\eta_2 + v_3^{(n)}},$$

and after the third step we obtain

$$\frac{A_n}{B_n} = \cfrac{\xi_1}{\eta_1 + \cfrac{\xi_{1,1}}{\eta_{1,1} + \cfrac{\xi_{1,1,1}}{\eta_{1,1,1} + w_4^{(n)}} + \cfrac{\xi_{1,1,2}}{\eta_{1,1,2} + v_5^{(n)}}} + \cfrac{\xi_{1,2}}{\eta_{1,2} + v_4^{(n)}}} + \cfrac{\xi_2}{\eta_2 + \cfrac{\xi_{2,1}}{\eta_{2,1} + w_4^{(n)}} + \cfrac{\xi_{2,2}}{\eta_{2,2} + v_5^{(n)}}},$$

etc. Using the method of mathematical induction we prove that after m steps, $1 < m < n$, we get

$$\frac{A_n}{B_n} = \mathop{\mathrm{D}}_{k=1}^{m} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}}, \tag{23}$$

where $\xi^*_{i(k)} = \xi_{i(k)}$, if $i_1 + i_2 + \ldots + i_k \leq m$ or $i_1 + i_2 + \ldots + i_k = m+1$ and $i_k = 2$; if $i_1 + i_2 + \ldots + i_k \leq m-1$, then $\eta^*_{i(k)} = \eta_{i(k)}$; if $i_k = 1$ and $i_1 + i_2 + \ldots + i_k = m$, then $\eta^*_{i(k)} = \eta_{i(k)} + w^{(n)}_{m+1}$; if $i_k = 2$ and $i_1 + i_2 + \ldots + i_k = m$, then $\eta^*_{i(k)} = \eta_{i(k)} + v^{(n)}_{m+1}$; if $i_k = 2$ and $i_1 + i_2 + \ldots + i_k = m+1$, then $\eta^*_{i(k)} = \eta_{i(k)} + v^{(n)}_{m+2}$. In all other cases $\dfrac{\xi^*_{i(k)}}{\eta^*_{i(k)}} \equiv \dfrac{0}{1}$.

Let us make the next, $m+1$, step. Let $i_1 + i_2 + \ldots + i_k = m$, $i_k = 1$, then

$$\eta^*_{i(k)} = \eta_{i(k)} + w^{(n)}_{m+1} = \eta_{i(k)} + \frac{b_{m+1}}{c_{m+1} + w^{(n)}_{m+2}} + \frac{a_{m+2}}{c'_{m+2} + v^{(n)}_{m+3}}$$

or by using (21), (22) we obtain

$$\eta^*_{i(k)} = \eta_{i(k)} + \frac{\xi_{i(k),1}}{\eta_{i(k),1} + w^{(n)}_{m+2}} + \frac{\xi_{i(k),2}}{\eta_{i(k),2} + v^{(n)}_{m+3}}.$$

If $i_1 + i_2 + \ldots + i_k = m$, $i_k = 2$, then

$$\begin{aligned}\eta^*_{i(k)} = \eta_{i(k)} + v^{(n)}_{m+1} &= \eta_{i(k)} + \frac{b'_{m+1}}{c_{m+1} + w^{(n)}_{m+2}} + \frac{a'_{m+2}}{c'_{m+2} + v^{(n)}_{m+3}} \\ &= \eta_{i(k)} + \frac{\xi_{i(k),1}}{\eta_{i(k),1} + w^{(n)}_{m+2}} + \frac{\xi_{i(k),2}}{\eta_{i(k),2} + v^{(n)}_{m+3}}.\end{aligned}$$

If $i_1 + i_2 + \ldots + i_k = m+1$, $i_k = 2$, then $\eta^*_{i(k)} = \eta_{i(k)} + v^{(n)}_{m+2}$.

Hense, we get the equality (23), where m is replaced by $m+1$.

Put $m = n-1$. Then, using the equalities (19) we obtain that $\eta^*_{i(k)} = \eta_{i(k)} + \dfrac{\xi_{i(k),1}}{\eta_{i(k),1}}$ if $i_1 + i_2 + \ldots + i_k = n-1$, and $\eta^*_{i(k)} = \eta_{i(k)}$ if $i_1 + i_2 + \ldots + i_k = n$, $i_k = 2$.

Thus,

$$\frac{A_n}{B_n} = \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{2} \frac{\xi^*_{i(k)}}{\eta^*_{i(k)}} = \widehat{f}_n.$$

□

Remark 1. *A BCF with two branches of branching with arbitrary complex elements*

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{2} \frac{\alpha_{i(k)}}{\beta_{i(k)}} \tag{24}$$

can not be transformed into the form (2)*, where the* $\xi_{i(k)}$*,* $\eta_{i(k)}$*,* $i(k) \in \mathcal{I}$*, are determined by formulas* (20)–(22)*. For calculating canonical numerators and canonical denominators of a BCF with* N *branches of branching,* $N > 1$*, the linear recurrent relations do not hold.*

Let us consider the nth $\mathcal{B}$-figured approximants of BCF (24) and (2). Let $n = 2$, then we get second $\mathcal{B}$-figured approximant of the BCF (24) $\widehat{g}_2 = \alpha_1 / \left(\beta_1 + \alpha_{1,1}\beta^{-1}_{1,1}\right) + \alpha_2/\beta_2$, and by using the formulas (20)–(22) and (15) we obtain second $\mathcal{B}$-figured approximant of the BCF (2) $\widehat{f}_2 = b_1 / \left(c_1 + b_2 c_2^{-1}\right) + a_2 / (c_1 c_2 + b_2)$. If we put $b_1 = \alpha_1$, $c_1 = \beta_1$, $a_2 = \alpha_2$, $b_2 = \alpha_{11}$, $c_2 = \beta_{12}$, then we get that the relation $\beta_1\beta_{12} + \alpha_{11} = \beta_2$ must hold. But the β_2 is arbitrary. Hense, this is the case that illustrates the truth of the Remark 1.

Theorem 2. *Let the coefficients* a_n*,* b_n*,* c_n*,* $n \geq 2$*, of equation* (1) *be positive real numbers such that*

$$\sum_{k=2}^{\infty} \mu_k = \infty, \tag{25}$$

where

$$\begin{aligned}
\mu_k &= \min_{k \le j \le 2k} \left\{ \frac{M_j}{R_{j+1}}, \frac{M_{j+1}}{R'_{j+2}} \right\}, \quad k \geq 2, \\
M_j &= c_j c'_j c_{j+1} c'_{j+2}, \quad j \geq 2, \\
R_j &= b_j c'_{j-1} c'_{j+1} + a_{j+1} c_{j-1} c_j, \quad j \geq 3, \\
R'_j &= b'_j c'_{j-1} c'_{j+1} + a'_{j+1} c_{j-1} c_j, \quad j \geq 4,
\end{aligned}$$

and a'_{i+2}*,* b'_{i+1}*,* c'_i*,* $i \geq 2$*, are determited by* (15)*. Then the BCF* (2)*, whose elements satisfy relations* (20)–(22)*,* $\mathcal{B}$*-figured converges.*

Proof. Let us show that the elements of the BFC (2) satisfy the conditions of the Theorem 3.11 [7, p. 85]. For this, we consider the following expressions $d_{i(k+1)} = \eta_{i(k)}\eta_{i(k+1)}/\xi_{i(k+1)}$, $i(k+1) \in \mathcal{I}$, $k \geq 2$. If we fix $i(k-1) \in \mathcal{I}$, $k \geq 2$, using the relations (21), (22), we obtain

$$d_{i(k-1),1,1} = \frac{c_j c_{j+1}}{b_{j+1}},\ d_{i(k-1),2,1} = \frac{c_{j+1} c_{j+2}}{b'_{j+2}},\ d_{i(k-1),1,2} = \frac{c'_j c'_{j+2}}{a_{j+2}},\ d_{i(k-1),2,2} = \frac{c'_{j+1} c'_{j+3}}{a'_{j+3}},$$

where $j = \sum_{l=1}^{k-1} i_l + 1$. From this we obtain

$$\min_{i(k+1)\in\mathcal{I},\, k\geq 2}\left\{d_{i(k+1)}\right\} = \min_{k\leq j\leq 2k,\, k\geq 2}\left\{\frac{c_j c_{j+1}}{b_{j+1}}, \frac{c_{j+1}c_{j+2}}{b'_{j+2}}, \frac{c'_j c'_{j+2}}{a_{j+2}}, \frac{c'_{j+1}c'_{j+3}}{a'_{j+3}}\right\}$$
$$\geq \min_{k\leq j\leq 2k,\, k\geq 2}\left\{\frac{M_j}{R_{j+1}}, \frac{M'_{j+1}}{R'_{j+2}}\right\} = \mu_k.$$

Now from (25) it follows that the elements of the BFC (2) satisfy the conditions of the Theorem 3.11 [7, p. 85]. This means that the BFC (2) converges.

Finally, by the Theorem 2.2 [7, p. 48], the BFC (2) $\mathcal{B}$-figured converges. $\square$

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Біланик І.Б., Боднар Д.І., Буяк Л.М. *Зображення відношення розв'язків чотиричленного лінійного рекурентного співвідношення у вигляді гіллястого ланцюгового дробу* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 33–41.

Відношення двох лінійно незалежних розв'язків чотиричленного лінійного рекурентного співвідношення за аналогією з неперервними дробами представлено у вигляду гіллястого ланцюгового дробу з двома гілками розгалуження. Знайдено формули частинних чисельників та частинних знаменників цього гіллястого ланцюгового дробу. Розв'язки різницевого рівняння є канонічними чисельниками і канонічними знаменниками $\mathcal{B}$-фігурних підхідних дробів. Часто використовують два типи фігурних підхідних дробів: $\mathcal{A}$-фігурні і $\mathcal{B}$-фігурні. n–ий $\mathcal{A}$-фігурний підхідний дріб гіллястого ланцюгового дробу отримується додаванням наступної частинної частки до $(n-1)$–го $\mathcal{A}$-фігурного підхідного дробу. n–ий $\mathcal{B}$-фігурний підхідний дріб гіллястого ланцюгового дробу є гіллястий ланцюговий дріб, що є його частиною і містить всі ті елементи, сума індексів яких менша, або рівна n. $\mathcal{A}$-фігурні підхідні дроби використовуються при доведенні формул для канонічних чисельників і знаменників у вигляді визначників, $\mathcal{B}$-фігурні підхідні дроби – у задачах відповідності між кратними степеневими рядами і гіллястими ланцюговими дробами. Загальний гіллястий ланцюговий дріб не можна звести до побудованого гіллястого ланцюгового дробу. Для обчислення канонічних чисельників і канонічних знаменників гіллястих ланцюгових дробів з N, $N > 1$, гілками розгалуження не справджуються лінійні рекурентні співвідношення. Досліджена $\mathcal{B}$-фігурна збіжність побудованого дробу у випадку, коли коефіцієнтами рекурентного співвідношення є дійсні додатні числа.

Ключові слова і фрази: гіллястий ланцюговий дріб, рекурентне співвідношення.

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NOTE ON BASES IN ALGEBRAS OF ANALYTIC FUNCTIONS ON BANACH SPACES

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of continuous algebraically independent homogeneous polynomials on a complex Banach space X. We consider the following question: Under which conditions polynomials $\{P_1^{k_1}\cdots P_n^{k_n}\}$ form a Schauder (perhaps absolute) basis in the minimal subalgebra of entire functions of bounded type on X which contains the sequence $\{P_n\}_{n=0}^{\infty}$? In the paper we study the following examples: when P_n are coordinate functionals on c_0, and when P_n are symmetric polynomials on ℓ_1 and on $L_\infty[0,1]$. We can see that for some cases $\{P_1^{k_1}\cdots P_n^{k_n}\}$ is a Schauder basis which is not absolute but for some cases it is absolute.

Key words and phrases: Schauder bases, analytic functions on Banach spaces, symmetric analytic functions.

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INTRODUCTION AND PRELIMINARIES

Let X be a complex Banach space. We recall that $H_b(X)$ is the algebra of all entire analytic functions on X which are bounded on bounded subsets. It is well known that $H_b(X)$ endowed with the metrisable topology generated by the countable family of norms

$$\|f\|_r = \sup_{\|x\|\le r} |f(x)|, \quad r \in \mathbb{Q}_+, f \in H_b(X),$$

is a Fréchet algebra and the space $\mathcal{P}(X)$ of all continuous polynomials on X is a dense subalgebra in $H_b(X)$.

Let $\mathbb{P} = \{P_n\}_{n=0}^{\infty}$ be a sequence of continuous algebraically independent homogeneous polynomials on X with $\|P_n\| = 1$ and $P_0 = 1$. We denote by $\mathcal{P}_{\mathbb{P}}(X)$ the algebra of all polynomials generated by the sequence $\mathbb{P}$ and by $H_{b\mathbb{P}}(X)$ its closure in $H_b(X)$.

Clearly,

$$\{P^{(k)} = P_1^{k_1}\cdots P_n^{k_n} : (k) = (k_1,\ldots,k_n), \quad n = 0,1,2,\ldots\}$$

is a linear basis in $\mathcal{P}_{\mathbb{P}}(X)$, and so the span of $P^{(k)}$ is dense in $H_{b\mathbb{P}}(X)$. Here we set $P_0 = 1$. This work is motivated by the following natural question: *Under which conditions $\{P^{(k)}\}$ is a*

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Schauder (perhaps absolute) basis in $H_{b\mathbb{P}}(X)$? The main result of this paper is that depending on the sequence $\mathbb{P}$ we can have different answers on this question. In the paper we study the following examples: when P_n are coordinate functionals on c_0, and when P_n are symmetric polynomials on ℓ_1 and on $L_\infty[0,1]$.

Let us recall some definitions in the theory of locally convex spaces (see e.g. [14]).

A sequence of subspaces $\{E_n\}_n$ of a locally convex space E is a *Schauder decomposition* of E if for each x in E there exists a unique sequence of vectors $(x_n)_n$, $x_n \in E_n$, such that

$$x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n$$

and the projections $(u_m)_{m=1}^{\infty}$ defined by

$$u_m\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{m} x_n$$

are continuous. A Schauder decomposition $\{E_n\}_n$ of a locally convex space E is *absolute* if for each semi-norm $p \in cs(E)$,

$$q\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} p(x_n)$$

defines a continuous semi-norm on E. Finally, a Schauder decomposition $\{E_n\}_n$ of a locally convex space E is *global* if for all $r > 0$, all $x = \sum_{n=1}^{\infty} x_n \in E$ with all $x_n \in E_n$

$$\sum_{n=1}^{\infty} r^n x_n \in E$$

and for each $p \in cs(E)$,

$$p_r\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} r^n p(x_n)$$

defines a continuous semi-norm on E.

If each E_n is a finite dimensional subspace, then the decomposition is called *finite dimensional*. If each E_n is one dimensional and e_n spans E_n, then $(e_n)_{n=1}^{\infty}$ is a Schauder basis.

1 MAIN RESULTS

Let $X = c_0$ and $P_n = e_n^*$ be the coordinate functionals on c_0. Then

$$P^{(k)}(x) = (e_1^*(x))^{k_1} \cdots (e_n^*(x))^{k_n} = x_1^{k_1} \cdots x_n^{k_n}, \quad n = 0, 1, 2, \ldots,$$

are so-called $k_1 + \ldots + k_n$-homogeneous monomials on c_0. Since every polynomial on c_0 can be approximated by polynomials of finite type and every polynomial of finite type belongs to linear span of monomials, we have that $H_{b\mathbb{P}}(c_0) = H_b(c_0)$. Moreover, in [8] it is proved that the monomials $\{P^{(k)}\}$ endowed with some special order form a Schauder basis for $H_b(c_0)$ which however is not absolute. Indeed, if it is absolute, then the subset of monomials $\{P^{(k)} \colon \deg P^{(k)} = m\}$ form an unconditional basis in the Banach space of all m-homogeneous polynomials $\mathcal{P}({}^m c_0)$. But it is not so for $m > 1$, according to [6].

Let now $\deg P_n = n$. So if $P \in \mathcal{P}_{\mathbb{P}}(X)$ and $\deg P = m$, then

$$P(x) = \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1\ldots k_n} P_1^{k_1}(x)\cdots P_n^{k_n}(x), \qquad a_{k_1\ldots k_n} \in \mathbb{C}. \tag{1}$$

We denote $\mathcal{P}_{\mathbb{P}}(^nX)$ the linear space of all n-homogeneous polynomials in $\mathcal{P}_{\mathbb{P}}(X)$. From (1) it follows that $\mathcal{P}_{\mathbb{P}}(^nX)$ is finite dimensional, polynomials $\{P_1^{k_1}\cdots P_n^{k_n} : k_1 + 2k_2 + \ldots + nk_n = n\}$ form a linear basis in $\mathcal{P}_{\mathbb{P}}(^nX)$ and $\dim \mathcal{P}_{\mathbb{P}}(^nX) = \mathfrak{p}(n)$, where $\mathfrak{p}(n)$ is the number of partitions of n.

Proposition 1. *Let $\deg P_n = n$. Then the sequence of spaces $\{\mathcal{P}_{\mathbb{P}}(^nX)\}_{n=0}^{\infty}$ is a global finite dimensional Schauder decomposition for $H_{b\mathbb{P}}(X)$. Here $\mathcal{P}_{\mathbb{P}}(^0X) = \mathbb{C}$.*

Proof. In [14] it is proved that $\{\mathcal{P}(^nX)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $H_b(X)$. Since $H_{b\mathbb{P}}(X)$ is a closed subspace of $H_b(X)$, $\mathcal{P}_{\mathbb{P}}(^nX) = \mathcal{P}(^nX) \cap H_{b\mathbb{P}}(X)$ is a global Schauder decomposition for $H_{b\mathbb{P}}(X)$. □

Note that in the general case the existence of a finite dimensional Schauder decomposition does not imply the existence of a Schauder basis (see [13]).

Algebras of symmetric functions on ℓ_1 or $L_1[0,1]$ deliver us interesting examples of $H_{b\mathbb{P}}(X)$. By a symmetric function on ℓ_1 we mean a function which is invariant under any reordering of the basis in ℓ_1. We use the notations $\mathcal{H}_{bs}(\ell_1)$ for the algebra of all symmetric analytic functions on ℓ_1 that are bounded on bounded sets.

In [12] it is proved that the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \qquad k = 1, 2, \ldots,$$

form an algebraic basis in the algebra of all symmetric polynomials on ℓ_1. This means that the polynomials $\{F_k\}$ are algebraically independent and their algebraic combinations coincide with the space of all symmetric polynomials $\mathcal{P}_s(\ell_1)$ on ℓ_1. Thus, $\{F^{(k)} = F_1^{k_1}\cdots F_k^{k_n}\}$ forms a linear basis in $\mathcal{P}_s(\ell_1)$ or, in other words, $\mathcal{H}_{bs}(\ell_1) = H_{b\mathbb{F}}(\ell_1)$.

The algebras $\mathcal{H}_{bs}(\ell_p)$ and their spectrum were investigated in [2–4, 10].

In [5] was constructed an example of a symmetric analytic function on ℓ_1 which is not of bounded type.

The algebra $\mathcal{P}_{bs}(\ell_1)$ has other natural algebraic bases. For us it is important the basis $\{G_n\}$:

$$G_n(x) = \sum_{k_1<\cdots<k_n}^{\infty} x_{k_1}\cdots x_{k_n}$$

and $G_0 := 1$. It is known [3] that $\|G_n\| = 1/n!$. By the Waring's formula we have

$$G_k = \sum_{\lambda_1+2\lambda_2+\ldots+k\lambda_k=k} (-1)^{k-(\lambda_1+\lambda_2+\ldots+\lambda_k)} \frac{1}{\lambda_1!1^{\lambda_1}\cdots\lambda_k!k^{\lambda_k}} F_1^{\lambda_1}\cdots F_k^{\lambda_k}.$$

Note that in the general case, algebra $\mathcal{P}_{\mathbb{P}}(X)$ admits a lot of algebraic bases of homogeneous polynomials and linear bases as well. Indeed, if $\deg P_n = n$, then we can set $Q_1 = a_{11}P_1$ and

$$Q_n = a_{n1}Q_{n-1}P_1 + a_{n2}Q_{n-2}P_1 + \cdots + a_{nn}P_n$$

for some complex numbers a_{ij} such that $a_{ii} \neq 0$. Then polynomials Q_n form an algebraic basis and $Q^{(k)} = Q_1^{k_1} \cdots Q_k^{k_n}$ form a linear basis in $\mathcal{P}_{\mathbb{P}}(X)$. Note that there is a linear basis of $\mathcal{P}_s({}^n\ell_1)$ which is not generated by an algebraic basis. For a given partition $(k) = (k_1, \ldots, k_n)$ such that $|(k)| = k_1 + \ldots + k_n = n$ we denote by $M^{(k)}(x) = \sum_{i_1 \neq \ldots \neq i_n} x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$. Then $\{M^{(k)}\}_{|k|=0}^{\infty}$ is a linear basis in $\mathcal{P}_s({}^n\ell_1)$.

We need the following simple lemma which probably is well known (c.f. [1, Theorem 2.1]).

Lemma 1. *Let $P_1, \ldots, P_N$ be algebraically independent polynomials from a Banach space X to $\mathbb{C}$ such that the map*

$$X \ni x \mapsto (P_1(x), \ldots, P_N(x)) \in \mathbb{C}^N$$

is onto. Then there is an isomorphism I_N from the minimal subalgebra of entire functions generated by $P_1, \ldots, P_N$ onto the algebra of all entire functions on $\mathbb{C}^N$, $H(\mathbb{C}^N)$ such that $I_N(P_k) = t_k$, $k = 1, \ldots, N$, $(t_1, \ldots, t_N) \in \mathbb{C}^N$.

Theorem 1. *Let $P_n = n!G_n$. Then $\{P^{(k)} = P_1^{k_1} \cdots P_k^{k_n}\}$ is a Schauder basis in $\mathcal{H}_{bs}(\ell_1)$.*

Proof. Let r_N be the operator of restriction onto subspace $V_N \subset \ell_1$ spanned on the standard basis vectors $e_1, \ldots, e_N$. Clearly that $r_N(G_k) = 0$ if $N < k$. Also, we know that $r_N(P_1), \ldots, r_N(P_N)$ are algebraically independent and the map

$$\ell_1 \ni x \mapsto (r_N(P_1), \ldots, r_N(P_N)) \in \mathbb{C}^N$$

is onto. So from Lemma 1 we have the isomorphism I_N from the minimal subalgebra of entire functions $H_s(V_N)$ on V_N, generated by $r_N(P_1), \ldots, r_N(P_N)$ to $H(\mathbb{C}^N)$. By the same reason, we have the isomorphism $\mathcal{I}_N$ from the minimal subalgebra of entire functions $H_s^N(\ell_1)$ on ℓ_1, generated by $P_1, \ldots, P_N$ to $H(\mathbb{C}^N)$. From here we have that the operator of restriction $r_N \colon \mathcal{H}_{bs}(\ell_1) \to H_s(V_N)$ is onto and $\mathcal{I}_N^{-1} \circ I_N$ is the "extension" isomorphism from $H_s(V_n)$ to $H_s^N(\ell_1)$. Also, we know [7, p. 240] that monomials on $t_1, \ldots, t_n$ form an absolute basis in $H(\mathbb{C}^N)$. Thus $P_1^{k_1} \cdots P_k^{k_n}$ for $k \leq N$ form an absolute basis in $H_s^N(\ell_1)$ and so all projections T_m to finite dimensional subspaces W_m generated by these basis vectors are continuous. Thus any projection u_m from $\mathcal{H}_{bs}(\ell_1)$ to W_m can be represented by

$$u_k = T_n \circ \mathcal{I}_N^{-1} \circ I_N \circ r_N$$

and so is continuous. □

Let us denote $\mathcal{A}_{us}(B_{\ell_1})$ the completion of $\mathcal{H}_{bs}(\ell_1)$ by the norm $\|\cdot\|_1$ that is, the sup-norm on the unit ball B_{ℓ_1} of ℓ_1. Such algebra consists of analytic and uniformly continuous functions on B_{ℓ_1} and was considered in [1].

Theorem 2. *$\{F^{(k)} = F_1^{k_1} \cdots F_k^{k_n}\}$ cannot be an absolute Schauder basis in $\mathcal{H}_{bs}(\ell_1)$ and cannot be an unconditional basis in $\mathcal{A}_{us}(\ell_1)$.*

Proof. Let us remind that a sequence $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis of a Banach space, if there exists a constant M such that for every $\sum_{n=1}^{m} a_n e_n$ and for every $\varepsilon_1, \ldots, \varepsilon_n$, $|\varepsilon_k| = 1$, we have

$$M\Big\|\sum_{n=1}^{m} a_n e_n\Big\| \geq \Big\|\sum_{n=1}^{m} \varepsilon_n a_n e_n\Big\|. \tag{2}$$

It is well known in combinatorics that

$$\sum_{\lambda_1+2\lambda_2+\ldots+k\lambda_k=k} \frac{1}{\lambda_1!1^{\lambda_1}\cdot\ldots\cdot\lambda_k!k^{\lambda_k}} = 1. \tag{3}$$

Let $g(x) = \sum_{n=0}^{\infty} G_n(x)$. Since $\|G_n\| = \frac{1}{n!}$, $g(x) \in \mathcal{H}_{bs}(\ell_1) \subset \mathcal{A}_{us}(\ell_1)$. According to the Waring's formula,

$$g(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} (-1)^{n-(k_1+k_2+\ldots+k_n)} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n}.$$

We set $\varepsilon_{(k)} = \varepsilon_{(k_1,\ldots,k_n)} = (-1)^{(k_1+k_2+\ldots+k_n+n)}$. According to (3) and $\|F_1^{k_1}\cdots F_n^{k_n}\|_1 = 1$ the series

$$\sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n}$$

diverges. It contradicts (2). Also, if $\{F^{(k)}\}$ is an absolute basis in $\mathcal{H}_{bs}(\ell_1)$, then the series

$$\sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \left\| (-1)^{n-(k_1+k_2+\ldots+k_n)} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}} F_1^{k_1}\cdots F_n^{k_n} \right\|_1$$
$$= \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!1^{k_1}\cdots k_n!n^{k_n}}$$

should be convergent. But it is not so. □

Algebra of symmetric analytic functions $H_{bs}(L_\infty[0,1])$ on $L_\infty[0,1]$ consists of analytic functions which are invariant with respect to all measurable automorphisms of $[0,1]$.

According to [9] polynomials $P_n = R_n$, where

$$R_n(x) = \int_{[0,1]} (x(t))^n dt, \quad n \in \mathbb{N},$$

form an algebraic basis in the algebra of all symmetric polynomials on $L_\infty[0,1]$. In [11] it is proved that $\{R^{(k)} = R_1^{k_1}\cdots R_k^{k_n}\}$ is an absolute basis in $H_{bs}(L_\infty[0,1])$.

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Нехай $\{P_n\}_{n=0}^{\infty}$ — послідовність неперервних алгебраїчно незалежних однорідних поліномів на комплексному банаховому просторі X. Розглянемо наступне питання: За яких умов поліноми $\{P_1^{k_1}\cdots P_n^{k_n}\}$ утворюють базис Шаудера (можливо абсолютний) в мінімальній підалгебрі цілих функцій обмеженого типу на X, які містять послідовність $\{P_n\}_{n=0}^{\infty}$? У роботі досліджуються наступні приклади: коли P_n є координатними функціоналами c_0, і коли P_n є симетричними поліномами на ℓ_1 і на $L_\infty[0,1]$. Ми бачимо, що у деяких випадках $\{P_1^{k_1}\cdots P_n^{k_n}\}$ є базисом Шаудера який не є абсолютним, але в деяких випадках є абсолютним.

Ключові слова і фрази: базис Шаудера, аналітичні функції на банахових просторах, симетричні аналітичні функції.

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SPECTRAL APPROXIMATIONS OF STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

We establish analytical estimates of spectral approximations errors for strongly degenerate elliptic differential operators in the Lebesgue space $L_q(\Omega)$ on a bounded domain Ω. Elliptic operators have coefficients with strong degeneration near boundary. Their spectrum consists of isolated eigenvalues of finite multiplicity and the linear span of the associated eigenvectors is dense in $L_q(\Omega)$. The received results are based on an appropriate generalization of Bernstein-Jackson inequalities with explicitly calculated constants for quasi-normalized Besov-type approximation spaces which are associated with the given elliptic operator. The approximation spaces are determined by the functional $E\,(t,u)$, which characterizes the shortest distance from an arbitrary function $u \in L_q(\Omega)$ to the closed linear span of spectral subspaces of the given operator, corresponding to the eigenvalues such that not larger than fixed $t>0$. Such linear span of spectral subspaces coincides with the subspace of entire analytic functions of exponential type not larger than $t>0$. The approximation functional $E\,(t,u)$ in our cases plays a similar role as the modulus of smoothness in the functions theory.

Key words and phrases: elliptic operators, spectral approximations.

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1 Introduction

We investigate the problem of best approximations in the Lebesgue space $L_q(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$ by using spectral subspaces $\mathcal{R}(A)$ of a strongly degenerate elliptic differential operator A. Our aims is to prove the inverse and direct theorems that give precise estimates of approximation errors and which are connected with appropriate estimations by Bernstein-Jackson type inequalities.

For this purpose we use the best approximation functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$ which characterizes a shortest distance from an arbitrary function $u \in L_q(\Omega)$ to the closed linear span $\mathcal{R}^t(A)$ of all spectral subspaces $\mathcal{R}_{\lambda_j}(A)$ of the given operator A, corresponding to the eigenvalues λ_j such that $|\lambda_j| < t$ with a fixed $t>0$.

This best approximation problem we solve by finding exact values of constants in the Bernstein-Jackson inequalities. Namely, we establish the Bernstein-Jackson inequalities with explicitly calculated constants, using the suitable generalization of Besov's space $\mathcal{B}^s_r(A,L_q(\Omega))$, determined by a given operator A and an appropriate functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$.

It is essentially to note that the approximation functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$ in these inequalities plays a similar role as the modulus of smoothness in the functions theory. Earlier applications of smoothness modulus to approximation problems can be found in [5–7].

In this paper we continue the research started in [3,4].

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2 STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

We shall follow the treatment given in [8, Sec. 6.2.1]. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with the infinitely smooth boundary $\partial\Omega$. As usual, $C^\infty(\Omega)$ denotes the space of all infinitely differentiable complex-valued functions defined on Ω. Suppose that $\rho(x) \in C^\infty(\Omega)$ is a positive function such that:

(i) for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ there exist positive numbers c_α such that

$$|D^\alpha \rho(x)| \le c_\alpha \, \rho^{1+|\alpha|}(x) \ \text{ for all } \ x \in \Omega;$$

(ii) for any positive number K there exist numbers $\varepsilon_K > 0$ and $r_K > 0$ such that $\rho(x) > K$, if $d(x) \le \varepsilon_K$ or $|x| \ge r_K$, $x \in \Omega$ (here, $d(x)$ is the distance to the boundary $\partial\Omega$).

In what follows, $S_{\rho(x)}(\Omega)$ denotes the locally convex space

$$S_{\rho(x)}(\Omega) = \Big\{ u : u \in C^\infty(\Omega), \ \|u\|_{l,\alpha} = \sup_{x\in\Omega} \rho^l(x) |D^\alpha u(x)| < \infty \text{ for all } \ \alpha \text{ and } \ l \in \mathbb{N}_0 \Big\}.$$

Let $m \in \mathbb{N}$, $\mu, \tau \in \mathbb{R}$ and $\tau > \mu + 2m$. We put

$$\aleph_l = \frac{1}{2m} \left(\tau \,(2m - l) \, + \, \mu \, l \right), \ \ l = 0, 1, \dots, 2m,$$

and consider the differential elliptic operator

$$Au = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \rho^{\,\aleph_{2l}}(x) \, b_\alpha(x) \, D^\alpha u + \sum_{|\beta|<2m} a_\beta(x) \, D^\beta u, \tag{1}$$

where $b_\alpha(x) \in C^\infty(\Omega)$ $(|\alpha| = 2l, \ l = 0, 1, \dots, m)$ are real functions, all derivatives of which (inclusively the functions themselves) are bounded in Ω. In sequel we assume that there exists a positive number C such that for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$

$$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x) \, \xi^\alpha \ge C \, |\xi|^{2m}, \quad b_{(0,\dots,0)}(x) \ge C,$$
$$(-1)^l \sum_{|\alpha|=2l} b_\alpha(x) \, \xi^\alpha \ge 0, \quad l = 1, \dots, m-1.$$

Moreover, let $a_\beta(x) \in C^\infty(\Omega)$ $(0 \le |\beta| < 2m)$ and there exists a positive number $\delta > 0$ such that $D^\gamma a_\beta(\xi) = O\left(\rho^{\,\aleph_{|\beta|} + |\gamma| - \delta}\right)$ for $0 \le |\beta| < 2m$ and for all multi-indices γ.

Let $1 < q < \infty$, $\tau \ge \mu + sq$, $s \in \mathbb{N}_0$ and $\tau, \mu \in \mathbb{R}$. Consider the weighted Sobolev space $W^s_q(\Omega; \rho^\mu; \rho^\tau)$ endowed with the norm (see [8, Thm 3.2.4/2])

$$\|u\|_{W^s_q(\Omega;\rho^\mu;\rho^\tau)} = \left[\int_\Omega \Big(\sum_{|\alpha|=s} \rho^\mu(x) |D^\alpha u(x)|^q + \rho^\tau(x) |u(x)|^q \Big) dx \right]^{\frac{1}{q}}.$$

Let $\tau > 0$, $1 < q < \infty$ and $\rho^{-a}(x) \in L_1(\Omega)$ for an appropriate number $a \ge 0$. Then A given by (1) with the domain $\mathfrak{D}(A) = W^{2m}_q(\Omega; \rho^{q\mu}; \rho^{q\tau})$ is the closed operator in $L_q(\Omega)$ (see [8, Thm 6.6.2]). The spectrum of A consists of isolated eigenvalues $\{\lambda_j \in \mathbb{C} \colon j \in \mathbb{N}\}$ of finite algebraic multiplicity and its eigenvectors belongs to $S_{\rho(x)}(\Omega)$, as well as, its linear span is dense in $S_{\rho(x)}(\Omega)$ and, as a consequence, it is dense in $L_q(\Omega)$.

Let $\mathcal{R}_{\lambda_j}(A) = \{u \in \mathfrak{D}^\infty(A) = \bigcap_{k\in\mathbb{N}} \mathfrak{D}^k(A) \colon (\lambda_j - A)^{r_j} u = 0\}$ be the spectral subspace, corresponding to the eigenvalue λ_j of multiplicity r_j. Denote by $\mathcal{R}^\nu(A)$ the complex linear span in $L_q(\Omega)$ of all spectral subspaces $\mathcal{R}_{\lambda_j}(A)$ such that $|\lambda_j| < \nu$. Following to [4], let $\mathcal{R}(A) := \bigcup_{\nu>0} \mathcal{R}^\nu(A)$ be endowed with the quasi-norm

$$|u|_{\mathcal{R}(A)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{R}^\nu(A)\}.$$

3 ANALYTICAL ESTIMATES OF SPECTRAL APPROXIMATIONS

Let us consider the subspace of all exponential type vectors $\mathcal{E}(A)$ of the elliptic operator A as the union $\bigcup_{\nu>0} \mathcal{E}^\nu(A)$ which is endowed with the quasi-norm

$$|u|_{\mathcal{E}(A)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{E}^\nu(A)\},$$

where for any $\nu > 0$ the subspace $\mathcal{E}^\nu(A) = \left\{u \in \mathcal{E}(A) \colon \|u\|_{\mathcal{E}^\nu(A)} < \infty\right\}$ is endowed with the norm $\|u\|_{\mathcal{E}^\nu(A)} = \sum_{k\in\mathbb{N}_0} \left\|(A/\nu)^k u\right\|_{L_q(\Omega)}$ (see [3,4]).

Let $0 < s < \infty$ and $0 < r \le \infty$ or $0 \le s < \infty$ and $r = \infty$. To investigate spectral approximation errors, we consider the appropriate Besov spaces

$$\mathcal{B}^s_r(A, L_q(\Omega)) = \left\{u \in L_q(\Omega) \colon |u|_{\mathcal{B}^s_r(A,L_q(\Omega))} < \infty\right\},$$

associated with the given operator A on the space $L_q(\Omega)$, which is endowed with the norm

$$|u|_{\mathcal{B}^s_r(A,L_q(\Omega))} = \begin{cases} \left(\displaystyle\int_0^\infty \left[t^s E(t,u;\mathcal{E}(A),L_q(\Omega))\right]^r \frac{dt}{t}\right)^{1/r}, & 0 < r < \infty, \\ \sup\limits_{t>0} t^s E(t,u;\mathcal{E}(A),L_q(\Omega)), & r = \infty, \end{cases}$$

where $E(t,u;\mathcal{E}(A),L_q(\Omega)) = \inf\left\{\|u - u^0\|_{L_q(\Omega)} \colon u^0 \in \mathcal{E}(A), |u^0|_{\mathcal{E}(A)} < t\right\}$ for all $u \in L_q(\Omega)$ and $t > 0$. Denote $E(t,u;\mathcal{R}(A),L_q(\Omega)) = \inf\left\{\|u - u^0\|_{L_q(\Omega)} \colon u^0 \in \mathcal{R}(A), |u^0|_{\mathcal{R}(A)} \le t\right\}$ for all $u \in L_q(\Omega)$.

Now, we consider the space $\mathcal{E}^\nu(D) = \{u \in C^\infty(\bar{\Omega}) \colon D^\alpha u \in L_q(\Omega), |\alpha| = k \in \mathbb{N}_0\}$ endowed with the norm $\|u\|_{\mathcal{E}^\nu(D)} = \sum_{k\ge0}\sum_{|\alpha|=k} \nu^{-k}\|D^\alpha u\|_{L_q(\Omega)}$. On $\mathcal{E}(D) = \bigcup_{\nu>0}\mathcal{E}^\nu(D)$ we define the quasi-norm $|u|_{\mathcal{E}(D)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{E}^\nu(D)\}$.

In [3, Thm 9] it is proved that $\mathcal{E}(D)$ coincides with the space $\mathcal{M}_q(\Omega) = \bigcup_{\nu>0}\mathcal{M}^\nu_q(\Omega)$ endowed with the quasi-norm

$$|u|_{\mathcal{M}_q(\Omega)} = \inf_{v|_\Omega = u,\, v\in L_q(\mathbb{R}^n)} \left\{\|v\|_{L_q(\mathbb{R}^n)} + \sup_{\zeta\in\operatorname{supp} Fv} |\zeta|\right\},$$

where $\operatorname{supp} Fv$ denotes the support of the Fourier-image Fv of a function $v \in L_q(\mathbb{R}^n)$ and $\mathcal{M}^\nu_q(\Omega)$ means the space of entire analytic functions $v(z)$ of the complex variable $z \in \mathbb{C}^n$ of an exponential type $\nu > 0$ which restrictions to Ω belong to $L_q(\Omega)$.

Taking into account [1, Sec. 7.2] or [8, Sec. 2.5.4] and the mentioned above equality $\mathcal{E}(D) = \mathcal{M}_q(\Omega)$, the classic Besov space $B^s_{q,r}(\Omega)$ over Ω can be endowed with the norm

$$\|u\|_{B^s_{q,r}(\Omega)} = \begin{cases} \left(\int_0^\infty \left[t^s E(t,u;\mathcal{E}(D),L_q(\Omega))\right]^r \frac{dt}{t}\right)^{1/r}, & 0<r<\infty, \\ \sup\limits_{t>0} t^s E(t,u;\mathcal{E}(D),L_q(\Omega)), & r=\infty. \end{cases}$$

In $B^s_{q,r}(\Omega)$ we consider the subspace which is associated with the function $\rho(x)$,

$$B^s_{q,r,\rho(x)}(\Omega) = \left\{u \in B^s_{q,r}(\Omega)\colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\}.$$

Theorem 1. *The following Bernstein-Jackson inequalities hold,*

$$\|u\|_{B^s_{q,r}(\Omega)} \le c_{s,r}|u|^s_{\mathcal{R}(A)}\|u\|_{L_q(\Omega)}, \quad u \in \mathcal{R}(A), \tag{2}$$

$$t^s E(t,u;\mathcal{R}(A),L_q(\Omega)) \le C_{s,r}\|u\|_{B^s_{q,r}(\Omega)}, \quad u \in B^s_{q,r,\rho(x)}(\Omega) \tag{3}$$

with the constants $c_{s,r} = \left(rs^{-1}(s+1)^2\right)^{1/r}$ *and* $C_{s,r} = 2^{s+1}\left(r^{-1}s(s+1)^{-2}\right)^{1/r}$ *if* $r < \infty$, $c_{s,\infty} = C_{s,\infty} = 1$. *In addition, for each* $u \in B^s_{q,r,\rho(x)}(\Omega)$,

$$\inf\left\{\|u-u^0\|_{L_q(\Omega)}\colon u^0 \in \mathcal{R}^\nu(A)\right\} \le \nu^{-s} C_{s,r}\,\|u\|_{B^s_{q,r}(\Omega)}. \tag{4}$$

Proof. First, note that applying [2, Thm 2.2], we get the following equalities

$$\mathcal{E}(A) = \mathcal{R}(A), \quad |u|_{\mathcal{E}(A)} = |u|_{\mathcal{R}(A)} \quad \text{for all } u \in \mathcal{E}(A). \tag{5}$$

Now, we show that the following linear topological isomorphism holds,

$$\mathcal{B}^s_r(A,L_q(\Omega)) = B^s_{q,r,\rho(x)}(\Omega). \tag{6}$$

Using [8, Thm 6.5.2/1, Thm 3.2.4/3], we have

$$\mathfrak{D}^\infty(A) = \bigcap \mathfrak{D}^k(A) = \bigcap W^{2mk}_q(\Omega;\rho^{q\mu k};\rho^{q\tau k}) = S_{\rho(x)}(\Omega),$$

where the locally convex space $\mathfrak{D}^\infty(A)$ endowed with the semi-norms $\|A^k u\|_{L_q(\Omega)}$ for all $k \in \mathbb{N}_0$. Above, the equality also must be understood as linear topological isomorphism.

Let us prove the equality

$$\mathcal{E}(A) = \left\{u \in \mathcal{E}(D)\colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\}. \tag{7}$$

Since $\|A^k u\|_{L_q(\Omega)} \le \nu^k\|u\|_{L_q(\Omega)} \le \nu^{2k}\left(\sum_{|\alpha|=k}\nu^{-k}\|D^\alpha u\|_{L_q(\Omega)} + \nu^{-k}\|u\|_{L_q(\Omega)}\right)$ for all $u \in \mathcal{E}^\nu(A)$, we get $\sum \nu^{-2k}\|A^k u\|_{L_q(\Omega)} \le \sum\left(\sum_{|\alpha|=k}\nu^{-k}\|D^\alpha u\|_{L_q(\Omega)} + \nu^{-k}\|u\|_{L_q(\Omega)}\right)$. Substituting $\sigma = \nu^2$ with $\nu > 1$, we have

$$\|u\|_{\mathcal{E}^\sigma(A)} \le \|u\|_{\mathcal{E}^\nu(D)} + \frac{\nu\|u\|_{L_q(\Omega)}}{\nu-1} \le \|u\|_{\mathcal{E}^\nu(D)} + \frac{\nu\|u\|_{\mathcal{E}^\nu(D)}}{\nu-1} = \frac{2\nu-1}{\nu-1}\|u\|_{\mathcal{E}^\nu(D)}.$$

It follows that $\left\{u \in \mathcal{E}^{\sqrt{\nu}}(D)\colon \sup_{x\in\Omega}\rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\} \subset \mathcal{E}^\nu(A)$.

On the other hand, applying [8, Thm 6.5.2/1, Lemma 6.2.3] for any $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \|A^k u\|_{L_q(\Omega)} &\geq c_k \|u\|_{W_q^{2mk}(\Omega;\rho^{q\mu k};\rho^{q\tau k})} \\ &= c_k \left[\int_\Omega \Big(\sum_{|\alpha|=2mk} \rho^{q\mu k}(x) |D^\alpha u(x)|^q + \rho^{q\tau k}(x)|u(x)|^q \Big) dx \right]^{\frac{1}{q}} \\ &\geq c_k c_\rho^k \left[\int_\Omega \Big(\sum_{|\alpha|=2mk} |D^\alpha u(x)|^q + |u(x)|^q \Big) dx \right]^{\frac{1}{q}} = c_k c_\rho^k \|u\|_{W_q^{2mk}(\Omega)}, \end{aligned}$$

where $c_\rho > 0$ does not depend on k. Thus,

$$\begin{aligned} \|A^{k+1}u\|_{L_q(\Omega)} = \|A^k(Au)\|_{L_q(\Omega)} &\geq c_k c_\rho^k \|Au\|_{W_q^{2mk}(\Omega)} \\ &= c_k c_\rho^k \Big(\sum_{|\alpha|=2mk} \|D^\alpha Au\|^q_{L_q(\Omega)} + \|Au\|^q_{L_q(\Omega)} \Big)^{\frac{1}{q}} \\ &\geq c_k c_\rho^k \Big(\sum_{|\alpha|=2mk} \|AD^\alpha u\|^q_{L_q(\Omega)} + \|Au\|^q_{L_q(\Omega)} \Big)^{\frac{1}{q}} \\ &\geq c_k c_1 c_\rho^{k+1} \Big(\sum_{|\alpha|=2mk} \|D^\alpha u\|^q_{W_q^{2m}(\Omega)} + \|u\|^q_{W_q^{2m}(\Omega)} \Big)^{\frac{1}{q}} = c_{k+1} c_\rho^{k+1} \|u\|_{W_q^{2m(k+1)}(\Omega)}, \end{aligned}$$

where $c_{k+1} = c_k c_1 = c_1^{k+1}$ by induction on k. Hence, for each $k \in \mathbb{N}$ and $u \in \mathfrak{D}^k(A)$, we have $\|A^k u\|_{L_q(\Omega)} \geq c_1^k c_\rho^k \|u\|_{W_q^{2mk}(\Omega)}$ for all $u \in \mathfrak{D}^k(A)$, where $c_1 > 0$ does not depend on k. This leads to the inequality $\sum \nu^{-k} \|A^k u\|_{L_q(\Omega)} \geq \sum ((c_1 c_\rho)^{-1}\nu)^{-k} \|u\|_{W_q^k(\Omega)}$ from which it follows that

$$\mathcal{E}^\nu(A) \subset \Big\{ u \in \mathcal{E}^{(c_1 c_\rho)^{-1}\nu}(D) \colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0 \Big\}.$$

Hence, equality (7) holds. Now applying [3, Thm 9], we obtain the required equality (6).

Using (5) and [4, Thm 2], as well as, taking into account (7), we obtain the required inequalities (2), (3), while (4) directly follows from (3) and [3, Thm 6]. □

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Дмитришин М.І., Лопушанський О.В. *Спектральні апроксимації сильно вироджених еліптичних диференціальних операторів* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 48–53.

Встановлено аналітичні оцінки помилок спектральних апроксимацій сильно вироджених еліптичних диференціальних операторів в просторі Лебега $L_q(\Omega)$ над обмеженою областю Ω. Такі еліптичні оператори характеризуються сильним виродженням їх коефіцієнтів поблизу границі, їх спектр складається із ізольованих власних значень скінченної алгебраїчної кратності, а лінійна оболонка власних і приєднаних векторів щільна в просторі $L_q(\Omega)$. Отримані результати ґрунтуються на відповідному узагальненні нерівностей Бернштейна і Джексона з обчисленням точних констант для квазінормованих апроксимаційних просторів типу Бєсова, асоційованих з даним еліптичним оператором. Апроксимаційні простори визначаються за допомогою функціоналу $E\,(t,u)$, який характеризує найкоротшу відстань від заданої функції $u \in L_q(\Omega)$ до замкненої лінійної оболонки спектральних підпросторів заданого оператора, що відповідають власним значенням, які за абсолютною величиною не перевищують фіксоване число $t > 0$. При цьому вказана лінійна оболонка спектральних підпросторів співпадає з підпростором цілих аналітичних функцій експоненціального типу, що не перевищує $t > 0$. Апроксимаційний функціонал $E\,(t,u)$ в нашому випадку відіграє роль, подібну модулю гладкості в теорії функцій.

Ключові слова і фрази: еліптичні оператори, спектральні апроксимації.

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ON SOME OF CONVERGENCE DOMAINS OF MULTIDIMENSIONAL *S*-FRACTIONS WITH INDEPENDENT VARIABLES

The convergence of multidimensional S-fractions with independent variables is investigated using the multidimensional generalization of the classical Worpitzky's criterion of convergence, the criterions of convergence of the branched continued fractions with independent variables, whose partial quotients are of the form $\frac{q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}(1-q_{i(k-1)})z_{i(k)}}{1}$, and the convergence continuation theorem to extend the convergence, already known for a small domain (open connected set), to a larger domain. It is shown that the union of the intersections of the parabolic and circular domains is the domain of convergence of the multidimensional S-fraction with independent variables, and that the union of parabolic domains is the domain of convergence of the branched continued fraction with independent variables, reciprocal to it.

Key words and phrases: multidimensional S-fraction with independent variables, convergence.

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1 INTRODUCTION

It is well known (see, for example [2, 7]) that the branched continued fractions with independent variables are an efficient tool for the approximation of analytic multivariable functions, which are represented by multiple power series. One of the important problem for these branched continued fractions is to establish the widest domains (open connected sets) of their convergence. Convergence domains have been given in [1,2,8,11] for multidimensional regular C-fractions with independent variables, in [4] for multidimensional regular S-fractions with independent variables, in [9] for multidimensional g-fractions with independent variables, in [6] for multidimensional associated fractions with independent variables and in [6, 10] for multidimensional J-fractions with independent variables.

Let N be a fixed natural number and

$$\mathcal{I}_k = \{i(k):\ i(k) = (i_1, i_2, \ldots, i_k),\ 1 \le i_p \le i_{p-1},\ 1 \le p \le k,\ i_0 = N\},\ k \ge 1,$$

be the sets of multiindices. In addition, let $i(0) = 0$ and $\mathcal{I}_0 = \{0\}$.

We investigate here the convergence of multidimensional S-fraction with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{c_{i(1)} z_{i_1}}{1} \underset{+}{} \sum_{i_2=1}^{i_1} \frac{c_{i(2)} z_{i_2}}{1} \underset{+}{} \sum_{i_3=1}^{i_2} \frac{c_{i(3)} z_{i_3}}{1} \underset{+}{} \cdots, \qquad (1)$$

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where the $c_{i(k)} > 0$ for all $i(k) \in \mathcal{I}_k$, $k \geq 1$, $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, and reciprocal to it

$$\frac{1}{1+}\sum_{i_1=1}^{N}\frac{c_{i(1)}z_{i_1}}{1}\;\;+\sum_{i_2=1}^{i_1}\frac{c_{i(2)}z_{i_2}}{1}\;\;+\sum_{i_3=1}^{i_2}\frac{c_{i(3)}z_{i_3}}{1}\;\;+\cdots. \tag{2}$$

2 CONVERGENCE

We introduce the following notation $Q^{(n)}_{i(n)}(\mathbf{z}) \equiv 1$, $i(n) \in \mathcal{I}_n$, $n \geq 1$, and

$$Q^{(n)}_{i(k)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k}\frac{c_{i(k+1)}z_{i_{k+1}}}{1}\;\;+\sum_{i_{k+2}=1}^{i_{k+1}}\frac{c_{i(k+2)}z_{i_{k+2}}}{1}\;\;+\cdots+\sum_{i_n=1}^{i_{n-1}}\frac{c_{i(n)}z_{i_n}}{1},$$

where $i(k) \in \mathcal{I}_k,\ 1 \leq k \leq n-1,\ n \geq 2$. It is clear that the following recurrence relations hold

$$Q^{(n)}_{i(k)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k}\frac{c_{i(k+1)}z_{i_{k+1}}}{Q^{(n)}_{i(k+1)}(\mathbf{z})} \text{ for all } i(k) \in \mathcal{I}_k,\ 1 \leq k \leq n-1,\ n \geq 2. \tag{3}$$

Let $f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^{N}(c_{i(1)}z_{i_1}/Q^{(n)}_{i(1)}(\mathbf{z}))$ be the nth approximant of (1), $n \geq 1$.

We shall prove the following result.

Theorem 1. *A multidimensional S-fraction with independent variables* (1)*, where the* $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, *satisfy the inequalities*

$$c_{i(k)} \leq q^{i_k}_{i(k)} q^{i_k - 1}_{i(k-1)}(1 - q_{i(k-1)}) \text{ for all } i(k) \in \mathcal{I}_k,\ k \geq 2, \tag{4}$$

where $\{q_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}}$ *is a sequence of real numbers such that*

$$0 < q_{i(k)} < 1 \text{ for all } \ i(k) \in \mathcal{I}_k,\ k \geq 1, \tag{5}$$

converges to a function holomorphic in the domain

$$P_M = \bigcup_{\alpha\in(-\pi/2,\pi/2)}\left\{\mathbf{z}\in\mathbb{C}^N:\ |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < 2\cos^2(\alpha),\ |z_k| < M,\ 1 \leq k \leq N\right\} \tag{6}$$

for every constant $M > 0$*. The convergence is uniform on every compact subset of* P_M.

Proof. Let α be an arbitrary number from the interval $(-\pi/2, \pi/2)$, n be an arbitrary natural number, and let $\mathbf{z}$ be an arbitrary fixed point from domain (6). By induction on k for each multiindex $i(k) \in \mathcal{I}_k$ we show that the following inequalities are valid

$$\operatorname{Re}(Q^{(n)}_{i(k)}(\mathbf{z})e^{-i\alpha}) > q^{i_k}_{i(k)}\cos(\alpha) > 0, \tag{7}$$

where $1 \leq k \leq n$.

It is clear that for $k=n$, $i(n)\in\mathcal{I}_n$, relations (7) hold. By induction hypothesis that (7) hold for $k=r+1$, $r\le n-1$, $i(r+1)\in\mathcal{I}_{r+1}$, we prove (7) for $k=r$ and for each $i(r)\in\mathcal{I}_r$. Indeed, use of relations (3) for the arbitrary multiindex $i(r)\in\mathcal{I}_r$ lead to

$$Q^{(n)}_{i(r)}(\mathbf{z})e^{-i\alpha}=e^{-i\alpha}+\sum_{i_{r+1}=1}^{i_r}\frac{c_{i(r+1)}z_{i_{r+1}}e^{-2i\alpha}}{Q^{(n)}_{i(r+1)}(\mathbf{z})e^{-i\alpha}}.$$

In the proof of lemma 4.41 [12] it is shown that if $x\ge c>0$ and $v^2\le 4u+4$,

$$\min_{-\infty<y<+\infty}\operatorname{Re}\left(\frac{u+iv}{x+iy}\right)=-\frac{\sqrt{u^2+v^2}-u}{2x}. \tag{8}$$

We set

$$u=\operatorname{Re}(z_{i_{r+1}}e^{-2i\alpha}),\ v=\operatorname{Im}(z_{i_{r+1}}e^{-2i\alpha}),\ x=\operatorname{Re}(Q^{(n)}_{i(r+1)}(\mathbf{z})e^{-i\alpha}),\ y=\operatorname{Re}(Q^{(n)}_{i(r+1)}(\mathbf{z})e^{-i\alpha}).$$

Then from (6) it is easy to show that $v^2\le 4u+4$ for each index i_{r+1}, $1\le i_{r+1}\le i_r$.

Now, using (4)–(8) and induction hypothesis, we obtain that

$$\begin{aligned}\operatorname{Re}(Q^{(n)}_{i(r)}(\mathbf{z})e^{-i\alpha})&\ge\cos(\alpha)-\sum_{i_{r+1}=1}^{i_r}\frac{q^{i_{r+1}}_{i(r+1)}q^{i_{r+1}-1}_{i(r)}(1-q_{i(r)})(|z_{i_{r+1}}|-\operatorname{Re}(z_{i_{r+1}}e^{-2i\alpha}))}{2\operatorname{Re}(Q^{(n)}_{i(r+1)}(\mathbf{z})e^{-i\alpha})}\\&>\cos(\alpha)-\sum_{i_{r+1}=1}^{i_r}q^{i_{r+1}-1}_{i(r)}(1-q_{i(r)})\cos(\alpha)=q^{i_r}_{i(r)}\cos(\alpha)>0.\end{aligned}$$

It follows from (7) that $Q^{(n)}_{i(k)}(\mathbf{z})\ne 0$ for all $i(k)\in\mathcal{I}_k$, $1\le k\le n$, $n\ge 1$, and for all $\mathbf{z}$ from domain (6). Thus, the approximants $f_n(\mathbf{z})$, $n\ge 1$, of (1) form a sequence of functions holomorphic in P_M.

Again, let α be an arbitrary number from the interval $(-\pi/2,\pi/2)$. And, let

$$P_{\alpha,\sigma,M}=\left\{\mathbf{z}\in\mathbb{C}^N:\ |z_k|-\operatorname{Re}(z_ke^{-2i\alpha})<2\sigma\cos^2(\alpha),\ |z_k|<\sigma M,\ 1\le k\le N\right\}, \tag{9}$$

where $0<\sigma<1$. We set $c=\max\{c_1,c_2,\ldots,c_N\}$.

Using (7), for the arbitrary $\mathbf{z}\in P_{\alpha,\sigma,M}$, $P_{\alpha,\sigma,M}\subset P_M$, we obtain for $n\ge 1$

$$|f_n(\mathbf{z})|\le 1+\sum_{i_1=1}^{N}\frac{c_{i(1)}|z_{i_1}|}{\operatorname{Re}(Q^{(n)}_{i(1)}(\mathbf{z})e^{-i\alpha})}<1+\sum_{i_1=1}^{N}\frac{c\sigma M}{q^{i_1}_{i(1)}\cos(\alpha)}=C(P_{\alpha,\sigma,M}),$$

where the constant $C(P_{\alpha,\sigma,M})$ depends only on the domain (9), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $P_{\alpha,\sigma,M}$.

Let K be an arbitrary compact subset of P_M. Let us cover K with domains of form (9). From this cover we choose the finite subcover $P_{\alpha_1,\sigma_1,M}, P_{\alpha_2,\sigma_2,M},\ldots,P_{\alpha_r,\sigma_r,M}$. We set

$$C(K)=\max\left\{C(P_{\alpha_1,\sigma_1,M}),C(P_{\alpha_2,\sigma_2,M}),\ldots,C(P_{\alpha_r,\sigma_r,M})\right\}.$$

Then for arbitrary $\mathbf{z}\in K$ we obtain $|f_n(\mathbf{z})|\le C(K)$, for $n\ge 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (6).

Let $b = \min\{1, M, q_1/(2c), q_2/(2^2c), \ldots, q_N/(2^Nc)\}$ and let

$$\Delta_R = \left\{\mathbf{z} \in \mathbb{R}^N : \ 0 < z_k < R < b, \ 1 \le k \le N\right\}.$$

Evidently $\Delta_R \subset P_M$ for each $0 < R < b$, in particular, say $\Delta_{b/2} \subset P_M$. Then for the arbitrary $\mathbf{z} \in \Delta_R$, $\Delta_R \subset P_M$, we obtain

$$|c_{i(1)}z_{i_1}| < bc \le 2^{-i_1}q_{i(1)}^{i_1} \text{ for all } i(1) \in \mathcal{I}_1,$$
$$|c_{i(k)}z_{i_k}| < q_{i(k)}^{i_k}q_{i(k-1)}^{i_k-1}(1 - q_{i(k-1)}) \text{ for all } i(k) \in \mathcal{I}_k, \ k \ge 2.$$

It follows from theorem 1 [8], with $q_0 = 1/2$, that (1) converges in the domain Δ_R. Hence, by theorem 2.17 [3, p. 66] (see also theorem 24.2 [13, pp. 108–109]) the multidimensional S-fraction with independent variables (1) converges uniformly on compact subsets of P_M to a holomorphic function. □

The following two theorem can be proved in much the same way as theorem 1 using theorem 1 and 5 [5], respectively.

Theorem 2. *A multidimensional S-fraction with independent variables* (2)*, where the* $c_{i(k)}$*,* $i(k) \in \mathcal{I}_k$*,* $k \ge 1$*, satisfy the inequalities* $c_{i(k)} \le q_{i(k)}^{i_k}q_{i(k-1)}^{i_k-1}(1 - q_{i(k-1)})$ *for all* $i(k) \in \mathcal{I}_k$*,* $k \ge 1$*, where* $\{q_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}_0}$ *is a sequence of real numbers such that* $0 < q_{i(k)} < 1$ *for all* $i(k) \in \mathcal{I}_k$*,* $k \ge 0$*, converges to a function holomorphic in the domain*

$$D = \bigcup_{\alpha\in(-\pi/2,\pi/2)} \left\{\mathbf{z} \in \mathbb{C}^N : \ |z_k| - \mathrm{Re}(z_k e^{-2i\alpha}) < 2\cos^2(\alpha), \ 1 \le k \le N\right\}. \tag{10}$$

The convergence is uniform on every compact subset of D*.*

Theorem 3. *A multidimensional S-fraction with independent variables* (2)*, where the* $c_{i(k)}$*,* $i(k) \in \mathcal{I}_k$*,* $k \ge 1$*, satisfy the inequalities* $c_{i(1)} \le q_{i(1)}^{i_1+1}$ *for all* $i(1) \in \mathcal{I}_1$ *and* $c_{i(k)} \le q_{i(k)}^{i_k}q_{i(k-1)}^{i_k-1}(1 - q_{i(k-1)})$ *for all* $i(k) \in \mathcal{I}_k$*,* $k \ge 2$*, where* $\{q_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}}$ *is a sequence of real numbers satisfying the inequalities* (5) *and* $\sum_{i_1=1}^{N} q_{i(1)} < 1$*, converges to a function holomorphic in the domain* (10)*. The convergence is uniform on every compact subset of* D*.*

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Дмитришин Р.І. *Про деякі області збіжності багатовимірних S-дробів з нерівнозначними змінними* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 54–58.

Досліджується збіжність багатовимірних S-дробів з нерівнозначними змінними із використанням багатовимірного узагальнення класичної ознаки збіжності Ворпітського, ознак збіжності для гіллястих ланцюгових дробів з нерівнозначними змінними, частинні ланки яких мають вигляд $\frac{q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}(1-q_{i(k-1)})z_{i(k)}}{1}$, і теореми про продовження збіжності із уже відомої малої області до більшої. Отримано, що об'єднання перетинів параболічних і кругових областей є областю збіжності багатовимірного S-дробу з нерівнозначними змінними, а об'єднання параболічних областей — областю збіжності оберненого до нього гіллястого ланцюгового дробу з нерівнозначними змінними.

Ключові слова і фрази: багатовимірний S-дріб з нерівнозначними змінними, збіжність.

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GHOSH A.

RICCI SOLITON AND RICCI ALMOST SOLITON WITHIN THE FRAMEWORK OF KENMOTSU MANIFOLD

First, we prove that if the Reeb vector field ξ of a Kenmotsu manifold M leaves the Ricci operator Q invariant, then M is Einstein. Next, we study Kenmotsu manifold whose metric represents a Ricci soliton and prove that it is expanding. Moreover, the soliton is trivial (Einstein) if either (i) V is a contact vector field, or (ii) the Reeb vector field ξ leaves the scalar curvature invariant. Finally, it is shown that if the metric of a Kenmotsu manifold represents a gradient Ricci almost soliton, then it is η-Einstein and the soliton is expanding. We also exhibited some examples of Kenmotsu manifold that admit Ricci almost solitons.

Key words and phrases: Kenmotsu manifold, Ricci almost soliton, warped product.

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INTRODUCTION

In the recent years, there has been a growing interest in the study of Riemannian manifolds endowed with a metric which satisfies some structural equations involving Ricci curvature and some globally defined vector fields. Sometimes these are also appear as a solution of some geometric flows [5] and [7]. For instance, a Ricci soliton appears as a special (self-similar) solution of the Hamilton's Ricci flow [11]:

$$\frac{\partial}{\partial t}g_{ij} = -2S_{ij}.$$

A Ricci soliton is a smooth manifold M together with a Riemannian metric g that satisfies

$$(\pounds_V g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y), \tag{1}$$

where V is a vector field known as the potential vector field, $\pounds_V$ denotes the Lie-derivative operator along a vector field V, S is the Ricci tensor, λ is a constant, and X, Y arbitrary vector fields on M. This is also considered as a generalized fixed point of the Hamilton's Ricci flow, viewed as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings.

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Recently, in [16] Pigola et al. extended the notion of Ricci soliton on a Riemannian manifold by allowing the constant λ to be smooth function in the defining condition of the Ricci soliton (1). It is said to be shrinking, steady, or expanding according as λ is positive, zero, or negative respectively. If the potential vector field V is the gradient of a potential function f, then g is called a *gradient Ricci almost soliton*. In this case, the soliton equation (1) transforms into

$$\nabla_X Df + QX = \lambda X, \tag{2}$$

where D is the gradient operator with respect to the metric g, Q is the Ricci operator asoociated with the Ricci tensor S, i.e., $S(X,Y) = g(QX,Y)$, X, Y are arbitrary vector fields on M. Both equations (1) and (2) can be considered as a generalization of the Einstein equation $S = \lambda g$ and reduce to this latter in case V or Df are Killing vector fields. When $V = 0$ or f is constant we say the underlying Einstein manifold a trivial Ricci soliton. On a compact Riemannian manifold a Ricci soliton is always a gradient Ricci soliton [15]. Ricci solitons are of interest to physists as well and are known as *quasi Einstein* metrics in the physics literature [8]. Some aspects of compact Ricci almost soliton may be found in [3,4,17]. In particular, in [4] it is proved that any compact Ricci almost soliton is gradient provided its scalar curvature is constant (see also [17]).

In [13], a new class of almost contact metric manifolds was introduced and studied, which is known as Kenmotsu manifold. This type of manifold is very closely related to the warped product spaces. Actually, the warped product space $\mathbb{R} \times_\sigma N^{2n}$ with the warping function $\sigma(t) = ce^t$ on the real line $\mathbb{R}$, and N^{2n} is K*ä*hler admits such structure. Conversely, every point of a Kenmotsu manifold has a neighbourhood which is locally a warped product $(-\varepsilon, \varepsilon) \times_\sigma N^{2n}$ where $\sigma(t) = ce^t$ is a function on the open interval. It is interesting to notice that a Kenmotsu manifold can not be compact because it satisfies $div\xi = 2n$. Recently, the author [9] studied Kenmotsu 3-metric as a Ricci soliton and proved that it is of constant negative curvature -1. The existence of such metric has also been confirmed on the warped product of a Riemann surface N of constant negative curvature (a K*ä*hler manifold) with the real line. For higher dimensions it is proved that *"If the metric of an η-Einstein Kenmotsu manifold is a Ricci soliton, then it is necessarily an Einstein manifold" (see [10])*. These results intrigues us to consider Kenmotsu metric as a Ricci soliton. The organization of the paper is as follows. After recalling some basic definitions and formulas in Section 2, we study Kenmotsu manifold satisfying $£_\xi Q = 0$ in Section 3. In Section 4, we consider Kenmotsu metric as Ricci soliton. Finally, we study Kenmotsu metric as Ricci almost solitons.

1 Preliminaries on Kenmotsu Manifold

In this section, we recall the definitions and fundamental formulas on Kenmotsu manifolds. Let M be a smooth manifold of dimension $(2n+1)$. Then M is said to be an almost contact manifold if there exists a $(1-1)$ tensor field φ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\varphi^2 X = -X + \eta(X)\xi.$$

From which it is easy to verify $\varphi\xi = 0, \eta o \varphi = 0$. A Riemannian metric g on M is said to be an associated metric if it satisfies

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X Y on M. A Riemannian manifold M^{2n+1} together with an almost contact metric structure (φ, ξ, η, g) is said to be an almost contact metric manifold. We remark that an almost contact metric structure on a Riemannian manifold M^{2n+1} may be regarded as a reduction of the structure group M to $U(n)$. For such a manifold, we can always define a 2-form φ by $\varphi(X,Y) = g(X, \varphi Y)$ which is known as the fundamental 2-form. The almost contact metric structure is said to be normal if $[\varphi, \varphi](X,Y) + 2d\eta(X,Y)\xi = 0$, for any vector field X,Y on M, where

$$[\varphi, \varphi](X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

An almost contact metric manifold is said to be an almost Kenmotsu manifold if it satisfies $d\eta = 0$ and $d\varphi = 2\eta \wedge \varphi$. Further, a normal almost Kenmotsu manifold is said to be a Kenmotsu manifold, and this normality condition is expressed as

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

for any vector field X, Y on M. The following formulas also hold for a Kenmotsu manifold [13]

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4}$$

$$Q\xi = -2n\xi, \tag{5}$$

where R is the curvature tensor.

Next, we recall the notion of a β-Kenmotsu manifold which is a slight extension of the Kenmotsu manifold. An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be β-Kenmotsu if it satisfies

$$(\nabla_X \varphi)Y = \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

for some smooth function β on M. Kenmotsu manifolds appear as particular case of β-Kenmotsu manifolds with $\beta = 1$. Regarding the existence of such manifold we recall the following (e.g, see [1]).

Lemma 1. *The warped product $M = \mathbb{R} \times_\sigma N$, is a β-Kenmotsu manifold with $\beta = \sigma'/\sigma$, where $\mathbb{R}$ is the real line and N is a Kähler manifold.*

Recall that a vector field V on a contact manifold is said to be a contact vector field if

$$£_V \eta = f\eta,$$

for some smooth function f on M. The contact vector field V is called strict when $f = 0$. Finally, we recall some formulas involving Lie-derivatives along an arbitrary vector field V. First of all, using the well-known commutation formula (see p. 23 of [19]):

$$(£_V \nabla_X g - \nabla_X £_V g - \nabla_{[V,X]} g)(Y,Z) = -g((£_V \nabla)(X,Y), Z) - g((£_V \nabla)(X,Z), Y),$$

we deduce

$$(\nabla_X £_V g)(Y,Z)) = g((£_V \nabla)(X,Y), Z) + g((£_V \nabla)(X,Z), Y). \tag{6}$$

The following formulas are also known (see [19, p.23]):

$$(£_V R)(X,Y)Z = (\nabla_X £_V \nabla)(Y,Z) - (\nabla_Y £_V \nabla)(X,Z), \tag{7}$$

$$£_V \nabla_X Y - \nabla_X £_V Y - \nabla_{[V,X]} Y = (£_V \nabla)(X,Y). \tag{8}$$

2 KENMOTSU MANIFOLD SATISFYING $\pounds_\xi Q = 0$

Recently, Cho-Kimura [6] proved that if the Reeb vector field of a 3-dimensional Kenmotsu manifold leaves the Ricci operator invariant, then it is locally isometric to $\mathbb{H}^3(-1)$. Extending this in higher diemnsions we prove

Theorem 1. *Let $M(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold of dimension $(2n+1)$. If the Reeb vector field ξ of M leaves the Ricci operator Q invariant, then M is Einstein.*

The proof of the theorem follows from the following lemma.

Lemma 2. *For any Kenmotsu manifold of dimension $(2n+1)$ the following are valid*

(i) $(\pounds_\xi Q)Y = -2QY - 4nY$,

(ii) $(\nabla_\xi Q)Y = -2QY - 4nY$.

Proof. By virtue of (3), we have

$$(\pounds_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) = 2\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{9}$$

Taking covariant differentiation of (9) along an arbitrary vector field X and the use of (3) gives

$$(\nabla_X \pounds_\xi g)(Y,Z) = 2\{2\eta(X)\eta(Y)\eta(Z) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}.$$

Making use of this in (6), we obtain

$$\begin{aligned} g((\pounds_\xi \nabla)(X,Y),Z) + g((\pounds_\xi \nabla)(X,Z),Y) &= 2\{2\eta(X)\eta(Y)\eta(Z) \\ &\quad - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}. \end{aligned}$$

By a combinatorial combination, we deduce

$$\begin{aligned} g((\pounds_\xi \nabla)(Y,Z),X) + g((\pounds_\xi \nabla)(Y,X),Z) &= 2\{2\eta(Y)\eta(Z)\eta(X) \\ &\quad - g(Y,Z)\eta(X) - g(Y,X)\eta(Z)\}, \end{aligned}$$

$$\begin{aligned} g((\pounds_\xi \nabla)(Z,X),Y) + g((\pounds_\xi \nabla)(Z,Y),X) &= 2\{2\eta(Z)\eta(X)\eta(Y) \\ &\quad - g(Z,X)\eta(Y) - g(Z,Y)\eta(X)\}. \end{aligned}$$

Subtracting the first equation from the addition of the last two equations provides

$$((\pounds_\xi \nabla)(Y,Z) = 2\{\eta(Y)\eta(Z) - g(Y,Z)\xi\}. \tag{10}$$

Differentiating (10) covariantly along X and using (3), we find

$$\begin{aligned} (\nabla_X \pounds_\xi \nabla)(Y,Z) = 2\{&g(X,Y)\eta(Z)\xi + g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \\ &- g(Y,Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)\xi\}. \end{aligned}$$

Utilizing this in (7), we obtain

$$(\pounds_\xi R)(X,Y)Z = 2\{g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \tag{11}$$

Contracting (11) over X with respect to an orthonormal frame of the tangent space of M, we deduce

$$(\pounds_\xi S)(Y,Z) = 4n\{\eta(Y)\eta(Z) - g(Y,Z)\}. \tag{12}$$

Next, we take the Lie derivative of $S(Y,Z) = g(QY,Z)$, to get

$$(\pounds_\xi S)(Y,Z) = (\pounds_\xi g)(QY,Z) + g((\pounds_\xi Q)Y,Z). \tag{13}$$

On the other hand, replacing Y by QY in (9) and using (5), we obtain

$$(\pounds_\xi g)(QY,Z) = 2\{g(QY,Z) + 2n\eta(Y)\eta(Z)\}. \tag{14}$$

Using (14) and (13) in (12) we obtain $(\pounds_\xi Q)Y = -2QY - 4nY$. This completes the proof of (i). Taking into account of (3) we observe that

$$\begin{aligned}(\pounds_\xi Q)Y &= \pounds_\xi QY - Q\pounds_\xi Y = \nabla_\xi QY - \nabla_{QY}\xi - Q\nabla_\xi Y + Q\nabla_Y \xi\\ &= (\nabla_\xi Q)Y - QY - 2n\eta(Y)\xi + QY + 2n\eta(Y)\xi = (\nabla_\xi Q)Y.\end{aligned}$$

Using this in (i) we complete the proof of (ii). □

For a 3-dimensional Riemannian manifold it is known that the Ricci curvature determines the curvature completely and the curvature tensor can be explicitely expressed as

$$\begin{aligned}R(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\\ - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned} \tag{15}$$

Now, if a 3-dimensional Kenmotsu manifold M satisfies $\pounds_\xi Q = 0$, then from (i) of the above Lemma $QX = -2X$. Using this in (15) we see that M^3 is of constant curvature -1. Thus, we have the following (see also [6]).

Corollary 1. *A 3-dimensional Kenmotsu manifold satisfies $\pounds_\xi Q = 0$ if and only if M^3 is locally isometric to a Hyperbolic space $\mathbb{H}^3(-1)$.*

3 KENMOTSU METRIC AS A RICCI SOLITON

In [9] the author proved that if a Kenmotsu 3-metric represents a Ricci soliton, then it is of constant curvature -1. Here we extend this result in higher dimensions and prove the following assertion.

Theorem 2. *Let $M(\varphi,\xi,\eta,g)$ be a Kenmotsu manifold of dimension $(2n+1)(>3)$. If g represents a Ricci soliton, then the soliton is expanding. Moreover, the soliton is trivial (Einstein) if either (i) V is a contact vector field, or (ii) the Reeb vector field ξ leaves the scalar curvature invariant.*

Proof. First, differentiating (5) along an arbitrary vector field X and recalling (3) we deduce

$$(\nabla_X Q)\xi = -QX - 2nX. \tag{16}$$

Differentiating (1), using it in (6) and by a straightforward combinatorial computation we obtain

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z).$$

Taking ξ instead of Y in the preceding equation and then recalling (16) and (ii) of Lemma 3.1 provides

$$(\pounds_V \nabla)(Y,\xi) = 2QY + 4nY. \tag{17}$$

Covariant differentiation of (17) along X and the use of (3), (17) leads to

$$(\nabla_X \pounds_V \nabla)(Y,\xi) + (\pounds_V \nabla)(Y,X) - 2\eta(X)(QY + 2nY) = 2(\nabla_X Q)Y.$$

Making use of this in (7) and since $\pounds_V \nabla$ is a symmetric operator, one can deduce

$$(\pounds_V R)(X,Y)\xi = 2\{\eta(X)QY - \eta(Y)QX + (\nabla_X Q)Y - (\nabla_Y Q)X\} \\ + 4n\{\eta(X)Y - \eta(Y)X\}.$$

We now set $Y = \xi$ in the foregoing equation and making use of (16) and (ii) of Lemma 3.1 to achieve $(\pounds_V R)(X,\xi)\xi = 0$. On the other hand, Lie differentiating the formula (follows from (4)): $R(X,\xi)\xi = -X + \eta(X)\xi$ along V provides

$$(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V \xi)\xi + R(X,\xi)\pounds_V \xi = \{(\pounds_V \eta)X\}\xi + \eta(X)\pounds_V \xi. \tag{18}$$

Moreover, from (4) it follows that $R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X$. Using this, (4) and $(\pounds_V R)(X,\xi)\xi = 0$ in (18), we ultimately obtain

$$g(X,\pounds_V \xi)\xi - 2\eta(\pounds_V \xi)X = \{(\pounds_V \eta)X\}\xi. \tag{19}$$

Now from the soliton equation (1) along with (5) shows that

$$(\pounds_V \eta)X - g(X,\pounds_V \xi) - 2(\lambda + 2n)\eta(X) = 0. \tag{20}$$

Also, Lie differentiating $g(\xi,\xi) = 1$ along V and taking into account (1), (5) provides

$$\eta(\pounds_V \xi) = \lambda + 2n. \tag{21}$$

Using (20) and (21) in (19) it follows that $\lambda = -2n$. Hence the soliton is expanding. Next, we assume that V is a contact vector field. Then by using $\pounds_V \eta = f\eta$, $\lambda = -2n$ and (20) it follows that $\pounds_V \xi = f\xi$. Scalar product of this with ξ and using (21) implies that $f = \lambda + 2n = 0$. Thus, $\pounds_V \xi = 0$, and hence V is strict. Replacing Y by ξ in (8) and using (3), $\pounds_V \xi = 0 = \pounds_V \eta$, we have

$$\begin{aligned}(\pounds_V \nabla)(X,\xi) &= \pounds_V \nabla_X \xi - \nabla_{[V,X]}\xi = \pounds_V (X - \eta(X)\xi) - \pounds_V X + \eta(\pounds_V X)\xi \\ &= \pounds_V X - \eta(X)\pounds_V \xi - \{(\pounds_V \eta)X\}\xi - \eta(\pounds_V X)\xi - \pounds_V X + \eta(\pounds_V X)\xi = 0.\end{aligned}$$

Finally, using (17), we can conclude that M is Einstein. This completes the proof of (i). Next, we assume that the the Reeb vector field ξ leaves the scalar curvature r invariant. This means that $\pounds_\xi r = \xi r = 0$. On the other hand, tracing (i) of Lemma 2, we have

$$\xi r = -2\{r + 2n(2n+1)\}.$$

From which it follows that $r = -2n(2n+1)$. Hence, the scalar curvature r is constant. Next, we recall the following integrability formula ([5], [17])

$$\pounds_V r = -\Delta r + 2\lambda r + 2\left|Q\right|^2, \tag{22}$$

for a Ricci soliton, where $\Delta r = -divDr$. Since r is constant (22) shows that $|Q|^2 = -\lambda r = -2nr$. By virtue of this and $r = -2n(2n+1)$, we compute

$$\begin{aligned}|Q+2nI|^2 &= |Q|^2 + 2nr + 2nr + 4n^2(2n+1)\\ &= 2nr + 4n^2(2n+1) = 2n\{r + 2n(2n+1)\} = 0.\end{aligned}$$

Since the length of the symmetric tensor $Q + 2nI$ vanishes, we must have $Q = -2nI$. This shows that M is Einstein and we complete the proof. $\square$

We have mentioned earlier that the warped product $\mathbb{R} \times_\sigma N^{2n}$, where N^{2n} is a K$\breve{a}$hler manifold of dimension $2n$ and $\sigma(t) = ce^{2t}$ is the warping function, naturally admits Kenmotsu structure. From this, we have the following result.

Corollary 2. *If the metric of the warped product $\mathbb{R} \times_\sigma N^{2n}$ represents a Ricci soliton then it is necessarily expanding.*

Remark 1. *The above corollary shows that there exists examples of non-compact expanding Ricci solitons. In dimension 3, this has been derived explicitely in [9].*

4 RICCI ALMOST SOLITON AND KENMOTSU METRIC

In this section, we study Kenmotsu metric as a Ricci almost soliton. First, we construct a Kenmotsu metric that admits a Ricci almost soliton. The existence of non-compact Ricci almost soliton has been established by Pigola et. al [16] on some certain class of warped product manifolds. Following Lemma 1.1 of [16] we can construct the following example of a Ricci almost soliton.

Example 1. *Consider the warped product $\mathbb{R} \times_{\sigma(t)} \mathbb{H}^n$ with metric $g = dt^2 + \sigma^2(t)g_0$, where g_0 is the standard metric on the hyperbolic space $\mathbb{H}^n$. Let $\sigma(t) = \cosh t$, then the warped product $\mathbb{R} \times_{\sigma(t)} \mathbb{H}^n$ becomes Einstein manifold with Ricci tensor $S^M = -ng$ and it admits a Ricci almost soliton $(g, \nabla f, \lambda)$ with $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t - n$.*

From this example we have the following one.

Example 2. *Let $M^{2n+1} = \mathbb{R} \times_{\cosh t} \mathbb{CH}^{2n}$ with metric $g = dt^2 + (\cosh^2 t)g_0$, where g_0 is the standard metric on the complex hyperbolic space $\mathbb{CH}^{2n}$. Then M^{2n+1} becomes Einstein manifold with Ricci tensor $S^M = -2ng$ (follows from Lemma 1.1 of [16]). Consequently, $(M^{2n+1}, g, \nabla f, \lambda)$ is a Ricci almost soliton with $f(x,t) = \sinh t$ and $\lambda(x,t) = \sinh t - 2n$.*

Example 3. *Let N^{2n} be a complete Einstein Kaehler manifold with $S^N = -(2n-1)g_0$. We consider the warped product $(M^{2n+1}, g) = (\mathbb{R} \times_{\cosh t} \mathbb{N}^{2n}, dt^2 + (\cosh t)^2 g_0)$. Then it follows from Lemma 1.1 of [16] that (M^{2n+1}, g) is Einstein with $S^M = -2ng$. Since M is complete, by the result of Kanai, there exists a function f on M without critical points satisfying $\nabla^2 f = -fg$ (see Theorem D of [12]). Now if we choose $\lambda = -2n - f$, then it is easy to see that $(M^{2n+1}, g, \nabla f, \lambda)$ is a nontrivial gradient Ricci almost soliton.*

From, the last two examples it is evident that the metrics are not Kenmotsu. In fact, by Lemma 1 it follows that the warped product $\mathbb{R} \times_{\cosh t} \mathbb{CH}^{2n}$ is a β-Kenmotsu manifold with $\beta = \tanh t$. So, we consider a D-conformal deformation [14] on the warped product to transform the warped product metric into a Kenmotsu metric. Let

$$\bar{g} = \sigma g + (1 - \sigma)\eta \otimes \eta, \tag{23}$$

where σ is positive function depending only on the direction of ξ. By virtue of Lemma 2.2 of ([2]) it is easy to see that the resulting manifold $\bar{M}(\varphi, \xi, \eta, \bar{g})$ is $\bar{\beta}$-Kenmotsu, where $\bar{\beta} = \beta + \frac{\xi\sigma}{2\sigma}$. Next, we choose $\bar{\beta}$ in such a way that the manifold $\bar{M}$ under consideration will be a Kenmotsu. So, we look for σ satisfying:

$$\beta + \frac{\xi\sigma}{2\sigma} = 1.$$

Making use of a local parametrization that is $\xi = \frac{\partial}{\partial t}$, the foregoing equation can be written as

$$\frac{\partial}{\partial t}(\ln \sigma) = 2(1 - \beta).$$

Solving this we get $\sigma = e^{2t}/(\cosh t)^2$. Finally, using this and (25), equation (26) takes the form

$$\bar{g} = e^{2t} g + dt^2.$$

This is the desired Kenmotsu metric.

The above example motivates us to consider Kenmotsu metric as a gradient Ricci almost soliton. Thus, the equation (2) holds for a smooth function λ.

Theorem 3. *If the metric of a Kenmotsu manifold $M(\varphi, \xi, \eta, g)$ of dimension $(2n+1)$ represents a gradient Ricci almost soliton, then it is η-Einstein and the soliton is expanding. Moreover, if M is complete and the Reeb vector field ξ leaves the scalar curvature r invariant, then M is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}$, and the potential (soliton) function, upto an additive constant, can be expressed as a linear combination of* cosh t *and* sinh t.

Proof. By virtue of (2) and the well known expression of the curvature tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_[X,Y]$, we deduce

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + (X\lambda)Y - (Y\lambda)X. \tag{24}$$

Taking ξ instead of Y in the foregoing equation, recalling (ii) of Lemma 3.1 and (16) provides

$$\begin{aligned} R(X,\xi)Df &= (\nabla_\xi Q)X - (\nabla_X Q)\xi + (X\lambda)\xi - (\xi\lambda)X \\ &= -2QX - 4nX + QX + 2nX + (X\lambda)\xi - (\xi\lambda)X \\ &= -QX - 2nX + (X\lambda)\xi - (\xi\lambda)X. \end{aligned}$$

Making use of $R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X$ (follows from (4)) the foregoing equation implies that

$$g(X, Df - D\lambda)\xi - \{(\xi f) - (\xi\lambda)\}X = -QX - 2nX. \tag{25}$$

Inner product of of this equation with ξ and using (5) shows

$$Df - D\lambda = \{(\xi f) - (\xi\lambda\}\xi. \tag{26}$$

Thus, using (26) in (25) provides

$$\{(\xi f) - (\xi\lambda\}\eta(X)\xi - \{(\xi f) - (\xi\lambda)\}X = -QX - 2nX. \tag{27}$$

Next, we contract equation (24) over X to deduce

$$QDf = \frac{1}{2}Dr - 2nD\lambda.$$

By virtue of this and (5) one can obtain $\frac{1}{4n}(\xi r) = (\xi\lambda) - (\xi f)$. But from (ii) of Lemma 3.1, we have $(\xi r) = -2(r + 2n(2n+1))$. Therefore, $(\xi f) - (\xi\lambda) = \frac{r}{2n} + 2n + 1$. Using this (27) can be expressed as

$$S(X,Y) = (\frac{r}{2n} + 1)g(X,Y) - (\frac{r}{2n} + 2n + 1)\eta(X)\xi. \tag{28}$$

This shows that M is η-Einstein. Now, if ξ leaves the scalar curvature r invariant, then $\xi r = 0$. Consequently, $r = -2n(2n+1)$. By virtue of this, (28) shows that $QX = -2nX$, i.e., M is Einstein. Since $r = -2n(2n+1)$, we have $(\xi\lambda) = (\xi f)$ and therefore $Df = D\lambda$. Hence, equation (2) may be exhibited as

$$\nabla_X D\lambda = (2n + \lambda)X, \tag{29}$$

Applying Tashiro's theorem [18] we conclude that M is isometric to the hyperbolic space $\mathbb{H}^{2n+1}$. Since $\nabla_\xi\xi = 0$ (follows from (3)), we deduce from (29) that $\xi(\xi\lambda) = 2n + \lambda$. But we know [13] that a Kenmotsu manifold M of dimension $2n+1$ is locally isometric to the warped product $(-\varepsilon, \varepsilon) \times_{ce^t} N$, where N is a Kähler manifold of dimension $2n$ and $(-\varepsilon, \varepsilon)$ is an open interval. Using the local parametrization: $\xi = \frac{\partial}{\partial t}$ (where t denotes the coordinate on $(-\varepsilon, \varepsilon)$) we obtain

$$\frac{d^2\lambda}{dt} = 2n + \lambda.$$

Its solution can be exhibited as $\lambda = A\ cosh\ t + B\ sinh\ t - 2n$, where A, B are constants on M. This completes the proof. □

Example 4. *Let (N, J, g_0) be a Kähler manifold of dimension $2n$. Consider the warped product $(M, g) = (\mathbb{R} \times_\sigma N, dt^2 + \sigma^2 g_0)$, where t is the coordinate on $\mathbb{R}$. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R} \times_\sigma N$ by $\varphi X = JX$ for vector field X on N and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Then it is easy to testify (see [13]) that the warped product $\mathbb{R} \times_\sigma N$, $\sigma^2 = ce^{2t}$, with the structure (φ, ξ, η, g) is an almost Kenmotsu manifold, by Lemma 1. Thus, if we take $\sigma(t) = ce^t$, then M becomes a Kenmotsu manifold. Further, we set $N = CH^{2n}$, then N being Einstein, the Ricci tensor of M such that $S = -2ng$. Define $f(t) = ke^t, k > 0$. Then it is easy to verify that $(M, \nabla f, g)$ is a Ricci almost soliton with $\lambda = ke^t - 2n$. Similarly, we may also construct many examples of Ricci almost solitons by taking different potential functions on the warped product.*

Next, we consider a special type of Ricci almost soliton on Kenmotsu manifolds in which the potential vector field V is a pointwise collinear with the Reeb vector filed ξ. This type of problem has been considered by the author within the framework of contact metric manifolds in [10].

Theorem 4. *Let $M(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold of diemnsion $(2n+1)$. If g represents a non-trivial Ricci almost soliton such that the potential vector field V is pointwise collinear with the Reeb vector field, then it is η-Einstein.*

Proof. By hypothesis $V = k\xi$, for some smooth function k on M. Differentiating this along an arbitrary vector field X and using (3), we have

$$\nabla_X V = (Xk)\xi + k(X - \eta(X)\xi). \tag{30}$$

In view of (30) the soliton equation transforms to

$$2S(X,Y) + 2(k-\lambda)g(X,Y) + (Xk)\eta(Y) + (Yk)\eta(X) - 2k\eta(X)\eta(Y) = 0. \tag{31}$$

Setting $X = Y = \xi$ in (31) and using (5), we have $\xi k = 2n + \lambda$. Further, taking ξ instead of Y, using (5) and $\xi k = 2n + \lambda$ shows that $Xk = (2n+\lambda)\eta(X)$. Making use of this in (31), we have

$$QX = (\lambda - k)X - (2n + \lambda - k)\eta(X)\xi. \tag{32}$$

Tracing the foregoing equation provides

$$\frac{r}{2n} = (\lambda - k) - 1. \tag{33}$$

By virtue of (33), (32) can be written as

$$QX = (\frac{r}{2n} + 1)X - (\frac{r}{2n} + 2n + 1)\eta(X)\xi.$$

This shows that M^{2n+1} is η-Einstein and we complete the proof. $\square$

Remark 2. *If k is constant, then from $Xk = (\lambda + 2n)\eta(X)$, we see that $\lambda = -2n$ and which is constant. Thus from (33) follows that $\xi r = 0$. Then from (ii) of Lemma 3.1, we have $(\xi r) = -2(r + 2n(2n+1))$. Hence $r = -2n(2n+1)$. Making use of this in (33) we see that $k = 0$, and in this case the soliton becomes trivial.*

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Ґош А. *Солітон Річчі і майже солітон Річчі в рамках многовиду Кенмоцу* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 59–69.

Ми доводимо, спочатку, що якщо векторне поле Ріба ξ многовида Кенмоцу M залишає оператор Q інваріантним, то M є Айнштайнівським. Далі ми вивчаємо многовид Кенмоцу, метрика якого зображує солітон Річчі, і доводимо, що він є поширюючим. Більше того, солітон є тривіальним (Айнштайнівським), якщо або (i) V є контактним векторним полем або (ii) векторне поле Ріба ξ залишає скалярну кривизну інваріантною. Нарешті, доведено, що якщо метрика многовиду Кенмоцу зображає деякий градієнтний майже солітон Річчі, то цей многовид є η-Айнштайнівським і цей солітон є поширюючим. Ми також демонструємо деякі приклади многовиду Кенмоцу, які допускають майже солітони Річчі.

Ключові слова і фрази: многовид Кенмоцу, майже солітон Річчі, викривлений добуток.

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INTERCONNECTION BETWEEN WICK MULTIPLICATION AND INTEGRATION ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS IN THE LÉVY WHITE NOISE ANALYSIS

We deal with spaces of nonregular generalized functions in the Lévy white noise analysis, which are constructed using Lytvynov's generalization of a chaotic representation property. Our aim is to describe a relationship between Wick multiplication and integration on these spaces. More exactly, we show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of an extended stochastic integral; establish an analog of this result for a Pettis integral (a weak integral); and prove a theorem about a representation of the extended stochastic integral via the Pettis integral from the Wick product of the original integrand by a Lévy white noise. As examples of an application of our results, we consider some stochastic equations with Wick type nonlinearities.

Key words and phrases: Lévy process, extended stochastic integral, Pettis integral, Wick product.

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INTRODUCTION

A theory of test and generalized functions depending on infinitely many variables (i.e., with arguments belonging to infinite-dimensional spaces) is highly sought in many areas of modern physics and mathematics. One of successful approaches to building of such a theory consists in introduction of spaces of the above-mentioned functions in such a way that the dual pairing between test and generalized functions is generated by integration with respect to some probability measure on a dual nuclear space. First it was the Gaussian measure, the corresponding theory is called the *Gaussian white noise analysis* (e.g., [2, 16, 26–28]), then it were realized numerous generalizations. In particular, important results can be obtained if one uses the Lévy white noise measure (e.g., [6, 7, 29]), the corresponding theory is called the *Lévy white noise analysis*.

A very important role in the Gaussian analysis belongs to a so-called *chaotic representation property* (CRP). This property consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated Itô's stochastic integrals from nonrandom functions (see, e.g., [30] for a detailed presentation). Using CRP, one can construct various spaces of test and generalized functions, introduce stochastic integrals and derivatives on these spaces, etc. In the Lévy analysis there is no CRP (more exactly, the only

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Lévy processes with CRP are Wiener and Poisson processes) [35]; but there are different generalizations of this property: Itô's generalization [18], Nualart-Schoutens' generalization [31, 32], Lytvynov's generalization [29], Oksendal's generalization [6, 7], etc. The interconnections between these generalizations are described in, e.g., [1, 6, 7, 21, 29, 34, 36]. Now, depending on problems under consideration, one can select a most suitable generalization of CRP, construct corresponding spaces of test and generalized functions, and introduce necessary operators and operations on these spaces.

In the present paper we deal with one of the most useful and challenging generalizations of CRP in the Lévy white noise analysis, which is proposed by E. W. Lytvynov [29] (see also [5]). The idea of this generalization is to decompose random variables, square integrable with respect to the Lévy white noise measure, in series of special orthogonal functions, by analogy with decompositions of random variables, square integrable with respect to the Gaussian measure, by Hermite polynomials (remind that the last decompositions are equivalent to the decompositions by repeated Itô's stochastic integrals). Like using CRP in the Gaussian analysis, one can use Lytvynov's generalization of CRP in order to construct and study spaces of regular and nonregular test and generalized functions [19], various operators and operations on these spaces, etc. In particular, the extended stochastic integral and the Hida stochastic derivative on the spaces of *regular* test and generalized functions are introduced and studied in [10, 19], operators of stochastic differentiation—in [8,9,13], some elements of a Wick calculus and its relationship with operators of stochastic differentiation— in [11]. As for the spaces of *nonregular* test and generalized functions—the corresponding results are presented in [19, 22–24].

As is well known, in the Gaussian white noise analysis, in the same way as in various versions of a non-Gaussian analysis, a natural multiplication on spaces of generalized functions is a so-called Wick multiplication. In particular, in many cases, using the Wick multiplication, one can take a time-independent multiplier out of the sign of an extended stochastic integral. Moreover, such a result holds true for a Pettis integral (a weak integral). Also, the extended stochastic integral can be presented as a Pettis integral (or a formal Pettis integral—depending on the concrete situation) from the Wick product of the original integrand by the derivative (in the sense of generalized functions) of the integrator. On the above-mentioned spaces of *regular* generalized functions in the Lévy analysis such results were obtained in [12]. The aim of the present paper is to transfer the results of [12] to the spaces of *nonregular* generalized functions, which are constructed using Lytvynov's generalization of CRP. In a sense, this paper is a continuation of the paper [22].

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a probability triplet connected with L, convenient for our considerations; then we describe Lytvynov's generalization of CRP; construct a nonregular rigging of the space of square integrable random variables (the positive and negative spaces of this rigging are the spaces of nonregular test and generalized functions respectively); describe the extended stochastic integral with respect to L on the spaces of nonregular generalized functions; and recall necessary notions of the Wick calculus. In the second section we show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of the extended stochastic integral and of the Pettis integral; prove a theorem about a representation of the extended stochastic integral via the Pettis integral; and consider examples.

1 PRELIMINARIES

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the real, i.e., bilinear scalar product in a space H; and by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H. Further, we use a designation pr lim (resp., ind lim) for a projective (resp., inductive) limit of a family of spaces, this designation implies that the limit space is endowed with the projective (resp., inductive) limit topology (see, e.g., [3] for a detailed description).

1.1 A Lévy process and its probability space

Denote $\mathbb{R}_+ := [0,+\infty)$. Let $L = (L_u)_{u\in\mathbb{R}_+}$ be a real-valued locally square integrable Lévy process (i.e., a continuous in probability random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [4] for details) without Gaussian part and drift. As is well known (e.g., [7]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_u}] = \exp\Big[u\int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x)\nu(dx)\Big], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra; $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$*, there exists* $\varepsilon > 0$ *such that* $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty$*, and* $\int_{\mathbb{R}} x^2\nu(dx) = 1$.

Define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [3]; see also Subsection 1.3). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega\in\mathcal{D}'$ and $\varphi\in\mathcal{D}$ denote $\omega(\varphi)$ by $\langle\omega,\varphi\rangle$. It is worth noting that $\mathcal{D}$ and $\mathcal{D}'$ are the positive and negative spaces of a chain

$$\mathcal{D}' \supset L^2(\mathbb{R}_+) \supset \mathcal{D}, \tag{2}$$

where $L^2(\mathbb{R}_+)$ is the space of (classes of) real-valued functions on $\mathbb{R}_+$, square integrable with respect to the Lebesgue measure (e.g., [3]), and therefore $\langle\cdot,\cdot\rangle$ is the dual pairing generated by the scalar product in $L^2(\mathbb{R}_+)$. The notation $\langle\cdot,\cdot\rangle$ will be preserved for dual pairings in tensor powers of the complexification of chain (2).

Definition 1. *A probability measure* μ *on* $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$*, where* $\mathcal{C}$ *denotes the cylindrical* σ*-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}}(e^{i\varphi(u)x}-1-i\varphi(u)x)du\nu(dx)\Big], \quad \varphi\in\mathcal{D}, \tag{3}$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [17]), see [29]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ.*

Denote by $(L^2) := L^2(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ the space of (classes of) complex-valued functions on $\mathcal{D}'$, square integrable with respect to μ (in what follows, this notation will be used very often). Let $f\in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k\in\mathcal{D})_{k\in\mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k\to\infty$ (remind that $\mathcal{D}$ is a dense set in $L^2(\mathbb{R}_+)$). One can show [6,7,21,29] that $\langle\circ,f\rangle := (L^2)\text{-}\lim_{k\to\infty}\langle\circ,\varphi_k\rangle$ is a well-defined element of (L^2).

Denote by 1_A the indicator of a set A, and put $1_{[0,0)} \equiv 0$. It follows from (1) and (3) that $\big(\langle \circ, 1_{[0,u)}\rangle\big)_{u\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space (triplet) $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$, see, e.g., [6,7]. So, for each $u \in \mathbb{R}_+$ we have $L_u = \langle \circ, 1_{[0,u)}\rangle \in (L^2)$.

Note that the derivative in the sense of generalized functions of a Lévy process (a Lévy white noise) is $\dot{L}_\cdot(\omega) = \langle \omega, \delta_\cdot\rangle \equiv \omega(\cdot)$, where δ is the Dirac delta-function. Therefore $\dot{L}$ is a generalized random process (in the sense of [14]) with trajectories from $\mathcal{D}'$, and μ is the measure of $\dot{L}$ in the classical sense of this notion [15].

Remark 1. *A Lévy process L without Gaussian part and drift is a* Poisson process *if its Lévy measure ν is a point mass at* 1*. This measure does not satisfy the assumptions accepted above (its support does not contain an infinite number of points); nevertheless, all results of the present paper have natural analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [21], Subsection 1.2.*

1.2 Lytvynov's generalization of CRP

Denote by $\widehat{\otimes}$ the symmetric tensor multiplication, by a subscript $\mathbb{C}$—complexifications of spaces. Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote by $\mathcal{P}$ the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)}\rangle, \quad \omega \in \mathcal{D}',\ f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n},\ N_f \in \mathbb{Z}_+,\ f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle \omega^{\otimes 0}, f^{(0)}\rangle := f^{(0)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [29]), therefore $\mathcal{P}$ is a dense set in (L^2) [33]. Denote by $\mathcal{P}_n$, $n \in \mathbb{Z}_+$, the set of polynomials of power smaller than or equal to n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)); put $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n. \tag{4}$$

Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define real (bilinear) scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)}\rangle::\langle \omega^{\otimes n}, g^{(n)}\rangle: \mu(d\omega). \tag{5}$$

The proof of the well-posedness of this definition coincides up to obvious modifications with the proof of the corresponding statement in [29].

Denote by $|\cdot|_{ext}$ the norms corresponding to scalar products (5), i.e., $|\cdot|_{ext} := \sqrt{(\cdot,\bar{\cdot})_{ext}}$. Let $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, be the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to these norms. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)}\rangle: \stackrel{\mathrm{def}}{=} (L^2)\text{-}\lim_{k\to\infty} :\langle \circ^{\otimes n}, f_k^{(n)}\rangle:$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$. The well-posedness of this definition can be proved by the method of "mixed sequences". It is easy to show that $:\langle \circ^{\otimes 0}, F^{(0)}\rangle: = \langle \circ^{\otimes 0}, F^{(0)}\rangle = F^{(0)}$ and $:\langle \circ, F^{(1)}\rangle: = \langle \circ, F^{(1)}\rangle$ (cf. [29]).

Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)}\rangle: | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}\}$ is dense in $\mathbf{P}_n$, the next statement from (4) follows.

Theorem 1 (Lytvynov's generalization of CRP, cf. [29]). *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)}\rangle: \tag{6}$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n!|F^{(n)}|_{ext}^2 < \infty.$$

Remark 2. *In this paper we do not use directly an explicit formula for the scalar products $(\cdot,\cdot)_{ext}$, and therefore we prefer not to write it down. But for the interested reader we note that such a formula is calculated in [29]; in another record form (more convenient for some calculations) it is given in, e.g., [9, 11, 13]. Also we note that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{ext}^{(n)}$ is the symmetric subspace of the space of (classes of) complex-valued functions on $\mathbb{R}_+^n$, square integrable with respect to a certain Radon measure.*

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (in what follows, this notation will be used very often). It follows from the explicit formula for $(\cdot,\cdot)_{ext}$ that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k, j \in \{1,\dots,n\}$: $k \neq j$, but $u_k = u_j$). In this sense the *space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$*, this explains why we use the subscript "*ext*" in our designations.

1.3 A nonregular rigging of (L^2)

Let T be the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Denote by $\mathcal{H}_\tau$ the real Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is the completion of $\mathcal{D}$ with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [3]) that $\mathcal{D} = \operatorname{pr\,lim}_{\tau\in T} \mathcal{H}_\tau$ (moreover, one can show that for any $n \in \mathbb{N}$ $\mathcal{D}^{\widehat{\otimes} n} = \operatorname{pr\,lim}_{\tau\in T} \mathcal{H}_\tau^{\widehat{\otimes} n}$), and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$. Therefore one can consider a chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [3] $\mathcal{D}' = \operatorname{ind\,lim}_{\tau\in T} \mathcal{H}_{-\tau}$ (it is convenient for us to consider $\mathcal{D}'$ as a topological space with the inductive limit topology). By analogy with [20] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.*

Denote the norms in $\mathcal{H}_{\tau,\mathbb{C}}$ and its symmetric tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)}, \overline{f^{(n)}})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}}$ (note that $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$ and $|f^{(0)}|_\tau = |f^{(0)}|$).

It follows from results of [19] that one can modify T again (it is necessary to remove from T some "bad" indexes) in order to obtain the following statement.

Proposition 1. *For each $\tau \in T$ and each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ $|f^{(n)}|_{ext}^2 \leq n! c(\tau)^n |f^{(n)}|_\tau^2$.*

Accept on default $q \in \mathbb{Z}_+$ and $\tau \in T$. Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Define real (bilinear) scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \; g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W$$

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}. \tag{7}$$

The well-posedness of this definition is proved in [22].

Let $\|\cdot\|_{\tau,q}$ be the norms corresponding to scalar products (7), i.e., $\|\cdot\|_{\tau,q} := \sqrt{(\cdot,\bar{\cdot})_{\tau,q}}$. Denote by $(\mathcal{H}_\tau)_q$ the completions of $\mathcal{P}_W$ with respect to these norms, and set $(\mathcal{H}_\tau) := \operatorname{pr}\lim_{q\to\infty}(\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \operatorname{pr}\lim_{\tau\in T, q\to\infty}(\mathcal{H}_\tau)_q$. As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be uniquely presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \; f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \tag{8}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty \tag{9}$$

(since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be uniquely presented in form (8) and norm (9) is finite for each $q \in \mathbb{Z}_+$ (for each $\tau \in T$ and each $q \in \mathbb{Z}_+$).

Proposition 2 ([19,22]). *For each $\tau \in T$ there exists $q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0(\tau)} := \{q_0(\tau), q_0(\tau)+1, \dots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

In view of this proposition one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \; \tau \in T, \; q \in \mathbb{N}_{q_0(\tau)}, \tag{10}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \operatorname{ind}\lim_{q'\to\infty}(\mathcal{H}_{-\tau})_{-q'}$ and $(\mathcal{D}') = \operatorname{ind}\lim_{\tilde{\tau}\in T, q'\to\infty}(\mathcal{H}_{-\tilde{\tau}})_{-q'}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to (L^2).

Definition 2. *Chain* (10) *is called a nonregular rigging of the space* (L^2)*. The positive spaces of this rigging* $(\mathcal{H}_\tau)_q$*,* $(\mathcal{H}_\tau)$ *and* $(\mathcal{D})$ *are called (Kondratiev-type) spaces of nonregular test functions. The negative spaces of this rigging* $(\mathcal{H}_{-\tau})_{-q}$*,* $(\mathcal{H}_{-\tau})$ *and* $(\mathcal{D}')$ *are called (Kondratiev-type) spaces of nonregular generalized functions.*

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(m)}_{\mathbb{C}} \supset \mathcal{H}^{(m)}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{(m)}_{ext} \supset \mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes} m}_{\mathbb{C}}, \tag{11}$$

$m \in \mathbb{N}$, where $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ and $\mathcal{D}'^{(m)}_{\mathbb{C}} = \operatorname{ind\,lim}_{\tilde{\tau}\in T} \mathcal{H}^{(m)}_{-\tilde{\tau},\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}}$ and $\mathcal{D}^{\widehat{\otimes} m}_{\mathbb{C}}$ with respect to $\mathcal{H}^{(m)}_{ext}$. Set $\mathcal{D}^{\widehat{\otimes} 0}_{\mathbb{C}} = \mathcal{H}^{\widehat{\otimes} 0}_{\tau,\mathbb{C}} = \mathcal{H}^{(0)}_{ext} = \mathcal{H}^{(0)}_{-\tau,\mathbb{C}} = \mathcal{D}'^{(0)}_{\mathbb{C}} := \mathbb{C}$. In what follows, we denote by $\langle\cdot,\cdot\rangle_{ext}$ the real (bilinear) dual pairings between elements of negative and positive spaces from chains (11), these pairings are generated by the scalar products in $\mathcal{H}^{(m)}_{ext}$.

The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [19,20]).

Proposition 3. *There exists a system of generalized functions*

$$\{:\langle\circ^{\otimes m}, F^{(m)}_{ext}\rangle :\in (\mathcal{H}_{-\tau})_{-q} \mid F^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}},\ m \in \mathbb{Z}_+\}$$

such that

1) *for* $F^{(m)}_{ext} \in \mathcal{H}^{(m)}_{ext} \subset \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ $:\langle\circ^{\otimes m}, F^{(m)}_{ext}\rangle:$ *is a Wick monomial that is defined in Subsection 1.2;*

2) *any generalized function* $F \in (\mathcal{H}_{-\tau})_{-q}$ *can be uniquely presented as a series*

$$F = \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, F^{(m)}_{ext}\rangle:,\ F^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}, \tag{12}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$*, i.e.,*

$$\|F\|^2_{-\tau,-q} := \|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} |F^{(m)}_{ext}|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} < \infty; \tag{13}$$

and, vice versa, any series (12) *with finite norm* (13) *is a generalized function from* $(\mathcal{H}_{-\tau})_{-q}$ *(i.e., such a series converges in* $(\mathcal{H}_{-\tau})_{-q}$*);*

3) *the dual pairing between* $F \in (\mathcal{H}_{-\tau})_{-q}$ *and* $f \in (\mathcal{H}_\tau)_q$ *that is generated by the scalar product in* (L^2)*, has the form*

$$\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F^{(m)}_{ext}, f^{(m)}\rangle_{ext}, \tag{14}$$

where $F^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ *and* $f^{(m)} \in \mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}}$ *are the kernels from decompositions* (12) *and* (8) *for* F *and* f *respectively.*

It is clear that $F \in (\mathcal{H}_{-\tau})$ ($F \in (\mathcal{D}')$) if and only if F can be uniquely presented in form (12) and norm (13) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

1.4 An extended stochastic integral on spaces of nonregular generalized functions

Decomposition (6) for elements of (L^2) defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism)

$$\mathbf{I} : (L^2) \to \mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)},$$

where $\mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$ is a weighted extended symmetric Fock space: for $F \in (L^2)$ of form (6) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \dots) \in \mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$. Denote by $\mathbf{1} : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ the identity operator. The operator $\mathbf{I} \otimes \mathbf{1} : (L^2) \otimes \mathcal{H}_{\mathbb{C}} \to (\mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}) \otimes \mathcal{H}_{\mathbb{C}} \cong \mathop{\oplus}_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$ is, obviously, an isometric isomorphism between the spaces $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$ and $\mathop{\oplus}_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$. It is clear that for arbitrary $m \in \mathbb{Z}_+$ and $F_{\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ a vector $(\underbrace{0, \dots, 0}_{m}, F_{\cdot}^{(m)}, 0, \dots)$ belongs to the space $\mathop{\oplus}_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$. Set

$$:\langle \circ^{\otimes m}, F_{\cdot}^{(m)} \rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1}(\underbrace{0, \dots, 0}_{m}, F_{\cdot}^{(m)}, 0, \dots) \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}. \tag{15}$$

By the construction elements $:\langle \circ^{\otimes n}, F_{\cdot}^{(n)} \rangle:$, $F_{\cdot}^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, form an orthogonal basis in the space $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$ in the sense that $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ if and only if F can be uniquely presented as $F \equiv F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_{\cdot}^{(n)} \rangle:$ (the series converges in $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$), with $\|F\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} n! |F_{\cdot}^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty$.

Since, obviously, the restrictions of the generalized Wiener-Itô-Sigal isomorphism $\mathbf{I}$ to the spaces $(\mathcal{H}_\tau)_q$ are isometric isomorphisms between $(\mathcal{H}_\tau)_q$ and weighted symmetric Fock spaces $\mathop{\oplus}_{n=0}^{\infty} (n!)^2 2^{qn} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ (cf. [25]), for arbitrary $n \in \mathbb{Z}_+$ and $f_{\cdot}^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ we have $:\langle \circ^{\otimes n}, f_{\cdot}^{(n)} \rangle: \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\mathbb{C}}$. Moreover, elements $:\langle \circ^{\otimes n}, f_{\cdot}^{(n)} \rangle:$, $f_{\cdot}^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, form orthogonal bases in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\mathbb{C}}$: $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\mathbb{C}}$ if and only if f can be uniquely presented as $f \equiv f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_{\cdot}^{(n)} \rangle:$ (the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\mathbb{C}}$), with $\|f\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f_{\cdot}^{(n)}|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}\otimes\mathcal{H}_{\mathbb{C}}} < \infty$.

Further, as in the case of spaces $(\mathcal{H}_{-\tau})_{-q}$, it follows from the general duality theory that there exists a system of orthogonal in each $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ generalized functions

$$\{:\langle \circ^{\otimes m}, F_{ext,\cdot}^{(m)} \rangle: \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}} \mid F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}},\ m \in \mathbb{Z}_+\} \tag{16}$$

such that for $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ $:\langle \circ^{\otimes m}, F_{ext,\cdot}^{(m)} \rangle:$ is given by (15); any generalized function $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ can be uniquely presented as a convergent in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ series

$$F \equiv F(\cdot) = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext,\cdot}^{(m)} \rangle:,\ F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}, \tag{17}$$

with

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{m=0}^{\infty} 2^{-qm} |F_{ext,\cdot}^{(m)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty; \tag{18}$$

and, vice versa, any series (17) with finite norm (18) is a generalized function from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$). So, system (16) is an orthogonal basis in each space $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$. Moreover, it is clear that $F \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}} := \operatorname{ind}\lim_{q\to\infty}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ ($F \in (\mathcal{D}') \otimes \mathcal{H}_{\mathbb{C}} := \operatorname{ind}\lim_{\tau\in T, q\to\infty}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$) if and only if F can be uniquely presented in form (17) and norm (18) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

Now our aim is to describe the construction of an extended stochastic integral with respect to a Lévy process L, that is based on decomposition (17). We need a small preparation.

Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes} m}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} m}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes} m}_{\mathbb{C}}, \tag{19}$$

$m \in \mathbb{N}$ (as is well known (cf. [3]), $\mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}}$ and $\mathcal{D}'^{\widehat{\otimes} m}_{\mathbb{C}} = \operatorname{ind}\lim_{\widetilde{\tau}\in T}\mathcal{H}^{\widehat{\otimes} m}_{-\widetilde{\tau},\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}}$ and $\mathcal{D}^{\widehat{\otimes} m}_{\mathbb{C}}$ respectively). Set $\mathcal{D}^{\widehat{\otimes} 0}_{\mathbb{C}} = \mathcal{H}^{\widehat{\otimes} 0}_{\tau,\mathbb{C}} = \mathcal{H}^{\widehat{\otimes} 0}_{\mathbb{C}} = \mathcal{H}^{\widehat{\otimes} 0}_{-\tau,\mathbb{C}} = \mathcal{D}'^{\widehat{\otimes} 0}_{\mathbb{C}} := \mathbb{C}$. Since the spaces of test functions in chains (19) and (11) coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)}_{\mathbb{C}} \to \mathcal{D}'^{\widehat{\otimes} m}_{\mathbb{C}}, \ m \in \mathbb{Z}_+,$$

such that for all $F^{(m)}_{ext} \in \mathcal{D}'^{(m)}_{\mathbb{C}}$ and $f^{(m)} \in \mathcal{D}^{\widehat{\otimes} m}_{\mathbb{C}}$

$$\langle F^{(m)}_{ext}, f^{(m)}\rangle_{ext} = \langle U_m F^{(m)}_{ext}, f^{(m)}\rangle. \tag{20}$$

It is easy to see that the restrictions of U_m to the spaces $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ are *isometric* isomorphisms between the spaces $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ and $\mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}}$.

Remark 3. *Since $\mathcal{H}^{(1)}_{ext} = \mathcal{H}_{\mathbb{C}}$, in the case $m = 1$ chains* (19) *and* (11) *coincide. Thus $U_1 = \mathbf{1}$ is the identity operator on $\mathcal{D}'^{(1)}_{\mathbb{C}} = \mathcal{D}'_{\mathbb{C}}$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{C}$.*

Definition 3. *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ and $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$. We define an extended stochastic integral with respect to a Lévy process $\int_\Delta F(u)\widehat{d}L_u \in (\mathcal{H}_{-\tau})_{-q}$ by setting*

$$\int_\Delta F(u)\widehat{d}L_u := \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}^{(m)}_{ext,\Delta}\rangle:, \tag{21}$$

where

$$\widehat{F}^{(m)}_{ext,\Delta} := U^{-1}_{m+1}\{Pr[(U_m \otimes \mathbf{1})F^{(m)}_{ext,\cdot}1_\Delta(\cdot)]\} \in \mathcal{H}^{(m+1)}_{-\tau,\mathbb{C}}, \tag{22}$$

Pr is the symmetrization operator (more exactly, the orthoprojector acting for each $m \in \mathbb{Z}_+$ from $\mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ to $\mathcal{H}^{\widehat{\otimes} m+1}_{-\tau,\mathbb{C}}$), $F^{(m)}_{ext,\cdot} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition (17) *for F.*

Since

$$|\widehat{F}^{(m)}_{ext,\Delta}|_{\mathcal{H}^{(m+1)}_{-\tau,\mathbb{C}}} = |Pr[(U_m \otimes \mathbf{1})F^{(m)}_{ext,\cdot}1_\Delta(\cdot)]|_{\mathcal{H}^{\widehat{\otimes} m+1}_{-\tau,\mathbb{C}}} \leq |(U_m \otimes \mathbf{1})F^{(m)}_{ext,\cdot}|_{\mathcal{H}^{\widehat{\otimes} m}_{-\tau,\mathbb{C}}\otimes\mathcal{H}_{\mathbb{C}}} = |F^{(m)}_{ext,\cdot}|_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}\otimes\mathcal{H}_{\mathbb{C}}}$$

and therefore by (21), (13) and (18)

$$\Big\| \int_\Delta F(u)\widehat{d}L_u \Big\|^2_{-\tau,-q} = \sum_{m=0}^{\infty} 2^{-q(m+1)} |\widehat{F}^{(m)}_{ext,\Delta}|^2_{\mathcal{H}^{(m+1)}_{-\tau,\mathbb{C}}}$$
$$\le 2^{-q} \sum_{m=0}^{\infty} 2^{-qm} |F^{(m)}_{ext,\cdot}|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}\otimes\mathcal{H}_\mathbb{C}} = 2^{-q}\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_\mathbb{C}},$$

this definition is well-posed and, moreover, the extended stochastic integral

$$\int_\Delta \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_\mathbb{C} \to (\mathcal{H}_{-\tau})_{-q} \tag{23}$$

is a linear *continuous* operator.

As appears from the above, an extended stochastic integral can be defined by (21), (22) as a linear continuous operator acting from $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_\mathbb{C}$ to $(\mathcal{H}_{-\tau})$, or from $(\mathcal{D}') \otimes \mathcal{H}_\mathbb{C}$ to $(\mathcal{D}')$. Exactly the integral

$$\int_\Delta \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_\mathbb{C} \to (\mathcal{H}_{-\tau}) \tag{24}$$

will be the object of our considerations in the forthcoming section.

Remark 4. *As easily appears from results of [19, 21], stochastic integral* (23) *and its extension* (24) *are generalizations of the extended Skorohod stochastic integral on* $(L^2) \otimes \mathcal{H}_\mathbb{C}$ *[21]. The last integral, in turn, is an extension of the Itô stochastic integral.*

Also we note that, in contrast to the regular case [9, 12, 13, 19], integrals (23) *and* (24) *cannot be naturally restricted to the spaces of nonregular test functions, see [23] for details.*

Remark 5. *It follows from the definition of the extended stochastic integral that for each* $\Delta \in \mathcal{B}(\mathbb{R}_+)$

$$\int_\Delta \circ(u)\widehat{d}L_u = \int_{\mathbb{R}_+} \circ(u)1_\Delta(u)\widehat{d}L_u. \tag{25}$$

One can use this representation for an important generalization. Let a function $F : \mathbb{R}_+ \to (\mathcal{H}_{-\tau})$ *be such that* $F \notin (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_\mathbb{C}$*, but for some* $\Theta \in \mathcal{B}(\mathbb{R}_+)$ *we have* $F(\cdot)1_\Theta(\cdot) \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_\mathbb{C}$*. It is clear that for any measurable* $\Delta \subseteq \Theta$ *we have now* $F(\cdot)1_\Delta(\cdot) \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_\mathbb{C}$*, therefore one can define* $\int_\Delta F(u)\widehat{d}L_u \in (\mathcal{H}_{-\tau})$ *by formula* (25).

Finally we note that the operator, adjoint to the extended stochastic integral, is called the *Hida stochastic derivative*. This derivative is closely connected with so-called operators of stochastic differentiation on spaces of nonregular test functions [24]. All the mentioned operators play an important role in the Lévy white noise analysis.

1.5 Elements of a Wick calculus

Let $F \in (\mathcal{H}_{-\tau})$. We define an S-transform $(SF)(\lambda)$, $\lambda \in \mathcal{D}_\mathbb{C}$, as a formal series

$$(SF)(\lambda) := \sum_{m=0}^{\infty} \langle F^{(m)}_{ext}, \lambda^{\otimes m}\rangle_{ext} \equiv F^{(0)}_{ext} + \sum_{m=1}^{\infty} \langle F^{(m)}_{ext}, \lambda^{\otimes m}\rangle_{ext}, \tag{26}$$

where $F^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$ are the kernels from decomposition (12) for F. In particular, $(SF)(0) = F^{(0)}_{ext}$, $S1 \equiv 1$.

Note that each term in series (26) is well-defined, but the series can diverge. However, the last is not an obstruction in order to construct the Wick calculus.

Definition 4. *For $F, G \in (\mathcal{H}_{-\tau})$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define a Wick product $F\Diamond G$ and a Wick version of h $h^{\Diamond}(F)$ by setting formally*

$$F\Diamond G := S^{-1}(SF \cdot SG), \quad h^{\Diamond}(F) := S^{-1}h(SF).$$

It is obvious that the Wick multiplication $\Diamond$ is commutative, associative and distributive over a field $\mathbb{C}$.

Note that a function h can be decomposed in a Taylor series

$$h(u) = \sum_{m=0}^{\infty} h_m \big(u - (SF)(0)\big)^m. \tag{27}$$

Using this decomposition, it is easy to calculate that $h^{\Diamond}(F) = \sum_{m=0}^{\infty} h_m \big(F - (SF)(0)\big)^{\Diamond m}$, where $F^{\Diamond m} := \underbrace{F\Diamond \cdots \Diamond F}_{m \text{ times}} = S^{-1}[(SF)^m]$, $F^{\Diamond 0} := 1$.

"Coordinate formulas" for the Wick product and for the Wick versions of holomorphic functions (i.e., representations of $F\Diamond G$, $F_1\Diamond \cdots \Diamond F_n$, $n \in \mathbb{N}$, and $h^{\Diamond}(F)$ via kernels from decompositions (12) for F, G, $F_1, \dots, F_n$, and coefficients from decomposition (27) for h) are given in [22]. Using these formulas, one can prove the following statement.

Theorem 2 ([22]). *1) Let $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})$, $n \in \mathbb{N}$. Then $F_1\Diamond \cdots \Diamond F_n \in (\mathcal{H}_{-\tau})$. Moreover, the Wick multiplication is continuous in the sense that*

$$\|F_1\Diamond \cdots \Diamond F_n\|_{-\tau,-q} \le \sqrt{\max_{m\in\mathbb{Z}_+} [2^{-m}(m+1)^{n-1}]}\, \|F_1\|_{-\tau,-(q-1)} \cdots \|F_n\|_{-\tau,-(q-1)},$$

where $q \in \mathbb{N}$ is such that $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})_{-(q-1)}$.

2) Let $F \in (\mathcal{H}_{-\tau})$ and a function $h : \mathbb{C} \to \mathbb{C}$ be holomorphic at $(SF)(0)$. Then $h^{\Diamond}(F) \in (\mathcal{H}_{-\tau})$.

Finally, we will write out a "coordinate formula" for $F\Diamond G$, $F, G \in (\mathcal{H}_{-\tau})$, which will be necessary in the next section. We need a small preparation: it is necessary to introduce an analog of the symmetric tensor multiplication on the spaces $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$, $m \in \mathbb{Z}_+$.

For $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $G^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$, $n, m \in \mathbb{Z}_+$, set

$$F^{(n)}_{ext} \diamond G^{(m)}_{ext} := U^{-1}_{n+m}\{Pr[(U_n F^{(n)}_{ext}) \otimes (U_m G^{(m)}_{ext})]\} \equiv U^{-1}_{n+m}\{(U_n F^{(n)}_{ext}) \widehat{\otimes} (U_m G^{(m)}_{ext})\} \in \mathcal{H}^{(n+m)}_{-\tau,\mathbb{C}}. \tag{28}$$

It follows from properties of operators U_m (see Subsection 1.4) and of the symmetric tensor multiplication that the multiplication $\diamond$ is commutative, associative and distributive over a field $\mathbb{C}$. One can show [22] that $|F^{(n)}_{ext} \diamond G^{(m)}_{ext}|_{\mathcal{H}^{(n+m)}_{-\tau,\mathbb{C}}} \le |F^{(n)}_{ext}|_{\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}} |G^{(m)}_{ext}|_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}}$, and for any $\lambda \in \mathcal{D}_{\mathbb{C}}$ $\langle F^{(n)}_{ext}, \lambda^{\otimes n}\rangle_{ext} \langle G^{(m)}_{ext}, \lambda^{\otimes m}\rangle_{ext} = \langle F^{(n)}_{ext} \diamond G^{(m)}_{ext}, \lambda^{\otimes n+m}\rangle_{ext}$.

Proposition 4 ([22]). *For $F, G \in (\mathcal{H}_{-\tau})$*

$$F\Diamond G = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \sum_{k=0}^{m} F^{(k)}_{ext} \diamond G^{(m-k)}_{ext}\rangle :, \tag{29}$$

where $F^{(k)}_{ext} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$, $G^{(m-k)}_{ext} \in \mathcal{H}^{(m-k)}_{-\tau,\mathbb{C}}$, are the kernels from decompositions (12) for F and G respectively. In particular, for $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $G^{(m)}_{ext} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}}$

$$:\langle \circ^{\otimes n}, F^{(n)}_{ext}\rangle :\Diamond: \langle \circ^{\otimes m}, G^{(m)}_{ext}\rangle : = :\langle \circ^{\otimes n+m}, F^{(n)}_{ext} \diamond G^{(m)}_{ext}\rangle :. \tag{30}$$

Remark 6. *It is relevant to note that the multiplication $\diamond$ is an extension of an analog of the symmetric tensor multiplication on the spaces $\mathcal{H}_{ext}^{(m)}$, $m \in \mathbb{Z}_+$ [22, 24]. Using this fact, one can show that the Wick products and the Wick versions of holomorphic functions, introduced on the spaces of regular and nonregular generalized functions (see [11] and [22] respectively), coincide on the intersections of the mentioned spaces. The interested reader can find a detailed information in [22].*

2 MAIN RESULTS AND EXAMPLES

2.1 The Wick multiplication under the sign of an integral

As is known, some properties of an extended stochastic integral differ from habitual properties of the Lebesgue integral. In particular, for $F \in (\mathcal{H}_{-\tau})$ and $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$\int_{\mathbb{R}_+} (F \otimes H^{(1)})(u)\widehat{d}L_u \equiv \int_{\mathbb{R}_+} F \cdot H^{(1)}(u)\widehat{d}L_u \neq F \cdot \int_{\mathbb{R}_+} H^{(1)}(u)\widehat{d}L_u,$$

generally speaking, although F does not depend on u. Moreover, in general, the pointwise product $F \cdot \int_{\mathbb{R}_+} H^{(1)}(u)\widehat{d}L_u$ is undefined. Note that these facts are not directly related with pecularities of the Lévy analysis, and hold true even in the classical Gaussian analysis.

But if one uses the Wick multiplication instead of the pointwise multiplication, it becomes possible to take a time-independent multiplier out of the sign of the extended stochastic integral, as in the Lebesgue integration theory (again, this statement holds true in the Gaussian analysis, in the same way as in the Lévy analysis on the spaces of *regular* generalized functions [12]). In this subsection we'll explain this in detail.

We begin with a preparation. Let $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $G_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$, $n, m \in \mathbb{Z}_+$. Using the notation of the previous section, define

$$F_{ext}^{(n)} \overline{\diamond} G_{ext,\cdot}^{(m)} := (U_{n+m}^{-1} \otimes \mathbf{1})\{(Pr \otimes \mathbf{1})[(U_n F_{ext}^{(n)}) \otimes ((U_m \otimes \mathbf{1})G_{ext,\cdot}^{(m)})]\} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}. \tag{31}$$

Remark 7. *Let $n, m \in \mathbb{Z}_+$, $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $G_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ and $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$. By* (31) *and* (28)

$$F_{ext}^{(n)} \overline{\diamond} (G_{ext}^{(m)} \otimes H^{(1)}) = (F_{ext}^{(n)} \diamond G_{ext}^{(m)}) \otimes H^{(1)} \tag{32}$$

(cf. [24]).

It is easy to estimate the norm of $F_{ext}^{(n)} \overline{\diamond} G_{ext,\cdot}^{(m)}$ in the space $\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}$: since operators $U_m : \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \to \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m}$, $m \in \mathbb{Z}_+$, are isometric isomorphisms (see Subsection 1.4), by (31) we obtain

$$\begin{aligned} |F_{ext}^{(n)} \overline{\diamond} G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}} &= |(Pr \otimes \mathbf{1})[(U_n F_{ext}^{(n)}) \otimes ((U_m \otimes \mathbf{1})G_{ext,\cdot}^{(m)})]|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\mathbb{C}}} \\ &\leq |U_n F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}} |(U_m \otimes \mathbf{1})G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\mathbb{C}}} = |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}} |G_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}. \end{aligned} \tag{33}$$

Definition 5. *Let $F \in (\mathcal{H}_{-\tau})$ and $G \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$. We define a Wick product $F \overline{\Diamond} G \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$, setting*

$$F \overline{\Diamond} G \equiv (F \overline{\Diamond} G)(\cdot) := \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \sum_{k=0}^{m} F_{ext}^{(k)} \overline{\diamond} G_{ext,\cdot}^{(m-k)} \rangle :, \tag{34}$$

where $F^{(k)}_{ext} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$ *and* $G^{(m-k)}_{ext,\cdot} \in \mathcal{H}^{(m-k)}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ *are the kernels from decompositions* (12) *and* (17) *for* F *and* G *respectively (cf.* (29)*).*

Using estimate (33) one can prove by analogy with [22] that this definition is well-posed, and the Wick multiplication $\overline{\Diamond}$ is continuous in the sense that for any $q \in \mathbb{N}$ such that $F \in (\mathcal{H}_{-\tau})_{-(q-1)}$ and $G \in (\mathcal{H}_{-\tau})_{-(q-1)} \otimes \mathcal{H}_{\mathbb{C}}$,

$$\|F\overline{\Diamond}G\|_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}} \leq \|F\|_{(\mathcal{H}_{-\tau})_{-(q-1)}}\|G\|_{(\mathcal{H}_{-\tau})_{-(q-1)}\otimes\mathcal{H}_{\mathbb{C}}}.$$

Remark 8. *Let* $F, G \in (\mathcal{H}_{-\tau})$ *and* $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$*. Using* (34)*,* (32) *and* (29)*, one can easily show that*

$$F\overline{\Diamond}(G \otimes H^{(1)}) = (F\Diamond G) \otimes H^{(1)}. \tag{35}$$

Theorem 3. *Let* $\Delta \in \mathcal{B}(\mathbb{R}_+)$*,* $F \in (\mathcal{H}_{-\tau})$ *and* $G \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$*. Then*

$$\int_\Delta F\overline{\Diamond}G(u)\widehat{d}L_u \equiv \int_\Delta (F\overline{\Diamond}G)(u)\widehat{d}L_u = F\Diamond \int_\Delta G(u)\widehat{d}L_u \in (\mathcal{H}_{-\tau}). \tag{36}$$

Remark 9. *It is possible to interpret* G *as a function acting from* $\mathbb{R}_+$ *to* $(\mathcal{H}_{-\tau})$ *and, taking into account the construction of the Wick multiplications* $\Diamond$ *and* $\overline{\Diamond}$*, rewrite equality* (36) *in a classical form* $\int_\Delta F\Diamond G(u)\widehat{d}L_u = F\Diamond \int_\Delta G(u)\widehat{d}L_u$*.*

Proof. It is sufficient to consider the case $\Delta = \mathbb{R}_+$ only: if $\Delta \neq \mathbb{R}_+$, it is necessary to substitute $G(\cdot)1_\Delta(\cdot)$ instead of G.

Let at first $F = :\langle \circ^{\otimes n}, F^{(n)}_{ext}\rangle:$, $G(\cdot) = :\langle \circ^{\otimes m}, G^{(m)}_{ext,\cdot}\rangle:$, $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$, $G^{(m)}_{ext,\cdot} \in \mathcal{H}^{(m)}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$, $n, m \in \mathbb{Z}_+$. By (34) we have $(F\overline{\Diamond}G)(\cdot) = :\langle \circ^{\otimes n+m}, F^{(n)}_{ext}\overline{\diamond}G^{(m)}_{ext,\cdot}\rangle:$, hence $\int_{\mathbb{R}_+}(F\overline{\Diamond}G)(u)\widehat{d}L_u = :\langle \circ^{\otimes n+m+1}, \widehat{F^{(n)}_{ext}\overline{\diamond}G^{(m)}_{ext,\mathbb{R}_+}}\rangle:$ (see (21), (22)). On the other hand, by (21)

$$\int_{\mathbb{R}_+} G(u)\widehat{d}L_u = :\langle \circ^{\otimes m+1}, \widehat{G}^{(m)}_{ext,\mathbb{R}_+}\rangle:,$$

therefore $F\Diamond \int_{\mathbb{R}_+} G(u)\widehat{d}L_u = :\langle \circ^{\otimes n+m+1}, F^{(n)}_{ext} \diamond \widehat{G}^{(m)}_{ext,\mathbb{R}_+}\rangle:$ (see (30)). So, we have to prove that

$$\widehat{F^{(n)}_{ext}\overline{\diamond}G^{(m)}_{ext,\mathbb{R}_+}} = F^{(n)}_{ext} \diamond \widehat{G}^{(m)}_{ext,\mathbb{R}_+} \tag{37}$$

in $\mathcal{H}^{(n+m+1)}_{-\tau,\mathbb{C}}$.

Using (22) and (31) we obtain

$$\begin{aligned}
\widehat{F^{(n)}_{ext}\overline{\diamond}G^{(m)}_{ext,\mathbb{R}_+}} &= U^{-1}_{n+m+1}\{Pr[(U_{n+m} \otimes \mathbf{1})(F^{(n)}_{ext}\overline{\diamond}G^{(m)}_{ext,\cdot})]\} \\
&= U^{-1}_{n+m+1}\big\{Pr\big[(U_{n+m} \otimes \mathbf{1})(U^{-1}_{n+m} \otimes \mathbf{1})\{(Pr \otimes \mathbf{1})[(U_nF^{(n)}_{ext}) \otimes ((U_m \otimes \mathbf{1})G^{(m)}_{ext,\cdot})]\}\big]\big\} \\
&= U^{-1}_{n+m+1}\{Pr[(U_nF^{(n)}_{ext}) \otimes ((U_m \otimes \mathbf{1})G^{(m)}_{ext,\cdot})]\},
\end{aligned}$$

whereas by (28) and (22)

$$\begin{aligned}
F^{(n)}_{ext} \diamond \widehat{G}^{(m)}_{ext,\mathbb{R}_+} &= U^{-1}_{n+m+1}\{Pr[(U_nF^{(n)}_{ext}) \otimes (U_{m+1}\widehat{G}^{(m)}_{ext,\mathbb{R}_+})]\} \\
&= U^{-1}_{n+m+1}\big\{Pr\big[(U_nF^{(n)}_{ext}) \otimes (U_{m+1}U^{-1}_{m+1}\{Pr[(U_m \otimes \mathbf{1})G^{(m)}_{ext,\cdot}]\})\big]\big\} \\
&= U^{-1}_{n+m+1}\{Pr[(U_nF^{(n)}_{ext}) \otimes ((U_m \otimes \mathbf{1})G^{(m)}_{ext,\cdot})]\}.
\end{aligned}$$

Therefore equality (37) is true, hence in our special case the theorem is proved. In the general case the statement follows from the just obtained result, continuity of the Wick multiplications $\overline{\Diamond}$ and $\Diamond$, and continuity of operator of stochastic integration (24). □

Now let us obtain an analog of property (36) for a so-called Pettis integral (i.e., for a weak integral) on the spaces of nonregular generalized functions. Denote by ρ the Lebesgue measure on $\mathbb{R}_+$ and consider $\Delta \in \mathcal{B}(\mathbb{R}_+)$ such that $\rho(\Delta) < \infty$. For any $G \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$ define the Pettis integral $\int_\Delta G(u)du \in (\mathcal{H}_{-\tau})$ as a unique element of $(\mathcal{H}_{-\tau})$ such that for each $f \in (\mathcal{H}_\tau)$

$$\langle\langle \int_\Delta G(u)du, f\rangle\rangle_{(L^2)} = \langle\langle G(\cdot), f \otimes 1_\Delta(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}. \tag{38}$$

Since by the generalized Cauchy-Bunyakovsky inequality for each $q \in \mathbb{N}_{q_0(\tau)}$ (see Proposition 2)

$$|\langle\langle G(\cdot), f \otimes 1_\Delta(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}| \leq \|G\|_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{\mathbb{C}}} \|f\|_{(\mathcal{H}_\tau)_q}\sqrt{\rho(\Delta)},$$

this definition is well-posed and the Pettis integral

$$\int_\Delta \circ(u)du : (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}} \to (\mathcal{H}_{-\tau}) \tag{39}$$

is a linear *continuous* operator.

Let $G \in (\mathcal{H}_{-\tau})$, $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$. Then

$$\int_\Delta (G \otimes H^{(1)})(u)du \equiv \int_\Delta G \cdot H^{(1)}(u)du = G \cdot \int_\Delta H^{(1)}(u)du. \tag{40}$$

In fact, for each $f \in (\mathcal{H}_\tau)$ by (38) we have

$$\begin{aligned}\langle\langle \int_\Delta G \cdot H^{(1)}(u)du, f\rangle\rangle_{(L^2)} &= \langle\langle G \otimes H^{(1)}(\cdot), f \otimes 1_\Delta(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \\ &= \langle\langle G, f\rangle\rangle_{(L^2)} \int_\Delta H^{(1)}(u)du = \langle\langle G \cdot \int_\Delta H^{(1)}(u)du, f\rangle\rangle_{(L^2)}.\end{aligned}$$

Let now $F, G \in (\mathcal{H}_{-\tau})$ and $H^{(1)} \in \mathcal{H}_{\mathbb{C}}$. Using (35) and (40) we obtain

$$\begin{aligned}\int_\Delta F\overline{\Diamond}\big((G \otimes H^{(1)})(u)\big)du &\equiv \int_\Delta \big(F\overline{\Diamond}(G \otimes H^{(1)})\big)(u)du = \int_\Delta \big((F\Diamond G) \otimes H^{(1)}\big)(u)du \\ &\equiv \int_\Delta (F\Diamond G) \cdot H^{(1)}(u)du = (F\Diamond G) \cdot \int_\Delta H^{(1)}(u)du \\ &= F\Diamond\big(G \cdot \int_\Delta H^{(1)}(u)du\big) = F\Diamond \int_\Delta G \cdot H^{(1)}(u)du \\ &\equiv F\Diamond \int_\Delta (G \otimes H^{(1)})(u)du.\end{aligned}$$

From here, by virtue of continuity of the Wick multiplications $\overline{\Diamond}$ and $\Diamond$, and continuity of Pettis integral (39), we obtain the following statement (cf. Theorem 3).

Theorem 4. *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) < \infty$, $F \in (\mathcal{H}_{-\tau})$ and $G \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$. Then*

$$\int_\Delta F\overline{\Diamond}G(u)du \equiv \int_\Delta (F\overline{\Diamond}G)(u)du = F\Diamond \int_\Delta G(u)du \in (\mathcal{H}_{-\tau}). \tag{41}$$

Remark 10. *As in the case of the extended stochastic integral, now one can interpret G as a function acting from $\mathbb{R}_+$ to $(\mathcal{H}_{-\tau})$, and rewrite equality* (41) *in a form $\int_\Delta F\Diamond G(u)du = F\Diamond \int_\Delta G(u)du$.*

2.2 A representation of the extended stochastic integral via the Pettis integral

It is well known that in the Gaussian analysis the extended stochastic integral on spaces of generalized functions can be presented as a Pettis integral:

$$\int_\Delta F(u)\widehat{d}W_u = \int_\Delta F(u)\Diamond \dot{W}_u du, \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{42}$$

Here W is a Wiener process, $\dot{W}$ is a Gaussian white noise. Depending on the spaces under consideration, equality (42) can be formal or can have a rigorous sense. In any case this equality is very useful for applications, in particular, for studying stochastic equations with Wick type nonlinearities.

Remark 11. *In a sense, equality* (42) *is an analog of a formula for replacement of a measure in the Lebesgue integration theory. In particular, $\dot{W}$ is an analog of a Radon-Nikodym derivative.*

In the Lévy analysis representation (42) for the extended stochastic integral holds true up to obvious modifications: it is necessary to substitute a Lévy process and a Lévy white noise instead of a Wiener process and a Gaussian white noise respectively. Now on the spaces of *regular* generalized functions the analog of (42) is a formal equality [12]; in the *nonregular* case the corresponding analog is a rigorous equality. Let us explain this in detail.

As we saw in Subsection 1.1, a Lévy white noise can be presented in a form $\dot{L}_u = \langle \circ, \delta_u\rangle$, $u \in \mathbb{R}_+$. As is well known (e.g., [3]), for each u the Dirac delta-function $\delta_u \in \mathcal{H}_{-\tau}$, therefore $\dot{L}_u = \langle \circ, \delta_u\rangle = {:}\langle \circ, \delta_u\rangle{:} \in (\mathcal{H}_{-\tau})$. Let $F \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$. In this subsection it will be convenient to interpret F as a function acting from $\mathbb{R}_+$ to $(\mathcal{H}_{-\tau})$, so, for ρ-almost all $u \in \mathbb{R}_+$ the Wick product $F(u)\Diamond \dot{L}_u$ is a well-defined element of $(\mathcal{H}_{-\tau})$ (remind that ρ is the Lebesgue measure on $\mathbb{R}_+$). For arbitrary $\Delta \in \mathcal{B}(\mathbb{R}_+)$ let us define the Pettis integral $\int_\Delta F(u)\Diamond \dot{L}_u du$ as a unique element of $(\mathcal{H}_{-\tau})$ such that for each $f \in (\mathcal{H}_\tau)$

$$\langle\!\langle \int_\Delta F(u)\Diamond \dot{L}_u du, f\rangle\!\rangle_{(L^2)} = \int_\Delta \langle\!\langle F(u)\Diamond \dot{L}_u, f\rangle\!\rangle_{(L^2)} du \tag{43}$$

(cf. (38)). Since it is possible now that $\rho(\Delta) = \infty$, we cannot use the reasoning from Subsection 2.1 and have to prove the correctness of this definition (simultaneously we'll obtain an analog of (42)). It is sufficient to consider the case $\Delta = \mathbb{R}_+$: if $\Delta \neq \mathbb{R}_+$, one has to substitute $F(\cdot)1_\Delta(\cdot)$ instead of F. By (29), (28) and Remark 3 for ρ-almost all $u \in \mathbb{R}_+$

$$\begin{aligned} F(u)\Diamond \dot{L}_u = F(u)\Diamond{:}\langle \circ, \delta_u\rangle{:} &= \sum_{m=0}^{\infty} {:}\langle \circ^{\otimes m+1}, F_{ext,u}^{(m)} \diamond \delta_u\rangle{:} \\ &= \sum_{m=0}^{\infty} {:}\langle \circ^{\otimes m+1}, U_{m+1}^{-1}\{Pr[(U_m F_{ext,u}^{(m)}) \otimes \delta_u]\}\rangle{:}, \end{aligned}$$

therefore by (8), (14) and (20) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}_+} \langle\!\langle F(u)\Diamond \dot{L}_u, f\rangle\!\rangle_{(L^2)} du \\
&\qquad = \int_{\mathbb{R}_+} \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, U_{m+1}^{-1}\{Pr[(U_m F_{ext,u}^{(m)}) \otimes \delta_u]\}\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :\rangle\!\rangle_{(L^2)} du \\
&\qquad = \int_{\mathbb{R}_+} \sum_{m=0}^{\infty} (m+1)! \langle U_{m+1}^{-1}\{Pr[(U_m F_{ext,u}^{(m)}) \otimes \delta_u]\}, f^{(m+1)}\rangle_{ext} du \\
&\qquad = \int_{\mathbb{R}_+} \sum_{m=0}^{\infty} (m+1)! \langle (U_m F_{ext,u}^{(m)}) \otimes \delta_u, f^{(m+1)}\rangle du \\
&\qquad = \int_{\mathbb{R}_+} \sum_{m=0}^{\infty} (m+1)! \langle U_m F_{ext,u}^{(m)}, f^{(m+1)}(\cdot_1, \dots \cdot_m, u)\rangle du \\
&\qquad = \sum_{m=0}^{\infty} (m+1)! \int_{\mathbb{R}_+} \langle U_m F_{ext,u}^{(m)}, f^{(m+1)}(\cdot_1, \dots \cdot_m, u)\rangle du \\
&\qquad = \sum_{m=0}^{\infty} (m+1)! \langle (U_m \otimes \mathbf{1}) F_{ext,\cdot}^{(m)}, f^{(m+1)}\rangle .
\end{aligned}
\tag{44}
$$

Note that the penultimate equality in (44) is valid because, as is easy to verify,

$$\int_{\mathbb{R}_+} \sum_{m=0}^{\infty} (m+1)! |\langle U_m F_{ext,u}^{(m)}, f^{(m+1)}(\cdot_1, \dots \cdot_m, u)\rangle| du < \infty .$$

On the other hand, by (21), (22), (8), (14) and (20) we obtain

$$
\begin{aligned}
&\langle\!\langle \int_{\mathbb{R}_+} F(u) \widehat{d} L_u, f\rangle\!\rangle_{(L^2)} \\
&\qquad = \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, U_{m+1}^{-1}\{Pr[(U_m \otimes \mathbf{1}) F_{ext,\cdot}^{(m)}]\}\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :\rangle\!\rangle_{(L^2)} \\
&\qquad = \sum_{m=0}^{\infty} (m+1)! \langle U_{m+1}^{-1}\{Pr[(U_m \otimes \mathbf{1}) F_{ext,\cdot}^{(m)}]\}, f^{(m+1)}\rangle_{ext} \\
&\qquad = \sum_{m=0}^{\infty} (m+1)! \langle (U_m \otimes \mathbf{1}) F_{ext,\cdot}^{(m)}, f^{(m+1)}\rangle .
\end{aligned}
\tag{45}
$$

Comparing (44) and (45) we conclude that for all $f \in (\mathcal{H}_\tau)$ and $\Delta \in \mathcal{B}(\mathbb{R}_+)$

$$
\begin{aligned}
\int_{\Delta} \langle\!\langle F(u)\Diamond \dot{L}_u, f\rangle\!\rangle_{(L^2)} du &= \int_{\mathbb{R}_+} \langle\!\langle \big(F(u) 1_\Delta(u)\big)\Diamond \dot{L}_u, f\rangle\!\rangle_{(L^2)} du \\
&= \langle\!\langle \int_{\mathbb{R}_+} F(u) 1_\Delta(u) \widehat{d} L_u, f\rangle\!\rangle_{(L^2)} = \langle\!\langle \int_{\Delta} F(u) \widehat{d} L_u, f\rangle\!\rangle_{(L^2)},
\end{aligned}
$$

therefore by (43) $\int_\Delta F(u)\Diamond \dot{L}_u du$ is a well-defined element of $(\mathcal{H}_{-\tau})$ and, moreover, we have the following statement.

Theorem 5. *For arbitrary $\Delta \in \mathcal{B}(\mathbb{R}_+)$ and $F \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$*

$$\int_{\Delta} F(u) \widehat{d} L_u = \int_{\Delta} F(u) \Diamond \dot{L}_u du. \tag{46}$$

Remark 12. *The extended stochastic integral can be defined by formulas* (21), (22) *with* $\Delta = \mathbb{R}_+$ *as a linear continuous operator acting from* $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ *to* $(\mathcal{H}_{-\tau})$, *cf. [23] (now it is impossible to define the integral by a set* $\Delta \neq \mathbb{R}_+$ *because a multiplication of an element of* $\mathcal{H}^{(m)}_{-\tau,\mathbb{C}} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ *or* $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ *by* 1_Δ *is undefined, cf.* (22), (25)*). It is easy to show that formulas* (36) *and* (46) *(with* $\Delta = \mathbb{R}_+$*) hold true for this integral.*

Finally we note that all results of Subsections 2.1 and 2.2 hold true for integrands and measurable sets Δ, satisfying the assumptions of Remark 5.

2.3 Examples

In order to illustrate possible applications of our results, we consider some stochastic equations with Wick type nonlinearities.

Example 1. *(a linear equation) Let us consider an integral stochastic equation*

$$X_t = X_0 + \int_0^t F\overline{\Diamond} X_u du + \int_0^t G\overline{\Diamond} X_u \widehat{d}L_u, \tag{47}$$

where $X_0, F, G \in (\mathcal{H}_{-\tau})$ *(we use here the classical notation* $\int_0^t \equiv \int_{[0,t)}$*). Applying to this equation the* S*-transform with regard to* (41), (36) *and* (46), *and solving the obtained nonstochastic equation, we get*

$$SX_t = SX_0 \cdot \exp\Big\{SFt + SG\int_0^t \lambda(u)du\Big\}.$$

Applying to this equality the inverse S*-transform, we obtain the solution of* (47)

$$X_t = X_0 \Diamond \exp^{\Diamond}\{Ft + G\Diamond L_t\} \in (\mathcal{H}_{-\tau}).$$

Example 2. *(a Verhulst type equation) Consider an integral stochastic equation*

$$X_t = X_0 + r\int_0^t X_u \Diamond (N - X_u)du + v\int_0^t X_u \Diamond (N - X_u)\widehat{d}L_u, \tag{48}$$

where $X_0 \in (\mathcal{H}_{-\tau})$, $N, r, v \in \mathbb{R}$, $N > 0$, $r > 0$, $(SX_0)(0) > 0$. *Here for* ρ*-almost all* $u \in \mathbb{R}_+$ *we interpret* X_u *as a generalized function, it follows from the solution of* (48) *(see below) that* $X_u \in (\mathcal{H}_{-\tau})$ *and all integrals in* (48) *are well defined. As in the previous example, applying to* (48) *the* S*-transform with regard to* (46), *solving the obtained equation, and applying the inverse* S*-transform, we get the solution*

$$X_t = N\Big[1 + (NX_0^{\Diamond(-1)} - 1)\Diamond \exp^{\Diamond}\big\{-N(rt + vL_t)\big\}\Big]^{\Diamond(-1)} \in (\mathcal{H}_{-\tau}),$$

where $Y^{\Diamond(-1)} := S^{-1}\big(\frac{1}{SY}\big)$.

Remark 13. *It is very easy to show that all results of this paper hold true (up to obvious modifications) if we consider the spaces* $(\mathcal{D}')$ *and* $(\mathcal{D}') \otimes \mathcal{H}_{\mathbb{C}}$ *instead of the spaces* $(\mathcal{H}_{-\tau})$ *and* $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$ *respectively.*

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Качановський М.О., Качановська Т.О. *Взаємозв'язок між віківським множенням та інтегруванням на просторах нерегулярних узагальнених функцій в аналізі білого шуму Леві* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 70–88.

Ми маємо справу з просторами нерегулярних узагальнених функцій в аналізі білого шуму Леві, які побудовані з використанням литвинівського узагальнення властивості хаотичного розкладу. Наша мета — описати взаємовідносини між віківським множенням та інтегруванням на цих просторах. Точніше, ми показуємо, що, використовуючи віківське множення, можна виносити незалежний від часу множник за знак розширеного стохастичного інтегралу; встановлюємо аналог цього результату для інтегралу Петтіса (слабкого інтегралу); та доводимо теорему про представлення розширеного стохастичного інтегралу через інтеграл Петтіса від віківського добутку вихідної підінтегральної функції на білий шум Леві. Як приклади застосування наших результатів ми розглядаємо деякі стохастичні рівняння з нелінійностями віківського типу.

Ключові слова і фрази: Процес Леві, розширений стохастичний інтеграл, інтеграл Петтіса, віківський добуток.

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ALGEBRAIC BASIS OF THE ALGEBRA OF BLOCK-SYMMETRIC POLYNOMIALS ON $\ell_1 \oplus \ell_\infty$

We concider so called block-symmetric polynomials on sequence spaces $\ell_1 \oplus \ell_\infty, \ell_1 \oplus c, \ell_1 \oplus c_0$, that is, polynomials which are symmetric with respect to permutations of elements of the sequences. It is proved that every continuous block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$ can be uniquelly represented as an algebraic combination of some special block-symmetric polynomials, which form an algebraic basis. It is interesting to note that the algebra of block-symmetric polynomials is infinite-generated while ℓ_∞ admits no symmetric polynomials. Algebraic bases of the algebras of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$ and $\ell_1 \oplus c_0$ are described.

Key words and phrases: symmetric polynomials, block-symmetric polynomials, algebraic basis, topological algebra.

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1 INRODUCTION

Algebras of polynomials and analytic functions on a Banach space X which are invariant with respect to a group or semigroup of linear operators acting on X were studied by many authors (see e.g. [1, 3–5, 9]). In oder to study spectra of such algebras it is impotant to figure out with their algebraic bases (if exist). Let S_∞ be the group of all permutations of the set of natural numbers $\mathbb{N}$. That is, S_∞ consists of all bijections of $\mathbb{N}$ to itself. Let S^0_∞ be the subgroup in S_∞ of all finite permutations. If X is a sequence Banach space and for each $x = (x_1, x_2, \ldots, x_n, \ldots) \in X$, $\sigma(x)\colon = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots) \in X$, $\sigma \in S_\infty$, then we can concider functions which are invariants with respect to the operators $\sigma(x)$. A function $f : X \to \mathbb{C}$ is called *symetric* if $f(\sigma(x)) = f(x)$ for every $x \in X$ and $\sigma \in S_\infty$. If it is true for all $\sigma \in S^0_\infty$ then f is called *finitely symmetric*. In [11] Nemirovskii and Semenov described algebraic bases of algebra of continuous symmetric polynomials on real spaces ℓ_p, where $1 \le p < \infty$. Their results were generalized by Gonzalez et al. [7] for real separable rearrangement-invariant sequence spaces. Also, in [7] it is proved that for $\ell_p, 1 \le p < \infty$, finitely symmetric polynomials are symmetric and c_0 does not admit finitely symmetric polynomials. In [8] it is proved that there are no symmetric polynomials on ℓ_∞ but we have a lot of finitely symmetric polynomials. It is not difficult to check that every symmetric (and finitely symmetric) polynomial on c can be generated by the following one

$$L(x) = \lim_{n\to\infty} x_n.$$

In [9, 10] were concidered *block-symmetric polynomials*, wich also are called MacMahon Polynomials on Banach spaces. The block-symmetric polynomials can be defined by the following

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way. Let $X_1, \dots, X_m$ be sequence spaces. Then every $x \in X_1 \times \dots \times X_m$ can be represented by $x = (x^1, \dots, x^m)$, where $x^j \in X_j$. For any $\sigma \in S_\infty$ we can define $\sigma(x) = (\sigma(x^1), \dots, \sigma(x^m))$ and a polynomial $P : X_1 \times \dots X_m$ is block-symmetric if $P(\sigma(x)) = P(x)$ for every $\sigma \in S_\infty$. In [10] algebra of block-symetric analytic functions on $\ell_1 \times \ell_1$ is investigated. In [9] constructed an algebraic basis of block-symmetric polynomials on $\underbrace{\ell_p \times \dots \times \ell_p}_{n} \simeq \ell_p(\mathbb{C}^n)$. In this paper we construct an algebraic basis on the algebra of all block-symmetric polynomials on $\ell_1 \times \ell_\infty$. It is interesting to note that the algebra of block-symmetric polynomials is infinite-generated while ℓ_∞ admits no symmetric polynomials. Also, we concider block-symmetric polynomials on $\ell_1 \times c_0$ and $\ell_1 \times c$.

2 MAIN RESULTS

Let us denote by $\ell_1 \oplus \ell_\infty$ the space with elements $\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \dots \right)$, where $(x_1, x_2, \dots, x_n, \dots) \in \ell_1$, $(y_1, y_2, \dots, y_n, \dots) \in \ell_\infty$. The space $\ell_1 \oplus \ell_\infty$ with norm

$$||(x,y)||_{\ell_1 \oplus \ell_\infty} = \sum_{i=1}^{\infty} |x_i| + \sup_{i \geq 1} |y_i|$$

is a Banach space.

A polynomial P on the space $\ell_1 \oplus \ell_\infty$ is called block-symmetric (or vector-symmetric) if

$$P\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \dots \right) = P\left(\begin{pmatrix} x_{\sigma(1)} \\ y_{\sigma(1)} \end{pmatrix}, \dots, \begin{pmatrix} x_{\sigma(m)} \\ y_{\sigma(m)} \end{pmatrix}, \dots \right),$$

for every permutation σ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{C}^2$.

Let us denote by $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ the algebra of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$; by $\mathcal{H}_{bvs}(\ell_1 \oplus \ell_\infty)$ the algebra of block-symmetric analytic functions of bounded type on $\ell_1 \oplus \ell_\infty$.

In [9] it was proved that polynomials $H^{k_1,\dots,k_n}(x) = \sum\limits_{j=1}^{\infty} \prod\limits_{\substack{s=1 \\ k_s>0}}^{n} (x_j^s)^{k_s}$, where $x = (x_1, x_2, \dots) \in \ell_1(\mathbb{C}^n)$, $x_j = (x_j^1, \dots, x_j^n) \in \mathbb{C}^n$ form an algebraic basis of the algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$.

For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ let $|k| = k_1 + k_2 + \dots + k_n$. For an arbitrary nonempty finite set $M \in \mathbb{Z}_+^n$ let us define a mapping $\pi_M : c_{00}(\mathbb{C}^n) \to \mathbb{C}^{|M|}$, where $|M|$ is the cardinality of M, by

$$\pi_M(x) = (H^{k_1,\dots,k_n}(x))_{(k_1,\dots,k_n) \in M}.$$

In [9] it was proved the following theorem.

Theorem 1 ([9]). *Let M be a finite nonempty subset of $\mathbb{Z}_+^n$ such that $|k| \geq 1$ for every $k \in M$.*

1. *There exists $m \in \mathbb{N}$, such that for every $\xi = (\xi_{(k_1,\dots,k_n)})_{(k_1,\dots,k_n) \in M} \in \mathbb{C}^{|M|}$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi) = \xi$, where $c_{00}^{(m)}(\mathbb{C}^n)$ is the space of all sequences $x = (x_1, \dots, x_m, 0, \dots)$, $x_1, \dots, x_m \in \mathbb{C}^n$;*

2. *There exists a constant $\rho_M > 0$ such that if $||\xi||_\infty < 1$, then $||x_\xi||_p \leq \rho_M$ for every $p \in [1, +\infty)$, where $||\xi||_\infty = \max\limits_{k \in M} |\xi_k|$.*

Let us denote by $(\ell_1 \oplus \ell_\infty)^{(m)}$ the space of all sequences

$$\begin{pmatrix} x \\ y \end{pmatrix}_m = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots \right),$$

where $(x_1, \ldots, x_m, 0 \ldots) \in \ell_1$, $(y_1, \ldots, y_m, 0 \ldots) \in \ell_\infty$. Clearly, that $c_{00}^{(m)}(\mathbb{C}^n) = (\ell_1 \oplus \ell_\infty)^{(m)}$.

For an arbitrary nonempty finite set $M \in \mathbb{Z}_+^2$ let us define a mapping $\pi_M : \ell_1 \oplus \ell_\infty \longrightarrow \mathbb{C}^{|M|}$ by

$$\pi_M((x,y)) = (H^{k_1,k_2}(x,y))_{(k_1,k_2) \in M}.$$

Corollary 1. *Let M be a finite nonempty subset of $\mathbb{Z}_+^2$ such that $k_1 + k_2 \geq 1$ for every $(k_1, k_2) \in M$.*

1. *There exists $m \in \mathbb{N}$, such that for every $\xi = (\xi_{(k_1,k_2)})_{(k_1,k_2) \in M} \in \mathbb{C}^{|M|}$ there exists $(x,y)_\xi \in (\ell_1 \oplus \ell_\infty)^{(m)}$ such that $\pi_M((x,y)_\xi) = \xi$;*
2. *There exists a constant $\rho_M > 0$ such that if $||\xi||_\infty < 1$, then $||(x,y)_\xi||_{\ell_1 \oplus \ell_\infty} \leq \rho_M$.*

For elements $\begin{pmatrix} x \\ y \end{pmatrix}_m, \begin{pmatrix} z \\ t \end{pmatrix}_m \in \ell_1 \oplus \ell_\infty$, let

$$\begin{pmatrix} x \\ y \end{pmatrix}_m \oplus \begin{pmatrix} z \\ t \end{pmatrix}_m = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots \right).$$

For $(x,y)^1, (x,y)^2, \ldots, (x,y)^r \in \ell_1 \oplus \ell_\infty$, let

$$\bigoplus_{j=1}^{r} (x,y)^j = (x,y)^1 \oplus (x,y)^2 \oplus \ldots \oplus (x,y)^r.$$

Obviously that

$$\left|\left| \bigoplus_{j=1}^{r} (x,y)^j \right|\right|_{\ell_1 \oplus \ell_\infty} \leq \sum_{j=1}^{r} \left|\left| (x,y)^j \right|\right|_{\ell_1 \oplus \ell_\infty}.$$

Also note that for every $(k_1, k_2) \in \mathbb{Z}_+^2$, such that $k_1 + k_2 \geq 1$,

$$H^{k_1,k_2} \left(\bigoplus_{j=1}^{r} (x,y)^j \right) = \sum_{j=1}^{r} H^{k_1,k_2}((x,y)^j). \tag{1}$$

For $N \in \mathbb{N}$ let M_N be a finite nonempty subset $\mathbb{Z}_+^2$ such that $1 \leq k_1 + k_2 \leq N$ for every $(k_1, k_2) \in M_N$.

By Corollary 1, for $M = M_N$ there exists $\rho = \rho_M$, such that $\pi_{M_N}(V_\rho)$ contains the open unit ball of the space $\mathbb{C}^{|M|}$ with norm $||\xi||_\infty$, where

$$V_\rho = \left\{ (x,y) \in \ell_1 \oplus \ell_\infty : ||(x,y)||_{\ell_1 \oplus \ell_\infty} \leq \rho \right\}.$$

Proposition 1. *Let $q(\xi_{(l_1,l_2)})_{(l_1,l_2) \in M_N}$ be a polynomial on $\mathbb{C}^{|M_N|}$. If q is bounded on $\pi_M(V_\rho)$, then q does not depend on $\xi_{(0,k)}$, $k \in \mathbb{N}$.*

Proof. Let $(0,k) \in \mathbb{Z}_+^2, k \in \mathbb{N}$. Let $K = \pi_{M_N}(V_\rho)$, $K_1 = \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$ and $\eta : K \to K_1$ be an orthogonal projection, defined by

$$\eta : (\xi_{(l_1,l_2)})_{(l_1,l_2)\in M_N} \mapsto (\xi_{(l_1,l_2)})_{(l_1,l_2)\in M_N\setminus\{(0,k)\}}.$$

Let us show that for every ball

$$B(u,r) = \left\{\xi \in \mathbb{C}^{|M_N\setminus\{(0,k)\}|} : ||\xi - u||_\infty < r\right\}$$

centered at $u = (u_{(l_1,l_2)})_{(l_1,l_2)\in M_N\setminus\{(0,k)\}} \in \mathbb{C}^{|M_N\setminus\{(0,k)\}|}$ and of radius $r > 0$ such that $B(u,r) \subset \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$, the set $\eta^{-1}(B(u,r))$ is unbounded. Since $u \in \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$, there exists $(x,y)_u \in V_\rho$ such that $\pi_{M_N}((x,y)_u) = u$ by Corollary 1. For $m \in \mathbb{N}$, we set $(x,y)_m = \oplus_{j=1}^m \left(\frac{1}{j^{\frac{1}{|k|}}}\right) a_k$, where $a_k = \frac{1}{k^{\frac{1}{k}}}\left(\begin{pmatrix} 0 \\ \alpha_{k,0}\end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \alpha_{k,k-1}\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix} \ldots\right)$, $\alpha_{m,j} = (\sqrt[m]{-1})_j$, $0 \le j \le m-1$.

Choose ε such that

$$0 < \varepsilon < \min\left\{1, \frac{\rho - ||(x,y)_u||_{\ell_1\oplus\ell_\infty}}{||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})}, \frac{r}{||a_k||_1^N \zeta(s-1+\frac{1}{k})}\right\},$$

where $\zeta(\cdot)$ is the Riemann zeta-function.

Let $(x,y)_{m,\varepsilon} = (\varepsilon(x,y)_m) \oplus (x,y)_u$. Let us show that $(x,y)_{m,\varepsilon} \in V_\rho$.

$$||(x,y)_m||_{\ell_1\oplus\ell_\infty} = \sum_{j=1}^m \left|\left|\frac{1}{j^{\frac{1}{k}}} a_k\right|\right|_{\ell_1\oplus\ell_\infty} = \sum_{j=1}^m \frac{1}{j^{\frac{1}{k}}}||a_k||_{\ell_1\oplus\ell_\infty} = ||a_k||_{\ell_1\oplus\ell_\infty}\sum_{j=1}^m \frac{1}{j^{\frac{1}{k}}} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k}).$$

Therefore, $||(x,y)_m||_{\ell_1\oplus\ell_\infty} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})$. Then

$$||(x,y)_{m,\varepsilon}||_{\ell_1\oplus\ell_\infty} \le \varepsilon||(x,y)_m||_{\ell_1\oplus\ell_\infty} + ||(x,y)_u||_{\ell_1\oplus\ell_\infty} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k}) + ||(x,y)_u||_{\ell_1\oplus\ell_\infty}.$$

Since $\varepsilon < \frac{\rho - ||(x,y)_u||_{\ell_1\oplus\ell_\infty}}{||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})}$, it follows that $||(x,y)_{m,\varepsilon}||_{\ell_1\oplus\ell_\infty} < \rho$. Hence, $(x,y)_{m,\varepsilon} \in V_\rho$.

Note that for arbitrary $(l_1,l_2) \in \mathbb{Z}_+^2$ such that $l_1 + l_2 \ge 1$, by equality (1),

$$\begin{aligned} H^{l_1,l_2}((x,y)_m) &= \sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}} H^{l_1,l_2}(a_k) = H^{l_1,l_2}(a_k)\sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}}, \\ H^{l_1,l_2}((x,y)_{m,\varepsilon}) &= \varepsilon^{l_1+l_2}H^{l_1,l_2}((x,y)_m) + H^{l_1,l_2}((x,y)_u) \\ &= \varepsilon^{l_1+l_2}H^{l_1,l_2}(a_k)\sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}} + H^{l_1,l_2}((x,y)_u). \end{aligned} \tag{2}$$

Let us show that $\pi_{M_N\setminus\{(0,k)\}}((x,y)_{m,\varepsilon}) \in B(u,r)$. For $(l_1,l_2) \in M_N \setminus \{(0,k)\}$, such that $l_2 \ne 0 \mod k$, $H^{l_1,l_2}(a_k) = 0$ (see [9, Prop. 3]) and therefore, by (2), $H^{l_1,l_2}((x,y)_u) = u_{(l_1,l_2)}$.

Let $(l_1,l_2) \in M_N \setminus \{(0,k)\}$ be such that $l_1 = 0$ and $l_2 \equiv 0 \mod k$. Then $l = l_2 = s\cdot k$, $s \ge 1$, $s \in \mathbb{N}$. Hence

$$\begin{aligned} \left|H^{0,l}((x,y)_{m,\varepsilon}) - u_{(0,l)}\right| < \varepsilon^l|H^{0,l}(a_k)|\sum_{j=1}^m \frac{1}{j^{\frac{l}{k}}} &< \varepsilon^l|H^{0,l}(a_k)|\sum_{j=1}^m \frac{1}{j^{s-1+\frac{1}{k}}} \\ &< \varepsilon^l|H^{0,l}(a_k)|\zeta(s-1+\frac{1}{k}). \end{aligned}$$

Since $||H^{0,l}|| \leq 1$ (see [9, Prop. 2]), $|H^{0,l}(a_k)| \leq ||a_k||_1^l$. Since $\varepsilon < 1$, and $\varepsilon^l < \varepsilon$, so

$$\varepsilon^l |H^{0,l}(a_k)| \zeta(s-1+\frac{1}{k}) < \varepsilon ||a_k||_1^l \zeta(s-1+\frac{1}{k}).$$

From the inequality $\varepsilon < \frac{r}{||a_k||_1^N \zeta(s-1+\frac{1}{k})}$, it follows that $\left| H^{0,l}((x,y)_{m,\varepsilon}) - u_{(0,l)} \right| < r$ and therefore $\pi_{M_N \setminus \{(0,k)\}}((x,y)_{m,\varepsilon}) \in B(u,r)$.

By [9, Prop. 3], $H^{0,k}(a_k) = 1$. Then

$$H^{0,k}((x,y)_{m,\varepsilon}) = \varepsilon^k \sum_{j=1}^{m} \frac{1}{j} + H^{(0,k)}((x,y)_u) \longrightarrow \infty$$

as $m \to \infty$. Hence, $\eta^{-1}(B(u,r))$ is unbounded. By [9, Lemma 11], q does not depend on $\xi_{(0,k)}$. $\square$

Theorem 2. *Polynomials*

$$H^{k_1,k_2}(x,y) = \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2},$$

form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$, where $k_1, k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$.

Proof. In [9] it was proved that polynomials $H^{k_1,k_2}(x,y) = \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2}$, where $k_1, k_2 \in \mathbb{N}$, $k_1 \geq 0$, $k_2 \geq 0$ form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_1)$. Thus they are algebraically independed. Let us show that $H^{k_1,k_2}(x,y) = \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2}$, where $k_1, k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$ are algebraically independed on $\ell_1 \oplus \ell_\infty$. Suppose the opposite. Then there exists $Q \neq 0$ such that $Q(H^{1,0}(x,y), H^{2,0}(x,y), H^{1,1}(x,y), \ldots, H^{k_1,k_2}(x,y)) = 0$. Let Q_0 be the restriction of Q on $\ell_1 \oplus \ell_1$. Then $Q_0(H^{1,0}(x,y), H^{2,0}(x,y), H^{1,1}(x,y), \ldots, H^{k_1,k_2}(x,y)) = 0$, where $Q_0 \neq 0$. But it cntradicts algebraically independed of polynomials H^{k_1,k_2} on $\ell_1 \oplus \ell_1$, where $k_1, k_2 \in \mathbb{N}$, $k_1 \geq 0$, $k_2 \geq 0$. So, polynomials H^{k_1,k_2}, $k_1, k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$ are algebraically independed.

Let us prove that $H^{k_1,k_2}(x,y)$ are continuous on $\ell_1 \oplus \ell_\infty$. Indead,

$$\left| H^{k_1,k_2}(x,y) \right| = \left| \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2} \right| \leq \sum_{i=1}^{\infty} |x_i|^{k_1} |y_i|^{k_1}.$$

Since $||(x,y)||_{\ell_1 \oplus \ell_\infty} = \sum_{i=1}^{\infty} |x_i| + \sup_{i \geq 1} |y_i| \leq 1$ then $\sum_{i=1}^{\infty} |x_i| \leq 1$ and $\sup_{i \geq 1} |y_i| \leq 1$.

Moreover $\sum_{i=1}^{\infty} |x_i|^{k_1} |y_i|^{k_2} \leq \sum_{i=1}^{\infty} |x_i|^{k_1} \cdot \left(\sup_{i \geq 1} |y_i| \right)^{k_2}$.

Hence

$$\left|\left| H^{k_1,k_2} \right|\right| = \sup_{||(x,y)|| \leq 1} \left| H^{k_1,k_2}(x,y) \right| \leq \sup_{||(x,y)|| \leq 1} \left(\sum_{i=1}^{\infty} |x_i|^{k_1} \cdot \left(\sup_{i \geq 1} |y_i| \right)^{k_2} \right) \leq 1.$$

Therefore $H^{k_1,k_2}(x,y)$ are bounded and so continuous on $\ell_1 \oplus \ell_\infty$.

Let us prove that every continuous block-symmetric polynomial $P \in \mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ can be represented as an algebraic combination of polynomials $H^{k_1,k_2}(x,y)$, $k_1, k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$.

Let $\tilde{P}$ be restriction of P on $\ell_1 \oplus \ell_1$. For polynomial $\tilde{P}$ there exists a unique polynomial $q : \mathbb{C}^{M_N} \to \mathbb{C}$ such that $\tilde{P} = q \circ \pi_{M_N}$. Since $\tilde{P}$ is continuous, $\tilde{P}$ is bounded on V_ρ, so q is bounded on $\pi_{M_N}(V_\rho)$.

By Proposition 1, a polynomial q does not depend on $\xi_{(0,k)}, k \in \mathbb{N}$.

Since polynomials H^{k_1,k_2}, where $(k_1,k_2) \in M_N \setminus \{(0,k)\}$ are well-defined and continuous on $\ell_1 \oplus \ell_\infty$, then $P = q \circ \pi_{M_N \setminus \{(0,k)\}}$.

Therefore $H^{k_1,k_2}(x,y)$, $k_1,k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$ form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$. □

Note that there are finitely symmetric polynomials on $\ell_1 \oplus \ell_\infty$ which are not symmetric. For example, let U be a free ultrafilter on $\mathbb{N}$. Then polynomials of the form

$$P_U(x,y) = \lim_U y_n \quad \text{and} \quad Q_{U,k}(x,y) = \lim_U \frac{\sum\limits_{n=1}^{m} y_n^k}{m}$$

are finitely symmetric but not symmetric (see [8]).

Since, $\ell_1 \oplus c_0 \subset \ell_1 \oplus \ell_\infty$, we can concider the algebra of block-symmetric polynomials on $\ell_1 \oplus c_0$, $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$.

Proposition 2. *The restriction $H_0^{k_1,k_2}$ of polynomials H^{k_1,k_2}, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$ onto $\ell_1 \oplus c_0$ form an algebraic basis in $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$.*

Proof. Since $\ell_1 \oplus \ell_1 \subset \ell_1 \oplus c_0 \subset \ell_1 \oplus \ell_\infty$ and the restriction of H^{k_1,k_2} onto $\ell_1 \oplus \ell_1$ are algebraically independed, so $H_0^{k_1,k_2}$ are algebraically independed. Let P be a symmetric polynomial on $\ell_1 \oplus c_0$ and $\tilde{P}$ its Aron-Berner extension (see [2]) to the second dual $(\ell_1 \oplus c_0)'' = \ell'_\infty \oplus \ell_\infty$. It is known that the map $P \mapsto \tilde{P}$ is an algebra homomorphism and $\tilde{P}$ is symmetric on $(\ell_1 \oplus c_0)''$ with respect to extension of operators $\sigma(x,y), \sigma \in S_\infty$ (see [6]). Let $\tilde{P}_1$ be the restriction of $\tilde{P}$ to $\ell_1 \oplus \ell_\infty = (\ell_1 \oplus c_0)''$. Then $\tilde{P}_1$ is symmetric and according to Theorem 2 can be represented by

$$\tilde{P}_1 = \sum_{l_1|k^1|+\ldots+l_r|k^r|=0}^{\infty} a_{k^1,\ldots,k^r,l_1,\ldots,l_r} \left(H^{k^1}\right)^{l_1} \ldots \left(H^{k^r}\right)^{l_r},$$

where $k^j = (k_1^j, k_2^j)$, $|k^j| = k_1^j + k_2^j$.

So P is the restriction of $\tilde{P}_1$ to $\ell_1 \oplus c_0$ and have the representation

$$P = \sum_{l_1|k^1|+\ldots+l_r|k^r|=0}^{\infty} a_{k^1,\ldots,k^r,l_1,\ldots,l_r} \left(H_0^{k^1}\right)^{l_1} \ldots \left(H_0^{k^r}\right)^{l_r},$$

where $k^j = (k_1^j, k_2^j)$, $|k^j| = k_1^j + k_2^j$.

Hence, $H_0^{k_1,k_2}$, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$ form an algebraic basis in $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$. □

Note that $\ell_1 \oplus c$ admits a block-symmetric polynomial

$$L(x,y) = \lim_{n\to\infty} y_n$$

wich can not be obtained by an algebraic combination of H^{k_1,k_2}, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$.

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Кравців В.В. *Алгебраїчний базис алгебри блочно-симетриних поліномів на $\ell_1 \oplus \ell_\infty$* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 89–95.

В робті розглянуто так звані блочно-симетричні поліноми на просторах послідовностей $\ell_1 \oplus \ell_\infty, \ell_1 \oplus c, \ell_1 \oplus c_0$, а саме, поліноми які є симетричними відносно перестановок елементів послідовностей. Доведено, що кожен неперервний блочно-симетричний поліном на $\ell_1 \oplus \ell_\infty$ може бути єдиним чином поданий як алгебраїчна комбінація деяких спеціальних блочно-симетричних поліномів, які утворюють алгебраїчний базис. Цікаво зауважити, що алгебра блочно-симетричних поліномів є нескінченно породжена, при цьому на ℓ_∞ не існує симетричних поліномів. У статті описано алгебраїчні базиси алгебр блочно-симетричних поліномів на $\ell_1 \oplus \ell_\infty$ та $\ell_1 \oplus c_0$.

Ключові слова і фрази: симетричні поліноми, блочно-симетрині поліноми, алгебраїчний базис, топологічна алгебра.

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THE RELATIONSHIP BETWEEN ALGEBRAIC EQUATIONS AND (n,m)-FORMS, THEIR DEGREES AND RECURRENT FRACTIONS

Algebraic and recursion equations are widely used in different areas of mathematics, so various objects and methods of research that are associated with them are very important. In this article we investigate the relationship between (n,m)-forms with generalized Diophantine Pell's equation, algebraic equations of n degree and recurrent fractions. The properties of the (n, m^n+1)-forms and their characteristic equation are considered. The author applied parafunctions of triangular matrices to the study of algebraic equations and corresponding recurrence equations. The form of adjacent roots of the annihilating polynomial of arbitrary (n,m)-forms over the field of rational numbers are explored.

The following question is very importan for some applied problems: Is a given form the largest by module among its adjacent roots? If it is so, then there is a periodic recurrence fraction of n-order that is equal to this (n,m)-form, and its mth rational shortening will be its rational approximation.The author has identified the class (nm)-forms with the largest module among their adjacent roots and showed how to find periodic recurrence fractions of n-order and rational approximations for them.

Key words and phrases: (n,m)-form, parapermanent, unit of field, Diophantine equation, recurrence fraction, rational approximation.

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1 PRELIMINARY CONCEPTS AND THEOREMS

1.1 Algebraic form of n order

Definition 1. *A real number*

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}},\ n \in \mathbb{N},\ s_i, m \in \mathbb{Q}, \tag{1}$$

or corresponding n-dimensional vector

$$x = (s_0, s_1, \ldots, s_{n-1}) \tag{2}$$

is called an algebraic (n,m)-form or briefly (n,m)-form.

It is known that the set of (n,m)-forms with the usual operations of addition and multiplication is a field.

We check the isomorphism of (n,m)-forms with some class matrices. For each (n,m)-form (1) we put in correspondence the circular n order matrix

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$$X=\begin{pmatrix} s_0 & s_{n-1}\sqrt[n]{m^{n-1}} & s_{n-2}\sqrt[n]{m^{n-2}} & \cdots & s_2\sqrt[n]{m^2} & s_1\sqrt[n]{m} \\ s_1\sqrt[n]{m} & s_0 & s_{n-1}\sqrt[n]{m^{n-1}} & \cdots & s_3\sqrt[n]{m^3} & s_2\sqrt[n]{m^2} \\ s_2\sqrt[n]{m^2} & s_1\sqrt[n]{m} & s_0 & \cdots & s_4\sqrt[n]{m^4} & s_3\sqrt[n]{m^3} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ s_{n-2}\sqrt[n]{m^{n-2}} & s_{n-3}\sqrt[n]{m^{n-3}} & s_{n-4}\sqrt[n]{m^{n-4}} & \cdots & s_0 & s_{n-1}\sqrt[n]{m^{n-1}} \\ s_{n-1}\sqrt[n]{m^{n-1}} & s_{n-2}\sqrt[n]{m^{n-2}} & s_{n-3}\sqrt[n]{m^{n-3}} & \cdots & s_1\sqrt[n]{m} & s_0 \end{pmatrix} \tag{3}$$

and for each (n,m)-form (2) we put in correspondence the matrix

$$X=\begin{pmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{pmatrix}. \tag{4}$$

Both matrices (3) and (4) are uniquely defined by their first columns.

The product of (n,m)-forms

$$x'=s'_0+s'_1\sqrt[n]{m}+\ldots+s'_{n-1}\sqrt[n]{m^{n-1}}, \tag{5}$$
$$x''=s''_0+s''_1\sqrt[n]{m}+\ldots+s''_{n-1}\sqrt[n]{m^{n-1}}$$

is the following (n,m)-form

$$x=s_0+s_1\sqrt[n]{m}+\ldots+s_{n-1}\sqrt[n]{m^{n-1}},$$

where

$$s_i=\sum_{j=0}^{i}s'_js''_{i-j}+m\sum_{j=i+1}^{n-1}s'_js''_{i-j},\ i=0,1,\ldots,n-1. \tag{6}$$

Thus, we have proved the following theorem.

Theorem 1. *The semigroups of* (n,m)*-forms* (1) *and* (2) *are isomorphic to the semigroups of matrices* (3) *and* (4)*, respectively.*

From the above it follows that k degree of (n,m)-form (5) is responsible k degree of matrix

$$X'=\begin{pmatrix} s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} & s'_{n-2}\sqrt[n]{m^{n-2}} & \cdots & s'_2\sqrt[n]{m^2} & s'_1\sqrt[n]{m} \\ s'_1\sqrt[n]{m} & s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} & \cdots & s'_3\sqrt[n]{m^3} & s'_2\sqrt[n]{m^2} \\ s'_2\sqrt[n]{m^2} & s'_1\sqrt[n]{m} & s'_0 & \cdots & s'_4\sqrt[n]{m^4} & s'_3\sqrt[n]{m^3} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ s'_{n-2}\sqrt[n]{m^{n-2}} & s'_{n-3}\sqrt[n]{m^{n-3}} & s'_{n-4}\sqrt[n]{m^{n-4}} & \cdots & s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} \\ s'_{n-1}\sqrt[n]{m^{n-1}} & s'_{n-2}\sqrt[n]{m^{n-2}} & s'_{n-3}\sqrt[n]{m^{n-3}} & \cdots & s'_1\sqrt[n]{m} & s'_0 \end{pmatrix}$$

or matrix

$$X'=\begin{pmatrix} s'_0 & ms_{n-1} & ms'_{n-2} & \cdots & ms'_2 & ms'_1 \\ s'_1 & s'_0 & ms'_{n-1} & \cdots & ms'_3 & ms'_2 \\ s'_2 & s'_1 & s'_0 & \cdots & ms'_4 & ms'_3 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ s'_{n-2} & s'_{n-3} & s'_{n-4} & \cdots & s'_0 & ms'_{n-1} \\ s'_{n-1} & s'_{n-2} & s'_{n-3} & \cdots & s'_1 & s'_0 \end{pmatrix}.$$

It is also obvious that if the last two matrices multiply by on the matrices columns

$$X'' = \begin{pmatrix} s_0'' \\ s_1'' \sqrt[n]{m} \\ \vdots \\ s_{n-2}'' \sqrt[n]{m^{n-2}} \\ s_{n-1}'' \sqrt[n]{m^{n-1}} \end{pmatrix}, \; X'' = \begin{pmatrix} s_0'' \\ s_1'' \\ \vdots \\ s_{n-2}'' \\ s_{n-1}'' \end{pmatrix},$$

then we get the matrices columns

$$X = \begin{pmatrix} s_0 \\ s_1 \sqrt[n]{m} \\ \vdots \\ s_{n-2} \sqrt[n]{m^{n-2}} \\ s_{n-1} \sqrt[n]{m^{n-1}} \end{pmatrix}, \; X = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{pmatrix},$$

where s_i are defined by (6).

For any (n, m)-form

$$x = s_0 + s_1 \sqrt[n]{m} + \ldots + s_{n-1} \sqrt[n]{m^{n-1}}$$

there exists a unique (n, m)-form

$$\overline{x} = \overline{s_0} + \overline{s_1} \sqrt[n]{m} + \ldots + \overline{s_{n-1}} \sqrt[n]{m^{n-1}},$$

such that product $x\overline{x}$ is a real number. The (n, m)-form $\overline{x}$ is called conjugated to (n, m)-form x, and their product is called a norm of x and denoted by $|(n, m)|$.

Let X and $\overline{X}$ are matrices that corresponds to (n, m)-form

$$x = s_0 + s_1 \sqrt[n]{m} + \ldots + s_{n-1} \sqrt[n]{m^{n-1}}$$

and conjugate (n, m)-form $\overline{x}$. Then

$$X \cdot \overline{X} = |(n, m)| \cdot E,$$

where E be the identity matrix. The norm of (n, m)-form x is equal to $\det X$, and matrix that corresponds to conjugated (n, m)-form $\overline{x}$ is inverse to the matrix X multiplied on the determinant of X.

Therefore, the equation

$$\begin{vmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{vmatrix} = \pm 1$$

is an n-dimensional generalization Pell's equation

$$\begin{vmatrix} s_0 & ms_1 \\ s_1 & s_0 \end{vmatrix} = s_0^2 - ms_1^2 = \pm 1.$$

Using the polynomial formula it is easy to prove the equality

$$(s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}})^k = \sum_{\substack{\lambda_0+\lambda_1+\ldots+\lambda_{n-1}=k \\ \lambda_1+2\lambda_2+\ldots+(n-1)\lambda_{n-1}=ns+i}} \frac{k!}{\lambda_0!\lambda_1!\cdot\ldots\cdot\lambda_{n-1}!} s_0^{\lambda_0}s_1^{\lambda_1}\cdot\ldots\cdot s_{n-1}^{\lambda_{n-1}} m^s m^{\frac{i}{n}}.$$

However, the above formula is inconvenient for the elevation of (n, m)-forms to the k degree, because it is associated with orderly partition number of n on integer nonnegative summands.

1.2 Parafunctions of triangular matrices (tables)

Let K be some field of numbers.

Definition 2 ([2]). *Triangular table*

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{7}$$

*of numbers in K is called a **triangular matrix.***

To every element a_{ij} of the triangular matrix (7) we put in correspondence the $(i - j + 1)$ elements a_{ik}, $k \in \{j, ..., i\}$ which are called *derived elements* of triangular matrix, generated by a *key element* a_{ij}. A key element of a triangular matrix is also a derived element. The product of all derived elements generated by a key element of a_{ij} is denoted by $\{a_{ij}\}$ and is called a *factorial* product of this key element, i.e.,

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 3 ([2]). *The paradeterminant and parapermanent of the triangular matrix (7) are the numbers*

$$\operatorname{ddet}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

$$\text{pper(A)} = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s, p_1+\ldots+p_{s-1}+1}\},$$

where the summation of over the set of natural solutions of the equality $p_1 + \ldots + p_r = n$.

Paradeterminants and parapermanents of triangular matrices can be used in Algebra, Number Theory and Combinatorics (see [2] for more details and examples).

1.3 One-periodic recurrence fractions

Let us consider the algebraic equations of nth order

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n, \tag{8}$$

where $a_n \neq 0$, and the expression

$$\left[\begin{array}{c|ccccccc}
a_1 & & & & & & & \\
\frac{a_2}{a_1} & a_1 & & & & & & \\
\vdots & \dots & \ddots & & & & & \\
\frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \dots & a_1 & & & & \\
\frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \dots & \frac{a_2}{a_1} & a_1 & & & \\
0 & \frac{a_n}{a_{n-1}} & \dots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\
\vdots & \dots & \dots & \dots & \dots & \dots & \ddots & \\
0 & 0 & \dots & \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \dots & a_1
\end{array}\right]_m \tag{9}$$

which is closely related to (8). The expression (9) looks like as a symbol fraction, the numerator of which is a parapermanent P_m of order m formed by the removal columns from the expression pipe and the denominator of which is a parapermanent Q_m of order $m-1$ without first column of parapermanent of numerator.

If in the expression (9) we direct to the limit as $m \to \infty$, we obtain an one-periodic recurrent fraction of order n

$$\left[\begin{array}{c|ccccccc}
a_1 & & & & & & & \\
\frac{a_2}{a_1} & a_1 & & & & & & \\
\vdots & \dots & \ddots & & & & & \\
\frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \dots & a_1 & & & & \\
\frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \dots & \frac{a_2}{a_1} & a_1 & & & \\
0 & \frac{a_n}{a_{n-1}} & \dots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\
\vdots & \dots & \dots & \dots & \dots & \dots & \ddots & \\
0 & 0 & \dots & \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \dots & a_1
\end{array}\right]_\infty . \tag{10}$$

The expression (9) is called the mth approximant of (10).

Theorem 2 ([3])**.** *Let* (8) *be an algebraic equation from pairwise different roots. If for the m-rational shortening of one-periodic recurrent fraction of nth order* (10) *a finite non-zero real limit exists as $m \to \infty$, i.e.,*

$$\lim_{m\to\infty} \frac{P_m}{Q_m} = x \neq 0,$$

then a recurrent fraction of order n is an image of the real root of algebraic equations (8) *with the largest module.*

More information about recurrent fractions can be found in [3].

2 RELATIONSHIP (n,m)-FORM WITH ALGEBRAIC EQUATIONS

Let us find the integer coefficients of equation

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x^1 + a_{n,n} \tag{11}$$

the root of which is the (n,m)-form

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}},$$

where $s_i \in \mathbb{Q}$, $m \in \mathbb{N}$.

The main minor of rth order of matrix

$$X = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{12}$$

is denoted by

$$X\begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix} = \begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_r} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_r} \\ \vdots & \ldots & \ldots & \vdots \\ a_{i_r,i_1} & a_{i_r,i_2} & \cdots & a_{i_r,i_r} \end{vmatrix},$$

where $i_1 < i_2 < \ldots < i_r$. The characteristic equation of matrix (12) is

$$\det(X - xE) = 0$$

or

$$x^n = \alpha_{n1}x^{n-1} + \alpha_{n2}x^{n-2} + \ldots + \alpha_{n,n-1}x^1 + \alpha_{nn},$$

where

$$\alpha_{nj} = (-1)^{j-1} \sum_{1\le i_1<i_2<\ldots<i_j\le n} X\begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix}. \tag{13}$$

According to theorem Hamilton-Cayley, each square matrix satisfies the characteristic equation, so

$$X^n = \alpha_{n1}X^{n-1} + \alpha_{n2}X^{n-2} + \ldots + \alpha_{n,n-1}X^1 + \alpha_{nn}, \tag{14}$$

with coefficients (13), where X is matrix (12).

If matrix X in (14) is given by (4), then the coefficients a_{nj} of equation (11), for which a (n,m)-form (2) is the root, can be found using the equalities (13). Thus, we prove

Theorem 3. *If the (n,m)-form*

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}}$$

is a root of equation

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x^1 + a_{nn},$$

then the coefficients of this equation are equal

$$a_{nj} = (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} X \begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix},$$

where

$$X \begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix}$$

are major minors of matrix

$$X = \begin{pmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{pmatrix}.$$

Theorem 4. *The* (n, m^n+1)*-form*

$$m^{n-1} + m^{n-2}\sqrt[n]{m^n+1} + \ldots + m\sqrt[n]{(m^n+1)^{n-2}} + \sqrt[n]{(m^n+1)^{n-1}}$$

is the root of an algebraic equation

$$x^n = \binom{n}{1} m^{n-1}x^{n-1} + \binom{n}{2} m^{n-2}x^{n-2} + \ldots + \binom{n}{n-1} mx + \binom{n}{n}.$$

Proof. Since all the major minors the same order of matrix

$$\begin{pmatrix} m^{n-1} & m^n+1 & m(m^n+1) & \ldots & m^{n-3}(m^n+1) & m^{n-2}(m^n+1) \\ m^{n-2} & m^{n-1} & m^n+1 & \ldots & m^{n-4}(m^n+1) & m^{n-3}(m^n+1) \\ m^{n-3} & m^{n-2} & m^{n-1} & \ldots & m^{n-5}(m^n+1) & m^{n-4}(m^n+1) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ m & m^2 & m^3 & \ldots & m^{n-1} & m^n+1 \\ 1 & m & m^2 & \ldots & m^{n-2} & m^{n-1} \end{pmatrix}$$

are equal, we find one of them. Let us find the major minor of matrix

$$\begin{pmatrix} m^{n-1} & m^n+1 & m(m^n+1) & \cdots & m^{s-2}(m^n+1) \\ m^{n-2} & m^{n-1} & m^n+1 & \cdots & m^{s-3}(m^n+1) \\ m^{n-3} & m^{n-2} & m^{n-1} & \cdots & m^{s-4}(m^n+1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^{n-s} & m^{n-s+1} & m^{n-s+2} & \ldots & m^{n-1} \end{pmatrix}.$$

We multiply the first column on $-m^r$, $r = 1, 2, \ldots, s-1$ and add it to the $(r+1)$ column; then we get the determinant of matrix

$$\begin{pmatrix} m^{n-1} & 1 & m & \cdots & m^{s-2} \\ m^{n-2} & 0 & 1 & \cdots & m^{s-3} \\ m^{n-3} & 0 & 0 & \cdots & m^{s-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^{n-s} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Decomposing above determinant by elements of the first column, we get $(-1)^{s+1}m^{n-s}$.

Thus, according to the Theorem 3, coefficients a_{ns} are equal to

$$(-1)^{s-1}(-1)^{s+1}m^{n-s}\binom{n}{s} = m^{n-s}\binom{n}{s}.$$

$\square$

3 Some calculations related to an algebraic equations of n degree

Theorem 5. *If*

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}$$

and

$$x^m = A_{m1}x^{n-1} + A_{m2}x^{n-2} + \ldots + A_{m,n-1}x + A_{mn},\ n \le m,$$

then for all $i = 1, 2, \ldots, n$

$$A_{mi} = \begin{bmatrix} a_{ni} & & & & & & & & & \\ \frac{a_{n,i+1}}{a_{n1}} & a_{n1} & & & & & & & & \\ \frac{a_{n,i+2}}{a_{n2}} & \frac{a_{n2}}{a_{n1}} & a_{n1} & & & & & & & \\ \vdots & \dots & \dots & \ddots & & & & & & \\ \frac{a_{nn}}{a_{n,n-i}} & \frac{a_{n,n-i}}{a_{n,n-i-1}} & \frac{a_{n,n-i-1}}{a_{n,n-i-2}} & \dots & a_{n1} & & & & & \\ 0 & \frac{a_{nn}}{a_{n,n-i}} & \frac{a_{n,n-i}}{a_{n,n-i-1}} & \dots & \frac{a_{n2}}{a_{n1}} & a_{n1} & & & & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \ddots & & & \\ 0 & \frac{a_{nn}}{a_{n,n-1}} & \frac{a_{n,n-1}}{a_{n,n-2}} & \dots & \frac{a_{n,i+1}}{a_{ni}} & \frac{a_{ni}}{a_{n,i-1}} & \dots & a_{n1} & & \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & 0 & \dots & \frac{a_{nn}}{a_{n,n-1}} & \frac{a_{n,n-1}}{a_{n,n-2}} & \dots & \frac{a_{n3}}{a_{n2}} & \frac{a_{n2}}{a_{n1}} & a_{n1} \end{bmatrix}_{m-n+1}. \tag{15}$$

Proof. Obviously, equality (15) is true at $m = n$. Let us show that the induction step is performed. We have

$$x^{m+1} = A_{m+1,1}x^{n-1} + \ldots + A_{m+1,i}x^{n-i} + \ldots + A_{m+1,n-1}x + A_{m+1,n}.$$

On the other side,

$$\begin{aligned} x^{m+1} &= A_{m1}x^n + A_{m2}x^{n-1} + \ldots + A_{m,n-1}x^2 + A_{mn}x \\ &= A_{m1}(a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}) + A_{m2}x^{n-1} + \ldots + A_{m,n-1}x^2 + A_{mn}x \\ &= (a_{n1}A_{m1} + A_{m2})x^{n-1} + \ldots + (a_{ni}A_{m1} + A_{m,i+1})x^{n-i} \\ &\qquad + \ldots + (a_{n,n-1}A_{m1} + A_{mn})x + a_{nn}A_{m1}. \end{aligned}$$

Thus

$$A_{m+1,i} = a_{ni}A_{m1} + A_{m,i+1}.$$

It is easy to see that decomposing the parapermanent $A_{m+1,i}$ by elements of the first column, we get $a_{ni}A_{m1} + A_{m,i+1}$. $\square$

Corollary 1. *If*
$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}$$
and
$$x^m = A_{m1}x^{n-1} + A_{m2}x^{n-2} + \ldots + A_{m,n-1}x + A_{mn},\ n \le m,$$
then coefficients A_{mi} can be found from the recurrence equations
$$A_{mi} = a_{ni}A_{m-1,1} + a_{n,i+1}A_{m-2,1} + \ldots + a_{nn}A_{m-n+i-1,1},\ i = 1, 2, \ldots, n,$$
where
$$A_{n1} = a_{n1}, A_{n-1,1} = 1, A_{n-2,1} = \ldots = A_{0,1} = 0.$$

Proof. The proof it follows from the decomposition of parapermanent (15) by the elements of the first column. □

Example 1. *If*
$$x^3 = a_{31}x^2 + a_{32}x^1 + a_{33}$$
and
$$x^m = A_{m1}x^2 + A_{m2}x^1 + A_{m3},\ m \ge 3,$$
then coefficients $A_{mi}, i = 1, 2, 3$ can be found from the recurrence equations
$$\begin{aligned} A_{m1} &= a_{31}A_{m-1,1} + a_{3,2}A_{m-2,1} + a_{33}A_{m-3,1}, \\ A_{m2} &= a_{32}A_{m-1,1} + a_{3,3}A_{m-2,1}, \\ A_{m3} &= a_{33}A_{m-1,1},\ m \ge 4, \end{aligned}$$
where $A_{31} = a_{31}, A_{21} = 1, A_{11} = A_{01} = 0$.

For comparison, let us consider a similar algorithm of Delone and Fadeev ([1, p. 73]). Let
$$\omega^3 = S\omega^2 + Q\omega + N$$
and
$$\omega^m = U_m\omega^2 + V_m\omega + W_m,$$
then the coefficients U_m, V_m, W_m can be found from relations
$$\begin{aligned} U_m &= \sum_{\alpha+2\beta+3\gamma=m-2} \frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!} S^\alpha Q^\beta N^\gamma, \\ V_m &= U_{m+1} - U_m S, \\ W_m &= U_{m+2} - U_{m+1}S - U_m Q. \end{aligned}$$

Note that similar algorithms with $n > 3$ were not considered.

Theorem 6. *If (n, k)-form looks like*
$$x = s_0 + s_1\sqrt[n]{k} + \cdots + s_{n-1}\sqrt[n]{k^{n-1}},$$
then others of adjacent roots of diriment polynomial over the field of rational numbers of this form are as follows
$$x_i = s_0 + s_1\varepsilon^i\sqrt[n]{k} + \cdots + s_{n-1}\varepsilon^{(n-1)i}\sqrt[n]{k^{n-1}},$$
where ε is the primitive root of degree n of 1 and $i = 1, \ldots, n-1$.

Proof. To unify notation we also denoted (n,k)-form x by x_n. We will show that for every k, $S_m = \sum_{i=1}^{n} x_i^m$ does not depend on radicals and belongs the field of rational numbers.

Let us consider, first, $\sum_{i=1}^{n} \varepsilon^{ip} = \varepsilon^p \sum_{i=0}^{n-1} \varepsilon^{ip}$. Since ε is the primitive root of degree n of unit, then

$$\sum_{i=0}^{n-1} \varepsilon^{ip} = \sum_{i=1}^{n} \varepsilon^{ip}, \quad \text{i.e.,} \quad \sum_{i=1}^{n} \varepsilon^{ip} = 0,$$

if p is not a multiple of n.

In formula x_i^m each summand will looks like

$$s_{p_1}\varepsilon^{ip_1}\sqrt[n]{k^{p_1}} \cdot \ldots \cdot s_{p_m}\varepsilon^{ip_m}\sqrt[n]{k^{p_m}} = (s_{p_1}\ldots s_{p_m})\varepsilon^{i(p_1+\cdots+p_m)}\sqrt[n]{k^{p_1+\cdots+p_m}},$$

where $p_1,\ldots,p_m \in \{0,1,2,\ldots,n-1\}$. Then in $S_m = \sum_{i=1}^{n} x_i^m$ we can regroup the terms in groups of sets with the same $p_1,\ldots,p_m$; each a such group has representation:

$$\sum_{i=1}^{n}(s_{p_1}\ldots s_{p_m})\varepsilon^{i(p_1+\cdots+p_m)}\sqrt[n]{k^{p_1+\cdots+p_m}} = (s_{p_1}\ldots s_{p_m})\left(\sum_{i=1}^{n}\varepsilon^{i(p_1+\cdots+p_m)}\right)\sqrt[n]{k^{p_1+\cdots+p_m}}.$$

Hence, if $p_1+\cdots+p_m$ is not a multiple of n, then this group of summands is equal to zero; if $p_1+\cdots+p_m$ is a multiple of n, then this group of summands is a rational number.

According to Newton's formulas, σ_m (elementary symmetric expressions of $x_1,\ldots,x_n$), $m = 1,\ldots,n$ are expressed through the S_q, $q \le m$, i σ_r, $r < m$. Since $\sigma_1 = S_1$ and all S_m is a rational number, then all σ_m, where $m = 1,\ldots,n$, also a rational number. According to the Viete formulas $(x-x_1)\ldots(x-x_n) \in \mathbb{Q}[x]$, that proves the theorem. ☐

For some applications it less important to known the view of adjacent roots of (n,k)-form then the answer to the question: Is this form the largest by module? This question is quite difficult and requires a special investigation, but we have an obvious consequence.

Corollary 2. *If $s_0, s_1, \ldots, s_{n-1}$, $n \in \mathbb{N}$ are nonnegative rational numbers, then (n,k)-form $x = s_0 + s_1\sqrt[n]{k} + \cdots + s_{n-1}\sqrt[n]{k^{n-1}}$ is the largest by module among its adjacent roots diriment polynomial over the field of rational numbers.*

Thus, using Theorem 3 for each (n,m)-form (1) we can write an algebraic equation of order n. According to Corollary 2 and Theorem 2, by rational approximations of recurrent fractions we can build a mth rational shortening (9) to the (n,m)-form (1).

Theorem 7. *If (n,m)-form (1) with nonnegative coefficients $s_i, i = 0,1,\ldots,n-1$ is the root of an algebraic equation (8), then recurrent fraction (10) is it's image, and it's mth approximant (9) is it's rational approximation.*

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Алгебраїчні та рекурентні рівняння мають широке застосування не тільки в алгебрі але й в інших розділах математики, чим викликають неабияке зацікавлення до різного роду об'єктів та методів дослідження пов'язаних із ними. В цій статті досліджено зв'язок (n,m)-форм з узагальненими рівняннями Пеля, алгебраїчними рівняннями n-ого степеня і рекурентними дробами. Розглянуто властивості (n,m^n+1)-форми і її характеристичного рівняння. Застосовано парафункції трикутних матриць до алгебраїчних рівнянь n-ого степеня та відповідних їм рекурентних рівнянь. Досліджено вигляд суміжних коренів анулюючого полінома довільної (n,m)-форми над полем раціональних чисел.

Для деяких прикладних задач велике значення має відповідь на питання: чи є дана (n,m)-форма найбільша за модулем серед своїх суміжних коренів? Тоді в цьому випадку існуватиме одноперіодичний рекурентний дріб n-ого порядку, який дорівнюватиме даній (n,m)-формі, а його m-те раціональне вкорочення буде її раціональним наближенням. Автор виділив клас (n,m)-форм, які є найбільшими за модулем серед своїх суміжних коренів, та показав як для них знайти одноперіодичні рекурентні дроби n-ого порядку й раціональні наближення.

Ключові слова і фрази: (n,m)-форма, параперманент, узагальнене рівняння Пеля, рекурентний дріб, раціональне наближення.

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INVERSE PROBLEM FOR $2b$-ORDER DIFFERENTIAL EQUATION WITH A TIME-FRACTIONAL DERIVATIVE

We study the inverse problem for a differential equation of order $2b$ with the Riemann-Liouville fractional derivative of order $\beta \in (0,1)$ in time and given Schwartz type distributions in the right-hand sides of the equation and the initial condition. The problem is to find the pair of functions (u, g): a generalized solution u to the Cauchy problem for such equation and the time dependent multiplier g in the right-hand side of the equation. As an additional condition, we use an analog of the integral condition

$$(u(\cdot,t), \varphi_0(\cdot)) = F(t), \quad t \in [0,T],$$

where the symbol $(u(\cdot,t), \varphi_0(\cdot))$ stands for the value of an unknown distribution u on the given test function φ_0 for every $t \in [0,T]$, F is a given continuous function.

We prove a theorem for the existence and uniqueness of a generalized solution of the Cauchy problem, obtain its representation using the Green's vector-function. The proof of the theorem is based on the properties of conjugate Green's operators of the Cauchy problem on spaces of the Schwartz type test functions and on the structure of the Schwartz type distributions.

We establish sufficient conditions for a unique solvability of the inverse problem and find a representation of an unknown function g by means of a solution of a certain Volterra integral equation of the second kind with an integrable kernel.

Key words and phrases: distribution, fractional derivative, inverse problem, Green vector-function.

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INTRODUCTION

Different initial and boundary value problems to differential and pseudo-differential equations with distributions in the right-hand sides are sufficiently investigated (see, for example, [1–9] and references therein).

Equations with fractional derivatives [10] and inverse problems to them are appearing in different branches of science and engineering, and the range of the applicability of the generated models is increase considerable. The conditions of classical solvability of the Cauchy and boundary value problems to equations with a time fractional derivative were obtained, for example, in [11–15]. The inverse boundary value problems to a time fractional diffusion equation with different unknown functions or parameters were investigated, for example, in [16–24]. Most papers were devoted to inverse problems with an unknown right-hand sides, mainly under regular data.

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In this paper for the equation

$$u_t^{(\beta)} - A(D)u = g(t)F_0(x), \quad (x,t) \in \mathbb{R}^n \times (0,T] := Q, \tag{1}$$

with the Riemann-Liouville fractional derivative of order $\beta \in (0,1)$ we study the inverse problem

$$u(x,0) = F_1(x), \ \ x \in \mathbb{R}^n, \tag{2}$$

$$(u(\cdot,t), \varphi_0(\cdot)) = F(t), \quad t \in [0,T], \tag{3}$$

of the determination the pair (u,g) where

$$A(D)u = \sum_{|\gamma| \le 2b} A_\gamma D^\gamma u$$

is a differential expression of order $2b$ with constants coefficients A_γ, $|\gamma| \le 2b$ such that

$$\frac{\partial u}{\partial t} - A(D)u$$

is the parabolic differential expression [7,12], F_j $(j = 0,1)$ are given Schwartz type distributions, F is a given continuous function, the symbol $(u(\cdot,t), \varphi_0(\cdot))$ stands for the value of an unknown distribution u on the given test function φ_0 for every $t \in [0,T]$.

Note that the conditions of the existence a regular solution for such fractional Cauchy problem, even with the variable coefficients $A_\gamma = A_\gamma(x)$, $|\gamma| \le 2b$, was obtained in [8] by M.I. Matijchuk. The inverse boundary value problems of finding a pair (u,g) for a time-fractional diffusion equations under regular given data in the right-hand sides and similar (integral) over-determination conditions were studied, for example, in [16,18]. The over-determination condition of kind (3), but with the scalar product (u,φ_0) in abstract Hilbert space, was used in [17]. The inverse problem of kind (1)–(3) with $(-\Delta)^{\gamma/2}$ $(\gamma > \beta)$ instead of $A(D)$ and distributions with compact supports in the right-hand sides was studied in [22].

1 Notations, definitions and auxiliary results

We use the following: $Q = \mathbb{R}^n \times (0,T]$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\bar\alpha = (\alpha_0, \alpha)$, $\alpha_j \in \mathbb{Z}_+$, $j \in \{0,1,\ldots,n\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$, $D^\alpha v(x,t) = D_x^\alpha v(x,t) = \frac{\partial^{|\alpha|} v(x,t)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $D^{\bar\alpha} v(x,t) = (\frac{\partial}{\partial t})^{\alpha_0} D^\alpha v(x,t)$, $\mathcal{S}(\mathbb{R}^n)$ is the space of indefinitely differentiable functions v in $\mathbb{R}^n$ such that $x^\gamma D^\alpha v$ are bounded in $\mathbb{R}^n$ for all multi-indexes α, γ (the Schwartz space of smooth rapidly decreasing functions), $S_\gamma(\mathbb{R}^n)$ $(\gamma > 0)$ is the space of type $\mathcal{S}(\mathbb{R}^n)$ (see [2, p. 201]):

$$S_\gamma(\mathbb{R}^n) = \{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \le C_\alpha e^{-a|x|^{\frac{1}{\gamma}}}, \ \ x \in \mathbb{R}^n, \ \ \forall\, \alpha\}$$

with some positive constants $C_\alpha = C_\alpha(v)$ and $a = a(v)$,

$$S_{\gamma,(a)}(\mathbb{R}^n) = \{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \le C_{\alpha,\delta}(v) e^{-(a-\delta)|x|^{\frac{1}{\gamma}}}, \ \ x \in \mathbb{R}^n, \ \ \forall\, \alpha, \ \ \forall\, \delta > 0\},\ a > 0,$$

$C^{\infty,(0)}(\bar Q) = \{v \in C^\infty(\overline{Q}) : (\frac{\partial}{\partial t})^k v|_{t=T} = 0, \ k \in \mathbb{Z}_+\}$, $\mathcal{S}(\bar Q)$ $(\mathcal{S}_\gamma(\bar Q), S_{\gamma,(a)}(\bar Q))$ is the space of functions $v \in C^{\infty,(0)}(\bar Q)$ such that $(\frac{\partial}{\partial t})^s v(\cdot,t) \in \mathcal{S}(\mathbb{R}^n)$ $(\mathcal{S}_\gamma(\mathbb{R}^n)$, $\mathcal{S}_{\gamma,(a)}(\mathbb{R}^n)$, respectively) for all $t \in [0,T]$, $s \in \mathbb{Z}_+$. By E' we denote the space of linear continuous functionals over E (the space of distributions). The symbol (f, φ) stands for the value of the distribution $f \in E'$ on the test function $\varphi \in E$,

$$S'_{\gamma,C}(\bar{Q}) = \{f \in S'_\gamma(\bar{Q}) : \ \big(f(x,\cdot), \varphi(x)\big) \in C[0,T] \quad \forall \varphi \in S_\gamma(\mathbb{R}^n)\},$$
$$S'_{\gamma,(a),C}(\bar{Q}) = \{f \in S'_{\gamma,(a)}(\bar{Q}) : \ \big(f(x,\cdot), \varphi(x)\big) \in C[0,T] \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n)\}, \ a > 0.$$

We denote by $(g \hat{*} \varphi)(x) = \big(g(\xi), \varphi(x+\xi)\big)$ the convolution of the distribution g and the test function φ, by $f*g$ the convolution of the distributions f and g: $(f*g, \varphi) = (f, g\hat{*}\varphi)$ for any test function φ, by fg the direct product of the distributions f and g: $(fg, \varphi) = (f(x), (g(t), \varphi(x,t))$ for any test function $\varphi(x,t)$, use the function

$$f_\lambda(t) = \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} \text{ for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \ \text{ for } \lambda \le 0,$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$, $f_\lambda \hat{*} f_\mu = f_{\lambda+\mu}$.

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined by the formula

$$v^{(\beta)}(t) = f_{-\beta}(t) * v(t),$$

the Djrbashian-Caputo (regularized) fractional derivative of order $\beta \in (0,1)$ is defined by

$$D^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} v'(\tau)d\tau,$$

and therefore $D^\beta v(t) = v^{(\beta)}(t) - f_{1-\beta}(t)v(0)$.

We denote

$$(Lv)(x,t) \equiv v_t^{(\beta)}(x,t) - (Av)(x,t),$$
$$(L^{reg}v)(x,t) \equiv D_t^\beta v(x,t) - (Av)(x,t),$$
$$(\widehat{L}v)(x,t) \equiv f_{-\beta}(t) \hat{*} v(x,t) - (Av)(x,t), \quad (x,t) \in Q.$$

The Green formula

$$\int_Q v(x,\tau)(\widehat{L}\psi)(x,\tau)dxd\tau = \int_Q (L^{reg}v)(x,\tau)\psi(x,\tau)dxd\tau + \int_Q v(x,0)f_{1-\beta}(\tau)\psi(x,\tau)dxd\tau,$$

$v, \psi \in \mathcal{S}(\bar{Q})$, holds (see, for example, [5]).

Definition 1. *The function $u \in \mathcal{S}'_{\gamma,(a),C}(\bar{Q})$ is called a solution of the Cauchy problem* (1), (2) *if the identity*

$$\int_0^T \big(u(\cdot,t), (\widehat{L}\psi)(\cdot,t)\big)dt = \int_0^T g(t)\big(F_0(\cdot), \psi(\cdot,t)\big)dt + \big(F_1(y)f_{1-\beta}(t), \psi(y,t)\big) \tag{4}$$

holds for all $\psi \in \mathcal{S}_{\gamma,(a)}(\bar{Q})$.

Definition 2. *The pair $(u,g) \in \mathcal{S}'_{\gamma,(a),C}(\bar{Q}) \times C[0,T]$ is called a solution of the problem* (1)–(3) *if the identity* (4) *and the condition* (3) *hold.*

It follows from (2) and (3) the compatibility condition

$$(F_1, \varphi_0) = F(0). \tag{5}$$

Definition 3. *The vector-function* $\big(G_0(x,t), G_1(x,t)\big)$ *is called a Green vector-function of the Cauchy problem* (2) *to the equation* $(Lu)(x,t) = \Phi(x,t)$, $(x,t) \in Q$, *and also of such problem to the equation*

$$(L^{reg}u)(x,t) = \Phi(x,t), \quad (x,t) \in Q, \tag{6}$$

if under rather regular Φ, F_1 *the function*

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} G_0(x-y, t-\tau)\Phi(y,\tau)dy + \int\limits_{\mathbb{R}^n} G_1(x-y,t)F_1(y)dy, \quad (x,t) \in \overline{Q}, \tag{7}$$

is the regular solution of the problem (6), (2).

Such Green vector-function exists [8] and has the following bounds:

$$\begin{aligned} |G_0(x,t)| &\le Ct^{-\frac{\beta n}{2b}+\beta-1} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}), \\ |G_1(x,t)| &\le Ct^{-\frac{\beta n}{2b}} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}), \end{aligned} \tag{8}$$

where $\Psi_m(z) = \Psi_m(1)$ for $|z| > 1$ and $\Psi_m(z) = \begin{cases} 1, & m < 0, \\ 1 + |ln|z||, & m = 0, \\ |z|^{-m}, & m > 0, \end{cases}$ for $|z| < 1$.

Hereinafter $c, C, c_k, \widehat{c}_k, d_k, \widehat{d}_k, C_k, \widehat{C}_k$ ($k \in \mathbb{Z}_+$) are positive constants. Let

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) = \int\limits_\tau^T dt \int\limits_{\mathbb{R}^n} \varphi(x,t) G_0(x-y, t-\tau)dx, (y,\tau) \in \bar{Q},$$

$$(\widehat{\mathcal{G}}_1\varphi)(y) = \int\limits_0^T dt \int\limits_{\mathbb{R}^n} \varphi(x,t) G_1(x-y,t)dx, y \in \mathbb{R}^n.$$

$$(\widehat{G_j}\varphi)(y,t) = \int\limits_{\mathbb{R}^n} G_j(x-y,t)\varphi(x)\, dx, \quad (y,t) \in \bar{Q}, \quad j = \overline{0,1}.$$

Lemma 1. *If* $a > 0$, $\gamma \ge 1 - \frac{\beta}{2b}$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ *then there exist numbers* $C > 0$, $a' \in (0, a]$ *such that for all* $k \in \mathbb{Z}_+$, *multi-index* κ, $|\kappa| = k$, $\delta > 0$ *the following bounds hold:*

$$\big|D_y^\kappa(\widehat{G_0}\varphi)(y,t)\big| \le c_k t^{\beta-1} e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha \varphi(x)| e^{(a'-\delta)|x|^{\frac{1}{\gamma}}}, (y,t) \in Q,$$

$$\big|D_y^\kappa(\widehat{G_1}\varphi)(y,t)\big| \le c_k e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha \varphi(x)| e^{(a'-\delta)|x|^{\frac{1}{\alpha}}}, (y,t) \in \bar{Q}.$$

Proof. We use the bounds (8). In the case $n > 2b$ for all multi-index α, $|\alpha| = k$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$

and $\delta' = \delta/a$ we have

$$\Big|\int_{\mathbb{R}^n} G_0(x-y,t-\tau)D^\alpha\varphi(x)dx\Big| \le \int_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} |G_0(x-y,t-\tau)|\,|D^\alpha\varphi(x)|dx$$

$$+ \int_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} |G_0(x-y,t-\tau)|\,|D^\alpha\varphi(x)|dx$$

$$\le C(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|D^\alpha\varphi(x)||x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}dx$$

$$+ \int_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}|D^\alpha\varphi(x)|dx\Big]$$

$$\le C_1(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}e^{-c(1-\delta')[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx$$

$$+ \int_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c\delta'[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}e^{-c(1-\delta')[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx\Big]\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_1(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}e^{-c(1-\delta')[|x-y|T^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx$$

$$+ \int_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c\delta'[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}e^{-c(1-\delta')[|x-y|T^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx\Big]\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}.$$

Putting $c_\gamma = 2^{1-\frac{1}{\gamma}}$ for $\gamma \in [1-\frac{\beta}{2b},1]$, $c_\gamma = 1$ for $\gamma \ge 1$, $a' = c_\gamma\min\{cT^{-\frac{\beta}{2b\gamma}},a\}$ and using the inequality [12, p. 25] $|A|^{\frac{1}{\gamma}} + |B|^{\frac{1}{\gamma}} \ge c_\gamma|A+B|^{\frac{1}{\gamma}}$ we get

$$c\big(|x-y|T^{-\frac{\beta}{2b}}\big)^{\frac{1}{\gamma}} + a|x|^{\frac{1}{\gamma}} \ge \min\{cT^{-\frac{\beta}{2b\gamma}},a\}[|x-y|^{\frac{1}{\gamma}}+|x|^{\frac{1}{\gamma}}] \ge a'|y|^{\frac{1}{\gamma}}.$$

Then

$$\Big|\int_{\mathbb{R}^n} G_0(x-y,t-\tau)D^\alpha\varphi(x)dx\Big| \le C_2\Big[\frac{1}{(t-\tau)}\int_0^{(t-\tau)^{\frac{\beta}{2b}}} r^{2b-1}dr$$

$$+ (t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\int_{t^{\frac{\beta}{2b}}}^{\infty} r^{n-1}e^{-c\delta'[r(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}dr\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_3(t-\tau)^{\beta-1}\Big[1+\int_1^{+\infty} z^{(1-\frac{\beta}{2b})n-1}e^{-c\delta' z}\,dz\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_4(t-\tau)^{\beta-1}e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}, \quad y\in\mathbb{R}^n, \ \ 0\le\tau<t\le T,$$

$$\Big|\int\limits_{\mathbb{R}^n} G_1(x-y,t)D^\alpha\varphi(x)dx\Big| \le C(t-\tau)^{-\frac{\beta n}{2b}}\Big[\int\limits_{\{x\in\mathbb{R}^n:|x-y|<t^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}dx$$

$$+\int\limits_{\{x\in\mathbb{R}^n:|x-y|>t^{\frac{\beta}{2b}}\}} e^{-c[|x-y|t^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}dx\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta)|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_5\Big[1+\int\limits_1^\infty z^{n-1-\frac{\beta n}{2b}}e^{-cz}dz\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$=C_6 e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}},\quad (y,t)\in\bar{Q}$$

and similarly for $n\le 2b$. Integrating by parts we finish the proof. □

Lemma 2. *If $a>0$, $\gamma\ge 1-\frac{\beta}{2b}$, $a'=c_\gamma\min\{cT^{-\frac{\beta}{2b\gamma}},a\}$, then*

$\widehat{G}_0: S_{\gamma,(a)}(\mathbb{R}^n)\to S_{\gamma,(a')}(\mathbb{R}^n)$, $\widehat{G}_1: S_{\gamma,(a)}(\mathbb{R}^n)\to S_{\gamma,(a')}(\mathbb{R}^n)$, for each $t\in[0,T]$,

$\widehat{\mathcal{G}}_0: S_{\gamma,(a)}(\bar{Q})\to S_{\gamma,(a')}(\bar{Q})$, $\widehat{\mathcal{G}}_1: S_{\gamma,(a)}(\bar{Q})\to S_{\gamma,(a')}(\mathbb{R}^n)$.

Proof. It follows from Lemma 1 the correctness of the mappings for $\widehat{G}_j$, $j=0,1$. Using the property of the convolution and convolution's differentiation we finish the proof. □

Lemma 3. *For $\gamma\ge 1$, $0<aT^{\frac{\beta}{2b\gamma}}\le c$, any $\psi\in S_{\gamma,(a)}(\bar{Q})$ the following relations hold:*

$$\begin{aligned}(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau)&=\psi(y,\tau),\quad (y,\tau)\in\bar{Q},\\ (\widehat{\mathcal{G}}_1(\widehat{L}\psi))(y)&=\big(f_{1-\beta}(\tau),\psi(y,\tau)\big),\quad y\in\mathbb{R}^n.\end{aligned}\tag{9}$$

Proof. For all $\psi\in S_{\gamma,(a)}(\bar{Q})$, $(y,s)\in\bar{Q}$ and multi-index α we have

$$(f_{-\beta}\hat{*}\psi)(y,s)=f'_{1-\beta}(s)\hat{*}\psi(y,s)=-\int_0^{T-s}\frac{q^{-\beta}}{\Gamma(1-\beta)}\frac{\partial}{\partial s}\psi(y,q+s)dq,$$

$$\Big(\frac{\partial}{\partial s}\Big)^k D_y^\alpha(f_{-\beta}\hat{*}\psi)(y,s)=(-1)^k\Big(f_{1-\beta}\hat{*}\Big(\frac{\partial}{\partial s}\Big)^{k+1}D_y^\alpha\psi\Big)(y,s).$$

Therefore, $f_{-\beta}\hat{*}\psi\in S_{\gamma,(a)}(\bar{Q})$ and $\widehat{L}\psi\in S_{\gamma,(a)}(\bar{Q})$. Then it follows from Lemmas 1 and 2 that $a'=a$ and $(\widehat{\mathcal{G}}_0(\widehat{L}\psi))\in S_{\gamma,(a)}(\bar{Q})$, $(\widehat{\mathcal{G}}_1(\widehat{L}\psi))\in S_{\gamma,(a)}(\mathbb{R}^n)$.

By [8], under rather regular (in particular, compactly supported) F_0, F_1, $g\in C[0,T]$ the unique regular solution (7) with $\Phi=F_0 g$ of the Cauchy problem (1), (2) exists. Substituting it in the Green formula (instead of v) we get

$$\int\limits_Q\Big(\int\limits_0^t d\tau\int\limits_{\mathbb{R}^n}G_0(x-y,t-\tau)F(y)g(\tau)dy\Big)(\hat{L}\psi)(x,t)dxdt$$

$$+\int\limits_Q\Big(\int\limits_{\mathbb{R}^n}G_1(x-y,t)F_1(y)dy\Big)(\hat{L}\psi)(x,t)dxdt$$

$$=\int\limits_Q F(x)g(t)\psi(x,t)dxdt+\int\limits_{\mathbb{R}^n}F_1(x)\big(f_{1-\beta}(t),\psi(x,t)\big)dx,$$

$$\int\limits_Q \Big(\int\limits_\tau^T dt \int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)(\hat{L}\psi)(x,t)dx \Big) F(y)g(\tau)dyd\tau$$

$$+ \int\limits_{\mathbb{R}^n} \Big(\int\limits_Q G_1(x-y,t)(\hat{L}\psi)(x,t)dxdt \Big) F_1(y)dy$$

$$= \int\limits_Q \psi(y,\tau)F(y,\tau)dyd\tau + \int\limits_{\mathbb{R}^n} \big(f_{1-\beta}(t), \psi(y,t)\big) F_1(y)dy,$$

and obtain the desirable formulas (9) after an arbitrariness of F_0, F_1, g. □

Lemma 4. *For any $\varphi \in S_{\gamma,(a)}(\bar{Q})$ there exists $\psi \in S_{\gamma,(a)}(\bar{Q})$ such that*

$$(\widehat{L}\psi)(x,t) = \varphi(x,t), \qquad (x,t) \in \bar{Q}.$$

Proof. As in [21], we show that

$$\psi(y,\tau) = \int\limits_\tau^T dt \int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)\varphi(x,t)dx, \ \ (y,\tau) \in \bar{Q}$$

is the unknown function. □

2 EXISTENCE AND UNIQUENESS THEOREM FOR THE CAUCHY PROBLEM

Theorem 1. *Assume that $\gamma \geq 1$, $0 < aT^{\frac{\beta}{2b\gamma}} \leq c$, $F_0, F_1 \in S'_{\gamma,(a)}(\mathbb{R}^n)$, $g \in C[0,T]$. Then there exists the unique solution $u \in S'_{\gamma,(a),C}(\bar{Q})$ of the Cauchy problem (1), (2). It is defined by*

$$\Big(u(\cdot,t), \varphi(\cdot)\Big) = \int\limits_0^t g(\tau)\Big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)d\tau + \Big(F_1(\cdot), (\widehat{G}_1\varphi)(\cdot,t)\Big)$$
$$\forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n), \ t \in [0,T]. \tag{10}$$

Moreover, for any $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ there exist positive constants $\widehat{d}_j = \widehat{d}_j(\varphi)$, $j = 0,1$ such that

$$\big|\big(u(\cdot,t), \varphi(\cdot)\big)\big| \leq \widehat{d}_0 t^\beta + \widehat{d}_1, \ \ t \in [0,T]. \tag{11}$$

Proof. Using Lemma 2 we get that for any $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ the right-hand side of (10) exists and belongs to $C[0,T]$. As in [22], we show that the function (10) satisfies the equality (4). For all $\psi \in S_{\gamma,(a)}(\overline{Q})$ we have

$$\begin{aligned}
\big(u, \widehat{L}\psi\big) &= \int\limits_0^T \Big(u(\cdot,t), (\widehat{L}\psi)(\cdot,t)\Big)dt \\
&= \int\limits_0^T \Big(\int\limits_0^t g(\tau)\Big(F_0(y), (\widehat{G}_0(\widehat{L}\psi))(y,t,\tau)\Big)d\tau\Big)dt + \int\limits_0^T \Big(F_1(y), (\widehat{G}_1(\widehat{L}\psi))(y,t)\Big)dt \\
&= \Big(F_0(y), \int\limits_0^T g(\tau)d\tau \int\limits_\tau^T (\widehat{G}_0(\widehat{L}\psi))(y,t,\tau)dt\Big) + \Big(F_1(y), \int\limits_0^T (\widehat{G}_1(\widehat{L}\psi))(y,t)dt\Big) \\
&= \Big(F_0(y) \int\limits_0^T g(\tau)(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau)d\tau\Big) + \Big(F_1(y), \widehat{\mathcal{G}}_1(\widehat{L}\psi)(y)\Big).
\end{aligned}$$

Using Lemma 3 we get the identity (4). By Definition 1 the function (10) is the solution of the problem (1), (2).

To prove the performance of (11) for the function (10) we use [2, p. 211] that

$$S_{\gamma,(a)}(\mathbb{R}^n) = \{v \in C^\infty(\mathbb{R}^n) : ||v||_{k,a} = \sup_{|\alpha| \le k, x \in \mathbb{R}^n} e^{a(1-\frac{1}{k})|x|^{\frac{1}{\gamma}}} |D^\alpha v(x)| < +\infty \ \ \forall k \in \mathbb{N},\ k \ge 2\}$$

and the sequence $v_m(x)$ converges to zero ($m \to +\infty$) in the space $S_{\gamma,(a)}(\mathbb{R}^n)$ if the sequence $D^\alpha v_m(x)$ converges to zero uniformly on an arbitrary compact $|x| \le C < +\infty$ for each multi-index α and the norms $||v_m||_{k,a}$ are limited at random $m, k \in \mathbb{N}$, $k \ge 2$. Note that

$$||v||_{k,a} \le ||v||_{k+p,a} \ \ \forall k, p \in \mathbb{N},\ \ k \ge 2,\ \ a > 0,\ \ v \in S_{\gamma,(a)}(\mathbb{R}^n).$$

We say (see [25, p. 151]) that the distribution $F \in S'_{\gamma,(a)}(\mathbb{R}^n)$ has the order $k \in \mathbb{Z}_+$ if there exists $C > 0$ such that

$$\big|(F, \varphi)\big| \le C||\varphi||_{k,a} \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n). \tag{12}$$

A distribution from $S'_{\gamma,(a)}(\mathbb{R}^n)$ has a finite order. Indeed, the functional F satisfying (12) is continuous on $S_{\gamma,(a)}(\mathbb{R}^n)$. Conversely, if $F \in S'_{\gamma,(a)}(\mathbb{R}^n)$ and (12) is incorrect, then for each $k \in \mathbb{N},\ k \ge 2$ there exists $\varphi_k \in S_{\gamma,(a)}(\mathbb{R}^n)$ such that $|(F, \varphi_k)| > k||\varphi_k||_{k,a}$. Then

$$|(F, \psi_k)| > 1, \ \ \text{where} \ \ \psi_k(x) = \frac{\varphi_k(x)}{k||\varphi_k||_{k,a}}, \ \ x \in \mathbb{R}^n.$$

By definition, $||\psi_k||_{k,a} \le \frac{1}{k}$, and the sequence $\psi_k \to 0$ ($k \to \infty$) in the space $S_{\gamma,(a)}(\mathbb{R}^n)$. We get a contradiction with the previous inequality $|(F, \psi_k)| > 1$ for all $k \in \mathbb{N},\ k \ge 2$.

So, there exist $k_j \in \mathbb{Z}_+$ and positive constants B_j such that

$$\big|(F_j, \varphi)\big| \le B_j||\varphi||_{k_j,a} \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n), \quad j = \overline{0,1}.$$

Using it and Lemma 1, for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ we get

$$\begin{aligned} \big|\big(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau)\big)\big| &\le B_0||(\widehat{G_0}\varphi)(\cdot, t-\tau)||_{k_0,a} \\ &\le B_0 c_{k_0}(t-\tau)^{\beta-1}||\varphi||_{k_0,a} \le \widehat{c}_0(\varphi)(t-\tau)^{\beta-1}||\varphi||_{k_0,a},\ 0 \le \tau < t \le T, \end{aligned}$$

$$\int\limits_0^t |g(\tau)|\ \big|\big(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau)\big)\big| d\tau \le d_0 t^\beta ||\varphi||_{k_0,a} \le \widehat{d}_0 t^\beta, \ \ \text{and similarly,}$$

$$\big|\big(F_1(\cdot), (\widehat{G}_1\varphi)(\cdot, t)\big)\big| \le B_1||(\widehat{G}_1\varphi)(\cdot, t)||_{k_1,a} \le d_1||\varphi||_{k_1,a} \le \widehat{d}_1, \quad t \in [0, T].$$

Therefore, we obtain (11) with $\widehat{d_j} = d_j||\varphi||_{k,a}$, $k = \max\{k_0, k_1\}$, and see that the solution u of the Cauchy problem has the order k for each $t \in [0, T]$.

If u_1, u_2 are two solutions of the problem (1), (2) then for $u = u_1 - u_2$ from (4) we obtain

$$(u, \widehat{L}\psi) = 0 \ \ \forall \psi \in S_{\gamma,(a)}(\overline{Q}).$$

By using Lemma 4 we get $(u(\cdot, t), \varphi(\cdot)) = 0$ for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $t \in [0, T]$. We obtain $u = 0$ in $S'_{\gamma,(a),C}(\overline{Q})$. $\square$

3 SOLUTION OF THE INVERSE PROBLEM

We pass to the problem (1)–(3).

Theorem 2. *Assume that* $\gamma \geq 1$, $0 < aT^{\frac{\beta}{2b\gamma}} \leq c$, $F_0, F_1 \in S'_{\gamma,(a)}(\mathbb{R}^n)$, $g, F, F^{(\beta)} \in C[0,T]$, $\varphi_0 \in S_{\gamma,(a)}(\mathbb{R}^n)$, $(F_0, \varphi_0) \neq 0$ *and* (5) *holds. Then there exists the unique solution* $(u,g) \in S'_{\gamma,(a),C}(\bar{Q}) \times C[0,T]$ *of the problem* (1)–(3)*: u is defined by (10) with*

$$g(t) = \left[F^{(\beta)}(t) - r(t)\right]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in [0,T], \tag{13}$$

where $r(t)$ *is the solution of the integral equation*

$$r(t) = -\int_0^t K(t,\tau) r(\tau) d\tau + v(t), \;\; t \in [0,T], \tag{14}$$

$$K(t,\tau) = \frac{\big(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)\big)}{(F_0, \varphi_0)}, \tag{15}$$

$$v(t) = \int_0^t K(t,\tau) F^{(\beta)}(\tau) d\tau + \big(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\big), \;\; t \in [0,T]. \tag{16}$$

Proof. Let $u \in S'_{\gamma,(a),C}(\bar{Q})$ be the solution of the problem (1), (2). The equation (1) implies

$$\big(u_t^{(\beta)}(\cdot, t), \varphi_0(\cdot)\big) = \big(u(\cdot, t), A\varphi_0(\cdot)\big) + (F_0, \varphi_0) g(t).$$

By the over-determination condition (3) we get

$$F^{(\beta)}(t) = \big(u(\cdot, t), A\varphi_0(\cdot)\big) + (F_0, \varphi_0) g(t).$$

Using the assumption we find

$$g(t) = \left[F^{(\beta)}(t) - \big(u(\cdot, t), A\varphi_0(\cdot)\big)\right]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in [0,T]. \tag{17}$$

By Theorem 1 the right-hand side of (17) is the continuous function on $[0,T]$. By substituting it in (10) instead of $g(t)$ and putting $\varphi = \varphi_0$ one obtains

$$\big(u(\cdot, t), A\varphi_0(\cdot)\big) = \frac{1}{(F_0, \varphi_0)} \int_0^t \left[F^{(\beta)}(\tau) - \big(u(\cdot, \tau), A\varphi_0(\cdot)\big)\right] \big(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)\big) d\tau$$
$$+ \big(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\big), \;\; t \in [0,T].$$

We denote

$$r(t) = \big(u(\cdot, t), A\varphi_0(\cdot)\big).$$

Then the previous equation takes the form of equation (14). As in the proof of Theorem 1 we get

$$\begin{aligned}\big|\big(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t, \tau)\big)\big| &\leq B_0 ||(\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)||_{k_0} \\ &\leq \widehat{C}_0 ||A\varphi_0(\cdot, t-\tau)||_{k_0} \leq \widehat{C}_0 \, (t-\tau)^{\beta-1} ||\varphi_0(\cdot, t-\tau)||_{k_0+2b}, \\ \big|\big(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\big)\big| &\leq B_1 ||(\widehat{G}_1 A\varphi_0)(\cdot, t)||_{k_1} \leq \widehat{C}_1 ||\varphi_0(\cdot, t)||_{k_1+2b}.\end{aligned}$$

So, the kernel (15) is integrable, the function (16) is continuous on $[0, T]$, and the second type Volterra integral equation (14) has the unique solution $r \in C[0, T]$.

Let r, g be defined by (14), (13), respectively. Then by Theorem 1 the function (10) is the solution of the Cauchy problem (1)–(2) with the known $g(t)$. Using the property

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^n} \varphi(x) G_1(x-y,t)dx = \varphi(0) \quad \forall \varphi \in S(\mathbb{R}^n)$$

and the condition (5) we get

$$(u(\cdot,0), \varphi_0(\cdot)) = \Big(F_1(\cdot), (\widehat{G}_1\varphi_0)(\cdot,0)\Big) = (F_1, \varphi_0) = F(0).$$

Show that the function (10) with $g(t)$ defined by (13) satisfies the condition (3). If $F^*(t) = \big(u(\cdot,t), \varphi_0(\cdot)\big)$ then $F^*(0) = F(0)$, and from the over-determination condition (3) we get

$$g(t) = \big[F^{*(\beta)}(t) - \big(u(\cdot,t), A\varphi_0(\cdot)\big)\big]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in [0, T]. \tag{18}$$

As in the previous reasoning we obtain that the function $\big(u(\cdot,t), A\varphi_0(\cdot)\big)$ satisfies the equation (14), and by uniqueness of a solution of this equation we obtain $\big(u(\cdot,t), A\varphi_0(\cdot)\big) = r(t)$ for all $t \in [0, T]$. Then it follows from (18) and (13) that $F^{*(\beta)}(t) = F^{(\beta)}(t)$, and therefore, $F^*(t) = F(t)$, $t \in [0, T]$. So, the pair (u, g) defined by (10) and (13), with r defined by (14), is the solution of the problem (1)–(3).

If (u_1, g_1), (u_2, g_2) are two solutions of the problem (1)–(3), then for $u = u_1 - u_2$, $g = g_1 - g_2$ we obtain the problem

$$Lu(x,t) = F_0(x)g(t), \quad (x,t) \in Q,$$
$$u(x,0) = 0, \quad x \in \mathbb{R}^n,$$
$$\big(u(\cdot,t), \varphi_0(\cdot)\big) = 0, \quad t \in [0, T].$$

As before, we find

$$\big(u(\cdot,t), \varphi(\cdot)\big) = -\int\limits_0^t r(\tau)\big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot, t-\tau)\big)d\tau \quad \forall \varphi \in S(\mathbb{R}^n),$$

$$g(t) = -\frac{r(t)}{(F_0, \varphi_0)}, \quad t \in [0, T],$$

where $r(t)$ is a solution of the second type homogeneous Volterra integral equation

$$r(t) = -\int\limits_0^t K(t,\tau) r(\tau) d\tau, \quad t \in [0, T].$$

By uniqueness of a solution of this equation we obtain $r(t) = 0$ for all $t \in [0, T]$. Then, from the previous equalities, $g(t) = 0$ for all $t \in [0, T]$ and $u = 0$ in $S'_{\gamma,(a),C}(\bar{Q})$. $\square$

4 CONCLUSIONS

We proved the solvability of an inverse problem of the determination a time-dependent continuous part of a source for a time fractional 2b-order equation with constant coefficients and Schwartz type distributions in the right-hand sides using the over-determination condition (3). In a such way, by using the results of [8] the obtained results extend to some case of the operator $A(x, D)$ with infinitely differentiable coefficients.

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Лопушанський А., Лопушанська Г. *Обернена задача для диференціального рівняння порядку* $2b$ *з дробовою похідною за часом* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 107–118.

Вивчаємо обернену задачу для диференціального рівняння порядку $2b$ з дробовою похідною порядку $\beta \in (0,1)$ за часом і заданими узагальненими функціями типу Шварца у правих частинах рівняння і початкової умови. Задача полягає у знаходженні пари функцій (u, g): узагальненого розв'язку u задачі Коші для такого рівняння і залежного від часу неперервного множника g у правій частині рівняння. Як додаткову умову використовуємо аналог інтегральної умови

$$(u(\cdot, t), \varphi_0(\cdot)) = F(t), \quad t \in [0, T],$$

де $(u(\cdot, t), \varphi_0(\cdot))$ — значення шуканого узагальненого розв'язку u задачі Коші на фіксованій основній функції $\varphi_0(x)$, $x \in \mathbb{R}^n$ для кожного значення t, F — задана неперервна функція.

Доводимо теорему існування і єдиності узагальненого розв'язку задачі Коші, одержуємо його зображення за допомогою вектор-функції Ґріна. Доведення теореми ґрунтується на властивостях спряжених операторів Ґріна задачі Коші на просторах типу Шварца основних функцій і структурі узагальнених функцій типу Шварца.

Встановлюємо достатні умови однозначної розв'язності оберненої задачі і знаходимо зображення невідомої функції g через розв'язок певного інтегрального рівняння Вольтерри другого роду з інтегровним ядром.

Ключові слова і фрази: узагальнена функція, похідна дробового порядку, обернена задача, вектор-функція Ґріна.

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SOME INEQUALITIES FOR STRONGLY (p, h)-HARMONIC CONVEX FUNCTIONS

In this paper, we show that harmonic convex functions f is strongly (p, h)-harmonic convex functions if and only if it can be decomposed as $g(x) = f(x) - c(\frac{1}{x^p})^2$, where $g(x)$ is (p, h)-harmonic convex function. We obtain some new estimates class of strongly (p, h)-harmonic convex functions involving hypergeometric and beta functions. As applications of our results, several important special cases are discussed. We also introduce a new class of harmonic convex functions, which is called strongly (p, h)-harmonic log-convex functions. Some new Hermite-Hadamard type inequalities for strongly (p, h)-harmonic log-convex functions are obtained. These results can be viewed as important refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

Key words and phrases: p-harmonic convex functions, h-convex functions, strongly convex functions, Hermite-Hadamard type inequalities.

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1 INTRODUCTION

Inequalities have played an important role in the developments of various fields of pure and applied sciences. Convexity theory an inequalities theory are closely related with each other. It is well known that a function is a convex function if and only if it satisfies the integral inequality which is known as the Hermite-Hadamard inequality. Hermite-Hadmard type inequalities are used to obtain the error bounds for energy functions in the material sciences. For applications and other aspects of these inequalities and their generalized invariant forms, see [3–5, 7, 9, 10, 23, 26, 27].

In recent years, convex functions have been extended and generalized in various directions using novel and innovative techniques. Varosanec [25] introduced and studied a new class of convex functions involving an arbitrary non-negative function $h(.)$, which is known as h-convex function. With an appropriate and suitable choice of arbitrary function $h(\cdot)$, one can obtain a wide class of convex functions. This idea has been used to introduce various classes of convex functions in other fields. Polyak [24] introduced the concept of strongly convex functions, which include the convex functions as special cases. Strongly convex functions played a crucial role in optimization and variational inequalities problem. Motivated and inspired by its applications, Angulo et al. [2] introduced the notion of strongly h-convex functions and have shown that this class unifies other known and new classes of strongly convex functions. The class of strongly beta-convex functions has introduced and investigated by Noor et al. [19]. They obtained some integral inequalities involving hypergeometric and beta functions. The

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harmonic convex functions were introduced and studied by Anderson et al. [1] and Iscan [6]. Noor et al. [15] have introduced a class of strongly harmonic convex functions and established some Hermite-Hadamard type integral inequalities. Noor et al. [11] also introduced the concept of p-harmonic means, which includes the harmonic means, arithmetic means and geometric mean as special cases. Using this concept, they introduced and investigated the properties of p-harmonic convex sets and the p-harmonic convex functions. It have been shown that the p-harmonic convex functions include the harmonic convex functions and convex functions as special cases. For recent developments and generalizations, see [12–14, 16, 17, 20, 21].

Inspired and motivated by the ongoing research, Noor et al. [22] have introduced a concept of strongly (p, h)-harmonic convex functions with respect to an arbitrary non-negative function $h(\cdot)$ and obtained the integral inequalities. This class is more general and contains several known and new classes of convex functions as special cases. In this paper, study those conditions under which a function $f(.)$ is a strongly (p, h)-harmonic convex function, if it can be decomposed as $g(x) = f(x) - (\frac{1}{x^p})^2$, where $g(.)$ is (p, h)-harmonic convex functions. Some new estimates for the integral $\int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x$ in terms of hypergeometric and beta functions are obtained. Some special cases are discussed as applications of these new estimates. In addition, we introduce and study the strongly (p, h)-harmonic log-convex functions, which is quite general and unifying one. Hermite-Hadamard type integral inequalities are obtained. We would like to emphasize that the ideas and technqiues of this paper may stimulate further research in this dynamic field.

2 PRELIMINARIES

In this section, we introduce some new classes of harmonic convex functions. Throughout the paper, we will take $p \in \mathbb{R}$ and $I = [a, b] \subset (0, \infty)$ be an interval, unless otherwise specified.

Definition 1 ([11]). *A set I is said to be a p-harmonic convex set, if*

$$\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}} \in I, \qquad \forall x, y \in I,\ t \in [0,1].$$

We would like to point out that if $p = 1$, then p-harmonic convex set becomes harmonic convex set. If $p = -1$, then p-harmonic convex set becomes convex set and if $p = 0$, then p-harmonic convex set becomes geometrically convex set. This shows that the concept of p-harmonic convex set is quite general and unifying one.

Definition 2 ([11]). *Let I be a p-harmonic convex set. A function $f : I \to \mathbb{R}$ is said to p-harmonic convex, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

Noor et al. [11] have obtained the Hermite-Hadamard inequality for p-harmonic convex functions, which may be regarded as a refinement of the concept of convexity. In particular, it has been shown that f is a p-harmonic convex function, if and only if,

$$\begin{aligned} f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) &\le \frac{1}{2}\left[f\left(\left[\frac{4a^pb^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{4a^pb^p}{a^p+3b^p}\right]^{\frac{1}{p}}\right)\right] \\ &\le \frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x \\ &\le \frac{1}{2}\left[f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{f(a)+f(b)}{2}\right] \le \frac{1}{2}[f(a)+f(b)]. \end{aligned} \tag{1}$$

The inequality (1) holds in reversed direction, if f is a p-harmonic concave function.

Definition 3 ([22]). *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be strongly (p,h)-harmonic convex function with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \le h(1-t)f(x)+h(t)f(y)-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2. \tag{2}$$

The function f is said to be strongly (p,h)-harmonic concave function, if and only if, $-f$ is strongly (p,h)-harmonic convex function. For $t=\frac{1}{2}$ in (2), we have

$$f\left(\frac{2x^py^p}{x^p+y^p}\right)^{\frac{1}{p}} \le h\left(\frac{1}{2}\right)[f(x)+f(y)]-\frac{c}{4}\left(\frac{x^p-y^p}{x^py^p}\right)^2, \quad x,y\in I. \tag{3}$$

The function f is called Jensen strongly (p,h)-harmonic convex function.

For $h(t)=h(t)h(1-t)$, in Definition 3, we obtain a new class of p-harmonic convex functions, called relative strongly p-harmonic tgs-convex functions.

Definition 4. *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be relative strongly p-harmonic tgs-convex with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \le h(t)h(1-t)[f(x)+f(y)]-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2.$$

Remark 1. *(i) If $p=1$ in Definition 3, then it reduces to strongly harmonic h-convex functions introduced by Noor et al.* [18].

(ii) If $p=-1$ in Definition 3, then it reduces to strongly h-convex functions [2].

(iii) If $p=0$ in Definition 3, then it reduces to strongly geometrically h-convex functions.

Definition 5. *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be strongly geometrically h-convex function with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(x^{1-t}y^t\right) \le h(1-t)f(x)+h(t)f(y)-ct(1-t)\left(\ln x-\ln y\right)^2.$$

Now we discuss some special cases of strongly (p,h)-harmonic convex functions, which appears to be new ones.

I. If $h(t)=t$ in Definition 3, then it reduces to:

Definition 6. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic convex with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)f(x) + tf(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2, \quad \forall x, y \in I, t \in (0,1).$$

II. If $h(t) = t^s$ in Definition 3, then it reduces to:

Definition 7. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic s-convex function in second sense with modulus $c > 0$, where $s \in [-1, 1]$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)^s f(x) + t^s f(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2, \quad \forall x, y \in I, t \in (0,1).$$

III. If $h(t) = t^s(1-t)^s$ in Definition 3, then it reduces to:

Definition 8. *A function $f : I \to \mathbb{R}$ is said to be generalized strongly p-harmonic s-convex with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le t^s(1-t)^s[f(x) + f(y) - c\left(\frac{x^p - y^p}{x^p y^p}\right)^2], \quad \forall x, y \in I, t \in (0,1).$$

IV. If $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$ in Definition 3, then it reduces to the definition of strongly p-harmonic beta-convex functions.

Definition 9. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic beta-convex with modulus $c > 0$, where $\mathfrak{p}, \mathfrak{q} > -1$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)^{\mathfrak{p}} t^{\mathfrak{q}} f(x) + t^{\mathfrak{p}}(1-t)^{\mathfrak{q}} f(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2$$
$$\forall x, y \in I,\ t \in (0,1).$$

If $p = 1, -1, 0$, then Definition 9 reduces to the definition of strongly harmonic beta-convex, strongly beta-convex functions and strongly geometrically beta-convex functions, respectively.

Since strongly (p,h)-harmonic convexity is a strengthening of the notion of (p,h)-harmonic convexity, some properties of strongly (p,h)-harmonic convex functions are just stronger version of known properties of (p,h)-harmonic convex functions. Using the technique of Nikodem [8] and Noor et al. [15], we prove the following result which shows the relationships between strongly (p,h)-harmonic convex (strongly (p,h)-harmonic mid-convex) and (p,h)-harmonic convex ((p,h)-harmonic mid-convex) functions.

Lemma 1. *i). Let a function $f : I \subset (0,\infty) \to \mathbb{R}$ be strongly (p,h)-harmonic convex function with modulus c. If $h(t) \le t$, then the function $g(x) = f(x) - c(\frac{1}{x^p})^2$ is (p,h)-harmonic convex.*
ii). Let a function $f : I \subset (0,\infty) \to \mathbb{R}$ be strongly (p,h)-harmonic mid convex with modulus c. If $h(t) \le t$, then the function $g(x) = f(x) - c(\frac{1}{x^p})^2$ is (p,h)-harmonic mid convex function.

Proof. i) Assume that f is strongly (p,h)-harmonic convex with modulus c. Using properties of the inner product, we have

$$\begin{aligned}
g\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) - c\left(\frac{tx^p+(1-t)y^p}{x^py^p}\right)^2 \\
&\le h(1-t)f(x)+h(t)f(y)- ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2 - c\left(\frac{tx^p+(1-t)y^p}{x^py^p}\right)^2 \\
&= h(1-t)f(x)+h(t)f(y)-c\Bigg(t(1-t)\left[\left(\frac{1}{x^p}\right)^2-\frac{2}{x^py^p}+\left(\frac{1}{y^p}\right)^2\right] \\
&\quad +(1-t)^2\left(\frac{1}{x^p}\right)^2+\frac{2t(1-t)}{x^py^p}+t^2\left(\frac{1}{y^p}\right)^2\Bigg) \\
&= h(1-t)f(x)+h(t)f(y)-c(1-t)\left(\frac{1}{x^p}\right)^2-ct\left(\frac{1}{y^p}\right)^2 \\
&\le h(1-t)f(x)+h(t)f(y)-ch(1-t)\left(\frac{1}{x^p}\right)^2-ch(t)\left(\frac{1}{y^p}\right)^2 \\
&= h(1-t)g(x)+h(t)g(y),
\end{aligned}$$

which gives that g is (p,h)-harmonic convex function.
ii) Let f be strongly (p,h)-harmonic mid convex with modulus c. Then

$$\begin{aligned}
g\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) - c\left(\frac{x^p+y^p}{2x^py^p}\right)^2 \\
&\le h\left(\frac{1}{2}\right)[f(x)+f(y)] - \frac{c}{4}\left(\frac{x^p-y^p}{x^py^p}\right)^2 - \frac{c}{4}\left(\frac{x^p+y^p}{x^py^p}\right)^2 \\
&= h\left(\frac{1}{2}\right)[f(x)+f(y)] - \frac{c}{4}\left[2\left(\frac{1}{x^p}\right)^2+2\left(\frac{1}{y^p}\right)^2\right] \\
&\le h\left(\frac{1}{2}\right)[f(x)+f(y)] - ch\left(\frac{1}{2}\right)\left[\left(\frac{1}{x^p}\right)^2+\left(\frac{1}{y^p}\right)^2\right] \\
&= h\left(\frac{1}{2}\right)[g(x)+g(y)],
\end{aligned}$$

which gives that g is (p,h)-harmonic mid convex function. □

Remark 2. *Under the condition $h(t) \ge t$, the converse of the Lemma 1 holds.*

The Euler Beta function is a special function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x,y>0,$$

where $\Gamma(\cdot) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$ is a gamma function. The integral form of hypergeometric function is defined as:

$$
{}_2F_1[a,b;c;z] = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t,
$$

where $|z| < 1$, $c > b > 0$.

3 INTEGRAL INEQUALITIES

Some new and interesting estimates of the integral via strongly (p,h)-harmonic convex functions are obtained. These estimates can be viewed as refined bounds of the quadrature formula of Guass-Jacobi type. The quadrature formula of Guass-Jacobi type has the form

$$
\int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + R_m[f],
$$

for some $B_{m,k}$, γ_k and the remainder term $R_m[f]$.

Lemma 2. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$, then the following equality holds for some fixed $\alpha, \beta > 0$.*

$$
\int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x = a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} f\left(\frac{ab}{A_t}\right) \mathrm{d}t,
$$

where $A_t = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and $L[a,b]$ is the space of Lebesque integrable functions on $[a,b]$.

Theorem 1. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$
\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x \right|
\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\omega_1 |f(a)| + \omega_2 |f(b)| - c \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \omega_3 \right),
$$

where

$$
\omega_1 = \int_0^1 \frac{t^\alpha (1-t)^\beta h(1-t)}{A_t^{\alpha+\beta+2}} \mathrm{d}t, \tag{4}
$$

$$
\omega_2 = \int_0^1 \frac{t^\alpha (1-t)^\beta h(t)}{A_t^{p\alpha+p\beta+p+1}} \mathrm{d}t, \tag{5}
$$

$$
\begin{aligned}
\omega_3 &= \int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}} \mathrm{d}t \\
&= \frac{B(\alpha+2, \beta+2)}{b^{p\alpha+p\beta+p+1}} {}_2F_1\left[\alpha+\beta+1+1/p, \alpha+2; \alpha+\beta+4, 1 - \frac{a^p}{b^p}\right].
\end{aligned} \tag{6}
$$

Proof. Using Lemma 2 and strongly (p,h)-harmonic convexity of $|f|$, we have

$$\begin{aligned}
&\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\
&\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\Big\{h(1-t)|f(a)| \\
&\quad + h(t)|f(b)| - ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\Big\}\mathrm{d}t \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\Bigg(|f(a)|\int_0^1 \frac{t^\alpha(1-t)^\beta h(1-t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t \\
&\quad + |f(b)|\int_0^1 \frac{t^\alpha(1-t)^\beta h(t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\Bigg) \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\omega_1|f(a)| + \omega_2|f(b)| - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\omega_3\right).
\end{aligned}$$

This completes the proof. □

Corollary 1. *Under the conditions of Theorem 1 with $p=1$ and $h(t)=t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, we have*

$$\begin{aligned}
&\left|\int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x\right| \\
&\le a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\left(|f(a)|\omega_1^* + |f(b)|\omega_2^* - c\left(\frac{a-b}{ab}\right)^2\omega_3^*\right),
\end{aligned}$$

where

$$\begin{aligned}
\omega_1^* &= \int_0^1 \frac{t^{\alpha+\mathfrak{q}}(1-t)^{\beta+\mathfrak{p}}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+\mathfrak{q}+1,\beta+\mathfrak{p}+1)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+\mathfrak{q}+1;\alpha+\beta+\mathfrak{p}+\mathfrak{q}+2,1-\frac{a}{b}\right],
\end{aligned} \tag{7}$$

$$\begin{aligned}
\omega_2^* &= \int_0^1 \frac{t^{\alpha+\mathfrak{p}}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+\mathfrak{p}+1,\beta+\mathfrak{q}+1)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+\mathfrak{p}+1;\alpha+\beta+\mathfrak{p}+\mathfrak{q}+2,1-\frac{a}{b}\right],
\end{aligned} \tag{8}$$

$$\begin{aligned}
\omega_3^* &= \int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+2,\beta+2)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+2;\alpha+\beta+4,1-\frac{a}{b}\right].
\end{aligned} \tag{9}$$

Theorem 2. *If $f: I\to\mathbb{R}$ is a function such that $f\in L[a,b]$ and $|f|^q$ is strongly (p,h)-harmonic convex function, $\alpha,\beta>0$, $q\ge 1$, then*

$$\begin{aligned}
&\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| \\
&\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_4)^{1-\frac{1}{q}}\left(|f(a)|^q\omega_1 + |f(b)|^q\omega_2 - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\omega_3\right)^{\frac{1}{q}},
\end{aligned}$$

where ω_1, ω_2 *and* ω_3 *are given by* (4), (5) *and* (6) *respectively and*

$$\begin{aligned}\omega_4 &= \int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\\ &= \frac{B(\alpha+1,\beta+1)}{b^{p\alpha+p\beta+p+1}}\,{}_2F_1\Big[\alpha+\beta+1+1/p,\alpha+1;\alpha+\beta+2,1-\frac{a^p}{b^p}\Big].\end{aligned}$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^{\mathfrak{q}}$ and power mean inequality, we have

$$\begin{aligned}&\left|\int_a^b (x^p-a^p)^\alpha(b^p-x^p)^\beta f(x)\mathrm{d}x\right|\\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\\ &\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|^{\mathfrak{q}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}\\ &\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\right)^{1-\frac{1}{\mathfrak{q}}}\\ &\quad\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left\{h(1-t)|f(a)|^{\mathfrak{q}}+h(t)|f(b)|^{\mathfrak{q}}-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right\}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}\\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\right)^{1-\frac{1}{\mathfrak{q}}}\\ &\quad\left(|f(a)|^{\mathfrak{q}}\int_0^1 \frac{t^\alpha(1-t)^\beta h(1-t)}{A_t^{p\alpha+p\beta+1+p}}\mathrm{d}t+|f(b)|^{\mathfrak{q}}\int_0^1 \frac{t^\alpha(1-t)^\beta h(t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\right.\\ &\quad\left.-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}\\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_4)^{1-\frac{1}{\mathfrak{q}}}\left(|f(a)|^{\mathfrak{q}}\omega_1+|f(b)|^{\mathfrak{q}}\omega_2-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\omega_3\right)^{\frac{1}{\mathfrak{q}}},\end{aligned}$$

which is the required result. □

Corollary 2. *Under the conditions of Theorem* 2 *with* $p=1$ *and* $h(t)=t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, *we have*

$$\begin{aligned}&\left|\int_a^b (x-a)^\alpha(b-x)^\beta f(x)\mathrm{d}x\right|\\ &\qquad\le a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}(\omega_4^*)^{1-\frac{1}{q}}\left(|f(a)|^q\omega_1^*+|f(b)|^q\omega_2^*-c\left(\frac{a-b}{ab}\right)^2\omega_3^*\right)^{\frac{1}{q}},\end{aligned}$$

where ω_1^*, ω_2^* *and* ω_3^* *are given by* (7), (8) *and* (9) *respectively, and*

$$\begin{aligned}\omega_4^* &= \int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\mathrm{d}t\\ &= \frac{B(\alpha+1,\beta+1)}{b^{\alpha+\beta+2}}\,{}_2F_1\Big[\alpha+\beta+2,\alpha+1;\alpha+\beta+2,1-\frac{a}{b}\Big].\end{aligned}$$

Theorem 3. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^q$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right|$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t)\mathrm{d}t - \frac{c}{6} \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ and

$$\begin{aligned} \omega_5 &= \int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t \\ &= \frac{B(\alpha r+1, \beta r+1)}{b^{(p\alpha+p\beta+p+1)r}} \, {}_2F_1[(\alpha+\beta+1+1/p)r, \alpha r+1; (\alpha+\beta)r+2, 1-\frac{a^p}{b^p}]. \end{aligned}$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^q$ and the Hölder's integral inequality, we have

$$\begin{aligned} &\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right| \\ &= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} \left| f\left(\frac{ab}{A_t} \right) \right| \mathrm{d}t \\ &\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t \right)^{\frac{1}{r}} \left(\int_0^1 \left| f\left(\frac{ab}{A_t} \right) \right|^q \mathrm{d}t \right)^{\frac{1}{q}} \\ &\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t \right)^{\frac{1}{r}} \\ &\quad \left(\int_0^1 \left\{ h(1-t)|f(a)|^q + h(t)|f(b)|^q - ct(1-t) \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right\} \mathrm{d}t \right)^{\frac{1}{q}} \\ &= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t)\mathrm{d}t - \frac{c}{6} \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. □

Corollary 3. *Under the conditions of Theorem 3 with $p = 1$ and $h(t) = t^p(1-t)^q$, we have*

$$\left| \int_a^b (x-a)^\alpha (b-x)^\beta f(x) \mathrm{d}x \right|$$
$$\leq a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t)\mathrm{d}t - \frac{c}{6} \left(\frac{a-b}{ab} \right)^2 \right)^{\frac{1}{q}},$$

where

$$\begin{aligned} \omega_5^* &= \int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(\alpha+\beta+2)r}} \mathrm{d}t \\ &= \frac{B(\alpha r+1, \beta r+1)}{b^{(\alpha+\beta+2)r}} \, {}_2F_1\left[(\alpha+\beta+2)r, \alpha r+1; (\alpha+\beta)r+2, 1-\frac{a}{b} \right]. \end{aligned}$$

Theorem 4. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^q$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left|\int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x\right| \le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1) \times \left(|f(a)|^q \omega_6 + |f(b)|^q \omega_7 - c\left(\frac{a^p - b^p}{a^p b^p}\right)^2 \omega_8\right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ and

$$\omega_6 = \int_0^1 \frac{h(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}}\mathrm{d}t,$$

$$\omega_7 = \int_0^1 \frac{h(t)}{A_t^{(p\alpha+p\beta+p+1)q}}\mathrm{d}t,$$

$$\omega_8 = \int_0^1 \frac{t(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}}\mathrm{d}t = \frac{1}{6b^{(p\alpha+p\beta+p+1)q}}\ {}_2F_1\left[(\alpha+\beta+1+1/p)q, 2; 4, 1 - \frac{a^p}{b^p}\right].$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^q$ and the Hölder's integral inequality, we have

$$\left|\int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x\right| = a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} \left|f\left(\frac{ab}{A_t}\right)\right| \mathrm{d}t$$

$$\le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r}(1-t)^{\beta r}\mathrm{d}t\right)^{\frac{1}{r}} \left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)q}} \left|f\left(\frac{ab}{A_t}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

$$\le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r}(1-t)^{\beta r}\mathrm{d}t\right)^{\frac{1}{r}}$$

$$\left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)q}} \left\{h(1-t)|f(a)|^q + h(t)|f(b)|^q - ct(1-t)\left(\frac{a^p - b^p}{a^p b^p}\right)^2\right\}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r}(1-t)^{\beta r}\mathrm{d}t\right)^{\frac{1}{r}}$$

$$\left(|f(a)|^q \int_0^1 \frac{h(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}}\mathrm{d}t + |f(b)|^q \int_0^1 \frac{h(t)}{A_t^{(p\alpha+p\beta+p+1)q}}\mathrm{d}t - c\left(\frac{a^p - b^p}{a^p b^p}\right)^2 \int_0^1 \frac{t(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}}\right)^{\frac{1}{q}}$$

$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1) \left(|f(a)|^q \omega_6 + |f(b)|^q \omega_7 - c\left(\frac{a^p - b^p}{a^p b^p}\right)^2 \omega_8\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 4. *Under the conditions of Theorem 4 with $p = 1$ and $h(t) = t^p(1-t)^q$, we have*

$$\left|\int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x\right| \le a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1) \left(|f(a)|^q \omega_6^* + |f(b)|^q \omega_7^* - c\left(\frac{a-b}{ab}\right)^2 \omega_8^*\right)^{\frac{1}{q}},$$

where

$$\omega_6^* = \int_0^1 \frac{t^{\mathfrak{q}}(1-t)^{\mathfrak{p}}}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{B(\mathfrak{p}+1,\mathfrak{q}+1)}{b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, \mathfrak{q}+1; \mathfrak{p}+\mathfrak{q}+2, 1-\frac{a}{b}\Big],$$

$$\omega_7^* = \int_0^1 \frac{t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{B(\mathfrak{p}+1,\mathfrak{q}+1)}{b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, \mathfrak{p}+1; \mathfrak{p}+\mathfrak{q}+2, 1-\frac{a}{b}\Big],$$

$$\omega_8^* = \int_0^1 \frac{t(1-t)}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{1}{6b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, 2; 4, 1-\frac{a}{b}\Big].$$

Theorem 5. *If $f: I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^{\mathfrak{q}}$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| \le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} (\omega_9)^{\frac{1}{r}}$$
$$\left(|f(a)|^q\omega_{10} + |f(b)|^q\omega_{11} - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 B(\alpha q+2, \beta q+2)\right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r}+\frac{1}{q} = 1$ and

$$\omega_9 = \int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)r}}\mathrm{d}t = \frac{{}_2F_1[(\alpha+\beta+1+1/p)r, 1; 2, 1-\frac{a^p}{b^p}]}{b^{(p\alpha+p\beta+p+1)r}},$$

$$\omega_{10} = \int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}} h(1-t)\mathrm{d}t, \qquad \omega_{11} = \int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}} h(t)\mathrm{d}t.$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^{\mathfrak{q}}$ and the Hölder's integral inequality, we have

$$\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| = a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t$$

$$\le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathfrak{r}}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{r}}} \left(\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}\left|f\left(\frac{ab}{A_t}\right)\right|^{\mathfrak{q}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}$$

$$\le a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathfrak{r}}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{r}}}$$
$$\left(\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}\left\{h(1-t)|f(a)|^{\mathfrak{q}} + h(t)|f(b)|^{\mathfrak{q}} - ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right\}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}$$

$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathfrak{r}}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{r}}}$$
$$\Bigg(|f(a)|^{\mathfrak{q}}\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(1-t)\mathrm{d}t + |f(b)|^{\mathfrak{q}}\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(t)\mathrm{d}t$$
$$- c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 \int_0^1 t^{\alpha\mathfrak{q}+1}(1-t)^{\beta\mathfrak{q}+1}\mathrm{d}t\Bigg)^{\frac{1}{\mathfrak{q}}}$$

$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p-a^p)^{\alpha+\beta+1} (\omega_9)^{\frac{1}{\mathfrak{r}}}$$
$$\left(|f(a)|^{\mathfrak{q}}\omega_{10} + |f(b)|^{\mathfrak{q}}\omega_{11} - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 B(\alpha\mathfrak{q}+2, \beta\mathfrak{q}+2)\right)^{\frac{1}{\mathfrak{q}}}.$$

This completes the proof. □

Corollary 5. *Under the conditions of Theorem 5 with $p = 1$ and $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, we have*

$$\left|\int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x\right|$$
$$\leq a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}(\omega_9)^{\frac{1}{r}}\Bigg(|f(a)|^q B(\alpha q+\mathfrak{q}+1,\beta q+\mathfrak{p}+1)$$
$$+|f(b)|^q B(\alpha q+\mathfrak{p}+1,\beta q+\mathfrak{q}+1)-c\left(\frac{a-b}{ab}\right)^2 B(\alpha q+2,\beta q+2)\Bigg)^{\frac{1}{q}},$$

where

$$\omega_9^* = \int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)r}}\mathrm{d}t = \frac{{}_2F_1[(\alpha+\beta+2)r,1;2,1-\frac{a}{b}]}{b^{(\alpha+\beta+2)r}}.$$

Remark 3. *For $p = -1$ and $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, our results reduces to the previously known results obtained by Noor et al.* [19] *for strongly beta-convex functions.*

4 STRONGLY p-HARMONIC log-CONVEX FUNCTIONS

In this section, we define the class of strongly (p,h)-harmonic log-convex functions and obtain the integral inequalities.

Definition 10. *Let $h : J = [0,1] \to \mathbb{R}$ a nonnegative function. A function $f : I \to (0,\infty)$ is said to be strongly (p,h)-harmonic* log-*convex function with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)\left(\frac{x^p-y^p}{x^p y^p}\right)^2.$$

For $t = \frac{1}{2}$, we have

$$f\left(\left[\frac{2x^p y^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) \leq \big(f(x)f(y)\big)^{h(\frac{1}{2})} - \frac{c}{4}\left(\frac{x^p-y^p}{x^p y^p}\right)^2, \qquad \forall x,y\in I.$$

The function f is called Jensen type strongly (p,h)-harmonic log-convex function.

Now we discuss some special cases of Definition 10.

I. If $p = 1$, then Definition 10 reduces to:

Definition 11. *Let $h : J = [0,1] \to \mathbb{R}$ a nonnegative function. A function $f : I \to (0,\infty)$ is said to be strongly h-harmonic* log-*convex function with modulus $c > 0$, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)\left(\frac{x-y}{xy}\right)^2.$$

II. If $p = -1$, then Definition 10 reduces to:

Definition 12. *Let $h : J = [0,1] \to \mathbb{R}$ a nonnegative function. A function $f : I \to (0,\infty)$ is said to be strongly h-*log-*convex function with modulus $c > 0$, if*

$$f((1-t)x+ty) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)(x-y)^2.$$

III. If $p=0$, then Definition 10 reduces to:

Definition 13. *Let $h: J=[0,1]\to\mathbb{R}$ a nonnegative function. A function $f: I\to(0,\infty)$ is said to be strongly h-geometrically* log-*convex function with modulus $c>0$, if*

$$f\left(x^{1-t}y^{t}\right)\leq\left(f(x)\right)^{h(1-t)}\left(f(y)\right)^{h(t)}-ct(1-t)\left(\ln x-\ln y\right)^{2}.$$

We now consider the following definitions of special means, which are used in our coming results. For arbitrary $a,b(a\neq b)\in(0,\infty)$, we have

1) the arithmetic mean
$$A(a,b)=\frac{a+b}{2};$$

2) the generalized logarithmic mean
$$L_{\rho}(a,b)=\begin{cases}\left[\frac{b^{\rho+1}-a^{\rho+1}}{(\rho+1)(b-a)}\right]^{\frac{1}{\rho}}, & \rho\neq-1,0,\\ \frac{a-b}{\log a-\log b}, & a\neq b,\ \rho=-1,\\ \frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}, & \rho=0.\end{cases}$$

Under the assumptions of Definition 10 with $h(t)=t$, we obtain the Hermite-Hadamard inequalities for strongly p-harmonic log-convex functions.

Theorem 6. *Let $f: I\to(0,\infty)$ be strongly p-harmonic* log-*convex function with modulus $c>0$. If $f\in L[a,b]$, then*

$$\frac{a^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1+p}}\mathrm{d}x\leq\frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}.$$

Proof. Let f be strongly p-harmonic log-convex function with modulus $c>0$. Then

$$\begin{aligned}\frac{a^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1+p}}\mathrm{d}x &= \int_{0}^{1}f\left(\left[\frac{a^{p}b^{p}}{ta^{p}+(1-t)b^{p}}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ &\leq \int_{0}^{1}\left[\left(f(a)\right)^{1-t}\left(f(b)\right)^{t}-ct(1-t)\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\right]\mathrm{d}t\\ &= f(a)\int_{0}^{1}\left(\frac{f(b)}{f(a)}\right)^{t}\mathrm{d}t-\int_{0}^{1}ct(1-t)\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\mathrm{d}t\\ &= \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\\ &= L(f(a),f(b))-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\\ &\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}.\end{aligned}$$

□

Theorem 7. *Let $f,g : I \to (0,\infty)$ be strongly p-harmonic* log*-convex functions with modulus $c>0$. If $f,g \in L[a,b]$, then*

$$\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x\\
&\quad\le \alpha\frac{f(a)+f(b)}{2}\Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}} + \beta\frac{g(a)+g(b)}{2}\Big[L_{(\frac{1}{\beta}-1)}(g(b),g(a))\Big]^{\frac{\beta}{1-\beta}}\\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]\\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b))-\ln(g(a))]^2}\big[A(g(a),g(b))-L(g(a),g(b))\big] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}.
\end{aligned}$$

Proof. Let f,g be strongly p-harmonic log-convex functions with modulus $c>0$. Then

$$\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x = \int_0^1 f\bigg(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\bigg)g\bigg(\Big[\frac{a^pb^p}{(1-t)a^p+tb^p}\Big]^{\frac{1}{p}}\bigg)\mathrm{d}t\\
&\le \int_0^1 \Big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t - ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\Big[\big(g(a)\big)^t\big(g(b)\big)^{1-t} - ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\mathrm{d}t\\
&= \int_0^1 \big(f(a)\big)^{1-t}\big(f(b)\big)^t\big(g(a)\big)^t\big(g(b)\big)^{1-t}\mathrm{d}t - c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)\int_0^1 t(1-t)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t\\
&\quad - c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 g(b)\int_0^1 t(1-t)\bigg(\frac{g(a)}{g(b)}\bigg)^t\mathrm{d}t + c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4\int_0^1 t^2(1-t)^2\mathrm{d}t\\
&\le \int_0^1 \alpha\big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t\big]^{\frac{1}{\alpha}} + \beta\big[\big(g(a)\big)^t\big(g(b)\big)^{1-t}\big]^{\frac{1}{\beta}}\mathrm{d}t - \frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]}\int_0^1 (2t-1)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t\\
&\quad - \frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 g(b)}{\ln[\frac{g(a)}{g(b)}]}\int_0^1 (2t-1)\bigg(\frac{g(a)}{g(b)}\bigg)^t\mathrm{d}t + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&= \alpha^2\big(f(a)\big)^{\frac{1}{\alpha}}\int_0^{\frac{1}{\alpha}}\bigg(\frac{f(b)}{f(a)}\bigg)^w\mathrm{d}w + \beta^2\big(g(b)\big)^{\frac{1}{\beta}}\int_0^{\frac{1}{\beta}}\bigg(\frac{g(a)}{g(b)}\bigg)^w\mathrm{d}w\\
&\quad - \frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]}\Bigg[\frac{f(a)+f(b)}{f(a)\ln[\frac{f(b)}{f(a)}]} - 2\int_0^1\frac{\big(\frac{f(b)}{f(a)}\big)^t}{\ln[\frac{f(b)}{f(a)}]}\mathrm{d}t\Bigg]\\
&\quad - \frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 g(b)}{\ln[\frac{g(a)}{g(b)}]}\Bigg[\frac{g(a)+g(b)}{g(b)\ln[\frac{g(a)}{g(b)}]} - 2\int_0^1\frac{\big(\frac{g(a)}{g(b)}\big)^t}{\ln[\frac{g(a)}{g(b)}]}\mathrm{d}t\Bigg] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&= \alpha^2\frac{\big(f(b)\big)^{\frac{1}{\alpha}} - \big(f(a)\big)^{\frac{1}{\alpha}}}{\ln(f(b))-\ln(f(a))} + \beta^2\frac{\big(g(b)\big)^{\frac{1}{\beta}} - \big(g(a)\big)^{\frac{1}{\beta}}}{\ln(g(b))-\ln(g(a))}\\
&\quad - c\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Bigg[\frac{f(a)+f(b)}{[\ln(f(b))-\ln(f(a))]^2} - \frac{2f(b)-2f(a)}{[\ln(f(b))-\ln(f(a))]^3}\\
&\quad + \frac{g(a)+g(b)}{[\ln(g(a))-\ln(g(b))]^2} - \frac{2g(a)-2g(b)}{[\ln(g(a))-\ln(g(b))]^3}\Bigg] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}
\end{aligned}$$

$$
\begin{aligned}
&=\alpha^2\frac{(f(b))^{\frac{1}{\alpha}}-(f(a))^{\frac{1}{\alpha}}}{f(b)-f(a)}L(f(b),f(a))+\beta^2\frac{(g(b))^{\frac{1}{\beta}}-(g(a))^{\frac{1}{\beta}}}{g(b)-g(a)}L(g(b),g(a))\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b))-\ln(g(a))]^2}\big[A(g(a),g(b))-L(g(a),g(b))\big]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&=\alpha\Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}}L(f(b),f(a))+\beta\Big[L_{(\frac{1}{\beta}-1)}(g(a),g(b))\Big]^{\frac{\beta}{1-\beta}}L(g(b),g(a))\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b))-\ln(g(a))]^2}\big[A(g(a),g(b))-L(g(a),g(b))\big]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&\le\alpha\frac{f(a)+f(b)}{2}\Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}}+\beta\frac{g(a)+g(b)}{2}\Big[L_{(\frac{1}{\beta}-1)}(g(b),g(a))\Big]^{\frac{\beta}{1-\beta}}\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b))-\ln(g(a))]^2}\big[A(g(a),g(b))-L(g(a),g(b))\big]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30},
\end{aligned}
$$

which is the required result. ☐

Theorem 8. *Let $f : I \to (0,\infty)$ be strongly p-harmonic* log*-convex function with modulus $c>0$. If $f\in L[a,b]$, then*

$$
\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b\frac{f(x)f\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x\\
&\qquad\le f(a)f(b)-\frac{4c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}.
\end{aligned}
$$

Proof. Let f be strongly harmonic log-convex functions with modulus $c>0$. Then

$$
\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b\frac{f(x)f\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x=\int_0^1 f\bigg(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\bigg)f\bigg(\Big[\frac{a^pb^p}{(1-t)a^p+tb^p}\Big]^{\frac{1}{p}}\bigg)\mathrm{d}t\\
&\le\int_0^1\Big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t-ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\Big[\big(f(a)\big)^t\big(f(b)\big)^{1-t}-ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\mathrm{d}t\\
&=\int_0^1 f(a)f(b)\mathrm{d}t-c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)\int_0^1 t(1-t)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t\\
&\qquad-c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(b)\int_0^1 t(1-t)\bigg(\frac{f(a)}{f(b)}\bigg)^t\mathrm{d}t+^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4\int_0^1 t^2(1-t)^2\mathrm{d}t\\
&=f(a)f(b)-\frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]}\int_0^1(2t-1)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(b)}{\ln[\frac{f(a)}{f(b)}]}\int_0^1 (2t-1)\left(\frac{f(a)}{f(b)}\right)^t \mathrm{d}t+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\\
&= f(a)f(b)-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]}\left[\frac{f(a)+f(b)}{f(a)\ln[\frac{f(b)}{f(a)}]}-2\int_0^1 \frac{\left(\frac{f(b)}{f(a)}\right)^t}{\ln[\frac{f(b)}{f(a)}]}\mathrm{d}t\right]\\
&-\frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(b)}{\ln[\frac{f(a)}{f(b)}]}\left[\frac{f(a)+f(b)}{f(b)\ln[\frac{f(a)}{f(b)}]}-2\int_0^1 \frac{\left(\frac{f(a)}{f(b)}\right)^t}{\ln[\frac{f(a)}{f(b)}]}\mathrm{d}t\right]+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\\
&= f(a)f(b)-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{f(a)+f(b)}{[\ln(f(b))-\ln(f(a))]^2}-\frac{2f(b)-2f(a)}{[\ln(f(b))-\ln(f(a))]^3}\right.\\
&\left.+\frac{f(a)+f(b)}{[\ln(f(a))-\ln(f(b))]^2}-\frac{2f(a)-2f(b)}{[\ln(f(a))-\ln(f(b))]^3}\right]+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30}\\
&= f(a)f(b)-\frac{4c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{[\ln(f(b))-\ln(f(a))]^2}\left[A(f(a),f(b))-L(f(a),f(b))\right]+\frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30},
\end{aligned}
$$

which is the required result. □

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Нур М.А., Нур К.І., Іфтіхар С. *Деякі нерівності для сильно (p,h)-гармонійних опуклих функцій* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 119–135.

У даній статті ми показуємо, що гармонійні опуклі функції f є сильно (p,h)-гармонійно опуклими функціями тоді і тільки тоді коли їх можна подати у вигляді $g(x) = f(x) - c(\frac{1}{x^p})^2$, де $g(x)$ є (p,h)-гармонійно опуклою функцією. Отримано деякі нові оцінки класу сильно (p,h)-гармонійно опуклих функцій, включаючи гіпергеометричні та бета-функції. Як застосування наших результатів розглянуто кілька важливих особливих випадків. Також введено новий клас гармонійних опуклих функцій, які називаються сильно (p,h)-гармонійними log-опуклими функціями. Отримано деякі нові нерівності типу Ерміта-Адамара для сильно (p,h)-гармонійних log-опуклих функцій. Ці результати можна розглядати як важливе уточнення і суттєве покращення нових і попередніх відомих результатів. Ідеї та методики цієї роботи можуть бути підґрунтям для подальших досліджень.

Ключові слова і фрази: p-гармонійно опуклі функції, h-опуклі функції, сильно опуклі функції, нерівності типу Ерміта-Адамара.

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CHARACTERIZATIONS OF REGULAR AND INTRA-REGULAR ORDERED Γ-SEMIHYPERGROUPS IN TERMS OF BI-Γ-HYPERIDEALS

The concept of Γ-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups. In this paper, we study the notion of bi-Γ-hyperideals in ordered Γ-semihypergroups and investigate some properties of these bi-Γ-hyperideals. Also, we define and use the notion of regular ordered Γ-semihypergroups to examine some classical results and properties in ordered Γ-semihypergroups.

Key words and phrases: ordered Γ-semihypergroup, Γ-hyperideal, bi-Γ-hyperideal.

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1 INTRODUCTION

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [24]. The notion of a Γ-semigroup was introduced by Sen and Saha [37] as a generalization of semigroups as well as of ternary semigroups. Since then, hundreds of papers have been written on this topic, see [6,7,16]. Many classical notions of semigroups have been extended to Γ-semigroups. Let $S = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. Then, S is called a Γ-*semigroup* if there exists a mapping from $S \times \Gamma \times S$ to S, written as $(a, \gamma, b) \to a\gamma b$, satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all a, b, c in S and α, β in Γ. In this case by (S, Γ) we mean S is a Γ-semigroup. By an *ordered semigroup*, we mean an algebraic structure $(S, \cdot, \leq)$, which satisfies the following conditions: (1) $(S, \cdot)$ is a semigroup; (2) S is a partial ordered set by $\leq$; (3) If a and b are elements of S such that $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ for all $c \in S$. Ordered semigroups have been studied extensively by Kehayopulu and Tsingelis, for example, see [27–29]. The notions of an ordered Γ-groupoid and an ordered Γ-semigroup were defined by Sen and Seth in [38]. Many authors studied different aspects of ordered Γ-semigroups, for instance, Abbasi and Basar [1], Chinram and Tinpun [7,8], Dutta and Adhikari [16,17], Hila [22], Iampan [25], Kehayopulu [26], Kwon [31], Kwon and Lee [32,33], and many others. Recall from [38], that an *ordered* Γ-*semigroup* $(S, \Gamma, \leq)$ is a Γ-semigroup (S, Γ) together with an order relation $\leq$ such that $a \leq b$ implies that $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups introduced by Chvalina [11] as a special class

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of hypergroups. Many authors studied different aspects of ordered semihypergroups, for instance, Davvaz et al. [15], Gu and Tang [19], Heidari and Davvaz [20], Tang et al. [39], and many others. Explicit study of ordered semihypergroups seems to have begun with Heidari and Davvaz [20] in 2011. Recall from [20], that an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation $\circ$, meaning that for any $x, y, z \in S$,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Recently, Davvaz et al. [4, 5, 13, 21, 23] studied the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They proved some results in this respect and presented many exmaples of Γ-semihypergroups. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups. The notion of a Γ-hyperideal of a Γ-semihypergroup was introduced in [4]. Davvaz et al. [5] introduced the notion of Pawlak's approximations in Γ-semihypergroups. Abdullah et al. [2] studied M-hypersystems and N-hypersystems in a Γ-semihypergroup. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of hyperstructure was first introduced by Marty [34] at the eighth Congress of Scandinavian Mathematicians in 1934. A comprehensive review of the theory of hyperstructures can be found in [9, 10, 12, 40]. Let S be a non-empty set and $P^*(S)$ be the family of all non-empty subsets of S. A mapping $\circ : S \times S \to P^*(S)$ is called a *hyperoperation* on S. A *hypergroupoid* is a set S together with a (binary) hyperoperation. In the above definition, if A and B are two non-empty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}}, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad B \circ x = B \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every x, y, z in S, $x \circ (y \circ z) = (x \circ y) \circ z$. That is,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

A non-empty subset K of a semihypergroup S is called a *subsemihypergroup* of S if $K \circ K \subseteq K$. A hypergroupoid $(S, \circ)$ is called a *quasihypergroup* if for every $x \in S$, $x \circ S = S = S \circ x$. This condition is called the reproduction axiom. The couple $(S, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A non-empty subset K of S is a *subhypergroup* of S if $K \circ a = a \circ K = K$, for every $a \in K$. A hypergroup $(S, \circ)$ is called *commutative* if $x \circ y = y \circ x$, for every $x, y \in S$.

2 REVIEW: ORDERED Γ-SEMIHYPERGROUPS

The notion of a Γ-semihypergroup was introduced by Davvaz et al. [4,5,21]. In [20], Heidari and Davvaz introduced the concept of ordered semihypergroups, which is a generalization of ordered semigroups. In this section, we recall the notion of an ordered Γ-semihypergroup and then we present some definitions and properties which we will need in this paper. Throughout this paper, unless otherwise stated, S is always an ordered Γ-semihypergroup $(S, \Gamma, \leq)$.

Definition 1 ([4,5]). *Let S and Γ be two non-empty sets. Then, S is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$, we have $x\alpha(y\beta z) = (x\alpha y)\beta z$. If every $\gamma \in \Gamma$ is an operation, then S is a Γ-semigroup. Let A and B be two non-empty subsets of S. We define*

$$A\Gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} = \bigcup_{\gamma\in\Gamma} A\gamma B.$$

A Γ-semihypergroup S is called commutative if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. A Γ-semihypergroup S is called a Γ-hypergroup if for every $\gamma \in \Gamma$, (S, γ) is a hypergroup.

Now, we consider the notion of an ordered Γ-semihypergroup.

Definition 2 ([30]). *An algebraic hyperstructure $(S, \Gamma, \leq)$ is called an ordered Γ-semihypergroup if (S, Γ) is a Γ-semihypergroup and $(S, \leq)$ is a partially ordered set such that for any $x, y, z \in S$ and $\gamma \in \Gamma$, $x \leq y$ implies $z\gamma x \leq z\gamma y$ and $x\gamma z \leq y\gamma z$. Here, if A and B are two non-empty subsets of S, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.*

Let S be an ordered Γ-semihypergroup. By a sub Γ-semihypergroup of S we mean a non-empty subset A of S such that $a\gamma b \subseteq A$ for all $a, b \in A$ and $\gamma \in \Gamma$.

Example 1 ([30]). *Let $(S, \circ, \leq)$ be an ordered semihypergroup and Γ a non-empty set. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup.*

Definition 3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A non-empty subset I of S is called a left Γ-hyperideal of S if it satisfies the following conditions:*

(1) $S\Gamma I \subseteq I$;

(2) When $x \in I$ and $y \in S$ such that $y \leq x$, imply that $y \in I$.

A right Γ-hyperideal of an ordered Γ-semihypergroup S is defined in a similar way. By *two-sided Γ-hyperideal* or simply *Γ-hyperideal*, we mean a non-empty subset of S which both left and right Γ-hyperideal of S. A Γ-hyperideal I of S is said to be *proper* if $I \neq S$.

Let K be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If H is a non-empty subset of K, then we define $(H]_K := \{k \in K \mid k \leq h \text{ for some } h \in H\}$. Note that if $K = S$, then we define $(H] := \{x \in S \mid x \leq h \text{ for some } h \in H\}$. For $H = \{h\}$, we write $(h]$ instead of $(\{h\}]$. Note that the condition (2) in Definition 3 is equivalent to $(I] \subseteq I$. If A and B are non-empty subsets of S, then we have

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) If $A \subseteq B$, then $(A] \subseteq (B]$;

(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(5) $((A]\Gamma(B]] = (A\Gamma B]$.

Lemma 1. *If I and J are Γ-hyperideals of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$, then $I \cap J$ is a Γ-hyperideal of S.*

Proof. Let $x \in I, y \in J$ and $\gamma \in \Gamma$. Then, $x\gamma y \subseteq I\Gamma J \subseteq I\Gamma S \subseteq I$ and $x\gamma y \subseteq I\Gamma J \subseteq S\Gamma J \subseteq J$. So, $x\gamma y \subseteq I \cap J$ and hence $\varnothing \neq I \cap J \subseteq S$. We have $(I \cap J)\Gamma S \subseteq I\Gamma S \subseteq I$ and $S\Gamma(I \cap J) \subseteq S\Gamma J \subseteq J$. Similarly, $(I \cap J)\Gamma S \subseteq J$ and $S\Gamma(I \cap J) \subseteq I$. So, we have $(I \cap J)\Gamma S \subseteq I \cap J$ and $S\Gamma(I \cap J) \subseteq I \cap J$. Now, let $x \in I \cap J, y \in S$ and $y \leq x$. Since I and J are Γ-hyperideals of S, we obtain $y \in I$ and $y \in J$. Thus, $y \in I \cap J$. This completes the proof. □

Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A subset A of S is called *idempotent* if $A = (A\Gamma A]$.

Lemma 2. *The Γ-hyperideals of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ are idempotent if and only if for any Γ-hyperideals I, J of S, we have $I \cap J = (I\Gamma J]$.*

Proof. The sufficiency is obvious. For the necessity, let I, J be Γ-hyperideals of S. We have $(I\Gamma J] \subseteq (I\Gamma S] \subseteq (I] = I$ and $(I\Gamma J] \subseteq (S\Gamma J] \subseteq (J] = J$. So, we have $(I\Gamma J] \subseteq I \cap J$. On the other hand, by Lemma 1, $I \cap J$ is a Γ-hyperideal of S. By assumption, we have $I \cap J = ((I \cap J)\Gamma(I \cap J)] \subseteq (I\Gamma J]$. This completes the proof. □

Theorem 1. *Let $(S, \Gamma, \leq)$ be a commutative ordered Γ-semihypergroup. If I is a Γ-hyperideal of S and A is a non-empty subset of S, then $(I : A) = \{x \in S \mid x\gamma a \subseteq I \text{ for all } a \in A \text{ and } \gamma \in \Gamma\}$ is a Γ-hyperideal of S.*

Proof. Suppose that $x \in (I : A), s \in S$ and $\delta \in \Gamma$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. We have $(s\delta x)\gamma a = s\delta(x\gamma a) \subseteq S\Gamma I \subseteq I$. So, we have $s\delta x \subseteq (I : A)$. In the similar way, we obtain $x\delta s \subseteq (I : A)$. Now, let $x \in (I : A), y \in S$ and $y \leq x$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Also, we have $y\gamma a \leq x\gamma a$ for all $a \in A$ and $\gamma \in \Gamma$, by hypothesis. So, for any $u \in y\gamma a$, $u \leq v$ for some $v \in x\gamma a \subseteq I$. Since I is a Γ-hyperideal of S, it follows that $u \in I$. So, we have $y\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Thus, we have $y \in (I : A)$. Therefore, $(I : A)$ is a Γ-hyperideal of S. □

3 BI-Γ-HYPERIDEALS

The study of ordered semihyperrings was first undertaken by Davvaz and Omidi [14]. In [35], Omidi, Davvaz and Corsini studied some properties of hyperideals in ordered Krasner hyperrings. The concept of a bi-ideal is a very interesting and important thing in semigroups and ordered semigroups. In 1952, Good and Hughes [18] introduced the notion of bi-ideals in semigroups. Recently, Davvaz et al. [4] introduced the notion of bi-Γ-hyperideal in Γ-semihypergroups (cf. [3]). In [36], Pibaljommee and Davvaz studied the properties of bi-hyperideals in ordered semihypergroups. The concept of bi-Γ-hyperideals of an ordered Γ-semihypergroup is a generalization of the concept of Γ-hyperideals (left Γ-hyperideals, right Γ-hyperideals) of an ordered Γ-semihypergroup. First, we define the concept of a bi-Γ-hyperideal in ordered Γ-semihypergroups.

Definition 4 ([30]). *A sub Γ-semihypergroup B of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called a bi-Γ-hyperideal of S if the following conditions hold:*

(1) $B\Gamma S\Gamma B \subseteq B$;

(2) When $x \in B$ and $y \in S$ such that $y \leq x$, imply that $y \in B$.

The concept of bi-Γ-hyperideals of an ordered Γ-semihypergroup is a generalization of the concept of Γ-hyperideals (left Γ-hyperideals, right Γ-hyperideals) of an ordered Γ-semihypergroup. Obviously, every left (right) Γ-hyperideal of an ordered Γ-semihypergroup S is a bi-Γ-hyperideal of S, but the the following example shows that the converse is not true in general case. Indeed, If I is a left (right) Γ-hyperideal of S, then $I\Gamma I \subseteq S\Gamma I \subseteq I$. Hence, I is a sub Γ-semihypergroup of S.

Example 2. *Let $S = \{a,b,c,d,e,f\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

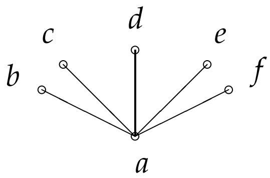
β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

Then S is a Γ-semihypergroup [41]. We have $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(a,f),(b,b),(c,c),(d,d),(e,e),(f,f)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b),(a,c),(a,d),(a,e),(a,f)\}.$$



Here,

(1) It is a routine matter to verify that $B_1 = \{a,b,c\}$ is a bi-Γ-hyperideal of S, but it is not a Γ-hyperideal of S.

(2) With a small amount of effort one can verify that $B_2 = \{a,b,c,f\}$ is a bi-Γ-hyperideal of S, but it is not a left Γ-hyperideal of S.

Lemma 3. *The intersection of any family of bi-Γ-hyperideals of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is a bi-Γ-hyperideal of S.*

Proof. Let $\{B_k \mid k \in \Lambda\}$ be a family of bi-Γ-hyperideals of S and $B = \bigcap_{k\in\Lambda} B_k$. It is easy to check that B is a sub Γ-semihypergroup of S. Now, let $x \in B\Gamma S\Gamma B$. Then, $x \in a\alpha s\beta b$ for some $a,b \in B$, $s \in S$ and $\alpha,\beta \in \Gamma$. Since each B_k is a bi-Γ-hyperideal of S, it follows that $a\alpha s\beta b \subseteq B_k\Gamma S\Gamma B_k \subseteq B_k$ for all $k \in \Lambda$. Then, $x \in B_k$ for all $k \in \Lambda$. So, we have $x \in \bigcap_{k\in\Lambda} B_k = B$. Since x was chosen arbitrarily, we have $B\Gamma S\Gamma B \subseteq B$. If $x \in B$ and $y \in S$ such that $y \leq x$, then $x \in B_k$ for all $k \in \Lambda$. Since each B_k is a bi-Γ-hyperideal of S, it follows that $y \in B_k$ for all $k \in \Lambda$. So, we have $y \in \bigcap_{k\in\Lambda} B_k = B$. Hence, B is a bi-Γ-hyperideal of S. □

Lemma 4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. If B is a bi-Γ-hyperideal of S and C is a bi-Γ-hyperideal of B, such that $C = (C\Gamma C]$, then C is a bi-Γ-hyperideal of S.*

Proof. By assumption, we have that

$$C\Gamma C = (C\Gamma C]\Gamma(C\Gamma C] \subseteq (C\Gamma(C\Gamma C\Gamma C)] \subseteq (C\Gamma C] = C,$$

which shows that C is a sub Γ-semihypergroup of S. On the other hand, we have $B\Gamma S\Gamma B \subseteq B$ and $C\Gamma B\Gamma C \subseteq C$. Thus, we have

$$\begin{aligned} C\Gamma S\Gamma C &= (C\Gamma C]\Gamma S\Gamma(C\Gamma C] = (C\Gamma C]\Gamma(S]\Gamma(C\Gamma C] \\ &\subseteq (C\Gamma C\Gamma S]\Gamma(C\Gamma C] \subseteq (C\Gamma(C\Gamma S\Gamma C)\Gamma C] \\ &\subseteq (C\Gamma(B\Gamma S\Gamma B)\Gamma C] \subseteq (C\Gamma B\Gamma C] \subseteq (C]_B \subseteq C. \end{aligned}$$

Now, let $c \in C$ and $x \leq c$, where $x \in S$. Since B is a bi-Γ-hyperideal of S and $C \subseteq B$, we get $x \in B$. On the other hand, C is a bi-Γ-hyperideal of B. It follows that $x \in C$. This completes the proof. □

Let A be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We denote by $L_S(A)$ (resp. $R_S(A)$, $I_S(A)$) the left (resp. right, two-sided) Γ-hyperideal of S generated by A.

Lemma 5. *If A is a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$, then the following hold:*

(1) $L_S(A) = (A \cup S\Gamma A]$;

(2) $R_S(A) = (A \cup A\Gamma S]$;

(3) $I_S(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$.

Proof. Since $A \subseteq L_S(A)$ and $S\Gamma A \subseteq L_S(A)$, it follows that $(A \cup S\Gamma A] \subseteq L_S(A)$. Clearly, $(A \cup S\Gamma A] \neq \varnothing$. We have

$$\begin{aligned} S\Gamma(A \cup S\Gamma A] &= (S]\Gamma(A \cup S\Gamma A] \subseteq (S\Gamma(A \cup S\Gamma A)] \\ &= (S\Gamma A \cup S\Gamma(S\Gamma A)] \subseteq (S\Gamma A] \subseteq (A \cup S\Gamma A]. \end{aligned}$$

Thus, $(A \cup S\Gamma A]$ is a left Γ-hyperideal of S containing A. This means that $L_S(A) \subseteq (A \cup S\Gamma A]$. This proves that (1) holds. The conditions (2) and (3) are proved similarly. □

Corollary 1. *Let a be an element of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then,*

(1) $L_S(a) = (a \cup S\Gamma a]$;

(2) $R_S(a) = (a \cup a\Gamma S]$;

(3) $I_S(a) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$.

Let A be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We define

$$\Theta = \{B \mid B \text{ is a bi-}\Gamma\text{-hyperideal of } S \text{ containing } A\}.$$

Since $S \in \Theta$, it follows that $\Theta \neq \varnothing$. We denote by $B_S(A)$ the bi-Γ-hyperideal of S generated by A. Clearly, $A \subseteq B_S(A) = \bigcap\limits_{B \in \Theta} B$. By Lemma 3, $B_S(A)$ is a bi-Γ-hyperideal of S.

Lemma 6. *Let A be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then,*

$$B_S(A) = (A \cup A\Gamma A \cup A\Gamma S\Gamma A].$$

Proof. Set $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. Clearly, $B \neq \varnothing$. We have

$$\begin{aligned} B\Gamma B &= (A \cup A\Gamma A \cup A\Gamma S\Gamma A]\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A] \\ &\subseteq ((A \cup A\Gamma A \cup A\Gamma S\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A)] \\ &\subseteq (A\Gamma S\Gamma A] \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]. \end{aligned}$$

Hence, B is a sub Γ-semihypergroup of S. Now,

$$\begin{aligned} B\Gamma S\Gamma B &= (A \cup A\Gamma A \cup A\Gamma S\Gamma A]\Gamma S\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A] \\ &\subseteq ((A \cup A\Gamma A \cup A\Gamma S\Gamma A)\Gamma S\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A)] \\ &\subseteq (A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]. \end{aligned}$$

Therefore, B is a bi-Γ-hyperideal of S, and hence $B_S(A) \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. Let C be a bi-Γ-hyperideal of S containing A. Then, $A\Gamma A \subseteq C$ and $A\Gamma S\Gamma A \subseteq C\Gamma S\Gamma C \subseteq C$. Thus, we have $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (C] = C$. Hence, B is the smallest bi-Γ-hyperideal of S containing A. Therefore, $B_S(A) = B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. □

Corollary 2. *Let a be an element of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then,*

$$B_S(a) = (a \cup a\Gamma a \cup a\Gamma S\Gamma a].$$

4 MAIN RESULTS

The concepts of regular (resp. intra-regular) ordered Γ-semihypergroups generalize the corresponding concepts of regular (resp. intra-regular) Γ-semihypergroups as each regular (resp. intra-regular) Γ-semihypergroup endowed with the order $\leq := \{(a, b) \mid a = b\}$ is a regular (resp. intra-regular) ordered Γ-semihypergroup. In this section, we introduce the notion of regular ordered Γ-semihypergroups and investigate some related results. We characterize regular ordered Γ-semihypergroups in terms of bi-Γ-hyperideals, left Γ-hyperideals and right Γ-hyperideals of ordered Γ-semihypergroups. In this paper, some well known results of ordered semihypergroups in case of ordered Γ-semihypergroups are examined.

Definition 5. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called regular if for every $a \in S$ there exist $x \in S$, $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. This is equivalent to saying that $a \in (a\Gamma S\Gamma a]$, for every $a \in S$ or $A \subseteq (A\Gamma S\Gamma A]$, for every $A \subseteq S$.*

Example 3. *Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e
a	$\{a, b\}$	$\{b, e\}$	c	$\{c, d\}$	e
b	$\{b, e\}$	e	c	$\{c, d\}$	e
c	c	c	c	c	c
d	$\{c, d\}$	$\{c, d\}$	c	d	$\{c, d\}$
e	e	e	c	$\{c, d\}$	e

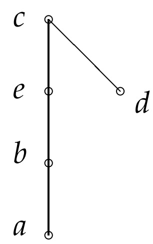
β	a	b	c	d	e
a	$\{b, e\}$	e	c	$\{c, d\}$	e
b	e	e	c	$\{c, d\}$	e
c	c	c	c	c	c
d	$\{c, d\}$	$\{c, d\}$	c	d	$\{c, d\}$
e	e	e	c	$\{c, d\}$	e

Then S is a Γ-semihypergroup [42]. We have $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a,a),(a,b),(a,c),(a,e),(b,b),(b,c),(b,e),(c,c),(d,c),(d,d),(e,c),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b),(b,e),(d,c),(e,c)\}.$$



We can easily verify that S is a regular ordered Γ-semihypergroup.

Lemma 7. *Every Γ-hyperideal I of a regular ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is a regular sub Γ-semihypergroup of S.*

Proof. Let $a \in I$. Since S is a regular ordered Γ-semihypergroup, there exist $x \in S$, $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \leq a\alpha x\beta a \leq a\alpha x\beta a\gamma x\delta a = a\alpha(x\beta a\gamma x)\delta a$. Since I is a Γ-hyperideal of S, it follows that $x\beta a\gamma x \subseteq S\Gamma I\Gamma S \subseteq I$. Thus, $a \leq t$ for some $t \in a\alpha(x\beta a\gamma x)\delta a \subseteq a\Gamma I\Gamma a$. So, we have $a \in (a\Gamma I\Gamma a]_I$. Therefore, I is a regular sub Γ-semihypergroup of S. □

Theorem 2. *If I and J are regular Γ-hyperideals of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$, then $I \cap J$ is also a regular Γ-hyperideal of S.*

Proof. Let I and J are regular Γ-hyperideals of S. By Lemma 1, $I \cap J$ is a Γ-hyperideal of S. By Lemma 7, I and J are regular sub Γ-semihypergroups of S. Now, let $a \in I \cap J$. Then, $a \leq a\alpha x\beta a$ and $a \leq a\gamma y\delta a$ for some $x, y \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. So, we have $a \leq a\alpha x\beta a \leq (a\alpha x\beta a)\mu s\lambda(a\gamma y\delta a) = a\alpha(x\beta a\mu s\lambda a\gamma y)\delta a$. Since I and J are Γ-hyperideals of S, we obtain $x\beta a\mu s\lambda a\gamma y \subseteq I \cap J$. Thus, we have $a \leq t$ for some $t \in a\alpha(x\beta a\mu s\lambda a\gamma y)\delta a \subseteq a\Gamma(I \cap J)\Gamma a$ which implies that $a \in (a\Gamma(I \cap J)\Gamma a]_I$. Hence, there exists $z \in I \cap J$ such that $a \leq a\alpha z\delta a$. Therefore, $I \cap J$ is a regular sub Γ-semihypergroup of S. □

We now prove the following theorem which is the crucial theorem in the establishment of our main theorems.

Theorem 3. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is regular if and only if for every right Γ-hyperideal R and every left Γ-hyperideal L of S, we have $R \cap L = (R\Gamma L]$.*

Proof. Let R be a right Γ-hyperideal and L a left Γ-hyperideal of S. As $R\Gamma L \subseteq S\Gamma L \subseteq L$ and $R\Gamma L \subseteq R\Gamma S \subseteq R$, we have $R\Gamma L \subseteq R \cap L$. So, $(R\Gamma L] \subseteq (R \cap L] \subseteq (R] \cap (L] \subseteq R \cap L$. Let S be regular; we need to prove that $R \cap L \subseteq (R\Gamma L]$. Since S is regular, we have

$$R \cap L \subseteq ((R \cap L)\Gamma S\Gamma(R \cap L)] \subseteq (R\Gamma S\Gamma(R \cap L)] \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L].$$

Conversely, suppose that $R \cap L = (R\Gamma L]$ for any right Γ-hyperideal R and any left Γ-hyperideal L of S. Let $a \in S$. Since $a \in R_S(a)$ and $a \in L_S(a)$, it follows that $a \in R_S(a) \cap L_S(a)$. By hypothesis, we have that

$$\begin{aligned} a \in (R_S(a)\Gamma L_S(a)] &= ((a \cup a\Gamma S]\Gamma(a \cup S\Gamma a]] \\ &\subseteq (a\Gamma a \cup a\Gamma S\Gamma a \cup a\Gamma S\Gamma S\Gamma a] \subseteq (a\Gamma a \cup a\Gamma S\Gamma a]. \end{aligned}$$

Hence, $a \leq t$ for some $t \in a\Gamma a \cup a\Gamma S\Gamma a$. If $u \in a\Gamma S\Gamma a$, then $a \leq a\alpha x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Thus, we have $a \in (a\Gamma S\Gamma a]$. Therefore, S is a regular ordered Γ-semihypergroup. If $u \in a\Gamma a$, then $a \leq a\alpha a \leq a\alpha(a\beta a)$. So, we have $a \in (a\Gamma S\Gamma a]$. Therefore, S is regular. □

Now, we obtain the following corollaries.

Corollary 3. *If $(S, \Gamma, \leq)$ is a regular ordered Γ-semihypergroup, then $S = (S\Gamma S]$.*

Corollary 4. *An ordered Γ-semihypergroup S is called fully Γ-hyperidempotent if every Γ-hyperideal of S is idempotent. If S is a regular ordered Γ-semihypergroup, then S is fully Γ-hyperidempotent.*

Theorem 4. *Let $(S, \Gamma, \leq)$ be a regular ordered Γ-semihypergroup. Then, B is a bi-Γ-hyperideal of S if and only if there exists a right Γ-hyperideal R and a left Γ-hyperideal L of S such that $B = (R\Gamma L]$.*

Proof. Let S be a regular ordered Γ-semihypergroup and B a bi-Γ-hyperideal of S. First, we show that $(B\Gamma S]$ is a right Γ-hyperideal of S. Let $y \in S$ and $x \in (B\Gamma S]$. Then, there exist $b \in (B\Gamma S]$, $c \in B$, $s \in S$ and $\alpha \in \Gamma$ such that $x \leq b \leq c\alpha s$. Since S is an ordered Γ-semihypergroup, it follows that $x\beta y \leq b\beta y \leq b \leq (c\alpha s)\beta y \subseteq B\Gamma S$, where $\beta \in \Gamma$. Hence, $x\beta y \subseteq (B\Gamma S]$. If $y \leq x$, then $y \leq x \leq b$, and so $y \in (B\Gamma S]$. Therefore, $(B\Gamma S]$ is a right Γ-hyperideal of S. Similarly, we can prove that $(S\Gamma B]$ is a left Γ-hyperideal of S. Now, we prove that $B = ((B\Gamma S]\Gamma(S\Gamma B]]$. Since S is regular, it follows that $B \subseteq (B\Gamma S\Gamma B]$, for every $B \subseteq S$. Since B is a bi-Γ-hyperideal of S, it follows that $B\Gamma S\Gamma B \subseteq B$. So, we have $(B\Gamma S\Gamma B] \subseteq (B] = B$. Hence, $B = (B\Gamma S\Gamma B]$. By Corollary 3, we have $S = (S\Gamma S]$. Hence,

$$\begin{aligned} B &= (B\Gamma S\Gamma B] = (B\Gamma(S\Gamma S]\Gamma B] = ((B]\Gamma((S\Gamma S]]\Gamma B] = ((B\Gamma S\Gamma S]\Gamma B] \\ &= ((B\Gamma S\Gamma S]\Gamma(B]] = ((B\Gamma S\Gamma S)\Gamma B] = ((B\Gamma S]\Gamma(S\Gamma B]]. \end{aligned}$$

Conversely, suppose that R is a right Γ-hyperideal and L a left Γ-hyperideal of S such that $B = (R\Gamma L]$. We prove that $(R\Gamma L]$ is a bi-Γ-hyperideal of S. We have

$$(R\Gamma L]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma(R\Gamma L)] = ((R\Gamma L\Gamma R)\Gamma L] \subseteq ((R\Gamma S\Gamma R)\Gamma L] \subseteq (R\Gamma L].$$

Then, $(R\Gamma L]$ is a sub Γ-semihypergroup of S. Also, we have

$$\begin{aligned} (R\Gamma L]\Gamma S\Gamma(R\Gamma L] &= (R\Gamma L]\Gamma(S]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma S]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma S\Gamma(R\Gamma L)] \\ &\subseteq (R\Gamma(L\Gamma S)\Gamma R\Gamma L] \subseteq ((R\Gamma S)\Gamma R\Gamma L] \subseteq (R\Gamma R\Gamma L] \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L]. \end{aligned}$$

Now, suppose that $y \in S$ and $x \in (R\Gamma L]$ such that $y \leq x$. Since $x \in (R\Gamma L]$, it follows that $x \leq a$ for some $a \in R\Gamma L$. Since $y \leq x$ and $x \leq a$, we get $y \leq a$. So, we have $y \in (R\Gamma L]$. Therefore, $(R\Gamma L]$ is a bi-Γ-hyperideal of S. □

Theorem 5. *An ordered* Γ*-semihypergroup* $(S, \Gamma, \leq)$ *is regular if and only if for every right* Γ*-hyperideal* R*, every left* Γ*-hyperideal* L *and every bi-*Γ*-hyperideal* B *of* S*, we have* $R \cap B \cap L \subseteq (R\Gamma B\Gamma L]$.

Proof. Let R be right Γ-hyperideal, L a left Γ-hyperideal and B a bi-Γ-hyperideal of S. By hypothesis, we have

$$\begin{aligned} R \cap B \cap L &\subseteq ((R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)] \\ &\subseteq ((R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)] \\ &\subseteq (R\Gamma S\Gamma B\Gamma S\Gamma B\Gamma S\Gamma L] = ((R\Gamma S)\Gamma(B\Gamma S\Gamma B)\Gamma(S\Gamma L)] \subseteq (R\Gamma B\Gamma L]. \end{aligned}$$

Conversely, suppose that $R \cap B \cap L \subseteq (R\Gamma B\Gamma L]$ for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S. Since S is a bi-Γ-hyperideal of S, we have $R \cap L = R \cap S \cap L \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L]$. By Theorem 3, S is regular. □

Definition 6. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semihypergroup. An element* $a \in S$ *is said to be intra-regular if there exist* $x, y \in S$*,* $\alpha, \beta, \gamma \in \Gamma$ *such that* $a \leq x\alpha a\beta a\gamma y$*. An ordered* Γ*-semihypergroup* S *is called intra-regular if all elements of* S *are intra-regular.*

Equivalent definitions*:*

(1) $a \in (S\Gamma a\Gamma a\Gamma S]$*, for all* $a \in S$*.*

(2) $A \subseteq (S\Gamma A\Gamma A\Gamma S]$*, for all* $A \subseteq S$*.*

Example 4. *Let* $S = \{a, b, c, d, e\}$ *and* $\Gamma = \{\gamma, \beta\}$ *be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

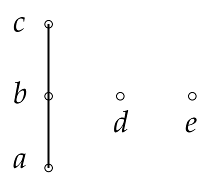
β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Then S *is a* Γ*-semihypergroup [41]. We have* $(S, \Gamma, \leq)$ *is an ordered* Γ*-semihypergroup where the order relation* $\leq$ *is defined by:*

$$\leq := \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c), (d,d), (e,e)\}.$$

The covering relation and the figure of S *are given by:*

$$\prec = \{(a,b), (b,c)\}.$$



Then, by routine calculations, $(S, \Gamma, \leq)$ *is intra-regular.*

Theorem 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, S is intra-regular if and only if for every right Γ-hyperideal R and every left Γ-hyperideal L of S, we have*

$$R \cap L \subseteq (L\Gamma R].$$

Proof. Let R be a right Γ-hyperideal and L a left Γ-hyperideal of S. Let S be intra-regular; we need to prove that $R \cap L \subseteq (L\Gamma R]$. Since S is intra-regular, we have

$$R \cap L \subseteq (S\Gamma(R \cap L)\Gamma(R \cap L)\Gamma S] \subseteq (S\Gamma L\Gamma R\Gamma S] \subseteq (L\Gamma R].$$

Conversely, suppose that $R \cap L \subseteq (L\Gamma R]$ for any right Γ-hyperideal R and any left Γ-hyperideal L of S. Let $a \in S$. Since $a \in R_S(a)$ and $a \in L_S(a)$, it follows that $a \in R_S(a) \cap L_S(a)$. By hypothesis, we have

$$\begin{aligned} a \in (L_S(a)\Gamma R_S(a)] &= ((a \cup S\Gamma a]\Gamma(a \cup a\Gamma S]] \\ &\subseteq (a\Gamma a \cup S\Gamma a\Gamma a \cup a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S]. \end{aligned}$$

Hence, $a \leq u$ for some $u \in a\Gamma a \cup S\Gamma a\Gamma a \cup a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S$. If $u \in S\Gamma a\Gamma a\Gamma S$, then $a \leq x\alpha a\beta a\gamma y$ for some $x, y \in S$, $\alpha, \beta, \gamma \in \Gamma$. Thus, we have $a \in (S\Gamma a\Gamma a\Gamma S]$. Therefore, S is intra-regular. If $u \in a\Gamma a$, then $a \leq a\alpha a \leq a\alpha(a\beta a) \leq a\alpha a\beta a\gamma a$. So, we have $a \in (S\Gamma a\Gamma a\Gamma S]$. Hence, S is intra-regular. If $u \in S\Gamma a\Gamma a$, then $a \leq x\alpha a\beta a \leq x\alpha(x\gamma a\delta a)\beta a$ for some $x \in S$, $\alpha, \beta, \gamma, \delta \in \Gamma$. So, we have $a \leq s\gamma a\delta a\beta a$. Hence, $a \in (S\Gamma a\Gamma a\Gamma S]$. If $u \in a\Gamma a\Gamma S$, in a similar way, we obtain $a \in (S\Gamma a\Gamma a\Gamma S]$. Therefore, S is intra-regular. □

Corollary 5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, the following statements are equivalent:*

(1) *S is regular and intra-regular.*

(2) *$(R\Gamma L] = R \cap L \subseteq (L\Gamma R]$ for every right Γ-hyperideal R and every left Γ-hyperideal L of S.*

Proof. It is immediately followed by Theorem 3 and Theorem 6. □

Theorem 7. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is intra-regular if and only if for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S, we have $R \cap B \cap L \subseteq (L\Gamma B\Gamma R]$.*

Proof. The proof is similar to the proof of Theorem 5. □

By routine verification we have the following theorem.

Theorem 8. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is both regular and intra-regular if and only if for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S, we have $R \cap B \cap L \subseteq (B\Gamma R\Gamma L]$.*

Our main aim in the following is to introduce and study the notion of simple ordered Γ-semihypergroups. Also, we characterize this type of ordered Γ-semihypergroups in terms of Γ-hyperideals.

Definition 7. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is said to be left (resp. right) simple if S has no proper left (resp. right) Γ-hyperideals. S is called a simple ordered Γ-semihypergroup if it does not contain proper Γ-hyperideals, i.e., for any Γ-hyperideal $I \neq \varnothing$ of S, we have $I = S$.*

Lemma 8. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, the following assertions hold:*

(1) *S is left simple if and only if $(S\Gamma a] = S$, for all $a \in S$.*

(2) *S is right simple if and only if $(a\Gamma S] = S$, for all $a \in S$.*

Proof. (1): Suppose that S is a left simple ordered Γ-semihypergroup and $a \in S$. We have

$$S\Gamma(S\Gamma a] = (S]\Gamma(S\Gamma a] \subseteq (S\Gamma(S\Gamma a)] = ((S\Gamma S)\Gamma a)] \subseteq (S\Gamma a].$$

Now, suppose that $x \in (S\Gamma a]$ and $y \in S$ such that $y \leq x$. Since $x \in (S\Gamma a]$, it follows that $x \leq u$ for some $u \in S\Gamma a$. Since $y \leq x$ and $x \leq u$, we get $y \leq u$. So, we have $y \in (S\Gamma a]$. Hence, $(S\Gamma a]$ is a left hyperideal of S. Since S is a left simple ordered Γ-semihypergroup, we have $(S\Gamma a] = S$.

Conversely, suppose that $(S\Gamma a] = S$ for all $a \in S$. Let L be a left hyperideal of S and $x \in L$. By assumption, we have $(S\Gamma x] = S$. If $s \in S$, then $s \in (S\Gamma x]$. So, $s \leq v$ for some $v \in S\Gamma x \subseteq L$. Since L is a left Γ-hyperideal of S, we have $s \in L$, and so $L = S$. Therefore, S is a left simple ordered Γ-semihypergroup.

(2): The proof is similar to the proof of (1). □

Theorem 9. *If $(S, \Gamma, \leq)$ is a left (right) simple ordered Γ-semihypergroup, then S is a simple ordered Γ-semihypergroup.*

Proof. It is straightforward. □

Theorem 10. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is left and right simple if and only if for every $a \in S$, we have $(S\Gamma a\Gamma S] = S$.*

Proof. Let S be left and right simple and $a \in S$. By Lemma 8, $a \in (S\Gamma a]$ and $a \in (a\Gamma S]$. We have

$$a \in (a\Gamma S] \subseteq ((S\Gamma a]\Gamma S] \subseteq (S\Gamma a\Gamma S],$$

and so $S \subseteq (S\Gamma a\Gamma S]$. Thus, $(S\Gamma a\Gamma S] = S$.

Conversely, suppose that $(S\Gamma a\Gamma S] = S$ for every $a \in S$. Let I be a Γ-hyperideal of S such that $I \subsetneq S$. Let $x \in I$. By assumption, we have $s \leq s\mu x\lambda s$ for every $s \in S$ and $\mu, \lambda \in \Gamma$. We have

$$s\mu x\lambda s \subseteq S\Gamma I\Gamma S \subseteq (S\Gamma I\Gamma S] \subseteq (I] = I.$$

Then, $S \subseteq I$, a contradiction. Therefore, S has no proper left and right Γ-hyperideals. This completes the proof. □

In what follows, we characterize simple ordered Γ-semihypergroups in terms of bi-Γ-hyperideals.

Theorem 11. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is left and right simple if and only if S does not contain proper bi-Γ-hyperideals.*

Proof. Suppose that S is a left and right simple ordered Γ-semihypergroup and B a bi-Γ-hyperideal of S. We claim that $S \subseteq B$. Consider $s \in S$ and $x \in B$. Since S is left simple, we get $S = (x \cup S\Gamma x]$. We can consider the following two cases:
Case 1. If $s \leq x$, then we have $s \in B$.
Case 2. Let $s \in (u\gamma x]$ for some $u \in S$ and $\gamma \in \Gamma$. By hypothesis, S is a right simple ordered Γ-semihypergroup. Then, we have $S = (x \cup x\Gamma S]$. Since $u \in S$, we have $u \leq x$ or $u \in (x\delta w]$ for some $w \in S$ and $\delta \in \Gamma$. By Lemma 8, we have $S = (x\Gamma S] = (S\Gamma x]$, and so $x \in (x\Gamma S] = (x\Gamma(S\Gamma x]] \subseteq (x\Gamma S\Gamma x]$. Then, S is a regular ordered Γ-semihypergroup. Thus, there exists $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x \in (x\alpha a\beta x]$. If $u \leq x$, then we have

$$(u\gamma x] \subseteq (x\gamma x] \subseteq (x\gamma x\alpha a\beta x] \subseteq (B\Gamma S\Gamma B] \subseteq B,$$

and so $s \in B$. If $u \in (x\delta w]$, then we have

$$(u\gamma x] \subseteq (x\delta w\gamma x] \subseteq (B\Gamma S\Gamma B] \subseteq B,$$

and so $s \in B$. Therefore, $S \subseteq B$.

Conversely, suppose that S does not contain proper bi-Γ-hyperideals. Let L be a left Γ-hyperideal of S. Then, L is a bi-Γ-hyperideal of S. By assumption, we have $S = L$. Therefore, S is a left simple ordered Γ-semihypergroup. Similarly, we can show that S is a right simple ordered Γ-semihypergroup. □

In the following, we study some properties of bi-Γ-hyperideals and minimal bi-Γ-hyperideals in ordered Γ-semihypergroups.

Definition 8. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is said to be B-simple if S does not contain any proper bi-Γ-hyperideals. A bi-Γ-hyperideal C of S is called a minimal bi-Γ-hyperideal of S if C does not properly contain any bi-Γ-hyperideal of S.*

Theorem 12. *Let B be a bi-Γ-hyperideal of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, $(u\Gamma B\Gamma v]$ is a bi-Γ-hyperideal of S for every $u, v \in S$. In particular, $(u\Gamma S\Gamma v]$ is a bi-Γ-hyperideal of S for every $u, v \in S$.*

Proof. The proof is similar to the proof of Theorem 2.2 in [8]. □

Corollary 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, S is B-simple if and only if $(u\Gamma S\Gamma u] = S$ for all $u \in S$.*

Proof. The necessity is obvious. For the sufficiency, let $(u\Gamma S\Gamma u] = S$ for all $u \in S$. We have

$$(u\Gamma S\Gamma u] \subseteq (S\Gamma u] \subseteq S \text{ and } (u\Gamma S\Gamma u] \subseteq (u\Gamma S] \subseteq S.$$

By assumption, we have $(S\Gamma u] = S$ and $(u\Gamma S] = S$ for all $u \in S$. Now, let B is a bi-Γ-hyperideal of S and $b \in B$. Then, $(S\Gamma b] = S = (b\Gamma S]$. So, we have

$$S = (b\Gamma S] = (b\Gamma(b\Gamma S]] \subseteq (b\Gamma S\Gamma b] \subseteq (B\Gamma S\Gamma B] \subseteq (B] \subseteq B.$$

This completes the proof. □

Corollary 7. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. If C is a minimal bi-Γ-hyperideal of S and B a bi-Γ-hyperideal of S, then $C = (c\Gamma B\Gamma d]$ for every $c, d \in C$.*

Proof. By Theorem 12, $(c\Gamma B\Gamma d]$ is a bi-Γ-hyperideal of S. Since C is a minimal bi-Γ-hyperideal of S and $(c\Gamma B\Gamma d] \subseteq (C\Gamma B\Gamma C] \subseteq (C\Gamma S\Gamma C] \subseteq (C] \subseteq C$, we obtain $C = (c\Gamma B\Gamma d]$. □

At the end of the paper, we prove the following theorem.

Theorem 13. *Let B be a bi-Γ-hyperideal of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, B is a minimal bi-Γ-hyperideal of S if and only if B is B-simple.*

Proof. Let B be a minimal bi-Γ-hyperideal of S. Then, B is a sub Γ-semihypergroup of S. Now, let C be a bi-Γ-hyperideal of B. Then, $C\Gamma B\Gamma C \subseteq C$. Put $K = (C\Gamma B\Gamma C]_C$. Then, $\varnothing \neq K \subseteq C \subseteq B$. Now, we prove that K is a bi-Γ-hyperideal of S. Let $k_1, k_2 \in K$, $x \in S$ and $\gamma, \delta \in \Gamma$. Then, $k_1 \leq c_1\alpha_1 b_1\beta_1 c_1'$ and $k_2 \leq c_2\alpha_2 b_2\beta_2 c_2'$ for some $c_1, c_1', c_2, c_2' \in C$, $b_1, b_2 \in B$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Gamma$. So, we have

$$k_1\gamma k_2 \leq c_1\alpha_1(b_1\beta_1 c_1'\gamma c_2\alpha_2 b_2)\beta_2 c_2'$$

and

$$k_1\gamma x\delta k_2 \leq c_1\alpha_1(b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2)\beta_2 c_2'.$$

Since $b_1\beta_1 c_1'\gamma c_2\alpha_2 b_2 \subseteq B\Gamma S\Gamma B \subseteq B$, it follows that $k_1\gamma k_2 \subseteq K\Gamma K \subseteq C\Gamma C \subseteq C$. So, $k_1\gamma k_2 \subseteq (C\Gamma B\Gamma C]_C = K$. Hence, K is a sub Γ-semihypergroup of S. Since $b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2 \subseteq B\Gamma S\Gamma B \subseteq B$, we get

$$c_1\alpha_1(b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2)\beta_2 c_2' \subseteq C\Gamma B\Gamma C \subseteq C.$$

Since C is a bi-Γ-hyperideal of B and $k_1\gamma x\delta k_2 \subseteq K\Gamma S\Gamma K \subseteq B\Gamma S\Gamma B \subseteq B$, we obtain $k_1\gamma x\delta k_2 \subseteq C$. So, we have $k_1\gamma x\delta k_2 \subseteq (C\Gamma B\Gamma C]_C = K$. Therefore, $K\Gamma S\Gamma K \subseteq K$. Now, let $y \in (K]$. Then, $y \leq k$ for some $k \in K$. Since $k \in K$, there exist $c, c' \in C$, $b \in B$ and $\mu, \lambda \in \Gamma$ such that $k \leq c\mu b\lambda c'$. Since $c\mu b\lambda c' \subseteq C\Gamma B\Gamma C \subseteq C \subseteq B$ and B is a bi-Γ-hyperideal of S, we get $k \in B$. Since B is a bi-Γ-hyperideal of S, we have $y \in B$. So, $y \leq z$ for some $z \in c\mu b\lambda c' \subseteq C\Gamma B\Gamma C \subseteq C$. Since C is a bi-Γ-hyperideal of B, we have $y \in C$. So, we have $y \in (C\Gamma B\Gamma C]_C = K$. Therefore, K is a bi-Γ-hyperideal of S. Since B is a minimal bi-Γ-hyperideal of S, it follows that $K = B$. So, we have $C = B$. Therefore, B is B-simple.

Conversely, assume that B is B-simple. Let C be a bi-Γ-hyperideal of S such that $C \subseteq B$. Then, $B \cap C \neq \varnothing$. Let $c \in B \cap C$. By Theorem 12, $(c\Gamma B\Gamma c]$ is a bi-Γ-hyperideal of B. Since B is B-simple, we obtain $(c\Gamma B\Gamma c] = B$. Now, we have

$$B = (c\Gamma B\Gamma c] \subseteq (C\Gamma B\Gamma C] \subseteq (C\Gamma S\Gamma C] \subseteq (C] = C.$$

Hence, $C = B$. Therefore, B is a minimal bi-Γ-hyperideal of S. □

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Оміді С., Давваз Б., Хіла К. *Характеристики регулярних і внутрішньо-регулярних впорядкованих* Γ*-напівгіпергруп в термінах bi-*Γ*-гіперідеалів* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 136–151.

Поняття Γ-напівгіпергруп є узагальненням напівгруп, узагальненням напівгіпергруп і узагальненням Γ-напівгруп. У даній роботі досліджується поняття бі-Γ-гіперідалів у впорядкованих Γ-напівгіпергрупах і досліджуються деякі властивості цих бі-Γ-гіперідеалів. Також ми визначаємо і використовуємо поняття регулярно впорядкованих Γ-напівгіпергруп для вивчення деяких класичних результатів і властивостей у впорядкованих Γ-напівгіпергрупах.

Ключові слова і фрази: упорядковані Γ-напівгіпергрупи, Γ-гіперідеали, bi-Γ-гіперідеали.

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ÖZARSLAN H.S.

ON A NEW APPLICATION OF QUASI POWER INCREASING SEQUENCES

In the present paper, absolute matrix summability of infinite series has been studied. A new theorem concerned with absolute matrix summability factors, which generalizes a known theorem dealing with absolute Riesz summability factors of infinite series, has been proved under weaker conditions by using quasi β-power increasing sequences. Also, a known result dealing with absolute Riesz summability has been given.

Key words and phrases: Riesz mean, almost increasing sequences, quasi power increasing sequences, Hölder inequality, Minkowski inequality, matrix transformation.

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INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots .$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [9])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |A_n(s) - A_{n-1}(s)|^k < \infty. \tag{1}$$

In the special case for $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, we obtain the $|\bar{N}, p_n|_k$ summability (see [2]). Also, it should be noted that for $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, the $\varphi - |A;\delta|_k$ summability reduces to $|\bar{N}, p_n; \delta|_k$ summability (see [3]).

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1 KNOWN RESULT

A positive sequence (h_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leq h_n \leq Lc_n$ (see [1]). By means of this sequence, Mazhar [7] has established following theorem.

Theorem 1. *If (X_n) is an almost increasing sequence and the conditions*

$$|\lambda_m|X_m = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{6}$$

are satisfied, where (t_n) is the nth $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2 MAIN RESULT

A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta,\gamma) \geq 1$ such that $Kn^\beta\gamma_n \geq m^\beta\gamma_m$ holds for all $n \geq m \geq 1$ (see [6]). It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. One can find some applications of quasi power increasing sequences (see [4–6, 10]). The purpose of this paper is to obtain a theorem which generalizes Theorem 1 for $\varphi - |A;\delta|_k$ summability using quasi β-power increasing sequence. Before giving this theorem, let us introduce some further notations.

Let $A = (a_{nv})$ be a normal matrix, $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots, \tag{8}$$

$\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{9}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{10}$$

Theorem 2. *Let $(\lambda_n) \in \mathcal{BV}$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{11}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v+1, \tag{12}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{13}$$

$$\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} = O(a_{nn}), \tag{14}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left(\varphi_v^{\delta k-1}\right) \quad as \quad m \to \infty, \tag{15}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O\left(\varphi_v^{\delta k}\right) \quad as \quad m \to \infty. \tag{16}$$

Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$ and $\varphi_n p_n = O(P_n)$. If conditions (2), (3) *of Theorem 1 and*

$$\sum_{n=1}^{m} \varphi_n^{\delta k} \frac{1}{n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{17}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{18}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Lemma 1. *([4]). Under the conditions of Theorem 2, we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{19}$$

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| < \infty. \tag{20}$$

3 PROOF OF THEOREM 2

Let (I_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v$$

by (9) and (10). Now, using Abel's transformation,

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{a}_{nv}\right) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v + \frac{n+1}{n} a_{nn} \lambda_n t_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, by (1), we will prove that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4.$$

For $r = 1$, applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| \, |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}.
\end{aligned}$$

By (7) and (8), we have

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}.$$

Thus using (7), (11) and (12)

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leq a_{nn}.$$

Hence,

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^k \, |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} |\lambda_v| |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\delta k-1} |t_r|^k + O(1)|\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by (2), (13), (15), (18) and (20). For $r = 2$, using Hölder's inequality, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v|\right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k\right) \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} v |\Delta\lambda_v| \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{1}{r} |t_r|^k + O(1) m |\Delta\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2\lambda_v| + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1) m |\Delta\lambda_m| X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (3), (13), (16), (17), (19) and (20).

Again, for $r = 3$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v}\right)^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v}\right) \left(\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v}\right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v}\right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |\lambda_{v+1}||t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{1}{r} |t_r|^k + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (2), (13), (14), (16), (17) and (20).

Finally, as in the process for $I_{n,1}$, by using Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \mid I_{n,4} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} a_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k-1} |\lambda_n||t_n|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (2), (13), (18) and (20). This completes the proof of Theorem 2.

If we take (X_n) as an almost increasing sequence, $\varphi_n = \frac{P_n}{p_n}$, $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$ in Theorem 2, then we get Theorem 1. In this case the conditions (14), (17) and (18) reduce to the conditions (4), (5) and (6), respectively. Also, if we take (X_n) as an almost increasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2, then we get a theorem dealing with $|\bar{N}, p_n; \delta|_k$ summability (see [8]).

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Озарслан Г. *Про нове застосування квазі-степеневих зростаючих послідовностей* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 152–157.

У роботі досліджено абсолютну матричну сумовність нескінченних послідовностей. Нову теорему, яка стосується умов абсолютної матричної сумовності і узагальнює відому теорему про умови абсолютної сумовності Ріса для нескінчнних послідовностей доведено за слабших умов з використанням квазі-β-степеневих зростаючих послідовностей. Також, отримано один відомий результат, який стосується абсолютної сумовності Ріса.

Ключові слова і фрази: середнє за Рісом, майже зростаючі послідовності, квазі-степеневі зростаючі послідовності, нерівність Гельдера, нерівність Мінковського, матричні перетворення.

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PRYIMAK H.M.

ON APPROXIMATION OF HOMOMORPHISMS OF ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

It is known due to R. Aron, B. Cole and T. Gamelin that every complex homomorphism of the algebra of entire functions of bounded type on a Banach space X can be approximated in some sens by a net of point valued homomorphism. In this paper we consider the question about a generalization of this result for the case of homomorphisms to any commutative Banach algebra A. We obtained some positive results if A is the algebra of uniformly continuous analytic functions on the unit ball of X.

Key words and phrases: analytic functions on Banach space, homomorphisms of algebras of analytic functions, approximation property.

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INTRODUCTION

Let X be a complex Banach space and $H_b(X)$ be the algebra of entire functions of bounded type on X, that is, $H_b(X)$ consists of all analytic functions on X which are bounded on all bounded sets. It is known that $H_b(X)$ is a Fréchet algebra with respect to the following family of norms

$$\|f\|_r = \sup_{\|x\|\leq r} |f(x)|, \quad f \in H_b(x),$$

where r is taken over the set of positive rational numbers. We denote by M_b the spectrum of $H_b(X)$, that is, the set of all continuous complex valued homomorphisms of $H_b(X)$. M_b is a topological space endowed with the Gelfand topology which is the weakest topology such that all mappings $\widehat{f}(\varphi) := \varphi(f)$ are continuous. Typical examples of elements in M_b are point evaluation functionals δ_x, $x \in X$ which are defined by $\delta_x(f) = f(x)$, $f \in H_b(X)$.

In [1] it was proved that for every complex homomorphism $\varphi \in M_b$ there exists a net $(x_\alpha) \subset X$ such that $\varphi(P) = \lim\limits_{\alpha} P(x_\alpha)$ for every $P \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the algebra of all continuous polynomials on X. This property was used for investigations of spectra in [8, 9, 6, 3]. Our task is to generalize this formula in the case of homomorphisms from $H_b(X)$ to some commutative Banach algebra A.

Let A be a complex commutative Banach algebra and $A \otimes_\pi X$ be the complete projective tensor product of A and X. Every element of $A \otimes_\pi X$ can be represented by the form $\overline{a} = \sum\limits_k a_k \otimes x_k$, where $a_k \in A,\ x_k \in X$.

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For every $f \in H_b(X)$ let us define a function $\overline{f} : A \otimes_\pi X \to A$ so that for every $\overline{a} \in A \otimes_\pi X$, $\overline{f}(\overline{a})$ is the "value" of f at $\overline{a}$ in the means of functional calculus for analytic functions on a Banach spaces ([5]). Then the mapping $f \mapsto \overline{f}$ is a homomorphism between algebras $H_b(X)$ and $(H_b(A \otimes_\pi X), A)$. For every fixed $\overline{a}$ we define $\theta_{\overline{a}}(f) = \overline{f}(\overline{a})$ and θ is a homomorphism from $H_b(X)$ to A (see also [7, 10]).

The article is motivated by the following general question: *under which conditions for an arbitrary homomorphism Φ from $H_b(X)$ to A there exists a net $(\overline{a}_\alpha) \subset A \otimes_\pi X$ such that*

$$\Phi(P) = \lim_\alpha \theta_{\overline{a}}(P) = \lim_\alpha \overline{P}(\overline{a}_\alpha), \ \forall P \in \mathcal{P}(X)? \tag{1}$$

We obtain some positive answers under assumption that X has the approximation property for the case when $A = \mathcal{H}^\infty_{uc}(B)$, where $\mathcal{H}^\infty_{uc}(B)$ is the algebra of all uniformly continuous analytic complex functions on closed unit ball $B := \{x \in X : ||x|| \leq 1\}$ with norm

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

For more definitions and properties of polynomials and entire functions of bounded type on Banach spaces we refer the reader to [4].

1 Main results

We consider case when $A = \mathcal{H}^\infty_{uc}(B)$. Also, we suppose, first that Φ is the identity mapping, that is, $\Phi = I\colon H_b(X) \hookrightarrow \mathcal{H}^\infty_{uc}(B)$ and $I(f)$ is the restriction of f to B.

Our destination is to show that under some conditions there exists a net $(\overline{a}_\alpha) \in \mathcal{H}^\infty_{uc}(B) \otimes_\pi X$ such that

$$\Phi(f)(x) = \lim_\alpha \overline{f}(\overline{a}_\alpha) \quad \forall\, f \in H_b(X) \tag{2}$$

for $\Phi = I$ and for a more general case of Φ.

Example 1. *Let us consider $X = \mathbb{C}^n$ and $\Phi = I\colon H_b(\mathbb{C}^n) = H(\mathbb{C}^n) \hookrightarrow \mathcal{H}^\infty_{uc}(B) = \mathcal{A}(B)$.*

Every element $x \in \mathbb{C}^n$ can be represented as

$$x = \sum_{k=1}^n e_k^*(x) e_k,$$

where $\{e_k\}_{k=1}^n$ is a basis in $\mathbb{C}^n$ and $\{e_k^*\}_{k=1}^n$ is the dual basis of the coordinate functionals. Then $\overline{a} \in \mathcal{A}(B) \otimes_\pi \mathbb{C}^n$, $\overline{a} = \sum\limits_{k=1}^n e_k^* \otimes e_k$, that is

$$\overline{a}(x) = \sum_{k=1}^n e_k^*(x) e_k = x.$$

On the other hand, in the sense of functional calculus we have:

$$I(f)(x) = I(f(x)) = f(x) = f\left(\sum_{k=1}^n e_k^*(x) e_k\right) = \overline{f}(\overline{a})(x) = \overline{f}\left(\sum_{k=1}^n e_k^* \otimes e_k\right)(x).$$

Thus, for the fixed homomorphism $\Phi = I$ we found an element $\overline{a}$ and an arbitrary functions $f \in \mathcal{A}(B)$ and we have equality:

$$I(f(x)) = \overline{f}(\overline{a})(x) = \theta_{\overline{a}}(f)(x).$$

Note that in this case we need just a single A-evaluation functional $\theta_{\overline{a}}$.

Let Φ be an arbitrary homomorphism from $H(\mathbb{C}^n)$ to $\mathcal{A}(B)$ such that there is an analytic automorphism $F : B \to B$ such that $\Phi = C_F \circ I$, where C_F is the composition operator, $C_F(f)(x) = f(F(x))$, $f \in \mathcal{A}(B)$, $x \in B$. We set

$$\overline{a} = \sum_{k=1}^{n} (e_k^* \circ F) \otimes e_k \ \in \mathcal{A}(B) \otimes \mathbb{C}^n.$$

Then $\Phi(f)(x) = \overline{f}(\overline{a})(x)$.

This example can be generalized to the case when X has a Schauder basis. Recall that the sequence $\{e_n\}_{n=1}^{\infty}$ in a Banach space is called a Schauder basis of X if for any $x \in X$ there exists a unique sequence of scalars $\{x_n\}_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

and the series converges by the norm of X, that is,

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} x_k e_k \right\| = 0.$$

We denote by e_n^* the coordinate functionals, $e_n^*(x) = x_n$.

Proposition 1. *Let X be a Banach space with a Schauder basis, $A = \mathcal{H}_{uc}^{\infty}(B)$, $\Phi = I\colon H_b(X) \to \mathcal{H}_{uc}^{\infty}(B)$. Then* (2) *holds for a sequence $\overline{a}_m \in \mathcal{H}_{uc}^{\infty}(B) \otimes_{\pi} X$.*

Proof. Let $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis in X. Then every element $x \in X$ can be represented as $x = \sum\limits_{k=1}^{\infty} e_k^*(x) e_k$. Consider

$$\overline{a}_m = \sum_{k=1}^{m} e_k^* \otimes e_k = \sum_{k=1}^{m} e_k^* e_k.$$

In the sense of functional calculus we have:

$$\overline{f}(\overline{a}_m)(x) = \overline{f}\left(\sum_{k=1}^{m} e_k^* \otimes e_k\right)(x) = f\left(\sum_{k=1}^{m} e_k^*(x) e_k\right) = f\left(\sum_{k=1}^{m} x_k e_k\right).$$

Since $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis, $\sum\limits_{k=1}^{m} x_k(e_k) \to x$ as $m \to \infty$. This means that

$$I(f)(x) = \lim_{m\to\infty} \overline{f}(\overline{a}_m)(x) = \lim_{m\to\infty} \theta_{\overline{a}_m}(f).$$

□

In the general case we consider the space with the approximation property.

Definition 1. *A Banach space X is said to have the approximation property in the sense of Grothendieck if for every compact set K in X and every $\varepsilon > 0$ there is an operator $T : X \to X$ of finite rank such that $\|Tx - x\| \le \varepsilon$ for every $x \in K$.*

Theorem 1. *Let X be a Banach space with the approximation property. Then for $\Phi = I$ equality* (2) *holds.*

Proof. Let $\mathfrak{A}$ be the following set of indexes: if $\alpha \in \mathfrak{A}$, then $\alpha = (K, \varepsilon, n)$, where K is a compact set in X, $\varepsilon > 0$ and $n \in \mathbb{N}$. We introduce a partial order on $\mathfrak{A}$ by the following way: $\alpha_1 \leq \alpha_2$ if and only if $K_1 \subset K_2$, $\varepsilon_1 \leq \varepsilon_2$ and $n_1 \leq n_2$. So $\mathfrak{A}$ is a directed set. Since X has the approximation property, for every $\alpha = (K_\alpha, \varepsilon_\alpha, n_\alpha) \in \mathfrak{A}$ there is an operator T_α with the rank n_α such that for every $x \in K_\alpha$, $\|T_\alpha x - x\| \leq \varepsilon_\alpha$. $(T_\alpha)_\alpha$ is a net and $x = \lim_\alpha T_\alpha x$ for every $x \in X$.

Let $\{\gamma_{k,\alpha}\}_{k=1}^{n_\alpha}$ be a basis in the range of T_α in X and $\{\gamma^*_{k,\alpha}\}_{k=1}^{n_\alpha} \in X'$ be linear functionals which are bi-orthogonal to $\{\gamma_{k,\alpha}\}_{k=1}^{n_\alpha}$. So $T_\alpha(x) = \sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha}(x)\gamma_{k,\alpha}$. Thus we can set

$$\overline{a}_\alpha = \sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha} \otimes \gamma_{k,\alpha},$$

Hence, for every $f \in H_b(X)$

$$I(f)) = \lim_\alpha \overline{f}(\overline{a}_\alpha) \in \mathcal{H}^\infty_{uc}(B)$$

and so equality (2) holds. □

It seems to be that the approximation property is to strong condition for our purpose. Let us consider the weak H_b topology on X as the the restriction of the Gelfand topology on X, that is, the weakest topology on X such that all $f \in H_b(X)$ are continuous.

Definition 2. *We say that X has the H_b-approximation property if for every compact set K in the weak $H_b(X)$ topology and every $\varepsilon > 0$ there exists a finite rank operator T such that*

$$|f(T(x) - x)| < \varepsilon$$

for every polynomial $f \in H_b(X)$ and every $x \in K$.

Doing the same work like in Theorem 1 we can prove the following theorem.

Theorem 2. *If X has the H_b-approximation property, then* (2) *holds.*

It is easy to see that every Banach space X with the approximation property has the H_b-approximation property but we do not know about the inverse implication. Also, we do not know any examples for which the property (2) is not true.

Let us consider (1) for more general case.

Theorem 3. *Let Φ be a homomorphism from $H_b(X)$ to $\mathcal{H}^\infty_{uc}(B)$ such that there is an analytic automorphisms $F : B \to B$ with $\Phi = C_F \circ I$, where C_F is the composition operator with F. Then* (2) *holds.*

Proof. Let

$$\sum_{k=1}^{n_\alpha} \gamma^*_{k,\alpha} \otimes \gamma_{k,\alpha}$$

be the net which approximate the identity map I as in the proof of Theorem 1. It enough to put

$$\overline{a}_\alpha = \sum_{k=1}^{n_\alpha} (\gamma^*_{k,\alpha} \circ F) \otimes \gamma_{k,\alpha}.$$

□

Note that in the general case, not every homomorphism Φ can be represented as $\Phi = C_F \circ I$. In [2] some related problems to the question about representation of homomorphisms by compositions operator were considered.

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Приймак Г.М. *Про наближення гомоморфізмів алгебри цілих функцій на банахових просторах* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 158–162.

Завдяки Р. Арону, Б. Коулу і Т. Гамеліну відомо, що кожен комплексний гомоморфізм алгебри цілих функцій обмеженого типу на банаховому просторі X можна наблизити в деякому сенсі за допомогою напрямленості поточкових гомоморфізмів. У даній роботі ми розглянемо питання про узагальнення цього результату для випадку гомоморфізмів зі значеннями у довільній комутативній банаховій алгебрі. Ми отримали деякі позитивні результати у випадку коли A — алгебра рівномірно неперервних аналітичних функцій на одиничній кулі простору X.

Ключові слова і фрази: аналітичні функції на банаховому просторі, гомоморфізми алгебри аналітичних функцій, властивість апроксимації.

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ON THE SOLUTIONS OF A CLASS OF NONLINEAR INTEGRAL EQUATIONS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

In this paper, we study the existence of the solutions of a class of functional integral equations by using some fixed point results in cone b-metric spaces over Banach algebras. In order to obtain these results we introduced and proved some properties of generalized weak φ-contractions, in which the φ are nonlinear weak comparison functions. The obtained results are generalizations of results of Van Dung N., Le Hang V. T., Huang H., Radenovic S. and Deng G. Also, some suitable examples are given to illustrate obtained results.

Key words and phrases: cone b-metric space over Banach algebra, φ-contraction, **c**-sequence, fixed point, integral equation.

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1 INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [5] introduced the concept of a cone metric space and generalized Banach fixed point theorem in such spaces. Afterwards, several authors published many papers on this topic. Aydi *et al.* [1, 2] proved some coupled coincidence point results on generalized distance in ordered cone metric spaces. Dordević *et al.* [4] considered fixed point and common fixed point results for maps in *tvs*-cone metric spaces under contractive conditions expressed in the terms of c-distance. By using an old Krein's result and a result concerning symmetric spaces, Jankovic *et al.* [10] showed in a very short way that fixed point results in cone metric spaces obtained recently, in which the assumption that the underlying cone is normal and solid is present, can be reduced to the corresponding results in metric spaces.

In 2013, Lia and Xu [12] introduced the notion of cone metric spaces over Banach algebras and defined a generalized Lipschitz contraction with vector contractive coefficient instead of usual real constant. The authors proved the existence of fixed points with the assumption that the underlying cone is normal. Furthermore, they explained by an example that the fixed point theorems in cone metric spaces over Banach algebra are not equivalent to those in metric spaces, and so, such generalizations are the genuine ones. Latter, Xu and Radenović [16] showed that the normality of the cone can be removed from the results of Liu and Xu [12]. In 2015, Huang and Radenović [6] introduced the notion of cone b-metric spaces over Banach algebra and presented some common fixed point theorems in such spaces. Subsequently, Huang and Radenović [7] considered the Banach type version of a fixed point result with the generalized Lipschitz constant k satisfying $\rho(k) \in [0, \frac{1}{s})$ where $\rho(k)$ is the spectral radius of k. In 2017, Huang *et al.* [8] generalized a famous result for Banach-type contractive map from $\rho(k) \in [0, \frac{1}{s})$

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to $\rho(k) \in [0,1)$ in cone b-metric spaces over Banach algebra with coefficient $s \geq 1$. Very recent, by using a nontrivial proof method Li and Huang [11] proved some fixed point results for weak φ-contractions in cone metric spaces over Banach algebras and applied to investigate the existence and uniqueness of a solution to two classes of equations. However, in the construction of such applications, the functions φ considered in φ-contractions are simple linear functions, for example see [6, Theorem 3.1] and [11, Theorem 3.2].

In this paper, we study the existence of the solutions of a class of functional integral equations by using some fixed point results in cone b-metric spaces over Banach algebras. In order to obtain these results we introduced and proved some properties of generalized weak φ-contractions, in which the φ are nonlinear weak comparison functions, and we also illustrated obtained results by suitable examples.

Now we recall definitions and properties which will be useful in what follows.

Definition 1 ([14, p. 245]). *Let $(\mathcal{A}, \|.\|)$ be a Banach space over the real field $\mathbb{R}$ in which a multiplication is defined that for all $x, y, z \in A$ and for all $\alpha \in \mathbb{R}$ satisfies*

1) $(xy)z = x(yz)$,

2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$,

3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

4) $\|xy\| \leq \|x\|\|y\|$,

5) there is a unit element e with $\|e\| = 1$ such that $xe = ex = x$.

Then $\mathcal{A}$ is called a Banach algebra.

Definition 2 ([7, p. 567]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and a zero element θ. A nonempty closed subset P of $\mathcal{A}$ is called a cone in $\mathcal{A}$ if*

1) $\{\theta, e\} \subset P$,

2) $\alpha P + \beta P \subset P$, for all $\alpha, \beta \in \mathbb{R}_+$,

3) $P^2 = PP \subset P$,

4) $P \cap (-P) = \{\theta\}$.

Definition 3 ([7, p. 567]). *Let $\mathcal{A}$ be a Banach algebra and P is a cone in $\mathcal{A}$. We say that*

1) P is a solid cone if $\operatorname{int} \mathrm{P} \neq \varnothing$, where $\operatorname{int} \mathrm{P}$ denotes the interior of P;

2) P is a normal cone if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq M\|y\|,$$

where $\|.\|$ is the norm in $\mathcal{A}$. The least positive value of M satisfying the above inequality is called the normal constant.

Note that, for any normal cone P we have $M \geq 1$ (see [13]).

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$, if $x \preceq y$ and $x \neq y$, and denote $x \ll y$ if and only if $y - x \in \text{int}P$.

In the sequel, unless otherwise specified, we always suppose that $\mathcal{A}$ is a Banach algebra, P is a solid cone in $\mathcal{A}$, and $\preceq$, $\ll$ are the above partial orderings with respect to P.

Definition 4 ([5]). *Let X be a nonempty set, $\mathcal{A}$ be a Banach algebra and $d : X \times X \to \mathcal{A}$ be a map such that for all $x, y, z \in X$*

1) $\theta \preceq d(x,y)$, and $d(x,y) = \theta$ if and only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,z) \preceq d(x,y) + d(y,z)$.

Then d is called a cone metric on X and $(X, \mathcal{A}, d)$ is called a cone metric space over Banach algebra.

Definition 5 ([5]). *Let $(X, \mathcal{A}, d)$ be a cone metric space over Banach algebra, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

1) $\{x_n\}$ converges to $x \in X$ if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$. Then, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;

2) $\{x_n\}$ is a Cauchy sequence if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

3) $(X, \mathcal{A}, d)$ is called complete if each Cauchy sequence is convergent in X.

Definition 6 ([7]). *Let X be a nonempty set, $s \geq 1$ be a constant, $\mathcal{A}$ be a Banach algebra and $d : X \times X \to \mathcal{A}$ be a map such that for all $x, y, z \in X$*

1) $0 \preceq d(x,y)$, and $d(x,y) = 0$ if only if $x = y$,

2) $d(x,y) = d(y,x)$,

3) $d(x,z) \preceq s[d(x,y) + d(y,z)]$.

Then d is called a cone b-metric on X and $(X, \mathcal{A}, d, s)$ is called a cone b-metric space over Banach algebra with the coefficient s.

Remark 1 ([7]). *A cone metric space over Banach algebra must be a cone b-metric space over Banach algebra. Conversely, it is not true. As a result, the notion of cone b-metric space over Banach algebra greatly generalizes the notion of cone metric space over Banach algebra.*

The following example shows that there exists a cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 1 ([7]). *Let $\mathcal{A} = C[0,1]$ be the usual Banach space with the supremum norm. Define multiplication in the usual way: $(xy)(t) = x(t)y(t)$, $t \in [0,1]$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Put $P = \{x \in \mathcal{A} : x(t) \geq 0, t \in [0,1]\}$ and $X = \mathbb{R}$. Define a map $d : X \times X \to \mathcal{A}$ by $d(x,y)(t) = |x-y|^p e^t$ for all $x, y \in X$, where $p > 1$ is a constant. This makes $(X, \mathcal{A}, d, s)$ into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra.*

Similar to Definition 5, we repeat the notions of convergent sequence, Cauchy sequence and complete space in cone b-metric space over Banach algebra.

Definition 7 ([7]). *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and $\{x_n\}$ be a sequence in X. We say that*

1) $\{x_n\}$ converges *to $x \in X$ if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$. Then, we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;*

2) $\{x_n\}$ is a Cauchy sequence if for each $c \in intP$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

3) $(X, \mathcal{A}, d)$ is a complete cone b-metric space if each Cauchy sequence in X is convergent.

Definition 8 ([4, Sect. 3.1]). *A sequence $\{u_n\} \subset P$ is called a* **c***-sequence if for each $c \in \operatorname{int} P$, there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n > N$.*

Lemma 1 ([7]). *Let P be a solid cone in a Banach algebra $\mathcal{A}$, $\{u_n\}$ and $\{v_n\}$ be two* **c***-sequences in P. If $\alpha, \beta \in P$ are two arbitrarily given vectors, then $\{\alpha u_n + \beta v_n\}$ is a* **c***-sequence.*

Lemma 2 ([14]). *Let $\mathcal{A}$ be a Banach algebra. Then the spectral radius of $k \in \mathcal{A}$ equals to $\rho(k) = \lim\limits_{n\to\infty} \|k^n\|^{\frac{1}{n}} = \inf\limits_{n\geq 1} \|k^n\|^{\frac{1}{n}}$.*

Lemma 3 ([6]). *Let $\mathcal{A}$ be a Banach algebra. Let $k \in \mathcal{A}$ and $\rho(k) < 1$. Then $\{k^n\}$ is a* **c***-sequence.*

Lemma 4 ([9]). *Let $\mathcal{A}$ be a Banach algebra and $u, v, w \in \mathcal{A}$. Then*

(1) if $u \preceq v$ and $v \ll w$, then $u \ll w$;

(2) If $u \ll v$ and $v \ll w$, then $u \ll w$;

(3) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$;

(4) $\alpha intP \subseteq intP$ for all $\alpha > 0$;

(5) If $c \in intP$, $\theta \preceq a_n$ and $\lim\limits_{n\to\infty} a_n = \theta$ then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n \ll c$.

Definition 9 ([9]). *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and $B \subseteq X$. An element $b \in B$ is called an interior point of B whenever there is $\theta \ll p$ such that $B_0(b, p) \subseteq B$, where $B_0(b, p) = \{y \in X : d(y, b) \ll p\}$.*

Definition 10 ([15, p. 246]). *A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if γ is increasing and $\lim\limits_{n\to\infty} \gamma^n(u) = 0$ for all $u \in \mathbb{R}_+$.*

The following some notions and property are well known in [11].

Definition 11 ([11]). *Let $\mathcal{A}$ be a Banach algebra and P be a cone in $\mathcal{A}$. A map $\varphi : P \to P$ is called a weak comparison if the following conditions hold*

(1) φ is nondecreasing with respect to $\preceq$, that is, for all $t_1, t_2 \in P$ and $t_1 \preceq t_2$, implies that $\varphi(t_1) \preceq \varphi(t_2)$;

*(2) $\{\varphi^n(t)\}$ is a **c**-sequence in P for all $t \in P$;*

*(3) if $\{u_n\}$ is a **c**-sequence in P, then $\{\varphi(u_n)\}$ is also a **c**-sequence in P.*

Definition 12 ([11]). *Let $(X, \mathcal{A}, d)$ be a cone metric space over Banach algebra and P be a cone in $\mathcal{A}$. Let $\varphi : P \to P$ be a weak comparison. Then a map $f : X \to X$ is called a weak φ-contraction if for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Theorem 1 ([11]). *Let $(X, \mathcal{A}, d)$ be a complete cone metric space over Banach algebra and $f : X \to X$ be a weak φ-contraction. Then f has a unique fixed point $u \in X$ and $\lim\limits_{n\to\infty} f^n(x) = u$ for each $x \in X$.*

2 FIXED POINT RESULTS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

First we extend the notion of weak φ-contraction in metric spaces to the setting of cone b-metric spaces over Banach algebra as follows.

Definition 13. *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra and P be a cone in $\mathcal{A}$. Let $\varphi : P \to P$ be a weak comparison. Then a map $f : X \to X$ is called a generalized weak φ-contraction if for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Lemma 5. *Let $(X, \mathcal{A}, d, s)$ be a cone b-metric space over Banach algebra, P be a cone in $\mathcal{A}$, and $f : X \to X$ be a generalized weak φ-contraction. Then,*

(1) for all $t_1, t_2 \in P$ with $t_1 \preceq t_2$ and all $n \in \mathbb{N}$, we have $\varphi^n(t_1) \preceq \varphi^n(t_2)$;

(2) for all $x, y \in X$ and all $n \in \mathbb{N}$, we have

$$d\big(f^n(x), f^n(y)\big) \preceq \varphi^n\big(d(x,y)\big).$$

Proof. (1). For any $t_1, t_2 \in P$ with $t_1 \preceq t_2$, since φ is a weak comparison, we have

$$\varphi(t_1) \preceq \varphi(t_2).$$

Then, we get

$$\varphi^2(t_1) = \varphi\big(\varphi(t_1)\big) \preceq \varphi\big(\varphi(t_2)\big) = \varphi^2(t_2).$$

Continuing the above process, we obtain that for all n,

$$\varphi^n(t_1) \preceq \varphi^n(t_2).$$

(2). For any $x, y \in X$, since f is a generalized weak φ-contraction, we have

$$d\big(f(x), f(y)\big) \preceq \varphi\big(d(x,y)\big).$$

Note that φ is a weak comparison, so we have

$$\varphi\Big(d\big(f(x), f(y)\big)\Big) \preceq \varphi\Big(\varphi\big(d(x,y)\big)\Big) = \varphi^2\big(d(x,y)\big). \tag{1}$$

Using f being a generalized weak φ-contraction again, we get

$$d\big(f^2(x), f^2(y)\big) \preceq \varphi\Big(d\big(f(x), f(y)\big)\Big). \tag{2}$$

From (1) and (2) we have

$$d\big(f^2(x), f^2(y)\big) \preceq \varphi^2\big(d(x,y)\big).$$

Continuing this process we obtain that for all n,

$$d\big(f^n(x), f^n(y)\big) \preceq \varphi^n\big(d(x,y)\big).$$

□

Now, we establish some results for generalized weak φ-contraction maps in complete cone b-metric space over Banach algebra.

Lemma 6. *Let $(X, \mathcal{A}, d, s)$ be a complete cone b-metric space over Banach algebra and $f : X \to X$ be a generalized weak φ-contraction. Then f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = u$.*

Proof. Let any $x \in X$ and put $x_0 = x$, $x_n = f^n(x)$ for all $n \geq 1$.

Then, by Definition 13, for each $c \in \operatorname{int}P$, exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(c) \ll s^{-1}c$. Using Lemma 5.(2), for every $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+n_0}) \preceq \varphi^n\Big(d\big(x_0, x_{n_0}\big)\Big). \tag{3}$$

Since $\{\varphi^n\big(d(x_0, x_{n_0})\big)\}$ is a **c**-sequence then by (3) and Lemma 4.(1), we have $\{d(x_n, x_{n+n_0})\}$ is also a **c**-sequence. Hence, exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+n_0}) \ll s^{-1}c - \varphi^{n_0}(c) \text{ for all } n \geq N_1.$$

Put

$$B(x_n, c) = \{y \in X : d(x_n, y) \ll c\} \text{ for all } n \geq N_1 - 1. \tag{4}$$

For each $n \geq N_1 - 1$, choosing $y \in B(x_n, c)$, by (3) and (4) we have

$$\begin{aligned} d(x_n, f^{n_0}y) &\preceq s[d(x_n, x_{n+n_0}) + d(x_{n+n_0}, f^{n_0}y)] \\ &\preceq s[s^{-1}c - \varphi^{n_0}(c) + \varphi^{n_0}(d(x_n, y))] \\ &\ll c - s\varphi^{n_0}(c) + s\varphi^{n_0}(c) \\ &= c. \end{aligned}$$

This implies that $B(x_n, c)$ is f^{n_0}-invariant. Hence, for each $k \in \mathbb{N}$, we have

$$d(x_n, x_{n+kn_0}) = d(x_n, f^{n_0}x_n) \ll c, \text{ for all } n \geq N_1 - 1. \tag{5}$$

Using Lemma 5.(2), for every $n \in \mathbb{N}$ we get

$$\begin{aligned} &sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0}) \\ &\qquad \preceq s\varphi^n\big(d(x_0, x_1)\big) + s^2\varphi^n\big(d(x_1, x_2)\big) + \cdots + s^{n_0}\varphi^n\big(d(x_{n_0-1}, x_{n_0})\big). \end{aligned} \tag{6}$$

For each $i = 0, 1, 2, ..., n_0$, we have $\{\varphi^n\big(d(x_i, x_{i+1})\big)\}$ is a **c**-sequences then by Lemma 1, $\{s\varphi^n\big(d(x_0, x_1)\big) + s^2\varphi^n\big(d(x_1, x_2)\big) + \cdots + s^{n_0}\varphi^n\big(d(x_{n_0-1}, x_{n_0})\big)\}$ is a **c**-sequence. Hence, by (6) and Lemma 4.(1), we have

$$\{sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0})\}$$

is also a **c**-sequence. So, for any $c \in \text{int}P$, exists $N_2 \in \mathbb{N}$ such that

$$sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{n_0} d(x_{n+n_0-1}, x_{n+n_0}) \ll c \tag{7}$$

for all $n \geq N_2$.

Denote $N = \max\{N_1, N_2\}$, for all $m,\ n > N$ we put

$$k_m = \Big[\frac{m - N}{n_0}\Big], \quad k_n = \Big[\frac{n - N}{n_0}\Big],$$

where $[\,.\,]$ stands for the integer part. Because

$$N \leq m - k_m n_0 < N + n_0, \quad N \leq n - k_n n_0 < N + n_0, \tag{8}$$

from (8) we find that

$$\big|(n - k_n n_0) - (m - k_m n_0)\big| < n_0.$$

Hence, from (7) we have

$$d(x_{n-k_n n_0}, x_{m-k_m n_0}) \preceq sd(x_{n-k_n n_0}, x_{n-k_n n_0+1}) + \cdots + s^{n_0} d(x_{n-k_n n_0+n_0-1}, x_{n-k_n n_0+n_0}) \ll c. \tag{9}$$

Hence, from (5) and (9) we find that

$$\begin{aligned} d(x_n, x_m) &\preceq sd(x_n, x_{n-k_n n_0}) + s^2 d(x_{n-k_n n_0}, x_{m-k_m n_0}) + s^2 d(x_{m-k_m n_0}, x_m) \\ &\ll (s + s^2 + s^2)c. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $(X, \mathcal{A}, d, s)$. Since $(X, \mathcal{A}, d, s)$ is complete there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$.

Next, we prove that u is the fixed point of f. Indeed, we have

$$\begin{aligned} d(fu, u) &\preceq sd(fu, x_n) + sd(x_n, u) \\ &= sd(fu, fx_{n-1}) + sd(x_n, u) \\ &\preceq s\varphi\big(d(u, x_{n-1})\big) + sd(x_n, u). \end{aligned} \tag{10}$$

Since $\{d(x_n, u)\}$ is a **c**-sequence and φ is weak comparison, then $\{\varphi\big(d(u, x_{n-1})\big)\}$ is also a **c**-sequence. Hence, by Lemma 1 we have $\{s\varphi\big(d(u, x_{n-1})\big) + sd(x_n, u)\}$ is a **c**-sequence. By (10), Lemma 4.(3) and $\{s\varphi(d(u,x_{n-1})) + sd(x_n,u)\}$ is a **c**-sequence, we find that $d(fu,u) = \theta$. This implies that u is a fixed point of f.

Finally, we prove that the fixed point is unique. Assume that v is another fixed point of f. Then we have

$$\theta \preceq d(u,v) = d\big(f^n(u), f^n(v)\big) \preceq \varphi^n\big(d(u,v)\big) \quad \text{for all } n \geq 1. \tag{11}$$

Since $\{\varphi^n\big(d(u,v)\big)\}$ is a **c**-sequence then by (11) and Lemma 4.(3), we have $d(u,v) = \theta$. This implies that $u = v$.

So, f have unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$. □

In Lemma 6, if we choose $\varphi : P \to P$ by $\varphi(t) = kt$, for all $t \in \mathcal{A}$ and $k \in P$ such that $\rho(k) < 1$, then we obtain the following.

Corollary 1 ([8]). *Let $(X, \mathcal{A}, d, s)$ be a complete cone b-metric space over Banach algebra and $f : X \to X$ be a map such that for all $x, y \in X$,*

$$d(f(x), f(y)) \preceq kd(x,y), \tag{12}$$

where $k \in P$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then, f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$.

By choosing $\mathcal{A} = \mathbb{R}$ and $P = \mathbb{R}_+$ in Lemma 6, then we obtain the following.

Corollary 2 ([3]). *Let (X, d, s) be a complete b-metric space and $f : X \to X$ be a map such that for all $x, y \in X$,*

$$d(f(x), f(y)) \leq \varphi(d(x,y)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function. Then, f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = u$.

The following example shows the superiority of the main result in the sense that there exist a complete cone b-metric space over Banach algebra and a map $f : X \to X$ such that Corollary 1 is not applicable to, while our result is.

Example 2. *Let $\mathcal{A} = \mathbb{R}^2$, $P = \{(x,y) \in \mathcal{A} : x, y \geq 0\}$, and $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$. Define*

(a) *the norm of $\mathcal{A}$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$;*

(b) *the multiplication of $\mathcal{A}$ by $xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$;*

(c) *$X = [0, \infty)$ and define $d : X \times X \to \mathcal{A}$ by $d(x,y) = \big(|x-y|^2, 0\big)$ for all $x, y \in X$;*

(d) *$f : X \to X$, $f(x) = \frac{x}{x+1}$ for all $x \in X$;*

(e) *$\varphi : P \to P$, $\varphi(z_1, z_2) = \left(\frac{z_1}{z_1+1}, 0\right)$ for all $(z_1, z_2) \in P$.*

Then

(1) *$\mathcal{A}$ is a Banach algebra with the identity element $e = (1,0)$ and $\theta = (0,0)$;*

(2) *for all* $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$, $x \succeq y$ *if and only if* $x_1 \geq y_1$ *and* $x_2 \geq y_2$*;*

(3) $(X, \mathcal{A}, d, s)$ *is a complete cone* b*-metric space over Banach algebra with* $s = 2$*;*

(4) *there does not exist* $k \in \mathcal{A}$ *with* $\rho(k) < 1$ *such that the condition* (12) *holds;*

(5) *all assumptions of Lemma 6 hold.*

Proof. (1). See [11, Theorem 3.1].

(2). Since $P = \{(x, y) \in \mathcal{A} : x, y \geq 0\}$, for any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{A}$, we have $x \succeq y$ if and only if $(x_1 - y_1, x_2 - y_2) \in P$. It is equivalent to $x_1 \geq y_1$ and $x_2 \geq y_2$.

(3). For any $x, y, z \in X = [0, \infty)$ we have

- $d(x, y) = \big(|x - y|^2, 0\big) \succeq (0, 0)$. So $d(x, y) \succeq \theta$, and $d(x, y) = \theta$ if and only if $x = y$;
- $d(x, y) = \big(|x - y|^2, 0\big) = \big(|y - x|^2, 0\big) = d(y, x)$.

Since $|x - z|^2 \leq 2\big(|x - y|^2 + |y - z|^2\big)$, we have $\big(|x - z|^2, 0\big) \leq 2\big[\big(|x - y|^2, 0\big) + \big(|y - z|^2, 0\big)\big]$. It implies that

$$d(x, z) \preceq 2\big(d(x, y) + d(y, z)\big).$$

By the above, d is a cone b-metric on X with $s = 2$.

Now for any Cauchy sequence $\{x_n\}$ in X and for each $c = (c_1, c_2) \in \operatorname{int}P$ there exists $m_0 \in \mathbb{N}$ such that for all $n, m > m_0$ we have

$$d(x_n, x_m) = \big(|x_n - x_m|^2, 0\big) \ll (c_1, c_2) = c.$$

This implies that for each $c_1 > 0$, we have $|x_n - x_m| \leq (c_1)^{\frac{1}{2}}$ for all $n, m > m_0$. It implies that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. So there exists $x \in \mathbb{R}$ such that $\lim\limits_{n \to \infty} |x_n - x| = 0$. Since $x_n \in X = \mathbb{R}_+$ for all n and $x_n \to x$ in $\mathbb{R}$, we have $x \in X$. This implies that for each $c = (c_1, c_2) \in \operatorname{int}P$, there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have $|x_n - x| \leq (c_1)^{\frac{1}{2}}$. Therefore, we get that for all $n > m_0$

$$d(x_n, x) = \big(|x_n - x|^2, 0\big) \ll (c_1, c_2) = c.$$

This proves that $\{x_n\}$ convergent to x in $(X, \mathcal{A}, d, s)$. So $(X, \mathcal{A}, d, s)$ is complete.

By the above, $(X, \mathcal{A}, d, s)$ is a complete cone b-metric space over Banach algebra with $s = 2$.

(4). Firstly, we observe that for $k = (k_1, k_2) \in P$, by induction we have

$$k^n = (k_1, k_2)^n = (k_1^n, n k_2 k_1^{n-1}).$$

It implies that

$$\rho(k) = \inf \|k^n\|^{\frac{1}{n}} = \inf \big(|k_1^n| + |n k_2 k_1^{n-1}|\big)^{\frac{1}{n}}. \tag{13}$$

Then, if $\rho(k) < 1$, by (13), we get that $k_1 < 1$.

On the contrary, suppose that there exists $k = (k_1, k_2) \in P$ with $\rho(k) < 1$ such that

$$d\big(f(x), f(y)\big) \preceq k d(x, y)$$

for all $x, y \in X$. Then for all $x, y \in X$,

$$\Big(\Big|\frac{x}{x+1} - \frac{y}{y+1}\Big|^2, 0\Big) \preceq (k_1, k_2)\big(|x - y|^2, 0\big).$$

For $x \neq 0$ and $y = 0$ we have

$$\Big(\Big|\frac{x}{x+1}\Big|^2, 0\Big) \preceq (k_1, k_2)\big(|x|^2, 0\big).$$

It is equivalent to

$$\Big(\Big|\frac{x}{x+1}\Big|^2, 0\Big) \preceq \big(k_1|x|^2, k_2|x|^2\big).$$

This implies that

$$\frac{|x|^2}{(x+1)^2} \leq k_1|x|^2.$$

Hence for all $x \neq 0$, we have

$$\frac{1}{(x+1)^2} \leq k_1. \tag{14}$$

Letting $x \to 0^+$ in (14) we get $1 \leq k_1$. This contradicts to the above observation.

(5). • For any $z = (z_1, z_2), t = (t_1, t_2) \in P$ with $z \preceq t$, that is, $0 \leq z_1 \leq t_1$ and $0 \leq z_2 \leq t_2$. Then we have

$$\frac{z_1}{z_1+1} \leq \frac{t_1}{t_1+1}.$$

It implies that

$$\varphi(z) = \Big(\frac{z_1}{z_1+1}, 0\Big) \preceq \Big(\frac{t_1}{t_1+1}, 0\Big) = \varphi(t).$$

So, for all $z, t \in P$ with $z \preceq t$, we have $\varphi(z) \preceq \varphi(t)$.

• Now for any $z = (z_1, z_2) \in P$ we have by induction that $\varphi^n(z) = \Big(\frac{z_1}{nz_1+1}, 0\Big)$. It follows that

$$(0,0) \leq \Big(\frac{z_1}{nz_1+1}, 0\Big) \text{ and } \lim_{n\to\infty} \frac{z_1}{nz_1+1} = 0.$$

This implies that

$$\theta \preceq \varphi^n(z) \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(z) = \theta.$$

Therefore, for each $c = (c_1, c_2) \in \text{int}P$, by Lemma 4.(5) there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have

$$\Big(\frac{z_1}{nz_1+1}, 0\Big) \ll (c_1, c_2) = c.$$

This implies that $\{\varphi^n(z)\}$ is a **c**-sequence in P.

• Suppose that $\{z_n\} = \{(z_1^{(n)}, z_2^{(n)})\}$ is a **c**-sequence in P, then for each $c = (c_1, c_2) \in \text{int}P$, there exists $k_0 \in \mathbb{N}$ such that for all $n > k_0$ we have $(z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2) = c$. This implies that

$$\varphi(z_n) = \Big(\frac{z_1^{(n)}}{z_1^{(n)}+1}, 0\Big) \preceq (z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2) = c, \text{ for all } n > k_0.$$

Therefore, $\{\varphi(z_n)\}$ is also a **c**-sequence in P.

Hence φ is a weak comparison.

Next, for any $x, y \in X$, we have

$$\begin{aligned}\Big(\Big|\frac{x}{x+1} - \frac{y}{y+1}\Big|^2, 0\Big) &= \Big(\Big|\frac{x-y}{xy+x+y+1}\Big|^2, 0\Big) \leq \Big(\Big|\frac{|x-y|}{|x-y|+1}\Big|^2, 0\Big) \\ &= \Big(\Big|\frac{|x-y|^2}{|x-y|^2+2|x-y|+1}\Big|, 0\Big) \leq \Big(\frac{|x-y|^2}{|x-y|^2+1}, 0\Big).\end{aligned} \tag{15}$$

Note that

$$d\big(f(x), f(y)\big) = d\Big(\frac{x}{x+1}, \frac{y}{y+1}\Big) = \Big(\Big|\frac{x}{x+1} - \frac{y}{y+1}\Big|^2, 0\Big)$$

and

$$\varphi(d(x,y)) = \Big(\frac{|x-y|^2}{|x-y|^2+1}, 0\Big).$$

So from (15) we find that $d\big(f(x), f(y)\big) \preceq \varphi(d(x,y))$ for all $x, y \in X$, and f is a generalized weak φ-contraction.

By the above, all assumptions of Lemma 6 hold. □

3 APPLICATIONS TO THE NONLINEAR INTEGRAL EQUATIONS

In this section, we apply Lemma 6 to study the existence and uniqueness of the solution to the nonlinear integral equations.

Lemma 7. *Let $C[a,b]$ be the set of all continuous functions on $[a,b]$, where $a, b \in \mathbb{R}$. Let $\mathcal{A} = \mathbb{R}^2$ and $P = \{(x,y) \in \mathcal{A} : x, y \geq 0\}$ with the same norm, the same multiplication, and the same partial order on $\mathcal{A}$ as stated in Example 2. Define $d : C[a,b] \times C[a,b] \to \mathcal{A}$ by*

$$d(x,y) = \Big(\sup_{t\in[a,b]} |x(t)-y(t)|^2, \sup_{t\in[a,b]} |x(t)-y(t)|^2\Big)$$

for all $x, y \in C[a,b]$. Then $\big(C[a,b], \mathcal{A}, d, s\big)$ is a complete cone b-metric space over Banach algebra with $s = 2$.

Proof. For any $x, y, z \in C[a,b]$ we have

$d(x,y) = \Big(\sup_{t\in[a,b]} |x(t)-y(t)|^2, \sup_{t\in[a,b]} |x(t)-y(t)|^2\Big) \geq (0,0)$. So $d(x,y) \succeq \theta$.

$d(x,y) = \theta$ if and only if $\sup_{t\in[a,b]} |x(t)-y(t)|^2 = 0$ if and only if $x(t) = y(t)$ for all $t \in [a,b]$, that is, $x = y$.

Since $\sup_{t\in[a,b]} |x(t)-y(t)|^2 = \sup_{t\in[a,b]} |y(t)-x(t)|^2$ for all $t \in [a,b]$, we get that $d(x,y) = d(y,x)$.

We have

$$|x(t)-z(t)|^2 \leq 2\big(|x(t)-y(t)|^2 + |y(t)-z(t)|^2\big) \text{ for all } t \in [a,b].$$

It implies that

$$\sup_{t\in[a,b]} |x(t)-z(t)|^2 \leq 2\Big(\sup_{t\in[a,b]} |x(t)-y(t)|^2 + \sup_{t\in[a,b]} |y(t)-z(t)|^2\Big) \text{ for all } t \in [a,b].$$

That is,

$$d(x,z) \preceq 2\big(d(x,y) + d(y,z)\big).$$

By the above, d is a cone b-metric on X with $s = 2$.

Now for any Cauchy sequence $\{x_n\}$ in $\big(C[a,b], \mathcal{A}, d, s\big)$ and for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $m_0 \in \mathbb{N}$ such that for all $n, m > m_0$ we have

$$d(x_n, x_m) = \big(\sup_{t\in[a,b]} |x_n(t)-x_m(t)|^2, \sup_{t\in[a,b]} |x_n(t)-x_m(t)|^2\big) \ll (c_1, c_2) = c. \quad (16)$$

This implies that

$$\sup_{t\in[a,b]} |x_n(t) - x_m(t)| \leq \sqrt{c_i},\ i = 1, 2,\ \text{for all } n, m > m_0. \tag{17}$$

So $\{x_n\}$ is a Cauchy sequence in $C[a,b]$. Since $C[a,b]$ with the sup-norm is complete, there exists $x \in C[a,b]$ such that $\lim_{n\to\infty} x_n = x$. Hence by (17) we have

$$\sup_{t\in[a,b]} |x_n(t) - x(t)| \leq \sqrt{c_i},\ i = 1, 2,\ \text{for all } n > m_0.$$

This implies by (16) that

$$d(x_n, x) = \Big(\sup_{t\in[a,b]} |x_n(t) - x(t)|^2, \sup_{t\in[a,b]} |x_n(t) - x(t)|^2\Big) \ll (c_1, c_2) = c,\ \text{for all } n > m_0.$$

This proves that $\{x_n\}$ converges to x in $\big(C[a,b], \mathcal{A}, d, s\big)$. So $\big(C[a,b], \mathcal{A}, d, s\big)$ is complete.

By the above, $\big(C[a,b], \mathcal{A}, d, s\big)$ is a complete cone b-metric space over Banach algebra with $s = 2$. □

Theorem 2. *Let $\big(C[a,b], \mathcal{A}, d, s\big)$ be a complete cone b-metric space over Banach algebra in Lemma 7. Consider a integral equation*

$$x(t) = \eta(t) + \int_a^b K\big(t, x(r)\big)dr, \quad t \in [a,b], \tag{18}$$

where $x \in C[a,b]$, $\eta \in C[a,b]$ and $K : [a,b] \times \mathbb{R} \to \mathbb{R}$. Assume that the following hypotheses hold:

1) *for each $t \in [a,b]$, $K\big(t, x(r)\big)$ is integrable with respect to r on $[a,b]$;*

2) *there exists a continuous function $\psi : [a,b] \times [a,b] \to \mathbb{R}$ with $\sup_{t\in[a,b]} \int_a^b \big|\psi(t,r)\big|dr \leq 1$ and there exists a comparison function γ such that for all $t, r \in [a,b]$ and all $x, y \in C[a,b]$,*

$$\Big|K\big(t, x(r)\big) - K\big(t, y(r)\big)\Big| \leq \big|\psi(t,r)\big|\gamma\big(|x(r) - y(r)|\big).$$

Then the integral equation (18) *has a unique solution $u \in C[a,b]$.*

Proof. Let $f : C[a,b] \to C[a,b]$ be a map defined by

$$\big(f(x)\big)(t) = \eta(t) + \int_a^b K\big(t, x(r)\big)dr, \quad x \in C[a,b],\ t \in [a,b].$$

For any $x, y \in C[a,b]$, we have

$$
\begin{aligned}
& d\Big(\big(f(x)\big)(t), \big(f(y)\big)(t)\Big) \\
&= \Big(\sup_{t\in[a,b]} \big|\big(f(x)\big)(t) - \big(f(y)\big)(t)\big|^2, \sup_{t\in[a,b]} \big|\big(f(x)\big)(t) - \big(f(y)\big)(t)\big|^2\Big) \\
&= \Big(\sup_{t\in[a,b]} \Big|\int_a^b \Big[K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big]dr\Big|^2, \sup_{t\in[a,b]} \Big|\int_a^b \Big[K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big]dr\Big|^2\Big) \\
&\leq \Big(\sup_{t\in[a,b]} \Big[\int_a^b \Big|K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big|dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b \Big|K\big(t,x(r)\big) - K\big(t,y(r)\big)\Big|dr\Big]^2\Big) \\
&\leq \Big(\sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(|x(r)-y(r)|\big)dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(|x(r)-y(r)|\big)dr\Big]^2\Big) \\
&\leq \Big(\sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(\sup_{r\in[a,b]} |x(r)-y(r)|\big)dr\Big]^2, \sup_{t\in[a,b]} \Big[\int_a^b |\psi(t,r)|\gamma\big(\sup_{r\in[a,b]} |x(r)-y(r)|\big)dr\Big]^2\Big) \\
&\leq \Big(\gamma^2\big(\sup_{r\in[a,b]} |x(r)-y(r)|\big), \gamma^2\big(\sup_{r\in[a,b]} |x(r)-y(r)|\big)\Big) \\
&= \Big(\gamma^2\Big(\sqrt{\sup_{r\in[a,b]} |x(r)-y(r)|^2}\Big), \gamma^2\Big(\sqrt{\sup_{r\in[a,b]} |x(r)-y(r)|^2}\Big)\Big) \\
&= \varphi\Big(\sup_{r\in[a,b]} |x(r)-y(r)|^2\big), \sup_{r\in[a,b]} |x(r)-y(r)|^2\big)\Big) = \varphi\big(d(x,y)\big),
\end{aligned}
$$

where $\varphi : P \to P$ defined by $\varphi(z) = \varphi(z_1, z_2) = \Big(\gamma^2(\sqrt{z_1}), \gamma^2(\sqrt{z_2})\Big)$ for all $z = (z_1, z_2) \in P$.

Now we prove that $\varphi(z)$ is a weak comparison.

- For any $z = (z_1, z_2)$, $t = (t_1, t_2) \in P$ with $z \preceq t$. Then we have $0 \leq z_1 \leq t_1$ and $0 \leq z_2 \leq t_2$. It follows that

$$0 \leq \gamma(\sqrt{z_1}) \leq \gamma(\sqrt{t_1}) \text{ and } 0 \leq \gamma(\sqrt{z_2}) \leq \gamma(\sqrt{t_2}).$$

This implies that

$$\gamma^2(\sqrt{z_1}) \leq \gamma^2(\sqrt{t_1}) \text{ and } \gamma^2(\sqrt{z_2}) \leq \gamma^2(\sqrt{t_2}).$$

Therefore, we get

$$\varphi(z) = \Big(\gamma^2(\sqrt{z_1}), \gamma^2(\sqrt{z_2})\Big) \preceq \Big(\gamma^2(\sqrt{t_1}), \gamma^2(\sqrt{t_2})\Big) = \varphi(t).$$

So, for all $z, t \in P$ with $z \preceq t$, we have $\varphi(z) \preceq \varphi(t)$.

- Since $z = (z_1, z_2) \in P$ and γ is the comparison function, we have

$$(0,0) \leq \Big(\gamma^{2n}(\sqrt{z_1}), \gamma^{2n}(\sqrt{z_2})\Big) \text{ and } \lim_{n\to\infty} \gamma^{2n}(\sqrt{z_1}) = \lim_{n\to\infty} \gamma^{2n}(\sqrt{z_2}) = 0.$$

This implies that

$$\theta \preceq \varphi^n(z) \text{ and } \lim_{n\to\infty} \varphi^n(z) = \theta.$$

Therefore, for each $c = (c_1, c_2) \in \text{int}P$, by Lemma 4.(5) there exists $m_0 \in \mathbb{N}$ such that for all $n > m_0$ we have

$$\varphi^n(z) \ll (c_1, c_2) = c.$$

This prove that $\{\varphi^n(z)\}$ is a **c**-sequence in P.

• Suppose that $\{z_n\} = \{(z_1^{(n)}, z_2^{(n)})\}$ is a **c**-sequence in P, then for each $c = (c_1, c_2) \in \mathrm{int}P$, there exists $l_0 \in \mathbb{N}$ such that for all $n > l_0$ we have $(z_1^{(n)}, z_2^{(n)}) \ll (c_1, c_2)$. Since γ is a comparison function, we find that

$$\begin{aligned} \varphi(z_n) &= \varphi(z_1^{(n)}, z_2^{(n)}) = \Big(\gamma^2\big(\sqrt{z_1^{(n)}}\big), \gamma^2\big(\sqrt{z_2^{(n)}}\big)\Big) \\ &\preceq \Big(\gamma\big(\sqrt{z_1^{(n)}}\big), \gamma\big(\sqrt{z_2^{(n)}}\big)\Big) \preceq \big(\sqrt{z_1^{(n)}}, \sqrt{z_2^{(n)}}\big) \ll (\sqrt{c_1}, \sqrt{c_2}). \end{aligned}$$

This implies that $\{\varphi(z_n)\}$ is also a **c**-sequence in P. Hence φ is a weak comparison.

Thus, all the conditions of Lemma 6 hold, and hence the integral equation (18) has a unique solution $u \in C[a,b]$. □

The following example guarantees the existence of the function K, ψ, γ and η that satisfies all assumptions in Theorem 2.

Example 3. *Let $C[0,1]$ be the set of all continuous functions on $[0,1]$. Consider the nonlinear integral equation*

$$x(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t + \int_0^1 r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big)dr, \quad t \in [0,1]. \tag{19}$$

Put

$$\eta(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t, \quad \psi(t,r) = r.\sin t \quad \textit{for all } t, r \in [0,1],$$

and

$$K\big(t, x(r)\big) = r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big) \quad \text{for all } x \in C[0,1] \quad \text{and all } t, r \in [0,1].$$

Then

(1) $\eta \in C[0,1]$ *and* $K\big(t, x(r)\big)$ *is integrable with respect to* r *on* $[0,1]$*;*

(2) $\psi(t,r)$ *is continuous on* $[0,1] \times [0,1]$ *and* $\sup\limits_{t\in[0,1]} \int_0^1 \mid \psi(t,r) \mid dr < 1$*;*

(3) *put* $\gamma(u) = \ln(1 + \frac{1}{2}u)$ *for all* $u \in \mathbb{R}_+$*, we have* γ *is a comparison function;*

(4) *for all* $t, r \in [0,1]$ *and* $x, y \in C[0,1]$*, we have*

$$\big|K\big(t, x(r)\big) - K\big(t, y(r)\big)\big| \le \big|\psi(t,r)\big|\gamma\big(|x(r) - y(r)|\big).$$

Proof. (1). Since $\eta(t) = t - \Big(\frac{3}{4} + \ln\frac{2\sqrt{6}}{9}\Big).\sin t$ for all $t \in [0,1]$, we have $\eta \in C[0,1]$. Since $x \in C[0,1]$, we have $K\big(t, x(r)\big) = r.\sin t.\ln\big(1 + \frac{1}{2}|x(r)|\big)$ is integrable with respect to r on $[0,1]$.

(2). It is easy to see that $\psi(t,r)$ is continuous on $[0,1] \times [0,1]$ and $\sup\limits_{t\in[0,1]} \int_0^1 \mid \psi(t,r) \mid dr < 1$.

(3). For all $u_1, u_2 \in \mathbb{R}_+$ and $u_1 \le u_2$, we have $\gamma(u_1) = \ln(1 + \frac{1}{2}u_1) \le \ln(1 + \frac{1}{2}u_2) = \gamma(u_2)$.

For any $u \in \mathbb{R}_+$, we have

$$\gamma(u) = \ln(1 + \frac{1}{2}u) \leq \frac{1}{2}u$$

and

$$\gamma^2(u) = \gamma(\gamma(u)) = \ln\left(1 + \frac{1}{2}\ln(1 + \frac{1}{2}u)\right) \leq \frac{1}{2}\ln(1 + \frac{1}{2}u) \leq \frac{1}{2^2}u.$$

Continuing the above process we obtain that for all n,

$$\gamma^n(u) \leq \frac{1}{2^n}u.$$

From the above, we have γ is increasing and $\lim\limits_{n\to\infty} \gamma^n(u) = 0$.

(4). Now let $x, y \in C[0,1]$. Then, for each $r, t \in [0,1]$, we have

$$\begin{aligned}
\mid K(t, x(r)) - K(t, y(r)) \mid &= \Big| r.\sin t.\ln\left(1 + \frac{1}{2}|x(r)|\right) - r.\sin t.\ln\left(1 + \frac{1}{2}|y(r)|\right)\Big| \\
&= |r.\sin t|.\Big| \ln\left(1 + \frac{1}{2}|x(r)|\right) - \ln\left(1 + \frac{1}{2}|y(r)|\right)\Big| \\
&= |r.\sin t|.\Big| \ln\Big(\frac{1 + \frac{1}{2}|x(r)|}{1 + \frac{1}{2}|y(r)|}\Big)\Big| \\
&= |r.\sin t|.\Big| \ln\Big(1 + \frac{\frac{1}{2}|x(r)| - \frac{1}{2}|y(r)|}{1 + \frac{1}{2}|y(r)|}\Big)\Big| \\
&\leq |r.\sin t|.\Big| \ln\Big(1 + \frac{\frac{1}{2}|x(r) - y(r)|}{1 + \frac{1}{2}|y(r)|}\Big)\Big| \\
&\leq |r.\sin t|.\Big| \ln\Big(1 + \frac{1}{2}\big|x(r) - y(r)\big|\Big)\Big| \\
&= \big|\psi(t, r)\big|.\gamma\big(\big|x(r) - y(r)\big|\big).
\end{aligned}$$

From the above, K, ψ, γ and η satisfy all assumptions of Theorem 2. Hence the integral equation (19) has a unique solution $u \in C[0,1]$. □

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Кван Л.Т., Ван Ан Т. *Про розв'язки деякого класу нелінійних інтегральних рівняннь в конічних b-метричних просторах над банаховими алгебрами* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 163–178.

У даній роботі ми вивчаємо існування розв'язків деякого класу функціональних інтегральних рівнянь з використанням деяких результатів про фіксовану точку у конічних b-метричних просторах над банаховими алгебрами. Для отримання цих результатів ми ввели і довели деякі властивості узагальнених слабких φ-скорочень, в яких φ є нелінійними слабкими функціями порівняння. Отримані результати є узагальненнями результатів Van Dung N., Le Hang V. T., Huang H., Radenovic S. і Deng G. Також, наведено деякі відповідні приклади для ілюстрації отриманих результатів.

Ключові слова і фрази: конічний b-метричний простір над банаховою алгеброю, φ-скорочення, $\mathbf{c}$-послідовність, нерухома точка, інтегральне рівняння.

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CLASSIFICATION OF GENERALIZED TERNARY QUADRATIC QUASIGROUP FUNCTIONAL EQUATIONS OF THE LENGTH THREE

A functional equation is called: *generalized* if all functional variables are pairwise different; *ternary* if all its functional variables are ternary; *quadratic* if each individual variable has exactly two appearances; *quasigroup* if its solutions are studied only on invertible functions. A *length* of a functional equation is the number of all its functional variables. A complete classification up to parastrophically primary equivalence of generalized quadratic quasigroup functional equations of the length three is given. Solution sets of a full family of representatives of the equivalence are found.

Key words and phrases: ternary quasigroup, quadratic equation, length of a functional equation, parastrophically primary equivalence.

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INTRODUCTION

We study functional equations which can be considered on an arbitrary set (a carrier) and therefore they have neither individual nor functional constants. Moreover, we focus our attention only on the solutions which are sequences of invertible functions (i.e., quasigroup functions) and such equations are called quasigroup equations. We do not pay attention to dependencies among functional variables. That is why, we consider generalized equations: all functional variables are pairwise different. The word 'ternary' means that every functional variable takes its value in the set Δ_3 of all ternary invertible operations defined on a carrier.

Every ternary invertible operation has three inverses: left, middle and right divisions and each of them is also invertible, etc. These operations are called parastrophes. Generally speaking, an arbitrary ternary invertible operation has $4! = 24$ parastrophes including itself and all of them are connected by some defining identities. These identities are true not only for all individual variables but for all functional variables provided they take their value in Δ_3. In other words, they are hyperidentities over the set Δ_3, and they are called primary. Renaming functional and individual variables and applying primary hyperidentities, one can transform one functional equation into some other equation. This relation between functional equations is an equivalence and is called a parastrophically primary equivalence. If two functional equations are parastrophically primarily equivalent, then there is an algorithm which transforms the solution set of the first equation into the solution set of the second one.

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The problem under consideration is *"Describe parastrophically primary equivalence on the set of all quasigroup functional equations, select all representatives (i.e., a maximal set of non-equivalent functional equations) and solve all of them"*.

This problem is discussed in A. Krapež [3], S. Krstić [15], A. Krapež and D. Živković [4], A. Ehsani, A. Krapež and Y. Movsisyan [5], F. Sokhatsky [8, 10], F. Sokhatsky and H. Krainichuk [6, 9], R. Koval' [14], H. Krainichuk [13] etc. for binary quasigroups. On ternary quasigroups, the parastrophically primary classification was carried out in the article [11], where a two-element transversal equivalence of the generalized non-trivial functional equations of the length one and the seven-element transversal of the equivalence of generalized non-trivial functional equations of the length two were singled out.

In this article, only quadratic generalized functional equations of the length three on invertible functions (i.e. quasigroup operations) are studied, that is, those equations in which each individual variable has exactly two appearances. If a quasigroup equation has one appearance of an individual variable, then it is trivial, i.e. it has solutions only on singletons.

In section 'Quasigroup solutions', general solutions of each element from a family of pairwise parastrophically primarily non-equivalent generalized quadratic functional equations of the length three on ternary quasigroups have been found in Theorems 2–5. In the next section 'Proof of Theorem 1', a full proof of the classification theorem is given.

1 PRELIMINARIES

1.1 Quasigroup

All operations considered in this article are defined on an arbitrary fixed set Q called a *carrier*. A binary operation is a mapping $g\colon Q^2 \to Q$, the set of all operations defined on Q is denoted by $\mathcal{O}_2$. A binary operation g is called *invertible*, if it is invertible in both monoids $(\mathcal{O}_2; \underset{1}{\oplus}, e_1)$ and $(\mathcal{O}_2; \underset{2}{\oplus}, e_2)$, where $e_1(x_1, x_2) := x_1$, $e_2(x_1, x_2) := x_2$ and

$$(g \underset{1}{\oplus} g_1)(x_1, x_2) := g(g_1(x_1, x_2), x_2), \qquad (g \underset{2}{\oplus} g_1)(x_1, x_2) := g(x_1, g_1(x_1, x_2)).$$

The operation g is the *main* one and its inverses in $(\mathcal{O}_2; \underset{1}{\oplus}, e_1)$ and $(\mathcal{O}_2; \underset{2}{\oplus}, e_2)$ are denoted by ${}^{\ell}g$ and ${}^{r}g$ and are called g's *left* and *right divisions* respectively. If an operation g is invertible, then the algebra $(Q; g, {}^{\ell}g, {}^{r}g)$ is called a *binary quasigroup* [10]. Usually, infix notations are used for binary operations. Therefore, an algebra $(Q; \circ, \overset{\ell}{\circ}, \overset{r}{\circ})$ is called a quasigroup if the identities

$$(x \overset{\ell}{\circ} y) \circ y = x, \quad (x \circ y) \overset{\ell}{\circ} y = x, \quad x \circ (x \overset{r}{\circ} y) = y, \quad x \overset{r}{\circ} (x \circ y) = y$$

hold.

Similarly, a *ternary operation* is a mapping $f\colon Q^3 \to Q$, the set of all ternary operations defined on Q is denoted by $\mathcal{O}_3$. A ternary operation f is called *invertible* if it is invertible in each of the monoids $(\mathcal{O}_3; \underset{i}{\oplus}, e_i)$, $i = 1, 2, 3$, where

$$(f \underset{1}{\oplus} f_1)(x_1, x_2, x_3) := f(f_1(x_1, x_2, x_3), x_2, x_3),$$
$$(f \underset{2}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, f_1(x_1, x_2, x_3), x_3),$$
$$(f \underset{3}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, x_2, f_1(x_1, x_2, x_3)),$$
$$e_i(x_1, x_2, x_3) := x_i, \ i = 1, 2, 3.$$

The operation f is the *main* one and its inverses in $(\mathcal{O}_3; \underset{1}{\oplus}, e_1)$, $(\mathcal{O}_3; \underset{2}{\oplus}, e_2)$, $(\mathcal{O}_3; \underset{3}{\oplus}, e_3)$ are denoted by ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ and they are called f's *left*, *middle* and *right divisions* respectively. In other words, the operation f is invertible if the identities

$$f({}^{(14)}f(x,y,z),y,z) = x, \qquad (1) \qquad\qquad {}^{(14)}f(f(x,y,z),y,z) = x, \qquad (4)$$
$$f(x,{}^{(24)}f(x,y,z),z) = y, \qquad (2) \qquad\qquad {}^{(24)}f(x,f(x,y,z),z) = y, \qquad (5)$$
$$f(x,y,{}^{(34)}f(x,y,z)) = z, \qquad (3) \qquad\qquad {}^{(34)}f(x,y,f(x,y,z)) = z \qquad (6)$$

hold. If an operation f is invertible, then the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ (in brief, $(Q; f)$) is called a *ternary quasigroup* [10]. It is easy to verify that all divisions of an invertible operation are also invertible and so are their divisions.

A *σ-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \quad :\Leftrightarrow \quad f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,$$

where S_4 denotes the group of all bijections of the set $\{1,2,3,4\}$. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide. If all parastrophes coincide, the quasigroup is called *totally symmetric*. Since parastrophes of a quasigroup satisfy the equalities

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \quad \text{and} \quad {}^{\iota}f = f, \tag{7}$$

then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f. Consequently, a totally symmetric quasigroup is a quasigroup whose parastrophic symmetry group is S_4. If the parastrophic symmetry group of a ternary quasigroup is trivial, then the quasigroup has 24 different parastrophes and it is called *asymmetric*.

An element e of $(Q; f)$ is called *neutral* if for all x from Q the equalities

$$f(x,e,e) = x, \qquad f(e,x,e) = x, \qquad f(e,e,x) = x$$

hold. In contrast to the binary case, a neutral element is not necessarily unique in a ternary quasigroup. A quasigroup is called a *loop* if it has a neutral element. For example, let $(Q; +)$ be a group of the exponent two and an operation f be defined by

$$f(x,y,z) := x + y + z.$$

It is easy to see that every element of the quasigroup is neutral in the ternary quasigroup $(Q; f)$. Such a quasigroup will be called universally neutral. Namely, a ternary quasigroup $(Q; f)$ will be called a *left*, *middle*, *right universally neutral* if the respective identity holds:

$$f(x,y,y) = x, \qquad f(y,x,y) = x, \qquad f(y,y,x) = x.$$

It will be called *universally neutral* if all three identities take place. Note, that the given example of the ternary quasigroup is not only universally neutral, but it is totally symmetric. A quasigroup which is both universally neutral and totally symmetric is called a *Steiner quasigroup* [2, 12]. Thus, every ternary Steiner quasigroup is a loop. Moreover, each of its elements is neutral.

An invertible operation f is called *repetition-free decomposable* if there exist two binary invertible operations g, h and bijection $\sigma \in S_3$ such that

$$f(x_1, x_2, x_3) = g(h(x_{1\sigma}, x_{2\sigma}), x_{3\sigma}).$$

Theorem 1 from [16] implies the following result.

Corollary 1. *If a ternary Steiner quasigroup $(Q; f)$ is repetition-free decomposable, then there is a group $(Q; +)$ of the exponent two such that*

$$f(x, y, z) = x + y + z.$$

1.2 Functional equations

Throughout the article, we will use the notion 'functional equation' in the following sense. Let T_1 and T_2 are second order terms which have only individual and functional variables. A formula $T_1 = T_2$ is called a *functional equation*, if it is universally quantified on all individual variables and has at least one free functional variable. Moreover, we consider only generalized ternary quadratic functional equations of the length three on quasigroups, where the notion 'ternary quasigroup equation' means that all functional variables take their values only in the set of ternary invertible functions; the word 'generalized' means that the variables are pairwise different; the word 'quadratic' means that every individual variable has exactly two appearances or none; the notion 'length of a functional equation' is the number of functional variables including their repetitions (see [1, 10]).

A subterm of an equation is a subterm of its left or right sides. A subterm of a term T is called *proper* if it coincides neither with T nor an individual variable. Let $F(t_1, t_2, t_3)$ be a term, then the function variable F is called *main*.

Let $T_1 = T_2$ be a ternary functional equation of the length three, (F, G_i, G_j) be the lexicographical sequence of its functional variables, i.e., $i < j$. A sequence (f, g, h) of invertible ternary functions defined on a set Q is called a *solution* of $T_1 = T_2$ if substituting f for F, g for G_1 and h for G_2, we obtain a true proposition $t_1 = t_2$, i.e., $t_1 = t_2$ is an identity. A quasigroup functional equation is called *trivial* if it has a solution only on a singleton.

Consequently, in an arbitrary non-trivial quasigroup functional equation, every individual variable has at least two appearances. In this article, we consider the case when every individual variable has exactly two appearances, these equations are called *quadratic*.

Let Δ_3 be the set of all invertible ternary functions defined on a carrier Q. The relationships (1)–(6) and (7) are true for all functions from Δ_3. In other words, the following *hyperidentities* are true over the set Δ_3:

$$\left.\begin{array}{ll} {}^{\sigma}({}^{\tau}F) = {}^{\sigma\tau}F, \quad {}^{\iota}F = F, & {}^{(14)}F(F(x,y,z),y,z) = x; \\ {}^{(24)}F(x, F(x,y,z), z) = y; & {}^{(34)}F(x,y,F(x,y,z)) = z, \\ F(x_1,x_2,x_3) = {}^{\sigma}F(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}), \quad \sigma \in S_3. & \end{array}\right\} \qquad (8)$$

The hyperidentities are called *primary*.

Two quasigroup functional equations are called: *equivalent over a set* Q if they have the same solution set over the carrier; *equivalent* if they are equivalent over each set.

Definition 1 ([8])**.** *Two functional equations are called parastrophically primarily equivalent if one can be obtained from the other in a finite number of the following steps: 1) replacing of the equation sides; 2) renaming of the functional variables; 3) renaming of the individual variables; 4) applying the hyperidentities* (8)*.*

A *lexicographical renaming of individual variables* is renaming all first appearances of these variables according to their lexicographical order.

Lemma 1. *Let $v = \omega$ and $v' = \omega'$ be generalized ternary functional equations of the length three. If they are parastrophically primarily equivalent, then there exists a bijection τ in S_3 and bijections $\sigma_1, \sigma_2, \sigma_3$ in S_4 such that for an arbitrary solution (f_1, f_2, f_3) of $v = \omega$ the sequence*

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is a solution of the equation $v' = \omega'$.

In this case, $(\tau, \sigma_1, \sigma_2, \sigma_3)$ is called a *defining bijection system* of the equations $v = \omega$ and $v' = \omega'$. This lemma implies a sufficient condition for parastrophically primary non-equivalence of ternary generalized functional equations of the length three. Namely, the following statement is valid.

Corollary 2. *If for every bijection τ in S_3 and bijections $\sigma_1, \sigma_2, \sigma_3$ in S_4 there exists a solution (f_1, f_2, f_3) of $v = \omega$ such that $({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$ is not a solution of $v' = \omega'$, then the functional equations $v = \omega$ and $v' = \omega'$ are not parastrophically primarily equivalent.*

A function f is called a *solution* of a functional equation if the sequence $(f, f, \ldots, f)$ is solution of the equation.

Corollary 3. *If a totally symmetric function is a solution of a functional equation but it is not a solution of another functional equation, then the equations are not parastrophically primarily equivalent.*

2 QUASIGROUP SOLUTIONS

Theorem 1 gives a full classification of generalized quadratic ternary quasigroup functional equations of the length three up to parastrophically primary equivalence. Also, all quasigroup solutions of all representatives (9)–(12) of the classification are proved in Theorem 2–5.

Theorem 1. *Every generalized quadratic ternary quasigroup functional equation of the length three is parastrophically primarily equivalent to exactly one of the following equations:*

$$F_1(z, x, F_2(x, y, y)) = F_3(z, u, u), \tag{9}$$

$$F_1(F_2(x, y, y), z, z) = F_3(x, u, u), \tag{10}$$

$$F_1(F_2(x, y, z), u, u) = F_3(x, y, z), \tag{11}$$

$$F_1(F_2(x, y, z), x, u) = F_3(y, z, u). \tag{12}$$

Lemma 2. *Let α, f be the unary and ternary invertible operations respectively. Then the equality*

$$f(x, y, y) = \alpha x \tag{13}$$

is equivalent to the existence of a left-universally neutral invertible operation g such that

$$f(x, y, z) = g(\alpha x, y, z). \tag{14}$$

Proof. Define operation g, by

$$g(x,y,z) := f(\alpha^{-1}x,y,z). \tag{15}$$

Since f is invertible and g is an isotope of f, the operation g is invertible. Taking into account (13), we have $x = f(\alpha^{-1}x,y,y) = g(x,y,y)$. Hence, the operation g is left-universally neutral. Applying (15), we obtain (14).

Conversely, let g be a left-universally neutral invertible operation and let the relationship (14) holds. Then $f(x,y,y) = g(\alpha x,y,y) = \alpha x$. □

Theorem 2. *A triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (9) if and only if there exist left-universally neutral invertible operations h_1, h_2, h_3 and bijections α, β such that*

$$f_1(x,y,z) = h_1(\alpha x,y,\beta^{-1}z), \tag{16}$$

$$f_2(x,y,z) = h_2(\beta x,y,z), \tag{17}$$

$$f_3(x,y,z) = h_3(\alpha x,y,z). \tag{18}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations defined on Q be a solution of the equation (9), i.e., for all x, y, z, u the identity

$$f_1(z,x,f_2(x,y,y)) = f_3(z,u,u) \tag{19}$$

holds. In particular, if $u = a \in Q$, we have

$$f_1(z,x,f_2(x,y,y)) = \alpha z, \tag{20}$$

where $\alpha z := f_3(z,a,a)$ is a bijection of Q because α is a left translation of the invertible operation f_3.

Also, from (20) and (19), we get the identity $f_3(z,u,u) = \alpha z$. According to Lemma 2, there exists a left-universally neutral invertible operation h_3 such that (18) holds.

Applying the definition of a parastrophe to the equality (20), we have

$$f_2(x,y,y) = {}^{(34)}f_1(z,x,\alpha z).$$

If $z = a \in Q$ and $\beta x :={}^{(34)} f_1(a,x,\alpha a)$, the equality is written as $f_2(x,y,y) = \beta x$. Note that β is bijective on Q since it is a translation of an invertible operation ${}^{(34)}f_1$. By Lemma 2, the latter relationship implies the existence of a left-universally neutral invertible operation h_2 such that (17) is true.

Replace $f_2(x,y,y)$ with βx in (20): $f_1(z,x,\beta x) = \alpha z$. Let $h_1(x,y,z) := f_1(\alpha^{-1}x,y,\beta z)$, then (16) holds and $h_1(x,y,y) = f_1(\alpha^{-1}z,x,\beta x) = \alpha\alpha^{-1}x = x$. Thus the operation h_1 is a left-universally neutral invertible.

Conversely, let the operations h_1, h_2, h_3 be left-universally neutral invertible operations and operations f_1, f_2, f_3 be defined by (16), (17), (18) for some bijections α, β of a set Q. Then

$$\begin{aligned} f_1(z,x,f_2(x,y,y)) &= h_1(\alpha z,x,\beta^{-1}h_2(\beta x,y,y) \\ &= h_1(\alpha z,x,\beta^{-1}\beta x) = h_1(\alpha z,x,x) = \alpha z \\ &= h_3(\alpha z,u,u) = f_3(z,u,u). \end{aligned}$$

Therefore, the triplet (f_1, f_2, f_3) is a solution of the equation (9). □

Theorem 3. *A triplet of ternary invertible operations (f_1, f_2, f_3) is a solution of the equation (10) if and only if there exist left-universally neutral invertible operations g_1, g_2, g_3 and bijections γ, δ such that*

$$f_1(x,y,z) = g_1(\gamma x, y, z), \tag{21}$$
$$f_2(x,y,z) = g_2(\delta x, y, z), \tag{22}$$
$$f_3(x,y,z) = g_3(\gamma\delta x, y, z). \tag{23}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (10), i.e., the identity

$$f_1(f_2(x,y,y),z,z) = f_3(x,u,u) \tag{24}$$

holds. In particular, if $y = u = a \in Q$, we have $f_1(f_2(x,y,y),a,a) = f_3(x,a,a)$. Then $\alpha f_2(x,y,y) = \beta x$, where $\alpha x := f_1(x,a,a)$ and $\beta x := f_3(x,a,a)$ are bijective since α and β are translations of the invertible operations f_1 and f_3 respectively. Therefrom

$$f_2(x,y,y) = \alpha^{-1}\beta x.$$

Defining $\delta := \alpha^{-1}\beta$, we have $f_2(x,y,y) = \delta x$. According to Lemma 2, there exists a left-universally neutral invertible operation g_2 such that the equality (22) holds.

Let us substitute δx in (24) for $f_2(x,y,y)$:

$$f_1(\delta x, z, z) = f_3(x,u,u).$$

Replace x with $\delta^{-1}x$ in the equality: $f_1(x,z,z) = f_3(\delta^{-1}x,u,u)$ for all x, z, u. In particular, when $u = a \in Q$, we have $f_1(x,z,z) = \gamma x$, where $\gamma x := f_3(\delta^{-1}x,a,a)$ is a bijection of the carrier Q, because γ is the left translation of the invertible operation f_3. Therefore, the relationship (21) holds for some left-universally neutral operation g_1. Applying (21) and (22) to (24), we have

$$\gamma\delta x = f_3(x,u,u).$$

According to Lemma 2, there exists a left-universally neutral invertible operation g_3 such that the equality (23) holds.

Vise versa, let the relationships (21), (22), (23) be true for some left-universally neutral operations g_1, g_2, g_3 and bijections γ, δ, then

$$\begin{aligned} f_1(f_2(x,y,y),z,z) &= g_1(\gamma g_2(\delta x,y,y),z,z) \\ &= g_1(\gamma\delta x,z,z) = \gamma\delta x = g_3(\gamma\delta x,u,u) = f_3(x,u,u). \end{aligned}$$

Thus, the triplet (f_1, f_2, f_3) is a solution of the equation (10). $\square$

Theorem 4. *A triplet (f_1, f_2, f_3) of ternary operations defined on a set Q is a quasigroup solution of the functional equation (11) if and only if the operation f_2 is invertible and there exists a bijection μ and a left-universally neutral operation g such that*

$$f_3(x,y,z) = \mu f_2(x,y,z), \qquad f_1(x,y,z) = g(\mu x, y, z). \tag{25}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations be a solution of the equation (11), i.e., for all x, y, z, u the identity

$$f_1(f_2(x,y,z),u,u) = f_3(x,y,z) \tag{26}$$

holds. In particular, when $u = a \in Q$ and $\mu x := f_1(x,a,a)$, we have the first identity from (25). Substituting μf_2 in (26) for f_3, we have

$$f_1(f_2(x,y,z),u,u) = \mu f_2(x,y,z).$$

Replacing $f_2(x,y,z)$ with x, we obtain $f_1(x,u,u) = \mu x$. According to Lemma 2, there exists a bijection μ and a left-universally neutral operation g such that the second relationship from (25) holds.

Conversely, let f_2 be invertible ternary operation and there exists a bijection μ and a left-universally neutral operation g such that the relationships (25) hold. Then

$$f_1(f_2(x,y,z),u,u) = g(\mu f_2(x,y,z),u,u) = g(f_3(x,y,z),u,u) = f_3(x,y,z).$$

Therefore, the triplet (f_1, f_2, f_3) is a quasigroup solution of the equation (11). □

Theorem 5. *A triplet (f_1, f_2, f_3) of ternary invertible operations defined on set Q is a solution of the functional equation (12) if and only if there exist binary invertible operations $\circ$, $*$, $\diamond$ on Q such that*

$$\begin{aligned} f_1(y,x,u) &= (x \diamond y) * u, \\ f_2(x,y,z) &= x \overset{r}{\diamond} (y \circ z), \\ f_3(y,z,u) &= (y \circ z) * u. \end{aligned} \tag{27}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (12), i.e., for all $x, y, z, u \in Q$ the identity:

$$f_1(f_2(x,y,z),x,u) = f_3(y,z,u) \tag{28}$$

holds. In particular, when $x = a \in Q$ and

$$y \circ z := f_2(a,y,z), \qquad t * u := f_1(t,a,u),$$

we have $(y \circ z) * u = f_3(y,z,u)$. Hence, we obtain the third relationship from (27). Note that $(\circ)$ and $(*)$ are invertible operations since they are retracts of ternary invertible operations f_2 and f_1. Applying the latter equality to (28), we get

$$f_1(f_2(x,y,z),x,u) = (y \circ z) * u. \tag{29}$$

Replace y with ${}^{(24)}f_2(x,y,z)$:

$$f_1(f_2(x, {}^{(24)}f_2(x,y,z),z),x,u) = ({}^{(24)}f_2(x,y,z) \circ z) * u.$$

Apply (2):

$$f_1(y,x,u) = ({}^{(24)}f_2(x,y,z) \circ z) * u.$$

Replacing z with a and denoting $x \diamond y := {}^{(24)}f_2(x,y,a) \circ a$, we obtain the first relationship from (27). Then (29) can be written as

$$(x \diamond f_2(x,y,z)) * u = (y \circ z) * u.$$

Since the operation $(*)$ is invertible, then

$$x \diamond f_2(x,y,z) = y \circ z.$$

Since the operation $(\diamond)$ is invertible, we can use the definition of the right division for binary operations. As a result, we obtain the second equality from (27).

Conversely, let $\circ$, $*$, $\diamond$ be invertible binary operations on Q. Then the ternary operations defined by the relationship (27) are invertible since they are repetition-free superpositions of binary invertible operations.

$$\begin{aligned} f_1(f_2(x,y,z),x,u) &= (x \diamond f_2(x,y,z)) * u \\ &= (x \diamond (x \overset{r}{\diamond} (y \circ z))) * u = (y \circ z) * u = f_3(y,z,u). \end{aligned}$$

Hence, for all x,y,z,u (28) holds. Therefore, the triplet (f_1, f_2, f_3) is a solution of (12). $\square$

3 PROOF OF THEOREM 1

Proof. Let $v = \omega$ be a generalized quadratic ternary quasigroup functional equation of the length three. Changing its sides if necessary, we obtain an equation which has one of the following forms:

$$i)\ F_i(\ldots,F_j(\ldots),\ldots) = F_k(\ldots), \qquad ii)\ F_i(\ldots,F_j(\ldots,F_k(\ldots),\ldots),\ldots) = t,$$
$$iii)\ F_i(\ldots,F_j(\ldots),\ldots,F_k(\ldots),\ldots) = t,$$

where t is an individual variable and $(\ldots)$ denotes some sequence of variables or an empty sequence.

When the equation has the form $ii)$ we substitute both sides of the equation for t' in the term ${}^{\sigma}F_i(\ldots,t',\ldots)$. As a result, we obtain

$${}^{\sigma}F_i(\ldots,F_i(\ldots,F_j(\ldots,F_k(\ldots),\ldots),\ldots),\ldots) = {}^{\sigma}F_i(\ldots,t,\ldots),$$

where ${}^{\sigma}F_i$ is a suitable division of F_i, i.e., σ is (14), (24) or (34). Applying the respective primary identity (1)–(6), we get

$$F_j(\ldots,F_k(\ldots)\ldots) = {}^{\sigma}F_i(\ldots,t,\ldots).$$

Therefore, every functional equation of the form $ii)$ is parastrophically primarily equivalent to an equation of the form $i)$.

If the functional equation has the form $iii)$, we substitute both sides of the equation for v in the term ${}^{\tau}F_i(\ldots,F_j(\ldots),\ldots,v,\ldots)$:

$$\begin{aligned} {}^{\tau}F_i(\ldots,F_j(\ldots),\ldots,F_i(\ldots,F_j(\ldots),\ldots,F_k(\ldots),\ldots),\ldots) \\ = F_i'(\ldots,F_j(\ldots),\ldots,t,\ldots), \end{aligned}$$

where ${}^{\tau}F_i$ is a suitable division of F_i. Applying one of the primary identities (1)–(6), we have

$$F_i'(\ldots, F_j(\ldots), \ldots, t, \ldots) = F_k(\ldots).$$

Thus, every functional equation is parastrophically primarily equivalent to a functional equation of the form $i)$.

Let a functional equation have the form $i)$. Applying a suitable transformation to a parastrophe, we obtain an equation of the form

$$F_i(\ldots, F_j(\ldots), \ldots) = F_k(\ldots).$$

Renaming its functional and individual variables in lexicographical order, we obtain

$$F_1(F_2(x, t_2, t_3), t_4, t_5) = F_3(t_6, t_7, t_8), \tag{30}$$

where $t_i \in \{x, y, z, u\}$. Denote a lexicographical order of individual variables by $\preccurlyeq$. If $t_2 \succcurlyeq t_3$, we replace the subterm $F_2(x, t_2, t_3)$ with the subterm ${}^{(23)}F_2(x, t_3, t_2)$, mutually rename the individual variables t_2 and t_3 and rename ${}^{(23)}F_2$ by F_2. As a result, we obtain the functional equation of the form (30) in which $t_2 \preccurlyeq t_3$.

Analogically, we suppose that $t_4 \preccurlyeq t_5$ and $t_6 \preccurlyeq t_7 \preccurlyeq t_8$. At last, we can put in order the second appearances of x, t_2, t_3. Namely, we rename them in a lexicographical order, then we transform them to the corresponding parastrophe of F_2. The same transformation holds for the pair t_4, t_5.

Thus, we have proved that every quadratic functional equation is parastrophically primarily equivalent to the equation (30) in which: 1) the first appearances of individual variables have a lexicographical order; 2) $t_2 \preccurlyeq t_3$, $t_4 \preccurlyeq t_5$ and $t_6 \preccurlyeq t_7 \preccurlyeq t_8$; 3) the second appearances of x, t_2, t_3 as well as the second appearances of t_4, t_5 are in the lexicographical order.

Hence, the proper subterm is

$$1)\ F_2(x, x, y) \quad \text{or} \quad 2)\ F_2(x, y, z).$$

The case $F_2(x, y, y)$ is impossible because the second appearances of x and y should be in a lexicographical order.

Let the proper subterm be $F_2(x, x, y)$. If $y \in \{t_4, t_5\}$, then t_4 is y and t_5 is z thus, we have the equation

$$F_1(F_2(x, x, y), y, z) = F_3(z, u, u).$$

Transform F_1 and F_2 to (13)-parastrophes of F_1 and F_2 in the equation. We obtain

$${}^{(13)}F_1(y, z, {}^{(13)}F_2(y, x, x)) = F_3(z, u, u).$$

Mutually renaming x and y and renaming the functional variables in a lexicographical order, we obtain the functional equation (9).

If $y \notin \{t_4, t_5\}$, then there are two possibilities for the pair (t_4, t_5): (z, z) and (z, u). Therefore, we have two equations:

$$F_1(F_2(x, x, y), z, z) = F_3(y, u, u), \tag{31}$$

$$F_1(F_2(x, x, y), z, u) = F_3(y, z, u). \tag{32}$$

The equation (31) is parastrophically primarily equivalent to (10) by means of transforming to (13)-parastrophe of F_2, by mutually renaming x and y and replacing ${}^{(13)}F_2$ with F_2.

Apply the hyperidentity (4) to (32):

$$^{(14)}F_1(F_3(y,z,u),z,u) = F_2(x,x,y),$$

then apply the hyperidentity (3):

$$F_2(x,x,{}^{(34)}F_3(y,z,u)) = {}^{(14)}F_1(y,z,u).$$

Transform F_2 to (13)-parastrophe of F_2 and rename the functional variables in a lexicographical order:

$$F_1(F_2(y,z,u),x,x) = F_3(y,z,u).$$

Renaming the individual variables according to the cycle $(yxuz)$, we obtain the functional equation (11).

Let the proper subterm be $F_2(x,y,z)$. Since the second appearances are ordered, then t_4 is x and t_5 is y or u. Consequently, we have two equations: equation (12) and

$$F_1(F_2(x,y,z),x,y) = F_3(z,u,u).$$

Apply (1) to the last functional equation:

$$F_3({}^{(14)}F_2(x,y,z),u,u) = F_1(z,x,y).$$

To obtain equation (11), transform F_1 to (312)-parastrophe of F_1 and rename the functional variables.

It remains to prove that the equations (9)–(12) are pairwise parastrophically primarily non-equivalent. According to Corollary 2, we can prove that for every pair of these equations and for every bijection σ_1, σ_2, σ_3, τ of the set $\{1,2,3\}$ there is a solution (f_1, f_2, f_3) of one equation such that $({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$ is not a solution of the other one. Note that all parastrophes of a totally symmetric quasigroup and, in particular of a Steiner quasigroup, coincide.

It is easy to verify that an arbitrary Steiner quasigroup is a solution of each of the functional equations (9), (10), (11). Suppose, a Steiner quasigroup $(Q;f)$ is a solution of the equation (12). Theorem 5 implies that f is a repetition-free superposition of two binary quasigroups. According to the definition, every Steiner quasigroup is a loop. Therefore, by Corollary 1 there is a group $(Q;+)$ of exponent two such that $f(x,y,z) = x+y+z$. There is no group of exponent two of the order 10 but Steiner quadruple systems exist (see [7]) thus, there exists a Steiner quasigroup of the order 10, but it can not be a solution of (12). Hence, according to Corollary 1, the functional equation (12) is not parastrophically primarily equivalent to any of the equations (9), (10), (11).

Let (f_1, f_2, f_3) be an arbitrary triplet of Steiner quasigroup operations defined on the same carrier Q. These operations can be isomorphic but all of them are pairwise different. It is easy to see that (f_1, f_2, f_3) is the solution of both functional equations: (9) and (10). Suppose $(f_{1\tau}, f_{2\tau}, f_{3\tau})$ is a solution of the functional equation (11) for some $\tau \in S_3$, i.e., the identity

$$f_{1\tau}(f_{2\tau}(x,y,z),u,u) = f_{3\tau}(x,y,z)$$

holds. Since $f_{1\tau}$ is a Steiner quasigroup operation, then $f_{2\tau} = f_{3\tau}$. There is a contradiction to the assumption. Thus, the triplet $(f_{1\tau}, f_{2\tau}, f_{3\tau})$ is not a solution of (11) for all $\tau \in S_3$. Therefore, the functional equation (11) is parastrophically primarily equivalent to neither (9) nor (10).

Hence, it remains to prove the parastrophically primary non-equivalence of the equations (9) and (10).

To avoid repetition, we will prove the following assertion.

Assertion. *Let $(Q;\cdot,e)$ be an arbitrary non-commutative group, ρ is its non-identical automorphism and*

$$f(x,y,z) := \rho x \cdot y \cdot z^{-1}. \tag{33}$$

If for a bijection $\sigma \in S_4$ there exists a bijection ν such that for all x, y, z

$${}^{\sigma}\!f(x,y,z) = \nu x, \tag{34}$$

then $\nu = \rho$ or $\nu = \rho^{-1}$.

To prove Assertion, consider the following notations:

$$t_{1\sigma} := x, \quad t_{2\sigma} := y, \quad t_{3\sigma} := y, \quad t_{4\sigma} := \nu x.$$

Then (34) can be written as ${}^{\sigma}\!f(t_{1\sigma}, t_{2\sigma}, t_{3\sigma}) = t_{4\sigma}$. According to the definition of σ-parastrophe, the equality is equivalent to $f(t_1, t_2, t_3) = t_4$. Using (33), we obtain $\rho t_1 \cdot t_2 \cdot t_3^{-1} = t_4$, i.e.

$$\rho t_1 \cdot t_2 = t_4 \cdot t_3. \tag{35}$$

We will analyze the relationship taking into account that two of the terms t_1, t_2, t_3, t_4 coincide with y.

If $t_1 = y$, then (35) with $y = e$ implies one of the following equalities: $\nu x \cdot x = e$ or $x = \nu x$. Consequently, $\nu e = e$. That is why, (35) with $x = e$ implies $\rho y \cdot y = e$ or $\rho y = y$. Since $(\cdot)$ is not commutative and ρ is a non-identical automorphism of $(\cdot)$, then neither $\rho y = y^{-1}$ nor $\rho y = y$ is true.

If $t_1 = x$, $t_2 = \nu x$, then (35) with $x = e$ implies $\nu e = y^2$. Therefrom when $y = e$ we have $\nu e = e$, therefore $y^2 = e$. But the group of exponent two is commutative. As a result we have a contradiction to the assumption.

If $t_1 = x$ and $t_2 = y$, then (35) with $y = e$ implies $\rho x = \nu x$ that is $\nu = \rho$.

Finally, let $t_1 = \nu x$, then (35) with $y = e$ implies one of the equalities $\rho\nu x \cdot x = e$ or $\rho\nu x = x$. The first equality follows from (35) when $t_2 = x$. Therefore, $y^2 = e$ and consequently, the group is commutative. As a result we have a contradiction to the assumption. The second equality implies $\nu = \rho^{-1}$.

Thus, Assertion has been proved.

We provide a proof of parastrophically primary non-equivalence of (9) and (10) by contradiction. Suppose (9) and (10) are parastrophically primarily equivalent. Denote the corresponding defining bijection sequence by $(\tau, \sigma_1, \sigma_2, \sigma_3)$.

Let $(Q;\cdot,e)$ be an arbitrary non-commutative group and γ, δ, $\gamma\delta$ be different non-identical automorphisms of $(Q;\cdot,e)$. Then, according to Theorem 3, the triplet (f_1, f_2, f_3) of operations defined by

$$f_1(x,y,z) := \gamma x \cdot y \cdot z^{-1}, \quad f_2(x,y,z) := \delta x \cdot y \cdot z^{-1}, \tag{36}$$
$$f_3(x,y,z) := \gamma\delta x \cdot y \cdot z^{-1}$$

is a solution of the equation (10). Lemma 1 implies that the triplet

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is a solution of the equation (9). By Theorem 2 there exist left-universally neutral operations h_1, h_2, h_3 and bijections α, β such that

$$\begin{aligned} {}^{\sigma_1}f_{1\tau}(x,y,z) &= h_1(\alpha x, y, \beta^{-1}z), \\ {}^{\sigma_2}f_{2\tau}(x,y,z) &= h_2(\beta x, y, z), \\ {}^{\sigma_3}f_{3\tau}(x,y,z) &= h_3(\alpha x, y, z). \end{aligned} \tag{37}$$

If $y = z$, the second and the third equations are

$${}^{\sigma_2}f_{2\tau}(x,y,y) = \beta x, \qquad {}^{\sigma_3}f_{3\tau}(x,y,y) = \alpha x.$$

Applying Assertion to these equalities, we have $\alpha, \beta \in \{\gamma, \gamma^{-1}, \delta, \delta^{-1}, \gamma\delta, \delta^{-1}\gamma^{-1}\}$. Replace z with βz in the first equality of (37): ${}^{\sigma_1}f_{1\tau}(x,y,\beta z) = h_1(\alpha x, y, z)$. If $y = z$, then

$${}^{\sigma_1}f_{1\tau}(x,y,\beta y) = \alpha x. \tag{38}$$

Introduce the notations: $t_{1\sigma_1} := x$, $t_{2\sigma_1} := y$, $t_{3\sigma_1} := \beta y$, $t_{4\sigma_1} := \alpha x$. Thus, (38) can be written as ${}^{\sigma_1}f_{1\tau}(t_{1\sigma_1}, t_{2\sigma_1}, t_{3\sigma_1}) = t_{4\sigma_1}$. Using the definition of a parastrophe, we have $f_{1\tau}(t_1, t_2, t_3) = t_4$. But $f_{1\tau}$ is one of the operations f_1, f_2, f_3, that is why we can apply the relationship (36): $\theta t_1 \cdot t_2 \cdot t_3^{-1} = t_4$, i.e.,

$$\theta t_1 \cdot t_2 = t_4 \cdot t_3,$$

where $\theta \in \{\gamma, \delta, \gamma\delta\}$.

If x has an appearance in θt_1, then we put $x = 0$. As a result, we obtain one of the equalities $y = \beta y$ or $0 = y \cdot \beta y$. The first equality is impossible, since the automorphisms γ, δ, $\gamma\delta$ are not identical. The second identity is impossible because the group is not commutative. If x has no appearance in θt_1, then we put $y = 0$ and obtain the same contradictions.

Thus, our assumption is not true, therefore, the equations (9) and (10) are not parastrophically primarily equivalent. Theorem 1 has been proved. □

4 CONCLUSION

There exist exactly four classes of generalized quadratic functional equations of the length three on invertible functions (i.e. quasigroup operations) concerning the parastrophically primary equivalence, (9)–(12) are their representatives whose solution sets are found in Theorems 2–5.

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Сохацький Ф.М., Тарасевич А.В. *Класифікація узагальнених тернарних квадратичних функційних рівнянь довжини три* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 179–192.

Функційне рівняння називається: *узагальненим*, якщо всі функційні змінні попарно різні; *тернарним*, якщо всі його функційні змінні є тернарними; *квадратичним*, якщо кожна предметна змінна має точно дві появи; *квазігруповим*, якщо його розв'язки вивчають лише на оборотних функціях. *Довжиною* функційного рівняння є кількість всіх його функційних змінних. Здійснено повну класифікацію з точністю до парастрофно-первинної рівносильності узагальнених квадратичних квазігрупових функційних рівнянь довжини три. Знайдено множини розв'язків повного набору представників.

Ключові слова і фрази: тернарна квазігрупа, квадратичне рівняння, довжина функційного рівняння, парастрофно-первинна рівносильність.

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ON INTEGRAL REPRESENTATION OF THE SOLUTIONS OF A MODEL $\vec{2b}$-PARABOLIC BOUNDARY VALUE PROBLEM

A general boundary value problem for Eidelman type $\overrightarrow{2b}$-parabolic system of equation without minor terms in the equations and boundary conditions, and with constant coefficients in the group of major terms is considered in the region $\{(t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | t \in (0, T], x_j \in \mathbb{R}, j \in \{1, \ldots, n-1\}, x_n > 0\}, T > 0, n \geq 2$. It is assumed that the boundary conditions are connected with the system of equations by the complementing condition, which is analogous to the Lopatynsky complementing condition. Integral representations of the solutions for such a problem are derived. The kernels of the integrals from this representation form the Green's matrix of the problem. It is revealed that, in general, not all the elements of the Green's matrix are ordinary functions. Some of them contain terms that are linear combinations of Dirac delta functions and their derivatives. This occurs in cases when the boundary conditions include derivatives with respect to the variables t and x_n of orders that are equal or greater than the highest orders of derivatives with respect to these variables in the equations of the system. The obtained results are important, in particular, for the establishing of the correct solvability and integral representation of solutions for more general $\overrightarrow{2b}$-parabolic boundary value problems.

Key words and phrases: Eidelman type $\overrightarrow{2b}$-parabolic system of equations, boundary value problem, integral representation of solutions, Green's matrix.

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INTRODUCTION

Nowadays, the general theory of boundary value problems for systems of equations that are parabolic in the sense of I. G. Petrovsky and for more general systems parabolic in the sense of V. A. Solonnikov is well known (see, for example, [2, 3, 5]). The parabolic boundary problems are determined by the parabolicity condition of the system of equations and the complementing condition for boundary differential expressions. We note that the conditions for the parabolicity of a problem are specified only by the groups of the major in the parabolic sense terms of the system of equations and the boundary conditions.

The theorems of the correct solvability in Hölder and Sobolev–Slobodetskii spaces for parabolic boundary value problems, in the framework of their general theory, (Schauder's theory and L_p-theory) are proved. It turned out that the a priori estimates of the solutions established in this case are necessary and sufficient conditions for the parabolicity of the problem.

An important step in the construction of the theory of parabolic boundary value problems is a detailed study of the so-called model problems, namely the problems in a half-spaces with

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respect to the spatial variables in which systems of equations and boundary conditions contain only the major terms in the parabolic sense, and their coefficients are constants.

If we consider the, so-called, $\overrightarrow{2b}$-parabolic systems defined in [2] by S. D. Eidelman, then the orders of such systems are vectorial and the group of their major terms includes the derivatives of different the highest orders with respect to different spatial variables, since spatial variables are not equal. Therefore, it is perhaps impossible to construct a general theory of boundary value problems for such systems, analogous to the above theory for the I. G. Petrovsky systems and for V. A. Solonnikov systems, in which all spatial variables are equal. But for S. D. Eidelman systems, one can construct a theory of model boundary-value problems in a half-spaces in which one of the spatial variables varies is in the interval $(0, \infty)$, and all the others are in the interval $(-\infty, \infty)$.

In the works of the authors [4,6,7], for a parabolic in the sense of S. D. Eidelman system of the first-order equations with respect to the time variable a model boundary-value problem in a half-space is considered in which only the last spatial variable varies in the interval $(0, \infty)$. For such a problem, the complementary condition is formulated. The problem is correctly posed when the boundary conditions satisfy this complementary condition. Thus, the definition of a model $\overrightarrow{2b}$-parabolic boundary-value problem (P problem) is given. For P problem the Poisson kernel and the homogeneous Green's matrix were constructed, their accurate estimates and the estimates of their derivatives were obtained, the divergent representation was received. Using these results, a theorem of the correct solvability of P problem in anisotropic Hölder spaces is proved. In this article we obtain the integral representation of solutions of the P problem and investigate the structures of the kernels of the integrals from the representation. These kernels form the Green's matrix of P problem.

1 P problem formulation, its homogeneous Green's matrix and Poisson kernels

We will use the following notation: $n, N, b_1, \ldots, b_n$ are given natural numbers; $\overrightarrow{2b} := (2b_1, \ldots, 2b_n)$; s is the least common multiple of numbers $b_1, \ldots, b_n$; $m_j := s/b_j$, $j \in \{1, \ldots, n\}$; $\mathbb{Z}^n_+$ is the set of all n-dimensional multi-indices $k := (k_1, \ldots, k_n)$; $\|k\| := \sum\limits_{j=1}^{n} m_j k_j$, if $k \in \mathbb{Z}^n_+$; $\|\bar{k}\| := 2sk_0 + \|k\|$, if $\bar{k} := (k_0, k)$, where $k_0 \in \mathbb{Z}^1_+, k \in \mathbb{Z}^n_+$; $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$; $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x_n > 0\}$, $\Pi^+_T := \{(t, x) \in \mathbb{R}^{n+1} | t \in (0, T], x \in \mathbb{R}^n_+\}$, $\Pi'_T := \{(t, x') | t \in (0, T], x' \in \mathbb{R}^{n-1}\}$, where T is given positive number; $\partial^k_x := \partial^{k_1}_{x_1} \ldots \partial^{k_n}_{x_n}$, $\partial^{\bar{k}}_{t,x} := \partial^{k_0}_t \partial^k_x$, if $\bar{k} = (k_0, k), k_0 \in \mathbb{Z}^1_+, k \in \mathbb{Z}^n_+, t \in \mathbb{R}^1$ i $x \in \mathbb{R}^n$. Here, as usual, $\mathbb{R}^n$ is the n-dimensional real Euclidean space, and $\partial^l_y := \frac{\partial^l}{\partial y^l}$, if l is a natural number and $y \in \mathbb{R}^1$.

In the region Π^+_T we will consider a boundary value problem:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n})u(t,x) := (I_N \partial_t - \sum_{\|k\|=2s} a_k \partial^k_x) u(t,x) = f(t,x), \quad (t,x) \in \Pi^+_T, \tag{1}$$

$$B^0_j(\partial_t, \partial_{x'}, \partial_{x_n})u(t,x)|_{x_n=0} := \sum_{\|\bar{k}\|=r_j} b_{j\bar{k}} \partial^{\bar{k}}_{t,x} u(t,x)|_{x_n=0} = g_j(t,x'), (t,x') \in \Pi'_T, \quad j \in \{1, \ldots, m\}, \tag{2}$$

$$u(t,x)|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n_+, \tag{3}$$

where u, f and φ are matrix columns of height N; a_k and $b_{j\bar{k}}$ are constant matrices of size $N \times N$ and $1 \times N$ respectively; I_N is a unit matrix of order N; $g_1, \dots, g_m$ are scalar functions; $r_1, \dots, r_m$ are non-negative integers.

We assume that the system of equations (1) is parabolic according to Eidelman [1]. The number of boundary conditions $m = b_n N$ and these boundary conditions satisfy the complementing condition from [6]. The problem (1)–(3) that satisfies these conditions, we will call a model $\overrightarrow{2b}$-parabolic boundary value problem or P problem.

For P problem, we will give the definitions and the results of the studying of the homogeneous Green's matrix and Poisson kernels from [4, 6] that are necessary for further investigation.

According to [2, 3], we define a homogeneous Green's matrix and Poisson kernels of P problem as a matrix $G_0(t, x, \xi), t \in \mathbb{R}^1 \backslash \{0\}, \{x, \xi\} \subset \mathbb{R}^n$ of the size $N \times N$ and matrices $G_j(t, x), t \in \mathbb{R}^1 \backslash \{0\}, x \in \mathbb{R}^n$ of the size $N \times 1$ that are the solutions of the following problems:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t, x, \xi) = I_N \delta(t, x - \xi),$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t, x, \xi)|_{x_n = 0} = 0, \quad j \in \{1, \dots, m\},$$

$$G_0(t, x, \xi) = 0 \quad \text{as } t < 0,$$

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_j(t, x) = 0,$$

$$B_l^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_j(t, x)|_{x_n = 0} = \delta_{lj} \delta(t, x'), \quad l \in \{1, \dots, m\},$$

$$G_j(t, x) = 0 \quad \text{as } t < 0, \quad j \in \{1, \dots, m\}$$

in spaces of generalized functions, where δ_{lj} is Kronecker symbol, $\delta(t, x - \xi)$ and $\delta(t, x')$ is Dirac delta functions with supports in points $t = 0, x = \xi$ and $t = 0, x' = 0$ respectively. Wherein $G_0(t, x, \xi)|_{t=0+} = I_N \delta(x - \xi)$.

From these definitions it follows that for an arbitrary smooth and finite functions $f, g_1, \dots, g_m$ and φ the solution of P problem (1)–(3) is represented in the form

$$u(t, x) = (\mathcal{G}_0 f + \sum_{j=1}^{m} \mathcal{G}_j g_j + \mathcal{G}_{m+1} \varphi)(t, x), \quad (t, x) \in \Pi_T^+,$$

where

$$(\mathcal{G}_0 f)(t, x) := \int_0^t d\tau \int_{\mathbb{R}_+^n} G_0(t - \tau, x, \xi) f(\tau, \xi) d\xi, \tag{4}$$

$$(\mathcal{G}_j g_j)(t, x) := \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_j(t - \tau, x - \xi') g_j(\tau, \xi') d\xi', \quad j \in \{1, \dots, m\}, \tag{5}$$

$$(\mathcal{G}_{m+1} \varphi)(t, x) := \int_{\mathbb{R}_+^n} G_0(t, x, \xi) \varphi(\xi) d\xi. \tag{6}$$

The existence of matrices $G_j, j \in \{0, 1, \dots, m\}$ and the correctness for their divergent representations

$$G_j = L^r(\partial_t, \partial_{x'}) G_j^{(r)}, \quad j \in \{0, 1, \dots, m\}, \tag{7}$$

where

$$L(\partial_t, \partial_{x'}) := \partial_t + a \sum_{j=1}^{n-1} (-1)^{b_j} \partial_{x_j}^{2b_j}, a > 0$$

and r is any non-negative number were proved in [4,6] and for $G_j^{(r)}$ the following estimates are fulfilled

$$\begin{aligned} |\partial_{t,x}^{\bar{k}} \partial_{\xi}^{l} G_0^{(r)}(t,x,\xi)| \le C_{\bar{k}l} t^{-M+r-(\|\bar{k}\|+\|l\|)/(2s)} E_c(t, x-\xi), \\ t > 0, \{x, \xi\} \subset \mathbb{R}_+^n, \bar{k} \in \mathbb{Z}_+^{n+1}, l \in \mathbb{Z}_+^n; \end{aligned} \tag{8}$$

$$\begin{aligned} |\partial_{t,x}^{\bar{k}} G_j^{(r)}(t,x)| \le C_{\bar{k}} t^{-M'+r-1+(r_j-\|\bar{k}\|)/(2s)} E_c(t,x), \\ t > 0, x \in \mathbb{R}_+^n, \bar{k} \in \mathbb{Z}_+^{n+1}, j \in \{1, \dots, m\}. \end{aligned} \tag{9}$$

In the estimates (8) and (9)

$$\begin{aligned} M := \textstyle\sum_{j=1}^{n} m_j/(2s),\ M' := \sum_{j=1}^{n-1} m_j/(2s), \\ E_c(t,x) := \exp\{-c \textstyle\sum_{j=1}^{n} t^{-1/(2b_j-1)} |x_j|^{2b_j/(2b_j-1)}\}, \end{aligned}$$

$C_{\bar{k}l}, C_{\bar{k}}$ and c are some positive constants.

2 REPRESENTATION OF SOLUTION FOR P PROBLEM WITH HOMOGENEOUS INITIAL CONDITION

Suppose that in the problem (1)–(3) f and $g_j, j \in \{1, \dots, m\}$ are sufficiently smooth functions such that they together with their derivatives are bounded and equal to zero as $t = 0$ and $\varphi = 0$. Let us find a formula for the solutions of P problem with these right-hand sides, namely for the following problem with zero initial condition:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) u(t,x) = f(t,x), \quad (t,x) \in \Pi_T^+, \tag{10}$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) u(t,x)|_{x_n=0} = g_j(t,x'), \quad (t,x') \in \Pi_T', \quad j \in \{1, \dots, m\}, \tag{11}$$

$$u(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}_+^n. \tag{12}$$

Consider the function

$$u_0(t,x) := (\mathcal{G}_0 f)(t,x), \quad (t,x) \in \Pi_T^+. \tag{13}$$

Based on the definition (4) of the operator $\mathcal{G}_0$ and the properties of the matrix G_0, the function u_0 is a solution of system (10) that satisfies the condition (12). In addition, if the order r_j of the differential expression $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$ is less than $2s$, then

$$\begin{aligned} &B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) u_0(t,x)|_{x_n=0} \\ &\qquad = \int_0^t d\tau \int_{\mathbb{R}_+^n} B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t-\tau, x, \xi)|_{x_n=0} f(\tau, \xi) d\xi = 0, \quad (t,x') \in \Pi_T'. \end{aligned}$$

In the case when $r_j \ge 2s$, it is impossible to apply the operation $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$ and pass to the limit as $x_n \to 0$ under the sign of the integral. In this case, we proceed as follows.

Consider $B_j^0(p, i\sigma', i\tau)$ and $A^0(p, i\sigma', i\tau)$, where i is the imaginary unit, as matrix polynomials of τ with fixed values of the parameters p and σ'. Based on the $\overrightarrow{2b}$-parabolicity of system (10), the determinant of the matrix, which is the coefficient at τ^{2b_n} in $A^0(p, i\sigma', i\tau)$, is nonzero (see Remark 1 in [4]). Therefore, there exist such matrix polynomials $C_j(p, i\sigma', i\tau)$ and $B_j'(p, i\sigma', i\tau)$ that their degrees on τ do not exceed $r_j - 2b_n$ and $2b_n - 1$ respectively, and they fulfill the equality

$$B_j^0(p, i\sigma', i\tau) = C_j(p, i\sigma', i\tau)A^0(p, i\sigma', i\tau) + B_j'(p, i\sigma', i\tau).$$

Turning to differential expressions, we obtain the equality

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) = C_j(\partial_t, \partial_{x'}, \partial_{x_n})A^0(\partial_t, \partial_{x'}, \partial_{x_n}) + B_j'(\partial_t, \partial_{x'}, \partial_{x_n}), \tag{14}$$

where C_j and B_j' are expressions containing differentiations on x_n of order not higher than $r_j - 2b_n$ and $2b_n - 1$, respectively.

For function (13), on the basis of equality (14) and the fact that $A^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0 = f$, we now get

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0} + B_j'(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0}.$$

Using representation (7) and estimates (8) for G_0 and integrating by parts, for a sufficiently large r, we obtain

$$B_j'(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = \int_0^t d\tau \int_{\mathbb{R}^n_+} B_j'(\partial_t, \partial_{x'}, \partial_{x_n})G_0^{(r)}(t-\tau, x, \xi)|_{x_n=0}L^r(\partial_\tau, \partial_{\xi'})f(\tau, \xi)d\xi.$$

Based on (14) and on the fact that $A^0(\partial_t, \partial_{x'}, \partial_{x_n})G_0^{(r)} = 0$, we replace B_j' by B_j^0 in the last integral. If we represent this integral as the limit of the integral over $\{\xi \in \mathbb{R}^n | \xi_n \geq \varepsilon\}, \varepsilon > 0$, as $\varepsilon \to 0$, and then integrate by parts of the expression $L^r(\partial_\tau, \partial_{\xi'})$ and use it to $G_0^{(r)}$, then we get that it is equal to zero. So,

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0}.$$

Note that $C_j = 0$ if the highest order of derivatives with respect to x_n in B_j^0 is less than $2b_n$.

Thus, the function (13) is a solution to the problem (10)–(12), in which g_j is replaced by $C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0}, j \in \{1, \dots, m\}$. Moreover, if the function f is finite in Π_T^+, then

$$C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0} = 0, \quad j \in \{1, \dots, m\}.$$

If, in the conditions (11), the functions $g_j, j \in \{1, \dots, m\}$, such as indicated at the beginning of this section, then using for G_j the representation (7) and the estimates (9) just as in [5], we prove that the function

$$u_1(t, x) := \sum_{j=1}^m \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_j(t-\tau, x-\xi')g_j(\tau, \xi')d\xi', \quad (t, x) \in \Pi_T^+,$$

is the solution to the problem

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n})u_1(t, x) = 0, \quad (t, x) \in \Pi_T^+,$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})u_1(t,x)|_{x_n=0} = g_j(t,x'), \quad (t,x') \in \Pi'_T, \quad j \in \{1,\dots,m\},$$
$$u_1(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}^n_+.$$

Therefore, for the functions f and $g_j, j \in \{1,\dots,m\}$ indicated at the beginning of Section 2, the solution of problem (10)–(12) is determined by the formula

$$u(t,x) = (\mathcal{G}_0 f)(t,x) + \sum_{j=1}^{m} (\mathcal{G}_j(g_j - C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0}))(t,x), \quad (t,x) \in \Pi^+_T. \tag{15}$$

3 THE GENERAL CASE OF P PROBLEM

Suppose now that in the problem (1)–(3) functions f, $g_j, j \in \{1,\dots,m\}$, and φ are sufficiently smooth in $\bar{\Pi}^+_T$, $\bar{\Pi}'_T$ and $\bar{\mathbb{R}}^n_+$ that are the closures of Π^+_T, Π'_T and $\mathbb{R}^n_+$, respectively and they together with their derivatives, are bounded and satisfy the corresponding matched conditions as $t=0$ and $x_n = 0$. Then, from the results of the paper [7], it follows that there exists a unique smooth solution u of the general P problem, defined in $\bar{\Pi}^+_T$ and bounded with all its derivatives. Now we were find the integral representation of this solution u.

Let us choose the infinitely differentiable function $\zeta(t), t \in \mathbb{R}^1$, that is equal to 1 for $t \geq 1$ and is equal to 0 for $t \leq 1/2$, and the function $v_h(t,x) := \zeta_h(t)u(t,x), (t,x) \in \bar{\Pi}^+_T$, where $\zeta_h(t) := \zeta(t/h)$, h is a sufficiently small positive number. Obviously, v_h has the same smoothness properties as the function u, and it is a solution to the problem

$$\begin{aligned} &A^0(\partial_t, \partial_{x'}, \partial_{x_n})v_h(t,x) = F_{0h}(t,x), \quad (t,x) \in \Pi^+_T, \\ &B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})v_h(t,x)|_{x_n=0} = F_{jh}(t,x'), \quad (t,x') \in \Pi'_T, \quad j \in \{1,\dots,m\}, \\ &v_h(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}^n_+, \end{aligned} \tag{16}$$

where

$$F_{0h}(t,x) := \zeta_h(t)f(t,x) + \zeta_h^{(1)}(t)u(t,x),$$

$$F_{jh}(t,x') := \zeta_h(t)g_j(t,x') + \sum_{\|\bar{k}\|=r_j} \sum_{\nu=0}^{k_0} C_{k_0}^{\nu} b_{j\bar{k}} \zeta_h^{(\nu)}(t) \partial_t^{k_0-\nu} u(t,x)|_{x_n=0}, \quad j \in \{1,\dots,m\}.$$

Here and further $C_{k_0}^{\nu} := \frac{k_0!}{\nu!(k_0-\nu)!}, \zeta_h^{(\nu)}(t) := \frac{d^\nu \zeta_h(t)}{dt^\nu}$.

Since the problem (16) is a problem with zero initial condition, then according to the result of Section 2, the representation of its solution could be written in the form (15), i.e.

$$v_h(t,x) = (\mathcal{G}_0 F_{0h})(t,x) + \sum_{j=1}^{m} (\mathcal{G}_j(F_{jh} - C_j(\partial_t, \partial_{x'}, \partial_{x_n})F_{0h}|_{x_n=0}))(t,x), \quad (t,x) \in \Pi^+_T. \tag{17}$$

Assuming that the point (t,x) is fixed from Π^+_T and $h \in (0,t)$, we pass to the limit as $h \to 0$ in (17). At the same time, we obtain $u(t,x)$ in the left-hand side. Further we find the limit of the right side.

We have

$$\mathcal{G}_0(\zeta_h f)(t,x) = (\mathcal{G}_0 f)(t,x) + \int_0^h d\tau \int_{\mathbb{R}^n_+} G_0(t-\tau,x,\xi)(\zeta_h(\tau)-1)f(\tau,\xi)d\xi \xrightarrow[h\to 0]{} (\mathcal{G}_0 f)(t,x). \tag{18}$$

Taking into account the properties of the function ζ_h and integrating by parts, we obtain

$$\begin{aligned}(\mathcal{G}_0(\zeta_h^{(1)}u))(t,x) &= \int\limits_{h/2}^{h} d\tau \int\limits_{\mathbb{R}^n_+} G_0(t-\tau,x,\xi)\zeta_h^{(1)}(\tau)u(\tau,\xi)d\xi \\ &= \int\limits_{\mathbb{R}^n_+} G_0(t-h,x,\xi)u(h,\xi)d\xi \\ &\quad - \int\limits_{h/2}^{h} d\tau \int\limits_{\mathbb{R}^n_+} \partial_\tau(G_0(t-\tau,x,\xi)u(\tau,\xi))\zeta_h(\tau)d\xi \xrightarrow[h\to 0]{} (\mathcal{G}_{m+1}\varphi)(t,x).\end{aligned} \tag{19}$$

Similarly we have

$$\begin{aligned}&(\mathcal{G}_jF_{jh})(t,x) = (\mathcal{G}_j(\zeta_h g_j))(t,x) + \sum_{\|\bar{k}\|=r_j} b_{j\bar{k}}\Big\{(\mathcal{G}_j(\partial^{\bar{k}}_{t,x}u|_{x_n=0}\zeta_h))(t,x) \\ &+\sum_{\nu=1}^{k_0}(-1)^{\nu-1}C^\nu_{k_0}\Big[\int\limits_{\mathbb{R}^{n-1}} \partial^{\nu-1}_\tau(G_j(t-\tau,x-\xi')\partial^{k_0-\nu}_\tau \partial^k_\xi u(\tau,\xi)|_{\xi_n=0})d\xi'\zeta_h(\tau)|^{\tau=h}_{\tau=h/2} \\ &-\int\limits_{h/2}^{h} d\tau \int\limits_{\mathbb{R}^{n-1}} \partial^{\nu}_\tau(G_j(t-\tau,x-\xi')\partial^{k_0-\nu}_\tau \partial^k_\xi u(\tau,\xi)|_{\xi_n=0})\zeta_h(\tau)d\xi'\Big]\Big\} \xrightarrow[h\to 0]{} (\mathcal{G}_jg_j)(t,x) \\ &+\sum_{\substack{\|\bar{k}\|=r_j\\(k_0>0)}} b_{j\bar{k}}\sum_{\nu=1}^{k_0}(-1)^{\nu-1}C^\nu_{k_0}\int\limits_{\mathbb{R}^{n-1}} \partial^{\nu-1}_\tau(G_j(t-\tau,x-\xi')\partial^{k_0-\nu}_\tau \partial^k_\xi u(\tau,\xi)|_{\xi_n=0})d\xi'|_{\tau=0}.\end{aligned} \tag{20}$$

Now consider $\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0})$. Using the record

$$C_j(\partial_t,\partial_{x'},\partial_{x_n}) = \sum_{\|\bar{k}\|\le\le r_j-2s} c_{j\bar{k}}\partial^{\bar{k}}_{t,x},$$

as above, using integration by parts, we obtain

$$\begin{aligned}&(\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0}))(t,x) \\ &= \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n-1}} G_j(t-\tau,x-\xi')\times\partial^{k_0}_\tau(\zeta_h(\tau)\partial^k_\xi f(\tau,\xi)+\zeta_h^{(1)}(\tau)\partial^k_\xi u(\tau,\xi))|_{\xi_n=0}d\xi' \\ &= \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n-1}} \partial^{k_0}_t G_j(t-\tau,x-\xi')(\zeta_h(\tau)\partial^k_\xi f(\tau,\xi)|_{\xi_n=0}+\zeta_h^{(1)}(\tau)\partial^k_\xi u(\tau,\xi)|_{\xi_n=0})d\xi'.\end{aligned}$$

The remaining terms are zero due to the properties of the functions ζ_h and G_j. Integrating by parts again and passing to the limit as $h\to 0$, we get

$$\begin{aligned}(\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0}))(t,x) \xrightarrow[h\to 0]{} \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\Big(\int\limits_0^t d\tau\int\limits_{\mathbb{R}^{n-1}} \partial^{k_0}_t G_j(t-\tau,x-\xi')\partial^k_\xi f(\tau,\xi)|_{\xi_n=0}d\xi' \\ +\int\limits_{\mathbb{R}^{n-1}} \partial^{k_0}_t G_j(t,x-\xi')\partial^k_\xi\varphi(\xi)|_{\xi_n=0}d\xi'\Big).\end{aligned} \tag{21}$$

From (17)–(21) it follows the formula

$$u(t,x) = (\mathcal{G}_0 f + \sum_{j=1}^{m} \mathcal{G}_j g_j + \mathcal{G}_{m+1}\varphi)(t,x)$$
$$+ \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t-\tau, x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0} d\xi' + \int_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t, x-\xi')\partial_\xi^k \varphi(\xi)|_{\xi_n=0} d\xi'$$
$$+ \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{m} \sum_{\substack{\|\bar{k}\|=r_j \\ (k_0>0)}} \sum_{\nu=1}^{k_0} (-1)^{\nu-1} C_{k_0}^{\nu} b_{j\bar{k}} \partial_\tau^{\nu-1} (G_j(t-\tau, x-\xi')\partial_\tau^{k_0-\nu}\partial_\xi^k u(\tau,\xi)|_{\xi_n=0}) d\xi'|_{\tau=0},\ (t,x)\in\Pi_T^+, \tag{22}$$

where

$$R_k(t,x) := \sum_{j=1}^{m} R_{jk}(t,x), \tag{23}$$

$$R_{jk}(t,x) := \begin{cases} \sum\limits_{k_0\le(r_j-\|k\|-2s)/(2s)} c_{j\bar{k}}\partial_t^{k_0} G_j(t,x), & \text{if } \|k\|\le r_j-2s, \\ 0, & \text{if } r_j-2s<\|k\|\le r_0, \end{cases} \tag{24}$$

$$r_0 := \max(0, r_1-2s, \dots, r_m-2s), \tag{25}$$

moreover $R_k = 0$, if the highest order of derivatives with respect to x_n in $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, $j \in \{1,\dots,m\}$, n_0 is less than $2b_n$.

All terms of the right-hand side of (22), except for the last, include only the right-hand sides of the problem (1)–(3). We transform the last term (denote it by D) in such way that it also will include only the right-hand sides of the problem (1)–(3). Note that the term D is absent if the expressions $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, $j\in\{1,\dots,m\}$, do not include differentiation with respect to t.

Using the Leibniz formula and changing the order of summation, we get

$$D = \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{m} \sum_{\mu=0}^{p_j-1} \sum_{\substack{k_0= \\ =\mu+1}}^{p_j} \sum_{\substack{\|k\|=r_j- \\ -2sk_0}} N_{k_0\mu} b_{j\bar{k}} \partial_\tau^{k_0-\mu-1} G_j(t-\tau, x-\xi')\partial_\tau^{\mu}\partial_\xi^k u(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi', \tag{26}$$

where p_j is the highest order of derivatives with respect to t in the expression $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, and

$$N_{k_0\mu} := \sum_{\nu=k_0-\mu}^{k_0} (-1)^{\nu-1} C_{k_0}^{\nu} C_{\nu-1}^{k_0-\mu-1}.$$

We will write the formula (26) in the form

$$D = \int_{\mathbb{R}^{n-1}} \sum_{\mu=0}^{p_0-1} \sum_{\|k\|\le r_0-2s\mu} Q_{\mu k}(t-\tau, x-\xi')\partial_\tau^{\mu}\partial_\xi^k u(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi', \tag{27}$$

where $p_0 := \max(p_1,\dots,p_m)$,

$$Q_{\mu k}(t,x) := \sum_{j=1}^{m} Q_{j\mu k}(t,x), \tag{28}$$

$$Q_{j\mu k}(t,x) :=$$
$$= \begin{cases} \sum\limits_{\substack{\mu+1\le k_0\le \\ \le (r_j-\|k\|-2s)/(2s)}} N_{k_0\mu} b_{j\bar{k}} \partial_t^{k_0-\mu-1} G_j(t,x), \text{if } 0 \le \mu \le p_j - 1,\, 2s\mu + \|k\| \le r_j - 2s, \\ 0, \qquad\qquad \text{if } p_j \le \mu \le p_0 - 1 \text{ or } r_j - 2s < 2s\mu + \|k\| \le r_0. \end{cases}$$

Using the system (1) and the condition (3) for $\mu > 0$ we obtain the representation

$$\partial_\tau^\mu \partial_\xi^k u(\tau,\xi)|_{\tau=0} = \sum_{\|\nu\|=2s\mu+\|k\|} A_{\mu k\nu} \partial_\xi^\nu \varphi(\xi) + \sum_{\substack{\|\bar{\nu}\|=2s(\mu-1)+\|k\| \\ (\nu_0\le\mu-1)}} B_{\mu k\bar{\nu}} \partial_{\tau,\xi}^{\bar{\nu}} f(\tau,\xi)|_{\tau=0}, \tag{29}$$

where $A_{\mu k\nu}$ and $B_{\mu k\bar{\nu}}$ are constant matrices of the size $N \times N$, wich compiled with coefficients a_k, $\|k\| = 2s$, from the system (1). Substituting expression (29) into (27), changing the order of summation and using the notation

$$V_\nu(t,x) := \sum_{\substack{2s\mu+\|k\|=\|\nu\| \\ (\mu\le p_0-1)}} Q_{\mu k}(t,x) A_{\mu k\nu},\ W_{\bar{\nu}}(t,x) := \sum_{\substack{2s\mu+\|k\|=\|\bar{\nu}\|+2s \\ (\mu\ge\nu_0+1)}} Q_{\mu k}(t,x) B_{\mu k\bar{\nu}}, \tag{30}$$

we get the following expression for D:

$$\begin{aligned} D = &\int\limits_{\mathbb{R}^{n-1}} \sum_{\|\nu\|\le r_0} V_\nu(t, x-\xi') \partial_\xi^\nu \varphi(\xi)|_{\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\substack{\|\bar{\nu}\|\le r_0-2s \\ (\nu_0\le p_0-2)}} W_{\bar{\nu}}(t-\tau, x-\xi') \partial_{\tau,\xi}^{\bar{\nu}} f(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi'. \end{aligned} \tag{31}$$

Therefore, from formulas (22) and (31) it follows the following representation of the solution of the general P problem:

$$\begin{aligned} u(t,x) =&(\mathcal{G}_0 f + \sum_{j=1}^m \mathcal{G}_j g_j + \mathcal{G}_{m+1}\varphi)(t,x) \\ &+ \int_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t-\tau, x-\xi') \partial_\xi^k f(\tau,\xi)|_{\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\substack{\|\bar{k}\|\le r_0-2s \\ (k_0\le p_0-2)}} W_{\bar{k}}(t-\tau, x-\xi') \partial_{\tau,\xi}^{\bar{k}} f(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} \big(R_k(t, x-\xi') + V_k(t, x-\xi')\big) \partial_\xi^k \varphi(\xi)|_{\xi_n=0} d\xi' \\ &=: I_1 + I_2 + I_3 + I_4, \quad (t,x) \in \Pi_T^+. \end{aligned} \tag{32}$$

Now, we rewrite this representation in another form. To do this, first we transform the addend I_3 from formula (32). Using the formula

$$W_{\bar{k}} \partial_\tau^{k_0} f = \sum_{l=0}^{k_0} (-1)^{k_0-l} C_{k_0}^l \partial_\tau^l (\partial_\tau^{k_0-l} W_{\bar{k}} f),$$

we get

$$I_3 = \int\limits_{\mathbb{R}^{n-1}} \sum_{k_0=0}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} \sum_{l=0}^{k_0} (-1)^{k_0-l} C_{k_0}^l \partial_\tau^l (\partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0})|_{\tau=0} d\xi'$$

$$= \int\limits_{\mathbb{R}^{n-1}} \sum_{l=0}^{p_0-2} (-1)^l \partial_\tau^l \Bigg[\sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0} C_{k_0}^l \partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0}\Bigg]\Bigg|_{\tau=0} d\xi'$$

$$= \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \sum_{l=0}^{p_0-2} \delta^{(l)}(\tau) \sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0} C_{k_0}^l \partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0}|_{\tau=0} d\xi'$$

$$= \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n_+} G_0''(t,x;\tau,\xi) f(\tau,\xi) d\xi,$$

where

$$G_0''(t,x;\tau,\xi) := \sum_{l=0}^{p_0-2} \delta^{(l)}(\tau) \sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0+|k|} C_{k_0}^l \partial_\tau^{k_0-l} \partial_{\xi'}^{k'} W_{\bar{k}}(t-\tau, x-\xi')\delta^{(k_n)}(\xi_n), \quad (33)$$

where $|k| := k_1 + \cdots + k_n$, $\delta^{(l)}(\tau)$ and $\delta^{(k_n)}(\xi_n)$ are the derivatives of delta functions concentrated at points $\tau = 0$ and $\xi_n = 0$ respectively.

Similarly transforming the addends I_2 and I_4 from (32) and taking into account the definitions (4)–(6), we write the representation (32) in the form

$$\begin{aligned} u(t,x) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n_+} \tilde{G}_0(t,x;\tau,\xi) f(\tau,\xi) d\xi + \sum_{j=1}^m \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} G_j(t-\tau, x-\xi') g_j(\tau,\xi') d\xi' \\ + \int\limits_{\mathbb{R}^n_+} \tilde{G}_{m+1}(t,x,\xi)\varphi(\xi) d\xi, \quad (t,x) \in \Pi_T^+, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \tilde{G}_0(t,x;\tau,\xi) &:= G_0(t-\tau,x,\xi) + G_0'(t-\tau,x,\xi) + G_0''(t,x;\tau,\xi), \\ \tilde{G}_{m+1}(t,x,\xi) &:= G_0(t,x,\xi) + G_0'(t,x,\xi) + +G_{m+1}'(t,x,\xi). \end{aligned} \quad (35)$$

Here

$$\begin{aligned} G_0'(t,x,\xi) &:= \sum_{\|k\|\le r_0} (-1)^{|k|} \partial_{\xi'}^{k'} R_k(t, x-\xi')\delta^{(k_n)}(\xi_n), \\ G_{m+1}'(t,x,\xi) &:= \sum_{\|k\|\le r_0} (-1)^{|k|} \partial_{\xi'}^{k'} V_k(t, x-\xi')\delta^{(k_n)}(\xi_n), \end{aligned} \quad (36)$$

and G_0'' is defined by the formula (33).

As a corollary we can get the following theorem from the results obtained above and from Theorem 2 [7] about the correct solvability of the P problem in Hölder spaces.

Theorem 1. *Any solution to the P problem* (1)–(3)*, that belongs to the Hölder space* $H^{2s+l+\lambda}(\bar{\Pi}^+_T, \mathbb{C}_{N1})$*, where* l *is an integer, such that* $l \geq r_0$ *and* $\lambda \in (0,1)$*, is represented in the form* (34)*. The kernels of this representation are defined by formulas* (33), (35) *and* (36)*. In these formulas* R_k*,* V_k *and* $W_{\bar{k}}$ *are defined by equalities* (23)–(25), (28) *and* (30)*. In all of these formulas,* G_0 *is a homogeneous Green's matrix, and* $G_j, j \in \{1,\ldots,m\}$*, are Poisson kernels of problem* (1)–(3)*. Moreover,* $G_0' = G_0'' = G_{m+1}' = 0$ *if* $2sp_0 + m_n n_0 < 2s$*, where* p_0 *and* n_0 *are the highest orders of derivatives with respect to* t *and* x_n *in boundary conditions* (2) *accordingly, and* $m_n = s/b_n$*.*

Definition 1. *The matrix composed of the elements of the matrices* $\tilde{G}_0, G_1, \ldots, G_m$ *and* $\tilde{G}_{m+1}$ *is called the Green's matrix of the problem* (1)–(3)*.*

So, the article describes the structure of the Green's matrix of the problem (1)–(3).

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В області $\{(t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | t \in (0,T], x_j \in \mathbb{R}, j \in \{1,\ldots,n-1\}, x_n > 0\}, T > 0, n \geq 2$, розглядається загальна крайова задача для $\overrightarrow{2b}$-параболічної за Ейдельманом системи рівнянь, в якій у рівняннях і крайових умовах відсутні молодші члени, а коефіцієнти групи старших членів сталі. Припускається, що крайові умови пов'язані з системою рівнянь умовою доповняльності, яка є аналогом умови доповняльності Лопатинського. Для розв'язків такої задачі виведено інтегральне зображення. Ядра інтегралів з цього зображення утворюють матрицю Ґріна задачі. Виявлено, що, взагалі кажучи, не всі елементи матриці Ґріна є звичайними функціями. Деякі з них містять доданки, які є лінійними комбінаціями дельта-функцій Дірака та їх похідних. Це виникає у випадках, коли в крайові умови входять похідні за змінними t і x_n порядків, рівних або більших за найвищі порядки похідних за цими змінними в рівняннях системи. Отримані результати є важливими, зокрема, для встановлення коректної розв'язності та інтегрального зображення розв'язків загальніших $\overrightarrow{2b}$-параболічних крайових задач.

Ключові слова і фрази: $\vec{2b}$-параболічна за Ейдельманом система рівнянь, крайова задача, інтегральне зображення розв'язків, матриця Ґріна.

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ЛОПУШАНСЬКОМУ ОЛЕГУ ВАСИЛЬОВИЧУ — 70 РОКІВ

13 березня 2019 року виповнилося 70 років відомому математику Лопушанському Олегу Васильовичу.

Олег Лопушанський — доктор фізико-математичних наук, професор, декан математично-природничого факультету Жешувського університету.

Народився Олег Васильович 13 березня 1949 року у селі Стрілки Старосамбірського району Львівської області. У 1971 році закінчив механіко-математичний факультет Львівського державного університету імені Івана Франка. У 1971–1973 роках служив в армії у Монголії. У 1973–1985 роках працював в обчислювальному центрі Інституту прикладних проблем механіки і математики АН України. З 1985 року перейшов працювати у відділ функціонального аналізу цього ж інституту, спочатку на посаду старшого наукового співробітника, а з 1990 року на посаду завідувача відділу. Під керівництвом І.Г. Журбенка, безпосереднього учня А.М. Колмогорова, у 1986 році в Білоруському університеті захистив кандидатську дисертацію на тему "Векторні борнології в спектральній теорії локально опуклих алгебр", а докторську дисертацію — у 1994 році у Львівському університеті на тему "Півобмежені та обмежені оператори в спектральній теорії локально опуклих алгебр". У 1996 році отримав вчене звання професора.

Упродовж 2008–2012 років працював у Прикарпатському національному університеті на посаді професора кафедри математичного і функціонального аналізу. За цей період розробив і читав курси "Теорія рівнянь Блека-Шоулса" та "Фінансова математика". У 2013 році, завдяки зокрема і його розробкам, було ліцензовано, пізніше успішно акредитовано магістерську програму з фінансової математики.

Упродовж роботи в Прикарпатському університеті Олег Васильович був виконавцем кількох науково-дослідних тем. За його безпосередньої участі в Прикарпатському університеті була відкрита і досі функціонує спеціалізована Вчена рада по захисту дисертацій із спеціальності "математичний аналіз". З 2009 року і до сьогодні Олег Лопушанський є заступником головного редактора фахового наукового журналу "Карпатські математичні публікації", котрий індексується багатьма наукометричними базами даних, серед яких Emerging Sources Citation Index (Web of Science) та Scopus. Під керівництвом професора О.В. Лопушанського було захищено 9 кандидатських дисертацій, з них шість — в Україні та три — у Польщі, для двох докторських дисертацій він був науковим консультантом. Четверо його учнів (2 кандидати наук та 2 доктори наук) зараз працюють у Прикарпатському національному університеті.

Завдяки сприянню О.В. Лопушанського студенти Прикарпатського університету спеціальностей "математика", "статистика", "фізика", "прикладна фізика", "філософія", "археологія", "фізичне виховання" мають змогу безкоштовно навчатися у Жешувському університеті за програмою подвійних магістерських дипломів. У 2016 році одностайним рішенням Вченої ради Лопушанському Олегу Васильовичу присвоєно звання Почесний професор Прикарпатського національного університету імені Василя Стефаника.

Наукові інтереси проф. О.В. Лопушанського лежать у сферах нелінійного функціонального аналізу, теорії операторів, теорії узагальнених функцій. Олег Лопушанський досягнув значного прогресу у розвитку теорії просторів Гарді для функцій від нескінченної кількості змінних, використовуючи інваріантні ймовірнісні міри, що задовольняють умови консистенції Колмогорова на нескінченновимірних унітарних групах, та дослідив їх зв'язки з симетричними просторами Фока. Це дозволяє використовувати методи нескінченновимірної голоморфності для функціонального представлення нескінченновимірних груп Гейзенберга. Професор О.В. Лопушанський запропонував новий підхід до побудови та дослідження операторного числення в класі симетричних основних та узагальнених функцій, використовуючи техніку тензорних добутків. Разом зі своїми учнями він розвинув поліноміальний аналог теорії двоїстості Ґротендіка та відповідний аналіз Фур'є-Лапласа для розподілів Шварца та ультрарозподілів Рум'є. Застосування цього аналізу було знайдено у побудові операторного числення для нескінченної кількості необмежених операторів у згорткових алгебрах поліноміальних (ультра)розподілів. Також він має вагомі результати в теорії спектральної апроксимації необмежених операторів в банахових просторах, де, використовуючи теорію інтерполяції, він описав простори типу Бєсова і довів нерівності Бернштейна-Джексона для довільних абстрактних операторів.

На честь 70-річчя професора О.В. Лопушанського у Прикарпатському національному університеті з 16 до 20 жовтня 2019 року пройде міжнародна математична конференція "Infinte dimensional analysis and topology". Запрошеними лекторами погодились бути наступні відомі математики: Тарас Банах (Львів, Україна), Андреас Дефант (Ольденбург, Німеччина), Сергій Фаворов (Харків, Україна), Пабло Ґаліндо (Валенсія, Іспанія), Володимир Кадець (Харків, Україна), Сергій Максименко (Київ, Україна), Мечислав Мастило (Познань, Польща), Михайло Попов (Івано-Франківськ, Україна), Влодзімєж Звонек (Краків, Польща). На вебсторінці `https://conference.pu.if.ua/idat` можна детальніше довідатись про конференцію та зареєструватись для участі.

У даний час Олег Васильович знаходиться у розквіті творчих сил і продовжує активно займатися науково-дослідною роботою. Від щирої душі вітаємо ювіляра та бажаємо йому міцного здоров'я, довгих років життя та творчого натхнення!

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Кириченко Володимир Васильович

17.06.1942 — 02.04.2019

Редколегія журналу з глибоким сумом сповіщає, що 2 квітня 2019 року пішов із життя видатний математик-алгебраїст, член редколегії, д.ф.-м.н, заслужений професор Київського національного університету імені Тараса Шевченка, лауреат Державної премії України в галузі науки і техніки Кириченко Володимир Васильович.

В.В. Кириченко народився 17 червня 1942 року у місті Пенза (Росія). У 1959 році, після здобуття середньої освіти, вступив на механіко-математичний факультет Київського державного університету імені Т.Г. Шевченка, який закінчив з відзнакою у 1964 році. В аспірантурі Інституту математики АН УРСР Володимир Васильович навчався з січня 1965 року по серпень 1967 року і у 1968 році, в Ленінградському відділенні Математичного інституту імені В.О. Стєклова, успішно захистив кандидатську дисертацію на тему "Зображення спадкових, цілком розкладних та басових порядків". Його науковим керівником був відомий фахівець в галузі алгебри, член-кореспондент АН СРСР, професор Д.К. Фаддєєв.

Педагогічну діяльність Володимир Васильович розпочав на механіко-математичному факультеті Київського національного університету імені Т.Г. Шевченка у вересні 1967 року. У 1986 році в Московському державному університеті імені М.В. Ломоносова він успішно захистив докторську дисертацію на тему "Модулі та структурна теорія кілець". З 1988 року по 2016 рік В.В. Кириченко працював на кафедрі геометрії механіко-математичного факультету Київського національного університету імені Т.Г. Шевченка, а згодом її і очолював. У 2016 – 2018 роках обіймав посаду старшого наукового співробітника науково-дослідної частини університету.

В.В. Кириченко є одним із засновників сучасної київської алгебраїчної школи. Його наукові результати стали класичними. Розроблена ним, Ю.А. Дроздом і А.В. Ройтером теорія басових та квазі-басових порядків стала відомою та розвивалася у закордонних алгебраїчних школах, зокрема, в Японії. В.В. Кириченку належить також провідна роль у розвитку структурної теорії кілець, де він ефективно застосував методи теорії зображень. Введені ним поняття сагайдака і первинного сагайдака кільця та розроблені методи дозволили отримати конструктивний опис низки важливих класів кілець, а згодом одержати розв'язок проблеми Л.А. Скорнякова про будову кілець спеціального типу. У 1976 році В.В. Кириченком спільно з О.Г. Завадським були введені так звані напівмаксимальні кільця. У 1993 році В.В. Кириченко довів важливу класифікаційну теорему сучасної теорії асоціативних кілець про будову напівпервинних нетерових напівдосконалих напівдистрибутивних кілець.

Володимир Кириченко є автором понад 250 наукових праць. Його наукові результати добре відомі у світі та ввійшли в монографічну літературу з теорії зображень та теорії кілець і модулів. У 2007 році професор В.В. Кириченко у складі колективу науковців механіко-математичного факультету та Інституту математики НАН України за цикл робіт «Зображення алгебраїчних структур і матричні задачі в лінійних та гільбертових просторах» став лауреатом Державної премії України в галузі науки і техніки. В.В. Кириченко активно працював над вихованням наукових кадрів. Під його керівництвом захищено 36 кандидатських та 8 докторських дисертацій.

В.В. Кириченко є одним із засновників регулярних міжнародних алгебраїчних конференцій в Україні, був редактором та членом редколегій багатьох наукових журналів, серед них "Algebra and Discrete Mathematics", "Карпатські математичні публікації", "Математичні студії" та ін.

Згадка про Володимира Васильовича зігріває душу. Він назавжди залишиться в пам'яті всіх, хто його знав, як високоінтелігентна, справедлива, чуйна людина, здатна завжди прийти на допомогу.

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Березанський Юрій Макарович

08.05.1925 — 07.06.2019

Уночі з 7 на 8 червня 2019 року відійшов у вічність Юрій Макарович Березанський — видатний український учений-математик, академік Національної академії наук України, один з фундаторів сучасного функціонального аналізу.

Народився Юрій Макарович 8 травня 1925 року у Києві в інтелігентній сім'ї. Його батько був агрономом-науковцем, мати — бібліотекарем. Дитинство і юність хлопця були дуже важкими — йому довелось пережити численні виснажливі хвороби, голодні роки, війну, окупацію. Як він згадував, німці влаштовували облави і відправляли непрацюючу молодь на роботи до Німеччини, але він мав довідку про роботу: батько влаштував його "на посаду опудала" — він мав ганяти птахів на дослідницькому полі цукрового інституту.

У 1943 році, після звільнення Києва від окупантів, кілька професорів і доцентів, що перебували в місті, намагаються відкрити Київський університет. Оголошено набір, вступні іспити складати не потрібно. Юрій встиг закінчити до війни лише 8 класів, але почав відвідувати лекції на фізико-математичному факультеті. Цей вибір був у певній мірі зумовлений інтересом до фізики, що його, у свою чергу, породив інтерес до радіо, але згодом хлопець вирішив стати математиком. Через багато років, вже будучи відомим на весь світ ученим, Юрій Макарович пояснював, що зробив вибір на користь математики, оскільки завдяки лекціям С.І. Зуховицького усвідомив, що математика — це наука, яка не залежить від політики, непідвладна жорстокостям режиму, а тому, займаючись нею, можна бути вільним, всупереч усьому. До того ж йому подобалось, що математика не вимагає гарної пам'яті: тут не треба багато запам'ятовувати, головне — логічне мислення.

У 1948 році Юрій Березанський закінчив з відзнакою Київський університет. Але шанси вступити до аспірантури у юнака, що не був комсомольцем, та ще й перебував під час окупації на окупованій території, були надто примарними. На допомогу прийшов С.Г. Крейн, який працював у Київському університеті з 1945 року та звернув увагу на талановитого студента. Разом з М.М. Боголюбовим він вмовив тодішнього директора Інституту математики АН УРСР М.О. Лаврентьєва допомогти Юрію Березанському стати аспірантом, та, разом зі своїм братом Марком Григоровичем, став його науковим керівником. З того часу і до кінця життя наукова діяльність Юрія Макаровича була пов'язана із Інститутом математики. Тут він захистив обидві свої дисертації: кандидатську "Гіперкомплексні системи з компактним і дискретним базисом" (1951 р.) та докторську "Деякі питання спектральної теорії рівнянь з частинними різницями і частинними похідними" (1955 р.), тут пройшов всі наукові посади від молодшого до головного наукового співробітника, був обраний членом-кореспондентом (1964) та академіком (1988) Національної академії наук УРСР (нині — НАН України), тут створив відділ математичного (з 1985 року — функціонального) аналізу, завідувачем якого був з 1960 по 2001 рік, тут започаткував нині всесвітньовідому київську школу з функціонального аналізу, яка успішно працює і розвивається до сьогодні.

Юрій Макарович Березанський віддав математиці понад 70 років свого життя. Він отримав низку фундаментальних результатів в області функціонального аналізу, теорії диференціальних рівнянь, математичної фізики. Його результати суттєво вплинули на розвиток таких напрямів математики, як спектральна теорія самоспряжених операторів та їх комутуючих сімей, теорія основних та узагальнених функцій, гармонійний аналіз, граничні задачі для диференціальних та різницевих рівнянь, обернені задачі спектрального аналізу та ін. Зокрема, у його роботах зі спектральної теорії операторів було завершено побудову теорії розкладів за узагальненими власними векторами абстрактних самоспряжених операторів, ним розвинуто теорію просторів з позитивними та негативними нормами, яка має велику кількість застосувань, і без якої неможливо уявити собі сучасний функціональний аналіз, ним разом із колегами та учнями вивчено нескінченновимірну проблему моментів та її узагальнення, пов'язані з квантовою теорією поля. В багатьох розділах функціонального аналізу, математичної фізики, та теорії випадкових процесів використовуються результати теорії основних та узагальнених функцій нескінченної кількості змінних, біля витоків якої стоять Ю.М. Березанський та його учні. Ця теорія дозволила, зокрема, побудувати цікаві та корисні узагальнення гауссівського аналізу білого шуму. Юрій Макарович зробив також суттєвий внесок у розвиток та вдосконалення методів нескінченновимірного аналізу. Цей перелік наукових досягнень ученого далеко не повний. Важко знайти розділ сучасного аналізу, у якому не застосовувались би результати наукової школи Ю.М. Березанського. Його особиста наукова спадщина складається з 7 монографій та близько 300 наукових статей, остання з яких вийде друком вже після його смерті. Аж до останніх днів свого життя він плідно займався улюбленою справою, зустрічався з колегами та учнями, генерував нові ідеї, та мав ще багато наукових планів на майбутнє.

Юрій Макарович був не тільки одним з найсильніших математиків сучасності, а й талановитим вчителем. Багато років він викладав у Київському університеті імені Тараса Шевченка, керував науковою роботою студентів і аспірантів, консультував докторантів та здобувачів наукового ступеня доктора наук. За своє життя підготував 44 кандидатів наук, був науковим консультантом 14 докторів наук. Двоє з його безпосередніх учнів стали членами-кореспондентами Національної академії наук України, одного з них обрано академіком НАНУ. На наукових семінарах, роботою яких керував Юрій Макарович, вважали за честь зробити доповідь не тільки молоді науковці, а й відомі та визнані фахівці.

Ю.М. Березанський був членом редколегій багатьох фахових математичних журналів, що видаються у різних країнах світу. А у 1996 році він створив науковий журнал "Methods of Functional Analysis and Topology" (нині один з провідних математичних журналів України) і очолював його редколегію, беручи активну участь у роботі, аж до кінця свого життя. Зокрема, зміст поточного номера журналу було сформовано під безпосереднім керівництвом Юрія Макаровича за кілька днів до його смерті.

Наукова діяльність Юрія Макаровича Березанського відзначена низкою нагород. Він — лауреат премій ім. М.М. Крилова (1980 р.), М.М. Боголюбова (1997 р.), М.В. Остроградського (2006 р.), М.Г. Крейна (2011 р.) НАН України, лауреатом Державної премії України в галузі науки і техніки (1998 р.), отримав звання заслуженого діяча науки і техніки України (2005 р.). Був членом Київського, Українського, Московського та Американського математичних товариств.

Варто сказати кілька слів про людські якості Ю.М. Березанського. Будучи безкомпромісним у науці, він був простим і доступним у спілкуванні. Студенти та аспіранти могли на рівних обговорювати з ним наукові питання. А якщо комусь із його оточення була потрібна допомога, Юрій Макарович завжди робив усе, щоб допомогти, іноді навіть всупереч власним інтересам.

А ще Юрій Макарович був справжнім патріотом України. Багато разів він отримував запрошення від закордонних наукових центрів, йому пропонували такі умови, про які в Україні годі й мріяти. Але він відхилив всі запрошення, бо хотів працювати на благо своєї Батьківщини, навчати саме українську молодь, підтримувати саме українську науку. Він ніколи не був ані комсомольцем, ані комуністом, бо не сприймав і вважав ворожою для України комуністичну ідеологію. І, незважаючи на можливі негативні наслідки для себе, ніколи не боявся відкрито виступати проти тих можновладців, яких вважав ворогами української держави.

Світла пам'ять про великого Ученого, талановитого Вчителя, добру і порядну Людину назавжди залишиться у серцях його учнів та колег.

Редакційна колегія