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ON A COMPLETE TOPOLOGICAL INVERSE POLYCYCLIC MONOID

We give sufficient conditions when a topological inverse λ-polycyclic monoid P_λ is absolutely H-closed in the class of topological inverse semigroups. For every infinite cardinal λ we construct the coarsest semigroup inverse topology τ_{mi} on P_λ and give an example of a topological inverse monoid S which contains the polycyclic monoid P_2 as a dense discrete subsemigroup.

Key words and phrases: inverse semigroup, bicyclic monoid, polycyclic monoid, free monoid, semigroup of matrix units, topological semigroup, topological inverse semigroup, minimal topology.

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In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [10, 12, 16, 31]. If A is a subset of a topological space X, then we denote the closure of the set A in X by $\operatorname{cl}_X(A)$. By $\mathbb{N}$ we denote the set of all positive integers and by ω the first infinite cardinal.

A semigroup S is called an *inverse semigroup* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

A map that associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *band* is a semigroup of idempotents. If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [35, Definition II.5.12] a chain L is called ω-chain if L is order isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [17] or [12, Section 2.1]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1; \quad a\mathscr{L}b \text{ if and only if } S^1a = S^1b;$$
$$\mathscr{D} = \mathscr{L}\circ\mathscr{R} = \mathscr{R}\circ\mathscr{L}; \quad \mathscr{H} = \mathscr{L} \cap \mathscr{R}.$$

The $\mathscr{R}$-class (resp., $\mathscr{L}$-, $\mathscr{H}$-, or $\mathscr{D}$–class) of the semigroup S which contains an element a of S will be denoted by R_a (resp., L_a, H_a, or D_a).

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The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [12]. Also the well known Andersen Theorem states that *a simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup* (see [2] and [12, Theorem 2.54]).

Let λ be a non-zero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation " $\cdot$ " as follows

$$(a,b) \cdot (c,d) = \begin{cases} (a,d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a,b) \cdot 0 = 0 \cdot (a,b) = 0 \cdot 0 = 0$ for $a,b,c,d \in \lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units* (see [12]).

In 1970 Nivat and Perrot proposed the following generalization of the bicyclic monoid (see [34] and [31, Section 9.3]). For a non-zero cardinal λ, the polycyclic monoid on λ generators P_λ is the semigroup with zero given by

$$P_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle.$$

If $\lambda = 1$ the semigroup P_1 is isomorphic to the bicyclic semigroup with adjoined zero. For every finite non-zero cardinal $\lambda = n$ the polycyclic monoid P_n is congruence free, combinatorial, 0-bisimple, 0-E-unitary inverse semigroup (see [31, Section 9.3]).

A *topological* (*inverse*) *semigroup* is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ an (*inverse*) *semigroup topology* on S. A *semitopological semigroup* is a Hausdorff topological space endowed with a separately continuous semigroup operation.

Let $\mathfrak{STSG}_0$ be a class of topological semigroups. A semigroup $S \in \mathfrak{STSG}_0$ is called *H-closed in* $\mathfrak{STSG}_0$, if S is a closed subsemigroup of any topological semigroup $T \in \mathfrak{STSG}_0$ which contains S both as a subsemigroup and as a topological space. The H-closed topological semigroups were introduced by Stepp in [39], and there they were called *maximal semigroups*. A topological semigroup $S \in \mathfrak{STSG}_0$ is called *absolutely H-closed in the class* $\mathfrak{STSG}_0$, if any continuous homomorphic image of S into $T \in \mathfrak{STSG}_0$ is H-closed in $\mathfrak{STSG}_0$. Absolutely H-closed topological semigroups were introduced by Stepp in [40], and there they were called *absolutely maximal*.

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [36] Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i.e., G is complete with respect to the two-sided uniformity. A topological group G is called *h-complete* if for every continuous homomorphism $h\colon G \to H$ the subgroup $f(G)$ of H is closed [13]. In our terminology such topological groups are called absolutely H-closed in the class of topological groups. The h-completeness is preserved under taking products and closed central subgroups [13]. H-closed paratopological and topological groups in the class of paratopological groups were studied in [37]. The paper [7] contains a sufficient condition for a quasitopological group to be H-closed, which allowed us to solve a problem by Arhangel'skii and Choban [3] and show that a topological group G is H-closed in the class of quasitopological groups if and only if G is Raikov-complete. In [18] it is proved that a topological group G is H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact.

In [40] Stepp studied H-closed topological semilattices in the class of topological semigroups. He proved that an algebraic semilattice E is algebraically h-complete in the class of topological semilattices if and only if every chain in E is finite. In [27] Gutik and Repovš studied the closure of a linearly ordered topological semilattice in a topological semilattice. They obtained a characterization of H-closed linearly ordered topological semilattices in the class of topological semilattices and showed that every H-closed linear topological semilattice is absolutely H-closed in the class of topological semilattices. Such semilattices were studied also in [11,20]. In [5] the closures of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ were described. In that paper the authors constructed an example of an H-closed topological semilattice in the class of topological semilattices, which is not absolutely H-closed in the class of topological semilattices. The constructed example gives a negative answer to Question 17 from [40]. H-closed and absolutely H-closed (semi)topological semigroups and their extensions in different classes of topological and semitopological semigroups were studied in [8, 18, 19, 21–26]

In [6] we showed that the λ-polycyclic monoid for an infinite cardinal $\lambda \geqslant 2$ has similar algebraic properties to that of the polycyclic monoid P_n with finitely many $n \geqslant 2$ generators. In particular we proved that for every infinite cardinal λ the polycyclic monoid P_λ is congruence-free, combinatorial, 0-bisimple, 0-E-unitary, inverse semigroup. Also we showed that every non-zero element $x \in P_\lambda$ is an isolated point in (P_λ, τ) for every Hausdorff topology on P_λ, such that P_λ is a semitopological semigroup; moreover, every locally compact Hausdorff semigroup topology on P_λ is discrete. The last statement extends results of the paper [32] treating topological inverse graph semigroups. We described all feebly compact topologies τ on P_λ such that (P_λ, τ) is a semitopological semigroup. Also in [6] we proved that for every cardinal $\lambda \geqslant 2$ any continuous homomorphism from a topological semigroup P_λ into an arbitrary countably compact topological semigroup is annihilating and there exists no Hausdorff feebly compact topological semigroup containing P_λ as a dense subsemigroup.

This paper is a continuation of [6]. In this paper we give sufficient conditions on a topological inverse λ-polycyclic monoid P_λ to be absolutely H-closed in the class of topological inverse semigroups. For every infinite cardinal λ we construct the coarsest semigroup inverse topology τ_{mi} on P_λ and give an example of a topological inverse monoid S which contains the polycyclic monoid P_2 as a dense discrete subsemigroup.

It is well known that for an arbitrary topological inverse semigroup S and every element $x \in S$ the continuity of the semigroup operation and the inversion in S implies that any $\mathscr{L}$-class L_x and any $\mathscr{R}$-class R_x which contain the element x are closed subsets in S. Indeed, the Wagner–Preston Theorem (see Theorem 1.17 from [12]) implies that $L_x = L_{x^{-1}x}$ and $R_x = R_{xx^{-1}}$ for arbitrary $x \in S$ and since the maps $\varphi\colon S \to E(S)$ and $\psi\colon S \to E(S)$ defined by the formulae

$$(x)\varphi = xx^{-1} \qquad \text{and} \qquad (x)\psi = x^{-1}x$$

are continuous, we get that $L_x = (x^{-1}x)\psi^{-1}$ and $R_x = (xx^{-1})\varphi^{-1}$ are closed subsets of the topological semigroup S. This implies that for any idempotents e and f of a topological inverse semigroup S the following $\mathscr{H}$-classes of S:

$$H_e = R_e \cap L_e \qquad \text{and} \qquad H_{e,f} = R_e \cap L_f$$

are closed subsets of the topological inverse semigroup S too. Moreover, the relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ are closed subsets in $S \times S$, but $\mathscr{D}$ and $\mathscr{J}$ are not necessary closed subsets in $S \times S$ for an arbitrary topological inverse semigroup S (see [15, Section II]).

The following proposition describes $\mathscr{D}$-equivalent $\mathscr{H}$-classes in an arbitrary topological inverse semigroup.

Proposition 1. *Let S be a Hausdorff topological inverse semigroup and a, c be $\mathscr{D}$-equivalent elements of S. Then there exists $b \in S$ such that $a\mathscr{R}b$ and $b\mathscr{L}c$ in S, and hence $as = b$, $bs' = a$, $tb = c$, $t'c = b$, for some $s, s', t, t' \in S$. The mappings $\mathfrak{f}_{a,c} \colon H_a \to H_c \colon x \mapsto txs$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a \colon x \mapsto t'xs'$ are continuous and mutually inverse, and hence are homeomorphisms of closed subspaces H_a and H_c of the topological space S. Moreover, if H_a and H_c are subgroups of S then H_a and H_c are topologically isomorphic closed topological subgroups in the topological inverse semigroup S.*

Proof. The above arguments imply that H_a and H_c are closed subspaces of S. Also, the algebraic part of the statement of our theorem follows from Theorem 2.3 of [12] and Theorem 1.2.7 from [28]. The continuity of the semigroup operation in S implies that the maps $\mathfrak{f}_{a,c} \colon H_a \to H_c$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a$ are continuous and hence are homeomorphisms. Now, the proof of Theorem 1.2.7 from [28] implies that in the case when H_a and H_c are subgroups of S, then there exist $u, u' \in S$ such that the maps $\mathfrak{f}_{a,c} \colon H_a \to H_c \colon x \mapsto uxu'$ and $\mathfrak{f}_{c,a} \colon H_c \to H_a \colon x \mapsto u'xu$ are mutually inverse isomorphisms and the continuity of the semigroup operation in S implies that so defined maps are topological isomorphisms. □

Remark 1. *The proof of Proposition 1 implies that any two $\mathscr{D}$-equivalent $\mathscr{H}$-classes of a Hausdorff semitopological semigroup S are homeomorphic subspaces in S, but they are not necessary closed subspaces in S, and a similar statement holds for maximal subgroups in S (see [18]).*

Lemma 1. *Let T and S be a Hausdorff topological inverse semigroup such that S is an inverse subsemigroup of T. Let G be an $\mathscr{H}$-class in S which is a closed subset of the topological inverse semigroup T and D_G be a $\mathscr{D}$-class of the semigroup S which contains the set G. Then every $\mathscr{H}$-class $H \subseteq D_G$ of the semigroup S is a closed subset of the topological space T.*

Proof. First we consider the case when G has an idempotent, i.e., G is a maximal subgroup of the semigroup S (see Theorem 2.16 of [12]).

In the case when the $\mathscr{H}$-class H contains an idempotent, Theorem 2.16 in [12] implies that H is a maximal subgroup of S and hence H is a subgroup of topological inverse semigroup T. We put e and f are unit elements of the groups G and H, respectively. Since the idempotents e and f are $\mathscr{D}$-equivalent in S, Proposition 3.2.5 of [31] implies that there exists $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$. Now by Proposition 3.2.11(5) of [31] the idempotents e and f are $\mathscr{D}$-equivalent in the semigroup T. Put H_e^T and H_f^T be the $\mathscr{H}$-classes of idempotents e and f in the semigroup T, respectively. We define the maps $\mathfrak{f}_{e,f} \colon T \to T$ and $\mathfrak{f}_{f,e} \colon T \to T$ by the formulae

$(x)\mathfrak{f}_{e,f} = a^{-1}xa$ and $(x)\mathfrak{f}_{f,e} = axa^{-1}$, respectively. Then for any $s \in H_e^T$ and $t \in H_f^T$ we get the equalities

$$\begin{aligned}
(s)\mathfrak{f}_{e,f}\big((s)\mathfrak{f}_{e,f}\big)^{-1} &= a^{-1}sa(a^{-1}sa)^{-1} = a^{-1}saa^{-1}s^{-1}a = a^{-1}ses^{-1}a = a^{-1}ss^{-1}a = a^{-1}ea\\
&= a^{-1}a = f,\\
\big((s)\mathfrak{f}_{e,f}\big)^{-1}(s)\mathfrak{f}_{e,f} &= (a^{-1}sa)^{-1}a^{-1}sa = a^{-1}s^{-1}aa^{-1}sa = a^{-1}s^{-1}esa = a^{-1}s^{-1}sa = a^{-1}ea\\
&= a^{-1}a = f,\\
(t)\mathfrak{f}_{f,e}\big((t)\mathfrak{f}_{f,e}\big)^{-1} &= ata^{-1}(ata^{-1})^{-1} = ata^{-1}at^{-1}a^{-1} = atft^{-1}a^{-1} = att^{-1}a^{-1} = afa^{-1}\\
&= aa^{-1} = e,\\
\big((t)\mathfrak{f}_{f,e}\big)^{-1}(t)\mathfrak{f}_{f,e} &= (ata^{-1})^{-1}ata^{-1} = at^{-1}a^{-1}ata^{-1} = at^{-1}fta^{-1} = at^{-1}ta^{-1} = afa^{-1}\\
&= aa^{-1} = e,\\
\big((s)\mathfrak{f}_{e,f}\big)\mathfrak{f}_{f,e} &= aa^{-1}saa^{-1} = ese = s,\\
\big((t)\mathfrak{f}_{f,e}\big)\mathfrak{f}_{e,f} &= a^{-1}ata^{-1}a = ftf = t,
\end{aligned}$$

because $aa^{-1} = ss^{-1} = s^{-1}s = e$, $ea = a$, $af = a$ and $a^{-1}a = tt^{-1} = t^{-1} = f$. Similarly, for arbitrary $s, v \in H_e^T$ and $t, u \in H_f^T$ we have that

$$(s)\mathfrak{f}_{e,f}(v)\mathfrak{f}_{e,f} = a^{-1}saa^{-1}va = a^{-1}seva = a^{-1}sva = (sv)\mathfrak{f}_{e,f}$$

and

$$(t)\mathfrak{f}_{f,e}(u)\mathfrak{f}_{f,e} = ata^{-1}aua^{-1} = atfua^{-1} = atua^{-1} = (tu)\mathfrak{f}_{f,e}.$$

Hence the restrictions $\mathfrak{f}_{e,f}|_{H_e^T} \colon H_e^T \to H_f^T$ and $\mathfrak{f}_{f,e}|_{H_f^T} \colon H_f^T \to H_e^T$ are mutually invertible group isomorphisms. Also, since $a \in S$ we get that the restrictions $\mathfrak{f}_{e,f}|_G \colon G \to H$ and $\mathfrak{f}_{f,e}|_H \colon H \to G$ are mutually invertible group isomorphisms too. This and the continuity of left and right translations in T imply that H is a closed subgroup of the topological inverse semigroup T.

Next we consider the case when the $\mathscr{H}$-class H contains no idempotents. Then there exists distinct idempotents $e, f \in S$ such that $ss^{-1} = e$ and $s^{-1}s = f$ for all $s \in H$. Suppose to the contrary that H is not a closed subset of the topological inverse semigroup T. Then there exists an accumulation point $x \in T \setminus H$ of the set H in the topological space T. Since every $\mathscr{H}$-class of a topological inverse semigroup T is a closed subset of T we get that H and x are contained in a same $\mathscr{H}$-class H_x of the semigroup T. Then $xx^{-1} = e$ and $x^{-1}x = f$. Now the $\mathscr{H}$-class H_e^T in T which contains the idempotent $e \in S$ is a topological subgroup of the topological inverse semigroup T and by Proposition 1 the subspace H_e^T of the topological space T is homeomorphic to the subspace H_x of T. Moreover, Theorem 1.2.7 from [28] implies that there exists a homeomorphism $\mathfrak{f} \colon H_x \ rightarrow H_e^T$ such that the image $(H)\mathfrak{f}$ is a topological subgroup of the topological inverse semigroup T and $(H)\mathfrak{f}$ is topologically isomorphic to the topological group G. Then $(H)\mathfrak{f}$ is not a closed subgroup of T which contradicts our above part of the proof.

Assume that G has no idempotents. By the previous part of the proof it suffices to show that there exists a maximal subgroup H_e with an idempotent e in the $\mathscr{D}$-class D_G such that H_e is a closed subgroup of topological semigroup T. Suppose to the contrary that every maximal subgroup in the $\mathscr{D}$-class D_G is not a closed in T. Fix and arbitrary subgroup H_e with an idempotent e in the $\mathscr{D}$-class D_G such that $xx^{-1} = e$ for all $x \in G$. Then Proposition 3.2.11(3) of [31] implies that there exist $\mathscr{H}$-classes H^T_G and H^T_e in the semigroup T which contain the set G and group H_e. Since in the topological semigroup T every $\mathscr{H}$-class is a closed subset in T, we have that G is a closed subset of the space H^T_G and H_e is not a closed subgroup of the topological group H^T_e. Then Proposition 3.2.11 of [31] and Proposition 1 imply that there exist $s, s', t, t^{\ prime} \in S$ such that the maps $\mathfrak{f}_e\colon H^T_e \to H^T_G\colon x \mapsto txs$ and $\mathfrak{f}_G\colon H^T_G \to H^T_e\colon x \mapsto t'xs'$ are mutually invertible homeomorphisms of the topological spaces H^T_e and H^T_G such that the restrictions $\mathfrak{f}_e|_{H_e}\colon H^T_e \to G$ and $\mathfrak{f}_G|_G\colon G \to H_e$ are mutually invertible homeomorphisms. This is a contradiction, because H_e is not a closed subset of H^T_e. This completes proof of the lemma. □

Lemma 1 implies the following corollary.

Corollary 1. *Let T and S be a Hausdorff topological inverse semigroup such that S is an inverse subsemigroup of T. Let G be a maximal subgroup in S which is H-closed in the class of topological inverse semigroups and D_G be a $\mathscr{D}$-class of the semigroup S which contains the group G. Then every $\mathscr{H}$-class $H \subseteq D_G$ of the semigroup S is a closed subset of the topological space T.*

Lemma 2. *Let S be a Hausdorff topological inverse semigroup such following conditions hold:*

(i) *every maximal subgroup of the semigroup S is H-closed in the class topological groups;*

(ii) *all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.*

If there exists a topological inverse semigroup T such that S is a dense subsemigroup of T and $T \setminus S \neq \varnothing$ then for every $x \in T \setminus S$ at least one of the points $x \cdot x^{-1}$ or $x^{-1} \cdot x$ belongs to $T \setminus S$.

Proof. First we consider the case when the topological semilattice $E(S)$ does not have the smallest element. Then the space $E(S)$ is discrete and Theorem 3.3.9 of [16] implies that $E(S)$ is an open subset of the topological space $E(T)$ and hence every point of the set $E(S)$ is isolated in $E(T)$. Also by Proposition II.3 [15] we have that $\mathrm{cl}_T(E(S)) = \mathrm{cl}_{E(T)}(E(S))$ and hence the points of the set $E(T) \setminus E(S)$ are not isolated in the space $E(T)$.

Fix an arbitrary point $x \in T \setminus S$. By Corollary 1 every $\mathscr{H}$-class is a closed subset of the topological inverse semigroup T. Since x is an accumulation point of the set S in the topological space T we have that every open neighbourhood $U(x)$ of the point x in T intersects infinitely many $\mathscr{H}$-classes of the semigroup S. By Proposition II.1 of [15] the inversion on T is a homeomorphism of the topological space T and hence $(U(x))^{-1}$ is an open neighbourhood of the point x^{-1} in T which intersects infinitely many $\mathscr{H}$-classes of the semigroup S. Then the continuity of the semigroup operations and the inversion in T implies that at least one of the sets $\Big(U(x)\,(U(x))^{-1}\Big) \cap E(T)$ or $\Big((U(x))^{-1}\,U(x)\Big) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point x in the topological semigroup T. This implies that at least one of $x\ cdot x^{-1}$ or $x^{-1} \cdot x$ is a non-isolated point in the topological space $E(T)$.

In the case when the semilattice $E(S)$ has a minimal idempotent the presented above arguments imply that for arbitrary point $x \in T \setminus S$ and every open neighbourhood $U(x)$ of the point x in T one of the sets $\Big(U(x)\,(U(x))^{-1}\Big) \cap E(T)$ or $\Big((U(x))^{-1}\,U(x)\Big) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point x in the topological semigroup T. Since H_e is a minimal ideal of S and it is a Raĭkov complete topological group. Then there exists an open neighborhood $U(x)$ of x in T, such that $U(x) \cap H_e = \varnothing$. If $xx^{-1} = e$ or $x^{-1}x = e$ then $x = xx^{-1}x \in H_e$, which contradicts that $x \in T \setminus S$. Hence $xx^{-1} \in T \setminus S$ or $x^{-1}x \in T \setminus S$. □

Lemma 2 implies the following two corollaries.

Corollary 2. *Let S be a Hausdorff topological inverse semigroup satisfying the following conditions:*

(i) *every maximal subgroup of the semigroup S and the semilattice $E(S)$ are H-closed in the class of topological inverse semigroups;*

(ii) *all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.*

Then S is H-closed in the class of topological inverse semigroups.

Corollary 3. *Let $\lambda \geqslant 2$ and let P_λ be a proper dense subsemigroup of a topological inverse semigroup S. Then either $xx^{-1} \in S \setminus P_\lambda$ or $x^{-1}x \in S \setminus P_\lambda$ for every $x \in S \setminus P_\lambda$.*

The following theorem gives sufficient condition when a topological inverse λ-polycyclic monoid P_λ is absolutely H-closed in the class of topological inverse semigroups.

Theorem 1. *Let λ be a cardinal $\geqslant 2$ and τ be a Hausdorff inverse semigroup topology on P_λ such that $U(0) \cap L$ is an infinite set for every open neighborhood $U(0)$ of zero 0 in (P_λ, τ) and every maximal chain L of the semilattice $E(P_\lambda)$. Then (P_λ, τ) is absolutely H-closed in the class of topological inverse semigroups.*

Proof. First we observe that the definition of the λ-polycyclic monoid P_λ implies that for every maximal chain L in $E(P_\lambda)$ the set $L \setminus \{0\}$ is an ω-chain. Then Theorem 2 of [5] implies that every maximal chain L in $E(P_\lambda)$ with the induced topology from (P_λ, τ) is an absolutely H-closed topological semilattice. Suppose that $E(P_\lambda)$ with the induced topology from (P_λ, τ) is not an H-closed topological semilattice. Then there exists a topological semilattice S which contains $E(P_\lambda)$ as a dense proper subsemilattice. Also the continuity of the semilattice operation in S implies that zero 0 of $E(P_\lambda)$ is zero in S. Fix an arbitrary element $x \in S \setminus E(P_\lambda)$. Then for an arbitrary open neighbourhood $U(x)$ of the point x in S such that $0 \notin U(x)$ the continuity of the semilattice operation in S implies that there exists an open neighbourhood $V(x)\ subseteq U(x)$ of x in S such that $V(x) \cdot V(x) \subseteq U(x)$. Now, the neighbourhood $V(x)$ intersects infinitely many maximal chains of the semilattice $E(P_\lambda)$, because all maximal chains in $E(P_\lambda)$ with the induced topology from (P_λ, τ) are absolutely H-closed topological semilattices. Then the semigroup operation of P_λ implies that $0 \in V(x) \cdot V(x) \subseteq U(x)$, which contradicts the choice of the neighbourhood $U(0)$. Therefore, $E(P_\lambda)$ with the induced topology from (P_λ, τ) is an H-closed topological semilattice.

Now, by Corollary 2 the topological inverse semigroup (P_λ, τ) is H-closed in the class of topological inverse semigroups. Since the λ-polycyclic monoid P_λ is congruence free, every continuous homomorphic image of (P_λ, τ) is H-closed in the class of topological inverse semigroups. Indeed, if $h\colon (P_\lambda, \tau) \to T$ is a continuous (algebraic) homomorphism from (P_λ, τ) into a topological inverse semigroup T, then the set $U(h(0)) \cap h(L)$ is infinite for every open neighbourhood $U(h(0))$ of the image zero $h(0)$ in T. Then the previous part of the proof implies that $h(P_\lambda)$ is a closed subsemigroup of T. □

Remark 2. *By Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) for every positive integer $n \geqslant 2$ any non-zero element x of the polycyclic monoid*

P_n has the form $u^{-1}v$, where u and v are elements of the free monoid $\mathscr{M}_n$, and the semigroup operation on P_n in this representation is defined in the following way:

$$a^{-1}b\cdot c^{-1}d=\begin{cases}(c_1a)^{-1}d, & \text{if } c=c_1b \quad \text{for some } c_1\in\mathscr{M}_n;\\ a^{-1}b_1d, & \text{if } b=b_1c \quad \text{for some } b_1\in\mathscr{M}_n;\\ 0, & \text{otherwise}\end{cases}$$
$$\text{and}\quad a^{-1}b\cdot 0=0\cdot a^{-1}b=0\cdot 0=0. \tag{1}$$

Then Lemma 2.4 of [6] implies that every any non-zero element x of the polycyclic monoid P_λ has the form $u^{-1}v$, where u and v are elements of the free monoid $\mathscr{M}_\lambda$, and the semigroup operation on P_λ in this representation is defined by formula (1).

Now we shall construct a topology τ_{mi} on the λ-polycyclic monoid P_λ such that $(P_\lambda,\tau_{\mathsf{mi}})$ is absolutely H-closed in the class of topological inverse semigroups.

Example 1. *We define a topology τ_{mi} on the polycyclic monoid P_λ in the following way. All non-zero elements of P_λ are isolated point in (P_λ,τ_{mi}). For an arbitrary finite subset A of $\mathscr{M}_\lambda$ put*

$$U_A(0)=\left\{a^{-1}b : a,b\in M_\lambda\setminus A\right\}.$$

We put $\mathscr{B}_{mi}=\{U_A(0)\colon A \text{ is a finite subset of } \mathscr{M}_\lambda\}$ to be a base of the topology τ_{mi} at zero $0\in P_\lambda$.

We observe that τ_{mi} is a Hausdorff topology on P_λ because $U_{\{a,b\}}(0)\not\ni a^{-1}b$ for every non-zero element $a^{-1}b\in P_\lambda$. Also, since $(U_A(0))^{-1}=U_A(0)$ for any $U_A(0)\in\mathscr{B}_{mi}$, the inversion is continuous in (P_λ,τ_{mi}). Fix an arbitrary $a^{-1}b\in P_\lambda$ and any basic neighbourhood $U_A(0)$ of zero in (P_λ,τ_{mi}). Let S_b be a set of all suffixes of the word b. Put $B=P_b\cup\{kb\in\mathscr{M}_\lambda\colon ka\in A\}$. It is obvious that the set B is finite and hence formula (1) implies that $a^{-1}b\cdot U_B(0)\subseteq U_A(0)$. Let S_a be a set of all suffixes of the word a. Put $D=S_a\cup\{ta\in\mathscr{M}_\lambda\colon tb\in A\}$. It is obvious that the set D is finite and hence formula (1) implies that $U_D(0)\cdot a^{-1}b\subseteq U_A(0)$. Also $U_T(0)\cdot U_T(0)\subseteq U_A(0)$ for $T=A\cup\{b\in\mathscr{M}_\lambda\colon\ b \text{ is a suffix of some } a\in A\}$. Therefore (P_λ,τ_{mi}) is a topological inverse semigroup.

Theorem 1 and Example 1 implies the following corollary.

Corollary 4. *The topological inverse semigroup $(P_\lambda,\tau_{\mathsf{mi}})$ is absolutely H-closed in the class of topological inverse semigroups.*

Definition 1 ([23])**.** *A Hausdorff topological (inverse) semigroup (S,τ) is said to be* minimal *if no Hausdorff semigroup (inverse) topology on S is strictly contained in τ. If (S,τ) is minimal topological (inverse) semigroup, then τ is called a* minimal (inverse) semigroup topology.

Minimal topological groups were introduced independently in the early 1970's by Doïtchinov [14] and Stephenson [38]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [9]). More than 20 years earlier L. Nachbin [33] had studied minimality in the context of division rings, and B. Banaschewski [4] investigated minimality in the more general setting of topological algebras. In [23] on the infinite semigroup of $\lambda\times\lambda$-matrix units B_λ the minimal semigroup and the minimal semigroup inverse topologies were constructed.

Theorem 2. *For any infinite cardinal λ, τ_{mi} is the coarsest inverse semigroup topology on P_λ, and hence $(P_\lambda, \tau_{\mathsf{mi}})$ is a minimal topological inverse semigroup.*

Proof. The definition of the topology τ_{mi} on P_λ implies that the subsemigroup of idempotents $E(P_\lambda)$ of the semigroup P_λ is a compact subset of the space $(P_\lambda, \tau_{\mathsf{mi}})$. By Proposition 3.1 of [6] every non zero-element of a semitopological monoid (P_λ, τ) is an isolated point in the space (P_λ, τ). This and above arguments imply that the topology τ_{mi} on P_λ induces the coarsest semigroup topology on the semilattice $E(P_\lambda)$. Also by Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) we have that every non-zero element of the semilattice $E(P_\lambda)$ can be represented in the form $a^{-1}a$ where a are elements of the free monoid $\mathscr{M}_n$, and the semigroup operation on $E(P_\lambda)$ in this representation is defined by formula (1).

Also, we observe that for any topological inverse semigroup S the following maps $\varphi\colon S \to E(S)$ and $\psi\colon S \to E(S)$ defines by the formulae $\varphi(x) = xx^{-1}$ and $\psi(x) = x^{-1}x$, respectively, are continuous. Since the inverse element of $u^{-1}v$ in P_λ is equal to $v^{-1}u$, we have that $U_A = P_\lambda \setminus \left(\varphi^{-1}(A) \cup \psi^{-1}(A)\right)$, for any finite subset A of the free monoid $\mathscr{M}_n$. This implies that $U_A(A) \in \tau$ for every inverse semigroup topology τ on P_λ, where A is an arbitrary finite subset of $\mathscr{M}_n$. Thus, τ_{mi} is the coarsest inverse semigroup topology on the λ-polycyclic monoid P_λ. $\square$

In the next example we construct a topological inverse monoid S which contains the polycyclic monoid $P_2 = \left\langle p_1, p_2 \mid p_1p_1^{-1} = p_2p_2^{-1} = 1, p_1p_2^{-1} = p_2p_1^{-1} = 0 \right\rangle$ as a dense discrete subsemigroup, i.e., the polycyclic monoid P_2 with the discrete topology is not H-closed in the class of topological inverse semigroups. Also, later we assume that the free monoid $\mathscr{M}_2$ is generated by two element p_1 and p_2.

Example 2. *Let $\mathscr{F}$ be the filter on the bicyclic semigroup $\mathscr{C}(p_1, p_1^{-1}) = \left\langle p_1, p_1^{-1} \mid p_1p_1^{-1} = 1 \right\rangle$, generated by the base $\mathscr{B} = \{U_n \colon n \in \mathbb{N}\}$, where $U_n = \left\{p_1^{-k}p_1^m \colon k, m > n\right\}$. We denote*

$$A = \left\{a^{-1}b \in P_2 \colon a \neq p_1a_1 \text{ and } b \neq p_1b_1 \text{ for any } a_1, b_1 \in \mathscr{M}_2\right\}.$$

For any element $a^{-1}b$ of the set A let $\mathscr{F}_{a^{-1}b}$ be the filter on P_2, generated by the base $\mathscr{B}_{a^{-1}b} = \{V_n \colon n \in \mathbb{N}\}$, where $V_n = a^{-1}U_nb = \left\{(p_1^ka)^{-1}p_1^mb \colon k, m > n\right\}$. It is obvious that $\mathscr{F} = \mathscr{F}_{1^{-1}1}$, where 1 is the unit element of the free monoid $\mathscr{M}_2$.

We extend the binary operation from P_2 onto $S = P_2 \cup \left\{\mathscr{F}_{a^{-1}b} \colon a^{-1}b \in A\right\}$ by the following formulae:

$$\text{(I)}\quad a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \begin{cases} \mathscr{F}_{(ea)^{-1}d}, & \text{if } c = eb; \\ \mathscr{F}_{(e)^{-1}d}, & \text{if } b = p_1^nc \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix} \\ & \quad \text{of } a \text{ such that } e \neq p_1f \text{ for some } f \in M_2; \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{(II)}\quad \mathscr{F}_{c^{-1}d} \cdot a^{-1}b = \begin{cases} \mathscr{F}_{c^{-1}eb}, & \text{if } d = ea; \\ \mathscr{F}_{c^{-1}e}, & \text{if } a = p_1^nd \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix} \\ & \quad \text{of } b \text{ such that } e \neq p_1f \text{ for some } f \in M_2; \\ 0, & \text{otherwise;} \end{cases}$$

$$\text{(III)}\quad \mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = \begin{cases} \mathscr{F}_{a^{-1}d}, & \text{if } b = c; \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that the subset $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from S is isomorphic to the semigroup of $\omega \times \omega$-matrix units B_ω and moreover we have that $(\mathscr{F}_{a^{-1}b})^{-1} = \mathscr{F}_{b^{-1}a}$ in T.

We determine a topology τ on the set S in the following way: assume that the elements of the semigroup P_2 are isolated points in (S,τ) and the family

$$\mathscr{B}(\mathscr{F}_{a^{-1}b}) = \{U_n(\mathscr{F}_{a^{-1}b}) : U_n \in \mathscr{B}_{a^{-1}b}\}$$

of the set $U_n(\mathscr{F}_{a^{-1}b}) = U_n \cup \{\mathscr{F}_{a^{-1}b}\}$ is a neighborhood base of the topology τ at the point $\mathscr{F}_{a^{-1}b} \in S$.

Now we show that so defined binary operation on (S,τ) is continuous.

In case (I) we consider three cases.

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = 0$ then we have that $a^{-1}b \cdot U_n(\mathscr{F}_{c^{-1}d}) = \{0\}$ for any positive integer n.

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{(ea)^{-1}d}$ then $c = eb$. We claim that $a^{-1}b \cdot U_n(\mathscr{F}_{c^{-1}d}) \subseteq U_n(\mathscr{F}_{(ea)^{-1}d})$ for any open basic neighbourhood $U_n(\mathscr{F}_{(ea)^{-1}d})$ of the point $\mathscr{F}_{(ea)^{-1}d}$ in (S,τ). Indeed, if $x \in U_n(\mathscr{F}_{c^{-1}d})$ then $x = (p_1^m c)^{-1} p_1^k d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^m c)^{-1} p_1^k d = a^{-1}b \cdot (p_1^m eb)^{-1} p_1^k d = (p_1^m ea)^{-1} p_1^k d \in U_n(\mathscr{F}_{(ea)^{-1}d}).$$

If $a^{-1}b \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{e^{-1}d}$, then e is the longest suffix of the word a in $\mathscr{M}_2$ which is not equal to the word $p_1 f$ for some $f \in \mathscr{M}_2$. This holds when $b = p_1^t c$ for some positive integer t. We claim that $a^{-1}b \cdot U_{n+t}(\mathscr{F}_{c^{-1}d}) \subseteq U_n(\mathscr{F}_{e^{-1}d})$ for any open basic neighbourhood $U_n(\mathscr{F}_{e^{-1}d})$ of the point $\mathscr{F}_{e^{-1}d}$ in (S,τ). Indeed, if $x \in U_{n+t}(\mathscr{F}_{c^{-1}d})$, then $x = (p_1^{m+t} c)^{-1} p_1^{k+t} d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = e^{-1} p_1^{-l} p_1^t c \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = (p_1^{m+l} e)^{-1} p_1^{k+t} d \in U_n(\mathscr{F}_{e^{-1}d}).$$

In case (II) the proof of the continuity of binary operation in (S,τ) is similar to case (I).

Now we consider case (III).

If $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = 0$ then $U_n(\mathscr{F}_{a^{-1}b}) \cdot U_n(\mathscr{F}_{c^{-1}d}) \subseteq \{0\}$, for any open basic neighbourhoods $U_n(\mathscr{F}_{a^{-1}b})$ and $U_n(\mathscr{F}_{c^{-1}d})$ of the points $\mathscr{F}_{a^{-1}b}$ and $\mathscr{F}_{c^{-1}d}$ in (S,τ), respectively.

If $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{c^{-1}d} = \mathscr{F}_{a^{-1}d}$ then $b = c$ and for every any open basic neighbourhood $U_n(\mathscr{F}_{a^{-1}d})$ of the point $\mathscr{F}_{a^{-1}d}$ in (S,τ) we have that $U_n(\mathscr{F}_{a^{-1}b}) \cdot U_n(\mathscr{F}_{b^{-1}d}) \subseteq U_n(\mathscr{F}_{a^{-1}d})$. Indeed if $(p_1^k a)^{-1} p_1^t b \in U_n(\mathscr{F}_{a^{-1}b})$ and $(p_1^l b)^{-1} p_1^m d \in U_n(\mathscr{F}_{b^{-1}d})$ then

$$(p_1^k a)^{-1} p_1^t b \cdot (p_1^l b)^{-1} p_1^m d = (p_1^k a)^{-1} p_1^t (b \cdot b^{-1}) p_1^{-l} p_1^m d = (p_1^s a)^{-1} p_1^z d,$$

for some positive integers $s, z > n$, and hence $(p_1^s a)^{-1} p_1^z d \in U_n(\mathscr{F}_{a^{-1}d})$.

Thus, we proved that the binary operation on (S,τ) is continuous. Taking into account that P_2 is a dense subsemigroup of (S,τ) we conclude that (S,τ) is a topological semigroup. Also, since $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from S is isomorphic to the semigroup of $\omega \times \omega$-matrix units B_ω we have that idempotents in S commute and moreover $\mathscr{F}_{a^{-1}b} \cdot \mathscr{F}_{b^{-1}a} \cdot \mathscr{F}_{a^{-1}b} = \mathscr{F}_{b^{-1}a}$. This implies that S is an inverse semigroup. Also, for every open basic neighbourhood $U_n(\mathscr{F}_{a^{-1}b})$ of the point $\mathscr{F}_{a^{-1}b}$ in (S,τ) we have that $(U_n(\mathscr{F}_{a^{-1}b}))^{-1} = U_n(\mathscr{F}_{b^{-1}a})$ for all $n \in \mathbb{N}$ and hence the inversion in (S,τ) is continuous.

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Вказано достатні умови, за яких топологічний інверсний λ-поліциклічний моноїд P_λ є абсолютно H-замкненим в класі топологічних інверсних напівгруп. Для довільного нескінченного кардиналу λ побудовано найслабшу напівгрупову інверсну топологію τ_{mi} на P_λ та наведено приклад топологічного інверсного моноїда S, що містить поліциклічний моноїд P_2 як щільну дискретну піднапівгрупу.

Ключові слова і фрази: інверсна напівгрупа, біциклічний моноїд, поліциклічний, вільний моноїд, напівгрупа матричних одиниць, топологічна напівгрупа, топологічна інверсна напівгрупа, мінімальна топологія.

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COUPLED FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS ALONG WITH CLR PROPERTY IN MENGER METRIC SPACES

Coupled fixed point problems have attracted much attention in recent times. The aim of this paper is to extend the notions of E.A. property, CLR property and $JCLR$ property for coupled mappings in Menger metric space and use this notions to establish common coupled fixed point results for four self mappings. Our work generalizes the recent results of Jian-Zhong Xiao [24] et al (2011). The main result is supported by a suitable example.

Key words and phrases: Menger metric space, t-norm of H-type, weak compatibility coupled common fixed point, CLR property, E.A. property, $JCLR$ property.

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1 INTRODUCTION

The concept of a probabilistic metric space was introduced and studied by Menger [3, 19]. Since then, many authors have studied the fixed point property for mappings defined on probabilistic metric spaces (see [2,4,7,12,24]). Jachymski [15] has proved some fixed point theorems for probabilistic nonlinear contractions with a gauge function φ and discussed the relations between several assumptions concerning φ.

Bhaskar and Lakshmikantham [24] introduced the notion of coupled fixed points and proved some coupled fixed point results in partially ordered metric spaces. The work [23] was illustrated by proving the existence and uniqueness of the solution for a periodic boundary value problem. These results were further extended and generalized by Lakshmikantham and Ćirić [8] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces.

Sedghi and al [5, 9–11] proved some coupled fixed point theorems under contractive conditions in fuzzy metric spaces. The results proved by Fang [1] for compatible and weakly compatible mappings under φ-contractive conditions in Menger spaces that provide a tool to Hu [6] for proving fixed points results for coupled mappings and these results are the genuine generalization of the result of [10].

Aamri and Moutawakil [22] introduced the concept of E.A. property in a metric space. Sintunavarat and Kuman [14] introduced a new concept of CLR property. Very recently, Chauhan et.al [13] introduced the notion of $JCLR$ property. The importance of CLR property ensures that one does not require the closeness of range subspaces.

In this paper, we give the concept of E.A. property, CLR property and $JCLR$ property for coupled mappings and prove a result which provides a generalization of the result of Zhong Xiao [24].

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2 PRELIMINARIES

We now state some basic concepts and results which will be used. In the standard notation, we suppose that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^+$ be the set of positive integers.

A function $F : \overline{\mathbb{R}} \to [0,1]$ is called a distribution function if it is non decreasing and left continuous with $F(-\infty) = F(+\infty) = 1$. The class of all distribution functions is denoted by D_∞.

Suppose that $D = \{F \in D_\infty : \inf F\, D_\infty^+ (t) = 0, \sup F(t) = 1\}$, $D_\infty^+ = \{F \in D_\infty : F(0) = 0\}$ and $D^+ = D \cap D_\infty^+$ (see [10, 17]).

A special element of D^+ is the Heaviside function H defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Definition 1 ([16, 17]). *A function $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a, b, c, d \in [0,1]$:*

(Δ-1) $\Delta(a, 1) = a$;

(Δ-2) $\Delta(a, b) = \Delta(b, a)$;

(Δ-3) $\Delta(a, b) \ge \Delta(c, d)$, *for $a \ge c, b \ge d$;*

(Δ-4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Two examples of t-norm are $\Delta_M(a, b) = \min\{a, b\}$ and $\Delta_P(a, b) = ab$. It is evident that, as regards the point wise ordering, $\Delta \le \Delta_M$ for each t-norm Δ.

Definition 2 ([16–18]). *A triplet (X, F, Δ) is called a generalized Menger probabilistic metric space if X is a non-empty set, Δ is t-norm and F is a mapping from $X \times X$ into D_∞^+ satisfying the following condition ($F(x, y)$ for $x, y \in X$ is denoted by $F_{x,y}$):*

(MS-1) $F_{x,y}(t) = H(t)$ *for all $t \in \mathbb{R}$ if and only if $x = y$;*

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ *for all $x, y \in X$ and $t \in \mathbb{R}$;*

(MS-3) $F_{x,y}(t + s) \ge T(F_{x,z}(t), F_{z,y}(s))$ *for all $x, y, z \in X$ and $t, s \in \mathbb{R}^+$.*

A Menger probabilistic metric space (for short, a Menger PM-space) is a generalized Menger space with $F(X \times X) \in D^+$.

Schweizer et al [1, 19] point out that if the t-norm T of a Menger PM-space satisfies the condition $\sup\limits_{0<a<1} \Delta(a, a) = 1$, then (X, F, Δ) is a first countable Hausdorff topological space in the (ε, λ) topology τ, i.e., the family of sets

$$\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in [0,1], \ (x \in X)\}$$

is the base of neighborhoods of point x for τ, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

By virtue of this topology τ a sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent to x (we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$) if $\lim\limits_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a Cauchy sequences in (X, F, Δ) if for any given $\varepsilon > 0$ and $\lambda \in [0,1]$, there exists $N = N(\varepsilon, \lambda) \in \mathbb{Z}^+$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \ge N$; (X, F, Δ) is said to be complete if each Cauchy sequence in X is convergent to some point in X.

In the sequel, we will always assume that (X, F, Δ) is a Menger space with the (ε, λ) topology.

Lemma 1. *Let (X, d) be a usual metric space. Define a mapping $F : X \times X \to D^+$ by*

$$F_{x,y}(t) = H(t - d(x, y)) \quad \text{for } x, y \in X \text{ and } t > 0.$$

Then (X, F, Δ_m) is a Menger PM-space. It is called the induced Menger PM space by (X, d) and it is complete if (X, d) is complete.

An arbitrary t-norm can be extended (by (Δ-3)) in a unique way to an n-ary operation. For $(a_1, a_2, \ldots, a_n) \in [0, 1]^n, n \in \mathbb{Z}^+$, the value $\Delta^n(a_1, a_2, \ldots, a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1, a_2, \ldots, a_n) = \Delta\left(\Delta^{n-1}(a_1, a_2, \ldots, a_{n-1}), a_n\right)$.

For each $a \in [0, 1]$, the sequence $\{\Delta^n(a)\}_{n=1}^{\infty}$ is defined by $\Delta^1(a) = a$ and $\Delta^n(a) = \Delta\left(\Delta^{n-1}(a), a\right)$.

Definition 3. *A t-norm Δ is said to be of H-type if the sequence of functions $\{\Delta^n(a)\}_{n=1}^{\infty}$ is equicontinuous at $a = 1$.*

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{0<a<1} \Delta(a, a) = 1$.

Lemma 2. *Let (X, F, Δ) be a Menger PM-space. For each $\lambda \in (0, 1]$, define a function $d_\lambda : X \times X \to \mathbb{R}^+$ by*

$$d_\lambda(x, y) = \inf\left\{t > 0 : F_{x,y}(t) > 1 - \lambda\right\}. \tag{1}$$

Then the following statements hold:

(1) $d_\lambda(x, y) < t$ if and only if $F_{x,y}(t) > 1 - \lambda$;

(2) $d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$ and $\lambda \in (0, 1]$;

(3) $d_\lambda(x, y) = 0$ for all $\lambda \in (0, 1]$ if and only if $x = y$.

Lemma 3. *Let (X, F, Δ) be a Menger PM-space and let $\{d_\lambda\}_{\lambda \in (0,1]}$ be a family of pseudo-metrics on X defined by (1).*

If Δ is a t-norm of H-type, then for each $\lambda \in (0, 1]$ there exists $\mu \in (0, \lambda]$ such that for each $m \in \mathbb{Z}^+$,

$$d_\lambda(x_0, x_m) \leq \sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}) \quad \text{for all } x_0, x_1, \ldots, x_m \in X.$$

Lemma 4. *Suppose that $F \in D^+$. For each $n \in \mathbb{Z}^+$, let $F_n : \mathbb{R} \to [0, 1]$ be nondecreasing, and $g_n : (0, +\infty) \to (0, +\infty)$ satisfies $\lim_{n \to +\infty} g_n(t) = 0$ for any $t > 0$. If*

$$F_n(g_n(t)) \geq F(t) \quad \text{for all} \quad t > 0,$$

then $\lim_{n \to +\infty} F_n(t) = 1$ for any $t > 0$.

Definition 4 ([20]). *An element $x \in X$ is called a common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if*

$$x = f(x, x) = g(x).$$

Definition 5 ([21]). *An element $(x, y) \in X \times X$ is called:*
(i) a coupled fixed point of the mapping $f : X \times X \to X$ if

$$f(x, y) = x, \;\; f(y, x) = y;$$

(ii) a coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$f(x, y) = g(x), \;\; f(y, x) = g(y);$$

(iii) a common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$x = f(x, y) = g(x), \;\; y = f(y, x) = g(y).$$

In [22], Abbas et al introduced the concept of weakly compatible mappings. Here we give a similar concept in Menger metric spaces as follows.

Definition 6. *Let (X, F, Δ) be a Menger metric space and let $f : X \times X \to X$ and $g : X \to X$ be two mappings. f and g are said to be weakly compatible (or w-compatible) if they commute at their coupled coincidence points, i.e.; if (x, y) is a coupled coincidence point of f and g, then*

$$g(f(x, y)) = f(g(x), g(y)), \qquad g(f(y, x)) = f(g(y), g(x)).$$

Definition 7 ([23]). *Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings. Then, the pairs (B, S) and (A, T) are said to have common coupled coincidence point if there exist a, b in X such that*

$$B(a, b) = S(a) = T(a) = A(a, b) \text{ and } B(b, a) = S(b) = T(b) = A(b, a).$$

3 MAIN RESULTS

Now, we introduce the following concepts.

Definition 8. *Let (X, F, Δ) be a Menger metric space and let mappings $A : X \times X \to X$ and $S : X \to X$ are said to satisfy the E.A. property if there exist sequences $\{x_n\}, \{y_n\} \in X$ such that*

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = x \text{ and } \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = y$$

for some $x, y \in X$.

Definition 9. *Let (X, F, Δ) be a Menger metric space and let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings.*

Then the pairs (B, T) and (A, S) are said to satisfy the common $E.A.$ property if there exist sequences $\{x_n\}, \{y_n\}, \{x_n'\}, \{y_n'\} \in X$ such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} B\left(x_n', y_n'\right) = \lim_{n\to\infty} T\left(x_n'\right) = x,$$

$$\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} B\left(y_n', x_n'\right) = \lim_{n\to\infty} T\left(y_n'\right) = y$$

for some $x, y \in X$.

Definition 10. *Let (X, F, Δ) be a Menger metric space. The mappings $A : X \times X \to X$ and $S : X \to X$ are said to satisfy the CLR_S property if there exist sequences $\{x_n\}, \{y_n\} \in X$ such that*

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = Sx \text{ and } \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = Sy$$

for some $x, y \in X$.

Definition 11. *Let (X, F, Δ) be a Menger metric space and let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings.*

Then the pairs (B, T) and (A, S) are said to satisfy the common CLR_{ST} property if there exist sequences $\{x_n\}, \{y_n\}, \{x'_n\}, \{y'_n\} \in X$ such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} B\left(x'_n, y'_n\right) = \lim_{n\to\infty} T\left(x'_n\right) = x,$$

$$\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} B\left(y'_n, x'_n\right) = \lim_{n\to\infty} T\left(y'_n\right) = y,$$

where $x, y \in S(X) \cap T(X)$.

Jian-Zhong Xiao [24] proved the following result.

Theorem 1. *Let (X, F, Δ) be a complete Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $T : X \to X$ be two mappings such that*

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Tx,Tu}(t),\, F_{Ty,Tv}(t)\right)\right]^{1/2}$$

for all $x, y, u, v \in X$ and $t > 0$, where $A(X \times X) \subseteq T(X)$, T is continuous and commutative with A. Then there exists a unique $u \in X$ such that $u = Tu = A(u, u)$.

We now give our main result which provides a generalization of Theorem 1.

Theorem 2. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Su}(t),\, F_{Sy,Sv}(t)\right)\right]^{1/2}; \tag{2}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{3}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$, and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (2) we have

$$\begin{aligned} F_{A(x_n,y_n),A(p,q)}(t) &\geq F_{A(x_n,y_n),A(p,q)}\left(\varphi^{n_0}(t)\right) \\ &\geq \left[\Delta\left(F_{S(x_n),S(p)}\left(\varphi^{n_0-1}(t)\right), F_{S(y_n),S(q)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),S(p)}\left(\varphi^{n_0-1}(t)\right) F_{S(y_n),S(q)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2}. \end{aligned} \tag{4}$$

Letting $n \to \infty$ in (4), we have $F_{S(p),A(p,q)}(t) = 1$, that is, $A(p,q) = S(p) = x$. Similarly, $S(q) = A(q,p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x,y) = S(x)$ and $A(y,x) = S(y)$. Hence A and S have a coupled coincidence point.

Step 2. To show that $S(x) = y, S(y) = x$.

In fact, from (2) we have

$$\begin{aligned} F_{S(x_n),S(y)}(\varphi(t)) = F_{A(x_n,y_n),A(y,x)}(\varphi(t)) &\geq \left[\Delta\left(F_{S(x_n),S(y)}(t), F_{S(y_n),S(x)}(t)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \end{aligned} \tag{5}$$

Similarly, we have

$$F_{S(x),S(y_n)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{6}$$

Suppose that $Q_n(t) = F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)$. By (5) and (6), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$ and hence,

$$Q_n(\varphi^n(t)) \geq Q_{n-1}\left(\varphi^{n-1}(t)\right) \geq \cdots \geq Q_0(t). \tag{7}$$

Furthermore, from (5)–(7) it follows that

$$F_{S(x_n),S(y)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \quad \text{and} \quad F_{S(x),S(y_n)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}. \tag{8}$$

It is evident that $[Q_0(t)]^{1/2} \in D^+$. Since $\lim_{n\to\infty} \varphi^n(t) = 0$, from (8) and Lemma 4 we have

$$\lim_{n\to\infty} S(x_n) = S(y) \quad \text{and} \quad \lim_{n\to\infty} S(y_n) = S(x).$$

This shows that $S(x) = y$ and $S(y) = x$. Hence, $A(x,y) = y$ and $A(y,x) = x$.

Step 3. Next we shall show that $x = y$.

By (2) we have

$$F_{x,y}(\varphi(t)) = F_{A(y,x),A(x,y)}(\varphi(t)) \geq \left[\Delta\left(F_{S(y),S(x)}(t), F_{S(x),S(y)}(t)\right)\right]^{1/2} \geq F_{x,y}(t). \tag{9}$$

From (9) we have $F_{x,y}(\varphi^n(t)) \geq F_{x,y}(t)$. Using Lemma 4, we have $F_{x,y}(t) = 1$, i.e., $x = y$. The uniqueness of x follows from (2). So, the proof of Theorem 2 is finished. $\square$

Theorem 3. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[F_{Sx,Su}(t)\ F_{Sy,Sv}(t)\right]^{1/2}; \tag{10}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{11}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

From (10) and $\varphi(t) < t$, we obtain

$$\begin{aligned} F_{S(x_n),A(p,q)}(t) &\geq F_{S(x_n),A(p,q)}(\varphi(t)) = F_{A(x_n,y_n),A(p,q)}(\varphi(t)) \\ &\geq \left[F_{S(x_n),S(p)}(t)\, F_{S(y_n),S(q)}(t)\right]^{1/2}. \end{aligned} \tag{12}$$

Letting $n \to \infty$ in (12), we have $\lim_{n\to\infty} S(x_n) = A(p, q)$. Hence, $S(p) = A(p, q) = x$. Similarly, we can show that $S(q) = A(q, p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x, y) = S(x)$, $A(y, x) = S(y)$.

Step 2. To show that $S(x) = y, S(y) = x$.

In fact, from (10) we have

$$F_{S(x_n),S(y)}(\varphi(t)) = F_{A(x_n,y_n),A(y,x)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{13}$$

Similarly, we have

$$F_{S(x),S(y_n)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{14}$$

Suppose that $Q_n(t) = F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)$. By (13) and (14), we have

$$\begin{aligned} &Q_n(\varphi^n(t)) \geq Q_{n-1}\left(\varphi^{n-1}(t)\right) \geq \cdots \geq Q_0(t); \\ &F_{S(x_n),S(y)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \quad \text{and} \quad F_{S(x),S(y_n)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}. \end{aligned}$$

Since $[Q_0(t)]^{1/2} \in D^+$ and $\lim_{n\to\infty} \varphi^n(t) = 0$, by Lemma 4 we conclude that

$$\lim_{n\to\infty} S(x_n) = S(y) \quad \text{and} \quad \lim_{n\to\infty} S(y_n) = S(x).$$

This shows that $S(x) = y$ and $S(y) = x$. Hence, $A(x,y) = y$ and $A(y,x) = x$.

Step 3. Finally, we prove that $x = y$.

By (10) we have

$$F_{x,y}(\varphi(t)) = F_{A(y,x),A(x,y)}(\varphi(t)) \geq \left[F_{S(y),S(x)}(t)\, F_{S(x),S(y)}(t)\right]^{1/2} \geq F_{x,y}(t). \tag{15}$$

From (15), we have $F_{x,y}(\varphi^n(t)) \geq F_{x,y}(t)$. Using Lemma 4, we have $F_{x,y}(t) = 1$, i.e., $x = y$. The uniqueness of x follows from (10). □

Theorem 4. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(t) \geq \min\{F_{Sx,Su}(\varphi(t)), F_{Sy,Sv}(\varphi(t))\}; \tag{16}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{17}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

From (16) and (17) it follows that

$$F_{S(x_n),A(p,q)}(t) = F_{A(x_n,y_n),A(p,q)}(t) \geq \min\left\{F_{S(x_n),S(p)}(\varphi(t)), F_{S(y_n),S(q)}(\varphi(t))\right\}. \tag{18}$$

Letting $n \to \infty$ in (18), we have $\lim_{n\to\infty} S(x_n) = A(p,q)$. Hence, $S(p) = A(p,q) = x$. Similarly, we can show that $S(q) = A(q,p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x,y) = S(x)$, $A(y,x) = S(y)$.

Step 2. We claim that $S(x) = y, S(y) = x$.

In fact, from (16) we have

$$F_{S(x_n),S(y)}(t) = F_{A(x_n,y_n),A(y,x)}(t) \geq \min\left\{F_{S(x_n),S(y)}(\varphi(t)), F_{S(y_n),S(x)}(\varphi(t))\right\}. \tag{19}$$

Similarly, we have

$$F_{S(x),S(y_n)}(t) \geq \min\left\{F_{S(x_n),S(y)}(\varphi(t)), F_{S(y_n),S(x)}(\varphi(t))\right\}. \tag{20}$$

Suppose that $M_n(t) = \min\left\{F_{S(x_n),S(y)}(t), F_{S(y_n),S(x)}(t)\right\}$. From (19) and (20) it follows that

$$M_n(t) \geq M_{n-1}(\varphi(t)) \geq \cdots \geq M_0(\varphi^n(t)).$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$M_0(\varphi^n(t)) = \min\left\{F_{S(x_0),S(y)}(t), F_{S(y_0),S(x)}(t)\right\} \to 1 \text{ as } n \to \infty.$$

This shows that $M_n(t) \to 1$ as $n \to \infty$, so

$$\lim_{n\to\infty} S(x_n) = S(y) \text{ and } \lim_{n\to\infty} S(y_n) = S(x).$$

Hence, $S(x) = y$ and $S(y) = x$.

Step 3. Finally, we prove that $x = y$.

By (16) we have

$$F_{x,y}(t) = F_{A(y,x),A(x,y)}(t) \geq \min\left\{F_{S(y),T(x)}(\varphi(t)), F_{S(x),T(y)}(\varphi(t))\right\} = F_{x,y}(\varphi(t)). \tag{21}$$

From (21), we have $F_{x,y}(t) \geq F_{x,y}(\varphi^n(t))$. Letting $n \to \infty$, we have $F_{x,y}(t) = 1$, i.e., $x = y$. Since the uniqueness of x follows from (16), the proof of Theorem 4 is completed. □

Now we give another generalization of Theorem 1.

Corollary 1. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$ and let $A : X \times X \to X, S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Su}(t),\ F_{Sy,Sv}(t)\right)\right]^{1/2};$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies E.A. property.

If $S(X)$ is a closed subspace of X, then A and S have a unique common fixed point in X.

Proof. Since A and S satisfy E.A. property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = x, \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = y$$

for some $x, y \in X$.

It follows from $S(X)$ being a closed subspace of X that $x = S(p)$, $y = S(q)$ for some $p, q \in X$ and then A and S satisfy CLR_S property. By Theorem 2, we get that A and S have a unique common fixed point in X. □

Corollary 2. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X, S : X \to X$ be two mappings satisfying the conditions of Corollary 1.*

Suppose that $A(X \times X) \subseteq S(X)$, if range of one of the maps A or S is a closed subspace of X, then A and S have a unique common fixed point in X.

Proof. It follows immediately from Corollary 1. □

Taking $S = I_X$ in Theorem 2, we obtain the following

Corollary 3. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X$ be a mapping satisfying the following condition, for all $x, y, u, v \in X$ and $t > 0$:*

(1)
$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{x,u}(t), F_{y,v}(t)\right)\right]^{1/2};$$

(2) there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} y_n = y$$
for some $x, y \in X$.

Then there exists a unique $z \in X$ such that $z = A(z, z)$.

Now, we prove Theorem 2, Theorem 3, Theorem 4 for four mappings satisfying CLR_{ST} property before proving our main theorems, we begin with the following observation.

Lemma 5. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the following conditions:*

(1) the pair (A, S) satisfies the CLR_S property (or the pair (B, T) satisfies the CLR_T property);

(2) $A(X \times X) \subseteq T(X)$ (or $B(X \times X) \subseteq S(X)$);

(3) $T(X)$ (or $S(X)$) is complete subspace of X;

(4) $B(x'_n, y'_n)$ converges for every sequences $\{x'_n\}$ and $\{y'_n\}$ in X whenever $T(x'_n), T(y'_n)$ converges (or $A(x_n, y_n)$ converges for every sequences $\{x_n\}$ and $\{y_n\}$ in X whenever $S(x_n), S(y_n)$ converges);

(5) for all $x, y, u, v \in X$ and $t > 0$
$$F_{A(x,y),B(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Tu}(t), F_{Sy,Tv}(t)\right)\right]^{1/2}. \tag{22}$$

Then (A, S) and (B, T) share the CLR_{ST} property.

Proof. Suppose the pair (A, S) satisfies the CLR_S property, then there exist $\{x_n\}$ and $\{y_n\}$ in X such that
$$\begin{aligned} \lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} S(x_n) = a \in S(X), \\ \lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} S(y_n) = b \in S(X). \end{aligned}$$

Since $A(X \times X) \subseteq T(X)$ (wherein $T(X)$ is complete), for each $\{x_n\}$, $\{y_n\}$ in X there correspond sequences $\{\acute{x_n}\}$ and $\{\acute{y_n}\}$ in X such that

$$A(x_n, y_n) = T\left(\acute{x_n}\right) \text{ and } A(y_n, x_n) = T\left(\acute{y_n}\right).$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} T\left(\acute{x_n}\right) = a, \\ \lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} T\left(\acute{y_n}\right) = b, \end{aligned}$$

where $a, b \in S(X) \cap T(X)$. Now, we prove that $B\left(\acute{x_n}, \acute{y_n}\right) \to a$ and $B\left(\acute{y_n}, \acute{x_n}\right) \to b$.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$, and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (22) we have

$$\begin{aligned} F_{A(x_n,y_n),B(\acute{x_n},\acute{y_n})}(t) &\geq F_{A(x_n,y_n),B(\acute{x_n},\acute{y_n})}(\varphi^{n_0}(t)) \\ &\geq \left[\Delta\left(F_{S(x_n),T(\acute{x_n})}\left(\varphi^{n_0+1}(t)\right), F_{S(y_n),T(\acute{y_n})}\left(\varphi^{n_0+1}(t)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),T(\acute{x_n})}\left(\varphi^{n_0+1}(t)\right) F_{S(y_n),T(\acute{y_n})}\left(\varphi^{n_0+1}(t)\right)\right]^{1/2}. \end{aligned} \tag{23}$$

Letting $n \to \infty$ in (23), we get $\lim_{n\to\infty} B\left(\acute{x_n}, y_n^2\right) = a$. Similarly, we can show $\lim_{n\to\infty} B\left(\acute{y_n}, \acute{x_n}\right) = b$.

Thus, the pairs (A, S) and (B, T) share the CLR_{ST} property. □

Theorem 5. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the inequality (22) of Lemma 5.*

If the pairs (A, S) and (B, T) share the CLR_{ST} property, then (A, S) and (B, T) have a coincidence point each. Moreover A, B, S and T have a unique common fixed point if both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since both the pairs (A, S) and (B, T) share the CLR_{ST} property, there exist four sequences $\{x_n\}, \{y_n\}, \{\acute{x_n}\}$ and $\{\acute{y_n}\}$ in X such that:

$$\begin{aligned} \lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T\left(\acute{x_n}\right) = \lim_{n\to\infty} B\left(\acute{x_n}, \acute{y_n}\right) = a, \\ \lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} T\left(\acute{y_n}\right) = \lim_{n\to\infty} B\left(\acute{y_n}, \acute{x_n}\right) = b, \end{aligned} \tag{24}$$

where $a \in S(X) \cap T(X)$ and $b \in S(X) \cap T(X)$. It implies that there exist points $r, s, p, q \in X$ such that

$$S(r) = a, \; S(s) = b, \; T(p) = a \text{ and } T(q) = b.$$

Step 1. We show that $B(p, q) = T(p)$ and $B(q, p) = T(q)$. Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$ and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (22) we have

$$\begin{aligned} F_{T(\acute{x_n}),B(p,q)}(t) &\geq F_{T(\acute{x_n}),B(p,q)}(\varphi^{n_0}(t)) = F_{A(x_n,y_n),B(p,q)}(\varphi^{n_0}(t)) \\ &\geq \left[\Delta\left(F_{S(x_n),T(p)}\left(\varphi^{n_0-1}(t)\right), F_{S(y_n),T(q)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),T(p)}\left(\varphi^{n_0-1}(t)\right) F_{S(y_n),T(q)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2}. \end{aligned} \tag{25}$$

Letting $n \to \infty$ in (25), we have $\lim_{n\to\infty} T\left(\acute{x_n}\right) = B\left(p,q\right)$. By (24), $T\left(p\right) = B\left(p,q\right) = a$. Similarly, we can show that $T\left(q\right) = B\left(q,p\right) = b$.

Since the pair (B,T) is weakly compatible, so $T\left(p\right) = B\left(p,q\right) = a$ implies $T\left(a\right) = B\left(a,b\right)$, similarly $T\left(b\right) = B\left(b,a\right)$.

Now, we show that: $S\left(r\right) = A\left(r,s\right)$ and $S\left(s\right) = A\left(s,r\right)$.

Since $\sum_{n=1}^{\infty} \varphi^n\left(t\right) < +\infty$, we have $\lim_{n\to\infty} \varphi^n\left(t\right) = 0$ and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}\left(t\right) < t$. Thus, from (22) we get

$$\begin{aligned} F_{A(r,s),S(x_n)}\left(t\right) &\geq F_{A(r,s),S(x_n)}\left(\varphi^{n_0}\left(t\right)\right) = F_{A(r,s),B\left(\acute{x_n},\acute{y_n}\right)}\left(\varphi^{n_0}\left(t\right)\right) \\ &\geq \left[\Delta\left(F_{S(r),T\left(\acute{x_n}\right)}\left(\varphi^{n_0-1}\left(t\right)\right), F_{S(s),T\left(\acute{y_n}\right)}\left(\varphi^{n_0-1}\left(t\right)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(r),T\left(\acute{x_n}\right)}\left(\varphi^{n_0-1}\left(t\right)\right) F_{S(s),T\left(\acute{y_n}\right)}\left(\varphi^{n_0-1}\left(t\right)\right)\right]^{1/2}. \end{aligned} \tag{26}$$

Letting $n \to \infty$ in (26), we have $\lim_{n\to\infty} S\left(x_n\right) = A\left(r,s\right)$. By (24), $S\left(r\right) = A\left(r,s\right) = a$. Similarly, we can show that $S\left(s\right) = A\left(s,r\right) = b$.

Since the pair (A,S) is weakly compatible, it follows that $A\left(a,b\right) = S\left(a\right)$, $A\left(b,a\right) = S\left(b\right)$.

Step 2. We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$.

In fact, from (22) we have

$$\begin{aligned} F_{T\left(\acute{y_n}\right),Ta}\left(\varphi\left(t\right)\right) = F_{A(y_n,x_n),B(a,b)}\left(\varphi\left(t\right)\right) &\geq \left[\Delta\left(F_{S(y_n),T(a)}\left(t\right), F_{S(x_n),T(b)}\left(t\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(y_n),T(a)}\left(t\right) F_{S(x_n),T(b)}\left(t\right)\right]^{1/2}. \end{aligned} \tag{27}$$

Similarly, we have

$$F_{T\left(\acute{x_n}\right),Tb}\left(\varphi\left(t\right)\right) \geq \left[F_{S(x_n),T(b)}\left(t\right) F_{S(y_n),Ta}\left(t\right)\right]^{1/2}. \tag{28}$$

Suppose that $Q_n(t) = F_{S(x_n),T(b)}(t) F_{S(y_n),Ta}(t)$. By (27) and (28), we have $Q_n\left(\varphi\left(t\right)\right) \geq Q_{n-1}\left(t\right)$, hence

$$Q_n\left(\varphi^n\left(t\right)\right) \geq Q_{n-1}\left(\varphi^{n-1}\left(t\right)\right) \geq \cdots \geq Q_0\left(t\right). \tag{29}$$

Furthermore, from (27)–(29) it follows that

$$F_{T\left(\acute{y_n}\right),Ta}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2} \quad \text{and} \quad F_{T\left(\acute{x_n}\right),Tb}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2}. \tag{30}$$

It is evident that $\left[Q_0\left(t\right)\right]^{1/2} \in D^+$. Since $\lim_{n\to\infty} \varphi^n\left(t\right) = 0$, from (30) and Lemma 4 we have

$$\lim_{n\to\infty} T\left(\acute{y_n}\right) = Ta \quad \text{and} \quad \lim_{n\to\infty} T\left(\acute{x_n}\right) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence $B\left(a,b\right) = b$ and $B\left(b,a\right) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence $A\left(a,b\right) = b$ and $A\left(b,a\right) = a$.

Step 3. Now we prove that $a = b$.

By (22) we have

$$F_{a,b}\left(\varphi\left(t\right)\right) = F_{A(b,a),B(a,b)}\left(\varphi\left(t\right)\right) \geq \left[\Delta\left(F_{S(b),T(a)}\left(t\right), F_{S(a),T(b)}\left(t\right)\right)\right]^{1/2} \geq F_{a,b}\left(t\right). \tag{31}$$

From (31), we have $F_{a,b}\left(\varphi^n\left(t\right)\right) \geq F_{a,b}\left(t\right)$. Using Lemma 4, we obtain $F_{a,b}\left(t\right) = 1$, i.e., $a = b$. The uniqueness of a follows from (22). So, the proof of Theorem 5 is finished. $\square$

Theorem 6. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the condition (1)–(5) of Lemma 1.*

Then A, B, S and T have a unique common fixed point if both the pairs (A, S) and (B, T) are w-compatible.

Proof. In view of Lemma 5, both the pairs (A, S) and (B, T) enjoy the CLR_{ST} property, therefore there exist two sequences $\{x_n\}, \{y_n\}, \{x'_n\}$ and $\{y'_n\}$ in X such that:

$$\begin{aligned}
\lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T\left(x'_n\right) = \lim_{n\to\infty} B\left(x'_n, y'_n\right) = a, \\
\lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} T\left(y'_n\right) = \lim_{n\to\infty} B\left(y'_n, x'_n\right) = b,
\end{aligned}$$

where $a \in S(X) \cap T(X)$ and $b \in S(X) \cap T(X)$.

The rest of the proof runs on the lines of the proof of Theorem 5. □

Similarly, we can prove Theorem 3 and Theorem 4 for four mappings using CLR_{ST} property.

Now, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results.

Example 1. *Let $X = \left[0, \frac{1}{2}\right) \cup \{1\}$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then (X, F, Δ) is a Menger metric space, but it is not complete.*

Obviously (X, F, Δ) is not complete. Define the mappings $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ by

$$A(x, y) = \begin{cases} 0 & \text{if } (x, y) = (1, 1), \\ \frac{x^2+y^2}{6} & \text{if } (x, y) \neq (1, 1), \end{cases}$$

$$B(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) = (1, 1), \\ \frac{x+y}{2} & \text{if } (x, y) \neq (1, 1), \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{12} & \text{if } x = 1, \\ \frac{x^2}{3} & \text{if } x \neq 1, \end{cases}$$

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1, \\ x & \text{if } x \neq 1. \end{cases}$$

It is noted that $A(X \times X) = \left[0, \frac{1}{12}\right) \nsubseteq T(X) = \left[0, \frac{1}{2}\right]$, $B(X \times X) = \left[0, \frac{1}{2}\right] \nsubseteq S(X) = \left[0, \frac{1}{12}\right]$ and $T(X)$ and $S(X)$ are complete.

Next, we show that our results can be used for this case.

Let us prove that A, B, S and T satisfy the CLR_{ST} property. Consider the sequences $\{x_n\}$, $\{y_n\}$, $\{x'_n\}$ and $\{y'_n\}$ in X which are defined by

$$x_n = \frac{1}{2n},\ y_n = \frac{1}{3n},\ x'_n = \frac{1}{4n} \text{ and } y'_n = \frac{1}{5n},\ n = 1, 2, 3, \ldots$$

Since

$$\begin{aligned}\lim_{n\to\infty} A\left(x_n, y_n\right) &= \lim_{n\to\infty} S\left(x_n\right) = \lim_{n\to\infty} T\left(\acute{x_n}\right) = \lim_{n\to\infty} B\left(\acute{x_n}, \acute{y_n}\right) = 0 \in S\left(X\right) \cap T\left(X\right), \\ \lim_{n\to\infty} A\left(y_n, x_n\right) &= \lim_{n\to\infty} S\left(y_n\right) = \lim_{n\to\infty} T\left(\acute{y_n}\right) = \lim_{n\to\infty} B\left(\acute{y_n}, \acute{x_n}\right) = 0 \in S\left(X\right) \cap T\left(X\right).\end{aligned}$$

Thus A, B, S and T satisfy the CLR_{ST} property with these sequences.

Next, we will show that the pairs (A, S) and (B, T) are w-compatible.

It is obtained that

1. $A(x,y) = S(x)$ and $A(y,x) = S(y)$ if and only if $x = y = 0$, since $A(S(0), S(0)) = S(A(0,0))$, mappings A and S are w-compatible, and

2. $B(x,y) = T(x)$ and $B(y,x) = T(y)$ if and only if $x = y = 0$, since $B(T(0), T(0)) = T(B(0,0))$, mappings B and T are w-compatible.

Finally, we prove that for $x, y, u, v \in X$,

$$F_{A(x,y),B(u,v)}\left(\varphi\left(t\right)\right) \geq \left[\Delta\left(F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right)\right]^{1/2}.$$

Let $\varphi : (0,\infty) \to (0,\infty)$ by $\varphi(t) = \frac{1}{2}t$. Then $\lim\limits_{n\to+\infty} \varphi^n(t) = 0$ for any $t > 0$. For $x, y, u, v \in X$, we distinguish the following cases.

Case 1. *$(x,y) \neq (1,1)$ and $(u,v) \neq (1,1)$. In this case we have*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x^2+y^2}{6} - \frac{u+v}{2}\right|} = \frac{t}{t + \left|\left(\frac{x^2}{3} - u\right) + \left(\frac{y^2}{3} - v\right)\right|} \\ &\geq \frac{t}{t + \left|\frac{x^2}{3} - u\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 2. *$(x,y) \neq (1,1)$ and $(u,v) = (1,1)$.*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x^2+y^2}{6} - \frac{1}{2}\right|} = \frac{t}{t + \left|\frac{x^2+y^2}{3} - 1\right|} \\ &\geq \frac{t}{t + \left|\frac{x^2}{3} - \frac{1}{2}\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 3. *$(x,y) = (1,1)$ and $(u,v) \neq (1,1)$.*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x+y}{2}\right|} = \frac{t}{t + |x+y|} \\ &\geq \frac{t}{t + \left|x - \frac{1}{12}\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 4. *$(x,y) = (1,1)$ and $(u,v) = (1,1)$.*

$$F_{A(x,y),B(u,v)}\left(kt\right) = \frac{\frac{t}{2}}{\frac{t}{2} + \frac{1}{2}} = \frac{t}{t + \frac{1}{2}} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.$$

Hence, all the hypotheses of Theorem 5 hold. Clearly $(0,0)$ is the unique common coupled fixed point of A, B, S and T.

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Бен Аоуа Л., Аліуче А. *Теореми про зчеплену нерухому точку для слабко сумісних відображень у сукупності з CLR властивістю в метричних просторах Менґера* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 195–210.

Проблеми зв'язної нерухомої точки привертають значну увагу в теперішній час. Мета цієї статті полягає у розширенні понять Е.А. властивості, CLR властивості та $JCLR$ властивості для зв'язних відображень в метричному просторі Менґера і використанні цих понять для дослідження загальних результатів про зв'язну нерухому точку для чотирьох власних відображень. Наша робота узагальнює результати Цян-Хжонг Ксяо [24] та ін. Основний результат наведено з використанням відповідного прикладу.

Ключові слова і фрази: метричний простір Менґера, t-норма типу H, слабка відповідність зв'язної нерухомої точки, CLR властивість, Е.А. властивість, $JCLR$ властивість.

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VASYLYSHYN T.V.

EXTENSIONS OF MULTILINEAR MAPPINGS TO POWERS OF LINEAR SPACES

We consider the question of the possibility to recover a multilinear mapping from the restriction to the diagonal of its extension to a Cartesian power of a space.

Key words and phrases: multilinear mapping, polarization formula.

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INTRODUCTION

Let X and Y be linear spaces over the same field $\mathbb{K}$. It is well-known (see e. g. [1, Theorem 1.10]) that every *symmetric* n-linear mapping $A : X^n \to Y$ can be recovered from its restriction to the diagonal $\widehat{A} : X \to Y$, $\widehat{A}(x) = A(x, \ldots, x)$, by means of the so-called Polarization Formula:

$$A(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1 \ldots \varepsilon_n \widehat{A}(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n).$$

But in general if A is non-symmetric, it cannot be recovered from $\widehat{A}$. For example, if A is alternating, then $\widehat{A}$ is equal to zero. Let us recall that A is called *alternating* if $A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma A(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$, where S_n is the group of all permutations of n elements and $(-1)^\sigma$ is the sign of the permutation σ.

In [1, p. 8] it has been introduced mappings between complex linear spaces, which are linear with respect to some arguments and antilinear with respect to other arguments. If such a mapping is symmetric with respect to “linear” and “antilinear” arguments separately, then it can be recovered from its restriction to the diagonal by means of polarization formulas, proved in [2] and [3]. Note that in this case there are no any requirements of symmetry between “linear” and “antilinear” arguments. In some cases for multilinear mappings there is a similar situation. For example, if $A : X^n \to Y$ is an n-linear mapping, then a mapping $\widetilde{A} : (X^n)^n \to Y$, defined by

$$\widetilde{A}(x_1, \ldots, x_n) = A(x_1^{(1)}, \ldots, x_n^{(n)}),$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(n)}) \in X^n$, $j \in \{1, \ldots, n\}$, is an n-linear mapping too (in general, non-symmetric) and its restriction to the diagonal $\widehat{\widetilde{A}}(x)$ is equal to $A(x^{(1)}, \ldots, x^{(n)})$ for $x = (x^{(1)}, \ldots, x^{(n)}) \in X^n$. Therefore, A and, consequently, $\widetilde{A}$, can be recovered from the restriction of $\widetilde{A}$ to the diagonal.

We consider the question of the possibility of recovering of a multilinear mapping from the restriction to the diagonal of its extension to a power of a space.

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1 THE MAIN RESULT

Let $M = (m_{ij})_{i,j=1}^{n}$ be a matrix of scalars from $\mathbb{K}$. Then for every n-linear mapping $A : X^n \to Y$ a mapping $E_M(A) : (X^n)^n \to Y$, defined by

$$E_M(A)(x_1,\ldots,x_n) = A(m_{11}x_1^{(1)} + \ldots + m_{1n}x_1^{(n)}, \cdots, m_{n1}x_n^{(1)} + \ldots + m_{nn}x_n^{(n)}),$$

where $x_1,\ldots,x_n \in X^n$, is an n-linear mapping. Its restriction to the diagonal is equal to

$$\widehat{E_M(A)}(x) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} m_{1k_1}\ldots m_{nk_n} A(x^{(k_1)},\ldots,x^{(k_n)}). \tag{1}$$

Note that if $m_{ij} = 1$, $i = 1,\ldots,n$, for the fixed $j \in \{1,\ldots,n\}$, then $E_M(A)$ is an extension of A.

Proposition 1.1. *For every n-linear alternating mapping $A : X^n \to Y$,*

$$\widehat{E_M(A)}(x) = \det(M) A(x^{(1)},\ldots,x^{(n)}),$$

where $x = (x^{(1)},\ldots,x^{(n)}) \in X^n$.

Proof. Since A is alternating, $A(x^{(k_1)},\ldots,x^{(k_n)}) = 0$ if $k_l = k_s$ for some $l \neq s$. Therefore, by (1),

$$\widehat{E_M(A)}(x) = \sum_{\sigma\in S_n} m_{1\sigma(k_1)}\ldots m_{n\sigma(k_n)} A(x^{(\sigma(1))},\ldots,x^{(\sigma(n))}).$$

Since $A(x^{(\sigma(1))},\ldots,x^{(\sigma(n))}) = (-1)^\sigma A(x^{(1)},\ldots,x^{(n)})$, therefore

$$\widehat{E_M(A)}(x) = \sum_{\sigma\in S_n} (-1)^\sigma m_{1\sigma(k_1)}\ldots m_{n\sigma(k_n)} A(x^{(1)},\ldots,x^{(n)}) = \det(M) A(x^{(1)},\ldots,x^{(n)}).$$

□

Let us consider recovering of multilinear mappings, which in general are neither symmetric nor alternating. It can be easily seen that if M is a diagonal matrix, then

$$\widehat{E_M(A)}(x) = m_{11}\ldots m_{nn} A(x^{(1)},\ldots,x^{(n)})$$

for every n-linear mapping A. Let us construct a non-diagonal matrix M' such that *every* n-linear mapping A can be recovered from $\widehat{E_{M'}(A)}$. Let

$$M' = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & -1 & 1 & \ldots & 1 \\ 1 & 1 & -1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & -1 \end{pmatrix}.$$

For $k \in \{1,\ldots,n\}$ let $i_k : X \to X^n$, $i_k(x) = (\underbrace{0,\ldots,0}_{k-1}, x, 0, \ldots, 0)$.

Theorem 1. *The n-linear mapping A can be recovered from $\widehat{E_{M'}(A)}$ by means of the formula:*

$$\begin{aligned} A(x_1,\ldots,x_n) = \frac{1}{2^{2n-1}} \sum_{j_2,\ldots,j_n=0}^{1} (-1)^{j_2+\ldots+j_n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)} \\ \times \left(\varepsilon_1 i_1(x_1) + \varepsilon_2 p^{(2)}_{j_2}(x_2) + \ldots + \varepsilon_n p^{(n)}_{j_n}(x_n)\right), \end{aligned} \tag{2}$$

where

$$p^{(k)}_{j_k}(x) = \begin{cases} i_1(x), & \text{if } j_k = 0, \\ i_k(x), & \text{if } j_k = 1 \end{cases}$$

for $k \in \{2,\ldots,n\}$.

Proof. Let $y_1 = i_1(x_1), y_2 = p^{(2)}_{j_2}(x_2),\ldots,y_n = p^{(n)}_{j_n}(x_n)$. Notice that

$$\begin{aligned} &\sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n) \\ &= \sum_{k_1,\ldots,k_n=1}^{n} E_{M'}(A)(y_{k_1},\ldots,y_{k_n}) \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n\varepsilon_{k_1}\ldots\varepsilon_{k_n} \end{aligned}$$

and

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n\varepsilon_{k_1}\ldots\varepsilon_{k_n} = \begin{cases} 2^n, & \text{if } k_1 \neq \ldots \neq k_n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{1}{2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1\ldots\varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n) = \sum_{\sigma\in S_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n)}).$$

For $\sigma \in S_n$ such that $\sigma(n) = n$ we have

$$\begin{aligned} &\sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},y_{\sigma(n)}) = \sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)\left(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},p^{(n)}_{j_n}(x_n)\right) \\ &= E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)) - E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_n(x_n)) \\ &= 2E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)). \end{aligned}$$

For $\sigma \in S_n$ such that $\sigma(n) \neq n$ we have

$$\begin{aligned} &\sum_{j_n=0}^{1} (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n)}) = \sum_{j_n=0}^{1} (-1)^{j_n} \left(y_{\sigma(1)},\ldots,p^{(n)}_{j_n}(x_n),\ldots,y_{\sigma(n)}\right) \\ &= E_{M'}(A)(y_{\sigma(1)},\ldots,i_1(x_n),\ldots,y_{\sigma(n)}) - E_{M'}(A)(y_{\sigma(1)},\ldots,i_n(x_n),\ldots,y_{\sigma(n)}) = 0. \end{aligned}$$

Therefore, the right-hand side of (2) is equal to

$$\frac{1}{2^{n-2}} \sum_{j_2,\ldots,j_{n-1}=0}^{1} (-1)^{j_2+\ldots+j_{n-1}} \sum_{\sigma\in S_n,\ \sigma(n)=n} E_{M'}(A)(y_{\sigma(1)},\ldots,y_{\sigma(n-1)},i_1(x_n)).$$

After applying this method $n-1$ times we obtain that the right-hand side of (2) is equal to $E_{M'}(A)(i_1(x_1),i_1(x_2),\ldots,i_1(x_n))$, which is equal to $A(x_1,x_2,\ldots,x_n)$. □

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Василишин Т.В. *Продовження мультилінійних відображень на степені лінійних просторів* // Карпатські матем. публ. — 2016. — Т.8, №2. — C. 211–214.

Розглянуто питання можливості відновлення мультилінійного відображення зі звуження на діагональ продовження цього відображення на деякий декартів степінь простору.

Ключові слова і фрази: мультилінійне відображення, поляризаційна формула.

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De N.

THE VERTEX ZAGREB INDICES OF SOME GRAPH OPERATIONS

Recently, Tavakoli et al. [6] introduced a new version of Zagreb indices, named as vertex Zagreb indices. In this paper explicit expressions of different graphs operations of vertex Zagreb indices are presented and also as an application, explicit formulas for vertex Zagreb indices of some chemical graphs are obtained.

Key words and phrases: degree, topological index, Zagreb index, graph operations.

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Introduction

In this paper, all the graphs are simple connected, having no directed or weighted edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Let the number of vertices and edges of G will be denoted by n and m respectively. Also let the edge connecting the vertices u and v is denoted by uv. The degree of a vertex v, is the number of first neighbors of v and is denoted by $d_G(v)$. Let $N(u)$ denotes the first neighbor set of u; then $|N(u)| = d_G(u)$. As usual P_n and C_n denote a path and cycle graph of order n respectively. Let, $\sum$ denotes the class of all graphs, then a function $T : \sum \to \mathbf{R}^+$ is known as a topological index if for every graph H isomorphic to G, $T(G) = T(H)$. Thus a topological index transforming chemical information of a molecular graph by means of a numeric parameter which characterize its topology and is necessarily invariant under automorphism of graphs.

The first and second Zagreb indices of a graph were introduced in 1972 [1], denoted by $M_1(G)$ and $M_2(G)$ and are respectively defined as

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2 = \sum_{uv\in E(G)} [d_G(u) + d_G(v)] \text{ and } M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

These indices are among one of the most important vertex-degree based topological indices and have good application, so that get lots of attention from chemists and mathematicians (see [2–5,7]).

There are various study of different versions of Zagreb indices. One of the modified versions of classical Zagreb indices, the vertex version of first and second Zagreb indices were introduced by Tavakoli et al. in [6] to calculate the eccentric connectivity index and Zagreb coindices of graphs under generalized hierarchical product and are defined as

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$$\bar{M}_1^*(G) = \sum_{\{u,v\}\subseteq V(G)} [d_G(u) + d_G(v)], \quad \bar{M}_2^*(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v).$$

In that paper, they also derived explicit expressions of first and second vertex Zagreb indices of generalized hierarchical product graphs. Till date, the study of these indices are largely limited and hence we have attracted in studying mathematical properties of these vertex version of Zagreb indices.

Graph operations played a very important role in chemical graph theory, as some chemically interesting graphs can be obtained by different graph operations on some general or particular graphs. In [7], Khalifeh et al. derived some exact formula for computing first and second Zagreb indices under some graph operations. In [8], Ashrafi et al. presented some explicit formulae of Zagreb coindices under some graph operations. In [9], Das et al. derived some upper bounds for multiplicative Zagreb indices for different graph operations. In [10] and [11], the present author obtained F-index and F-coindex of different graph operations. In [12] the present author found reformulated first Zagreb index under different graph operations. In [13], Azari and Iranmanesh presented explicit formulas for computing the eccentric-distance sum of different graph operations. There are several other results regarding various topological indices under different graph operations are available in the literature (for details see [14–23]). In this paper, we derive some exact expression of the first and second vertex Zagreb indices of different graph operations such as union, join, Cartesian product, composition and corona product of graphs.

1 MAIN RESULTS

In this section, we study the first and second vertex Zagreb indices under union, join, Cartesian product, composition and corona product of graphs. All these operations are binary, and the join and Cartesian product of graphs are commutative operations, whereas the composition and corona product operations are noncommutative. Let G_1 and G_2 be two simple connected graphs, so that their vertex sets and edge sets are represented as $V(G_i)$ and $E(G_i)$ respectively, for $i \in \{1,2\}$. Also let, n_i and m_i denote the number of vertices and edges of G_i respectively, for $i \in \{1,2\}$.

1.1 Union

Definition 1.1. *The union of two graphs G_1 and G_2 is the graph denoted by $G_1 \cup G_2$ with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. In this case we assume that $V(G_1)$ and $V(G_2)$ are disjoint.*

The degree of a vertex v of $G_1 \cup G_2$ is equal to degree of that vertex in the component G_i, $i = 1,2$, that contains it. In the following we calculate the first and second vertex Zagreb indices of $G_1 \cup G_2$.

Theorem 1. *Let G_1 and G_2 be two connected graphs, then*

$$\bar{M}_1^*(G_1 \cup G_2) = \bar{M}_1^*(G_1) + \bar{M}_1^*(G_2) + 2n_2m_1 + 2n_1m_2.$$

Proof. From definition, it is clear that, the vertex Zagreb index of $G_1 \cup G_2$ is equal to the sum of the vertex Zagreb index of the components G_i, in addition to that the contributions of the missing edges between the components, which makes the edge set of the complete bipartite graph K_{n_1,n_2}. Thus we have

$$\begin{aligned}\bar{M}_1^*(G_1 \cup G_2) &= \sum_{\{u,v\}\in V(G_1)} \left[d_{G_1}(u) + d_{G_1}(v)\right] + \sum_{\{u,v\}\in V(G_2)} \left[d_{G_2}(u) + d_{G_2}(v)\right] \\ &+ \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left[d_{G_1}(u) + d_{G_2}(v)\right],\end{aligned}$$

which proves the desired result. □

Theorem 2. *Let G_1 and G_2 be two connected graphs, then*

$$\bar{M}_2^*(G_1 \cup G_2) = \bar{M}_2^*(G_1) + \bar{M}_2^*(G_2) + 4m_1m_2.$$

Proof. From definition, similar to last theorem, we have

$$\begin{aligned}\bar{M}_2^*(G_1 \cup G_2) &= \sum_{\{u,v\}\in V(G_1)} d_{G_1}(u)d_{G_1}(v) + \sum_{\{u,v\}\in V(G_2)} d_{G_2}(u)d_{G_2}(v) \\ &+ \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} d_{G_1}(u)d_{G_2}(v),\end{aligned}$$

which proves the desired result. □

1.2 Join

Definition 1.2. *The join of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph denoted by $G_1 + G_2$ with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.*

Thus in the sum of two graphs all the vertices of one graph are connected with all the vertices of the other graph, keeping all the edges of both graphs. So, the degree of the vertices of $G_1 + G_2$ is given by

$$d_{G_1+G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, & v \in V(G_1) \\ d_{G_2}(v) + n_1, & v \in V(G_2). \end{cases}$$

In the following Theorem the first vertex Zagreb index of $G_1 + G_2$ is calculated.

Theorem 3. *The first vertex Zagreb index of $G_1 + G_2$ is given by*

$$\bar{M}_1^*(G_1 + G_2) = \bar{M}_1^*(G_1) + \bar{M}_1^*(G_2) + 2n_1m_2 + 2n_2m_1 + 2n_1n_2(n_1 + n_2 - 1).$$

Proof. Using definition of first vertex Zagreb index, we have

$$\bar{M}_1^*(G_1+G_2) = \sum_{\{u,v\}\subseteq V(G_1+G_2)} \left[d_{G_1+G_2}(u) + d_{G_1+G_2}(v)\right]$$
$$= \sum_{\{u,v\}\in V(G_1)} \left[d_{G_1+G_2}(u) + d_{G_1+G_2}(v)\right] + \sum_{\{u,v\}\subseteq V(G_2)} \left[d_{G_1+G_2}(u) + d_{G_1+G_2}(v)\right]$$
$$+ \sum_{u\in V(G_1),v\in V(G_2)} \left[d_{G_1+G_2}(u) + d_{G_1+G_2}(v)\right]$$
$$= \sum_{\{u,v\}\subseteq V(G_1)} \left[d_{G_1}(u) + d_{G_1}(v) + 2n_2\right] + \sum_{\{u,v\}\subseteq V(G_2)} \left[d_{G_2}(u) + d_{G_2}(v) + 2n_1\right]$$
$$+ \sum_{u\in V(G_1),v\in V(G_2)} \left[d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1\right]$$
$$= \sum_{\{u,v\}\subseteq V(G_1)} \left[d_{G_1}(u) + d_{G_1}(v)\right] + 2n_2.\frac{n_1(n_1-1)}{2}$$
$$+ \sum_{\{u,v\}\subseteq V(G_2)} \left[d_{G_2}(u) + d_{G_2}(v)\right] + 2n_1.\frac{n_2(n_2-1)}{2} + n_1n_2(n_1+n_2) + 2n_1m_2 + 2n_2m_1,$$

which proves the desired result. □

In the following, next we calculate the second vertex Zagreb index of $G_1 + G_2$.

Theorem 4. *The second vertex Zagreb index of $G_1 + G_2$ is given by*

$$\bar{M}_2^*(G_1+G_2) = \bar{M}_2^*(G_1) + n_2\bar{M}_1^*(G_1) + \bar{M}_2^*(G_2) + n_1\bar{M}_1^*(G_2) + \frac{1}{2}n_1{n_2}^2(n_1-1)$$
$$+ \frac{1}{2}{n_1}^2n_2(n_2-1) + 4m_1m_2 + 2n_1n_2(m_1+m_2) + {n_1}^2{n_2}^2.$$

Proof. Using definition of first vertex Zagreb index, we have

$$\bar{M}_2^*(G_1+G_2) = \sum_{\{u,v\}\subseteq V(G_1+G_2)} d_{G_1+G_2}(u)d_{G_1+G_2}(v)$$
$$= \sum_{\{u,v\}\subseteq V(G_1)} d_{G_1+G_2}(u)d_{G_1+G_2}(v) + \sum_{\{u,v\}\subseteq V(G_2)} d_{G_1+G_2}(u)d_{G_1+G_2}(v)$$
$$+ \sum_{u\in V(G_1),v\in V(G_2)} d_{G_1+G_2}(u)d_{G_1+G_2}(v) = \sum_{\{u,v\}\subseteq V(G_1)} (d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2)$$
$$+ \sum_{\{u,v\}\subseteq V(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1) + \sum_{u\in V(G_1),v\in V(G_2)} (d_{G_1}(u)+n_2)(d_{G_2}(v)+n_1)$$
$$= \sum_{\{u,v\}\subseteq V(G_1)} d_{G_1}(u)d_{G_1}(v) + n_2 \sum_{\{u,v\}\subseteq V(G_1)} \left[d_{G_1}(u) + d_{G_1}(v)\right]$$
$$+ {n_2}^2.\frac{n_1(n_1-1)}{2} + \sum_{\{u,v\}\subseteq V(G_2)} d_{G_2}(u)d_{G_2}(v) + n_1 \sum_{\{u,v\}\subseteq V(G_2)} \left[d_{G_2}(u) + d_{G_2}(v)\right]$$
$$+ {n_1}^2.\frac{n_2(n_2-1)}{2} + \sum_{u\in V(G_1),v\in V(G_2)} (d_{G_1}(u)d_{G_2}(v) + n_1d_{G_1}(u) + n_2d_{G_2}(v) + n_1n_2)$$
$$= \bar{M}_2^*(G_1) + n_2\bar{M}_1^*(G_1) + \frac{1}{2}n_1{n_2}^2(n_1-1) + \bar{M}_2^*(G_2) + n_1\bar{M}_1^*(G_2)$$
$$+ \frac{1}{2}{n_1}^2n_2(n_2-1) + 4m_1m_2 + 2n_1n_2m_2 + 2n_1n_2m_1 + {n_1}^2{n_2}^2,$$

from where the desired result follows. □

Example 1. *The complete bipartite graph $K_{p,q}$ can be defined as $K_{p,q} = \bar{K}_p + \bar{K}_q$. So its vertex Zagreb indices can be calculated from the previous theorem as*

(i) $\bar{M}_1^*(K_{p,q}) = 2pq(p+q-1)$,

(ii) $\bar{M}_2^*(K_{p,q}) = pq\left[2pq - \frac{1}{2}(p+q)\right]$.

The suspension of a graph G is defined as sum of G with a single vertex. So from the previous proposition the following corollary follows.

Corollary 1.1. *The first and second vertex Zagreb indices of suspension of a graph G is given by*

(i) $\bar{M}_1^*(G+K_1) = \bar{M}_1^*(G) + 2n^2 + 2m$,

(ii) $\bar{M}_2^*(G+K_1) = \bar{M}_1^*(G) + \bar{M}_2^*(G) + 2mn + \frac{1}{2}n(3n-1)$.

Example 2. *The star graph S_n with n vertices is the suspension of empty graph $\bar{K}_{n-1}$. So its first and second vertex Zagreb indices can be respectively calculated from the previous corollary as*

(i) $\bar{M}_1^*(S_n) = 2(n-1)^2$,

(ii) $\bar{M}_2^*(S_n) = \frac{3}{2}(n-1)^2 - \frac{1}{2}(n-1)$.

Example 3. *The wheel graph W_n on $(n+1)$ vertices is the suspension of C_n. So from the previous corollary its first and second vertex Zagreb indices are given by*

(i) $\bar{M}_1^*(C_n+K_1) = 4n^2$,

(ii) $\bar{M}_2^*(C_n+K_1) = \frac{15}{2}n^2 - \frac{9}{2}n$.

Example 4. *The fan graph F_n on (n+1) vertices is the suspension of P_n. So from the previous corollary its first and second vertex Zagreb indices are given by*

(i) $\bar{M}_1^*(P_n+K_1) = 2n(2n-1)$,

(ii) $\bar{M}_2^*(P_n+K_1) = 2n(2n-1)$.

1.3 The Cartesian product

Definition 1.3. *Let G_1 and G_2 be two connected graphs. The Cartesian product of G_1 and G_2 denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if [$u_p = u_q \in V(G_1)$ and $v_r v_s \in E(G_2)$] or [$v_r = v_s \in V(G_2)$ and $u_p u_q \in E(G_1)$] and $r, s = 1, 2, ..., |V(G_2)|$.*

In the following Theorem we express the first and second vertex Zagreb indices of the Cartesian product of graphs.

Theorem 5. *Let G_1 and G_2 be two connected graphs, then*

(i) $\bar{M}_2^*(G_1 \times G_1) = 2n_1n_2(n_1m_2 + n_2m_1) - 2m_1n_2 - 2m_2n_1$,

(ii) $\bar{M}_2^*(G_1 \times G_2) = 2(n_1m_2 + n_2m_1)^2 - 4m_1m_2 - \frac{1}{2}n_2M_1(G_1) - \frac{1}{2}n_1M_1(G_2)$.

The proof of the above Theorem follows by applying Theorem 1 and 4 of [7] and Proposition 13 of [8] respectively and using the fact that $M_1(G) = \bar{M}_1^*(G) - \bar{M}_1(G)$ and $M_2(G) = \bar{M}_2^*(G) - \bar{M}_2(G)$.

Example 5. *The Ladder graph L_n , made by n square and $(2n+2)$ vertices is the cartesian product of P_2 and P_{n+1}. So the first and second vertex Zagreb indices of L_n are given by*

(i) $\bar{M}_1^*(L_n) = 2(6n^2 + 5n + 1)$,

(ii) $\bar{M}_2^*(L_n) = 3(6n^2 + n + 1)$.

Example 6. *We have C_4-nanotorus $TC_4(m,n) = C_n \times C_m$. So its first and second vertex Zagreb indices are given by*

(i) $\bar{M}_1^*(TC_4(m,n)) = 4mn(mn-1)$,

(ii) $\bar{M}_2^*(TC_4(m,n)) = 8mn(mn-1)$.

Example 7. *We have C_4-nanotube $TUC_4(m,n) = P_n \times P_m$. So from the last theorem, its first and second vertex Zagreb indices are given by*

(i) $\bar{M}_1^*(TUC_4(m,n)) = 2(2mn - n - m)(mn - 1)$,

(ii) $\bar{M}_2^*(TUC_4(m,n)) = 2(2mn - n - m) - 4(n-1)(m-1) - (4mn - 3(n+m))$.

1.4 Composition

Definition 1.4. *The composition or lexicographic product of two graphs G_1 and G_2 is denoted by $G_1[G_2]$ and any two vertices (u_1, u_2) and $(v_1, v_2)$are adjacent if and only if $u_1v_1 \in E(G_1)$or $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$.*

The vertex set of $G_1[G_2]$ is $V(G_1) \times V(G_2)$ and the degree of a vertex (a,b) of $G_1[G_2]$ is given by $d_{G_1[G_2]}(a,b) = n_2 d_{G_1}(a) + d_{G_2}(b)$.

The proof of the next Theorem follows similarly from the expressions of Zagreb indices and Zagreb coindices of composition of graphs from Theorem 3 and 6 of [7] and Proposition 18 of [8] respectively.

Theorem 6. *Let G_1 and G_2 be two connected graphs, then the first and second vertex Zagreb indices of $G_1[G_2]$ is given by*

$$(i)\ \bar{M}_1^*(G_1[G_2]) = 2n_1n_2(n_1m_2 + {n_2}^2m_1) - 2m_1(n_1 + {n_2}^2),$$

$$(ii)\ \bar{M}_2^*(G_1[G_2]) = 2m_1{n_2}^2(2n_1m_2 + {n_2}^2m_1) + 2{n_1}^2{m_2}^2 - 4m_1m_2n_2 - \frac{1}{2}{n_2}^3M_1(G_1) - \frac{1}{2}n_1M_1(G_2).$$

Example 8. *The fence graph is defined as $P_n[P_2]$. So from the last theorem its first and second vertex Zagreb indices are given by*

(i) $\bar{M}_1^*(P_n[P_2]) = 18n^2 - 22n + 8$,

(ii) $\bar{M}_2^*(P_n[P_2]) = 50n^2 - 105n + 64$.

Example 9. *The closed fence graph is defined as $C_n[P_2]$ so that from the last theorem its first and second vertex Zagreb indices are given by*

(i) $\bar{M}^*{}_1(C_n[P_2]) = 18n^2 - 8n$,

(ii) $\bar{M}_2^*(C_n[P_2]) = 18n^2 + 7n$.

1.5 Corona Product

The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1 and n_1 copies of G_2 and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq n_1$. Thus, the corona product of G_1 and G_2 has total $(n_1n_2 + n_1)$ number of vertices and $(m_1 + n_1m_2 + n_1n_2)$ number of edges. A variety of topological indices under the corona product of graphs have already been studied by researchers [24, 26]. The degree of a vertex v of $G_1 \circ G_2$ is given by

$$d_{G_1 \circ G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, & v \in V(G_1) \\ d_{G_2}(v) + 1, & v \in V(G_{2,i}),\ i = 1, 2, \ldots, n_1, \end{cases}$$

where, $G_{2,i}$ is the i-th copy of the graph G_2. In the following theorem, the first and second vertex Zagreb indices of the corona product of two graphs are computed. The proof of the following theorem follows by manipulating the definition of corona product of graphs and hence we omit it.

Theorem 7. *The first and second vertex Zagreb indices of $G_1 \circ G_2$ is given by*

$$\begin{aligned} (i)\ \bar{M}_1^*(G_1 \circ G_2) &= \bar{M}_1^*(G_1) + n_1\bar{M}_1^*(G_2) + 2n_1n_2[(n_2 + m_2)(n_1 - 1) + m_1 + n_1 + n_2 - 1] \\ &\quad + 2m_2{n_1}^2, \\ (ii)\ \bar{M}_2^*(G_1 \circ G_2) &= \bar{M}_2^*(G_1) + n_1\bar{M}_2^*(G_2) + 2{n_1}^2(n_2 + m_2)^2 + 2n_1m_1(n_2 + m_2) - 2n_1{m_2}^2 \\ &\quad - 2(n_1m_2 + n_2m_1) - \frac{1}{2}n_1n_2(n_2 + 1). \end{aligned}$$

Let for a graph G, n and m are number of vertices and edges of G, respectively. If degree of any end vertex of an edge is one then it is call a thorn or pendent edge. The t-thorny graph G^t of a given graph G is obtained by joining t-number of thorns to each vertex of G. Different topological indices of thorn graphs have already been studied by researcher (see [14,25,27,28]). We know that, the t-thorny graph of G is defined as the corona product of G and complement of complete graph with t vertices $\bar{K}_t$. So, from the previous theorem we get the following corollary.

Corollary 1.2. *The first and second vertex Zagreb indices of the t-thorny graph are given by*

(i) $\bar{M}_1^*(G^t) = \bar{M}_1^*(G) + 2nt(nt + n + m - 1)$,

(ii) $\bar{M}_2^*(G^t) = \bar{M}_2^*(G) + 2n^2t^2 - \frac{1}{2}nt^2 + 2mt(2n - 1) - \frac{1}{2}nt$.

where, n and m are number of vertices and edges of G, respectively.

Example 10. *The first and second vertex Zagreb indices of t-thorny graph of C_n are given by*

(i) $\bar{M}_1^*({C_n}^t) = 2n(n - 1) + 2nt(nt + 2n - 1)$,

(ii) $\bar{M}_2^*({C_n}^t) = 2n(n - 1) + nt(6n - \frac{1}{2}t - \frac{5}{2})$.

Example 11. *The first and second vertex Zagreb indices of t-thorny graph of P_n are given by*

(i) $\bar{M}_1^*({P_n}^t) = 2(n - 1)^2 + 2nt(nt + 2n - 2)$,

(ii) $\bar{M}_2^*(P_n{}^t) = 2n^2 - 6n + 2n^2t^2 + 4n^2t - \frac{1}{2}nt^2 - \frac{13}{2}nt + 2t + 5.$

Example 12. *Let, n and m are the number of vertices and edges of G, respectively. One of the hydrogen suppressed molecular graph is the bottleneck graph (B) of a given graph G, which is defined as the corona product of K_2 and G. Using last theorem, the first and second vertex Zagreb indices of bottleneck graph of G are given by*

(i) $\bar{M}_1^*(B) = 2\bar{M}_1^*(G) + 8n^2 + 4nm + 8n + 8m + 2,$

(ii) $\bar{M}_2^*(B) = 2\bar{M}_2^*(G) + 7n^2 + 4m^2 + 16nm + 5n + 4m + 1.$

2 CONCLUSION

In this paper, we have studied the first and second vertex Zagreb indices of different graph operations. Also we apply our results to compute the vertex Zagreb indices for some special classes of graphs and nano-structures. For further study, vertex Zagreb indices of some other graph operations and for different composite graphs can be computed.

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Де Н. *Індекси Загреба вершин для деяких операцій з графами* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 215–223.

Нещодавно Таваколі М. ввів новий клас індексів Загреба, які називаються індекси Загреба вершин. У цій статті подані явні вирази для різних операцій з графами та отримані формули для обчислення індексів Загреба вершин для деяких хімічних графів.

Ключові слова і фрази: степінь, топологічний індекс, індекс Загреба, операції з графами.

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ON THE INTERSECTION OF WEIGHTED HARDY SPACES

Let $H^p_\sigma(\mathbb{C}_+)$, $1 \leq p < +\infty$, $0 \leq \sigma < +\infty$, be the space of all functions f analytic in the half plane $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ and such that

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r |\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

We obtain some properties and description of zeros for functions from the space $\bigcap_{\sigma>0} H^p_\sigma(\mathbb{C}_+)$.

Key words and phrases: zeros of functions, weighted Hardy space, angular boundary values.

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INTRODUCTION

Let $H^p(\mathbb{C}_+)$, $1 \leqslant p < +\infty$, be the Hardy space of holomorphic in $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ functions f such that

$$||f||^p = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\} < +\infty.$$

Let $H^p_\sigma(\mathbb{C}_+)$, $1 \leq p < +\infty$, $0 \leq \sigma < +\infty$, be the space of all functions f analytic in the half plane $\mathbb{C}_+$ and such that

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r |\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

We denote by $H^\infty_\sigma(\mathbb{C}_+)$, $0 \leq \sigma < +\infty$, the space of all functions analytic in the right half-plane satisfying the condition

$$||f|| := \sup_{z \in \mathbb{C}_+} \left\{ |f(z)| e^{-\sigma |\mathrm{Im} z|} \right\} < +\infty.$$

The space $H^p_\sigma(\mathbb{C}_+)$, $1 \leq p \leq +\infty$, $0 \leq \sigma < +\infty$, is a weighted Hardy space, as it follows from results of A. M. Sedletskii [9]. The theory of weighted Hardy space for the case if the weight is an exponential function considered in [2, 3, 10–13]. Functions $f \in H^p_\sigma(\mathbb{C}_+)$ have angular boundary values almost everywhere on $\partial\mathbb{C}_+$ (we denote the extension by the same

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symbols f) and $f \in L^p(\partial\mathbb{C}_+)$. Thus, the space $H^p_\sigma(\mathbb{C}_+)$, $p \geq 1$, is a Banach space. For functions $f \in H^p_\sigma(\mathbb{C}_+)$ there exists [4,12] an integral boundary function defined by the equality

$$h(t_2) - h(t_1) = \lim_{x\to 0+} \int_{t_1}^{t_2} \ln|f(x+it)|dt - \int_{t_1}^{t_2} \ln|f(it)|dt, \quad t_1 < t_2$$

up to an additive constant and to values at continuity points. The integral boundary function h is nonincreasing on $\mathbb{R}$ and $h'(t) = 0$ almost everywhere on $\mathbb{R}$. The interest to the space $H^p_\sigma(\mathbb{C}_+)$ is generated by studies of completeness [3], by the theory of integral operators and the shift operator [1,8].

A number of papers have been devoted to the intersection of Hardy and related spaces (see [5,7]). The aim of our research is to describe some properties of the following space

$$H^p_\cap(\mathbb{C}_+) = \bigcap_{\sigma>0} H^p_\sigma(\mathbb{C}_+).$$

Obviously, $H^p_\cap(\mathbb{C}_+) \supset H^p(\mathbb{C}_+)$ and $H^p_\cap(\mathbb{C}_+) \subset H^p_\varepsilon(\mathbb{C}_+)$ for all ε.

1 The main results

Theorem 1. $H^p_\cap(\mathbb{C}_+) \neq H^p(\mathbb{C}_+)$.

Proof. Let $f(z) = e^{-z\sqrt{\ln(z+2)}}$. We choose the branch of the logarithm that $\ln 1 = 0$ and $\sqrt{1} = 1$. Let us prove that the function f belongs to $H^p_\sigma(\mathbb{C}_+)$ for all $\sigma > 0$. Indeed,

$$\ln|f(re^{i\varphi})| = -r\sqrt[4]{\ln^2\sqrt{4r\cos\varphi + r^2 + 4} + \operatorname{arctg}^2\frac{r\sin\varphi}{r\cos\varphi + 2}}$$
$$\times\left(\cos\varphi\cos\frac{\operatorname{arctg}\frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2} - \sin\varphi\sin\frac{\operatorname{arctg}\frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2}\right)$$
$$\leq r\sqrt[4]{\ln^2\sqrt{4r\cos\varphi + r^2 + 4} + \operatorname{arctg}^2\frac{r\sin\varphi}{r\cos\varphi + 2}}\sin\varphi\sin\frac{\operatorname{arctg}\frac{\operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi+2}}{\ln\sqrt{4r\cos\varphi+r^2+4}}}{2}$$
$$\leq \frac{r}{2}\varphi\sin\varphi\frac{1}{\sqrt{\ln r}}, \quad r\to+\infty.$$

It follows easily that $f \in H^p_\sigma(\mathbb{C}_+)$. Consequently, $f \in H^p_\cap(\mathbb{C}_+)$.

Let us show that $f(z) = e^{-z\sqrt{\ln(z+2)}} \notin H^p(\mathbb{C}_+)$. Indeed,

$$\ln|f(iy)| = y\sqrt[4]{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2\frac{y}{2}}\sin\frac{\operatorname{arctg}\frac{\operatorname{arctg}\frac{y}{2}}{\ln\sqrt{4+y^2}}}{2}$$
$$= \frac{y}{\sqrt{2}}\sqrt[4]{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2\frac{y}{2}}\sqrt{1 - \frac{\ln\sqrt{4+y^2}}{\sqrt{\ln^2\sqrt{4+y^2} + \operatorname{arctg}^2\frac{y}{2}}}}$$
$$\geq \frac{y}{\sqrt{2\ln(4+y^2)}} \quad \text{for} \quad y \geq C > 0.$$

Therefore $f(iy) \notin L^p(0;+\infty)$. Hence, $f \notin H^p(\mathbb{C}_+)$. $\square$

Proposition 1. *Suppose that $f \in H^p_\cap(\mathbb{C}_+)$, $1 \le p \le \infty$. Then the following conditions are fulfilled :*

a) angular boundary values exist almost everywhere on $i\mathbb{R}$;

b) $|f(it)|e^{-\varepsilon|t|} \in L^p(\mathbb{R})$ for any $\varepsilon > 0$;

c) $H^p_\cap(\mathbb{C}_+)$ is a Banach space for uniform convergence on compact sets.

Proof. Let $f \in H^p_\cap(\mathbb{C}_+)$, then $f \in H^p_\varepsilon(\mathbb{C}_+)$ for some $\varepsilon > 0$. In [11] B. V. Vinnitskii proved that a function $f \in H^p_\sigma(\mathbb{C}_+)$, $p \in (1;+\infty)$, has almost everywhere on $i\mathbb{R}$ angular boundary values $f(iy)$ and $f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R})$. Therefore $f(iy)e^{-\varepsilon|y|} \in L^p(\mathbb{R})$ for some positive ε.

In [10] B. V. Vinnitskii showed that a function $f \in H^\infty_\sigma(\mathbb{C}_+)$ has almost everywhere on $i\mathbb{R}$ angular boundary values $f(it)$ and $f(it)e^{-\varepsilon|t|} \in L^\infty(\mathbb{R})$ for all ε. In [11] inequality

$$|f(z)| \le \frac{c_2 \exp(c_2|z|)}{\operatorname{Re}(z)^{\frac{1}{p}}}$$

proved for each function f belonging to $H^p_\sigma(\mathbb{C}_+)$. Furthermore, $H^p_\cap(\mathbb{C}_+)$ is a Banach space with respect to uniform convergence on compact sets. □

Let B is a class of continuous, increasing functions $\eta : [0;+\infty) \to (0;+\infty)$ such that $\eta(r) = o(r)$ as $r \to +\infty$. We denote by $H^p_\ominus(\mathbb{C}_+)$ the space of functions analytic in $\mathbb{C}_+$ for which there exists $\eta \in B$

$$\sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr \right\}^{\frac{1}{p}} < +\infty,$$

where $\eta \in B$.

Theorem 2. *If $f \in H^p_\ominus(\mathbb{C}_+)$, then $f \in H^p_\cap(\mathbb{C}_+)$.*

Proof. Let $f \in H^p_\ominus(\mathbb{C}_+)$, then $f \in H^p_\sigma(\mathbb{C}_+)$ for all $\sigma > 0$. Furthermore,

$$\int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr = \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} e^{-pr\sigma|\sin\varphi| + \eta(r)|\sin\varphi|} dr.$$

Since $-pr\sigma|\sin\varphi| + \eta(r)|\sin\varphi| = |\sin\varphi|(-pr\sigma + \eta(r)) < 0$ as $r > r_0$, we have

$$\int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \le \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr < +\infty.$$

This implies that

$$\sup \left\{ \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \right\} \le \sup \left\{ \int_{r_0}^{+\infty} |f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|} dr \right\}$$

and

$$\sup\left\{\int\limits_0^{r_0}|f(re^{i\varphi})|^p e^{-\eta(r)|\sin\varphi|}dr\right\}\le \sup\left\{\int\limits_0^{r_0}|f(re^{i\varphi})|^p \exp\{\min_{r\in[0;r_0]}\{-\eta(r)\}|\sin\varphi|\}dr\right\}$$
$$\le \sup\left\{\exp\{\min_{r\in[0;r_0]}\{-\eta(r)\}|\sin\varphi|\}\int\limits_0^{r_0}|f(re^{i\varphi})|^p dr\right\}\le c_1\int\limits_0^{r_0}|f(re^{i\varphi})|^p dr<+\infty.$$

In particular, choosing $c_2=\frac{2p\sigma r_0}{\eta(0)}$ we can achieve that

$$\int\limits_0^{+\infty}|f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}dr\le c_2<+\infty.$$

It follows that $f\in H^p_{\ominus}(\mathbb{C}_+)$. □

B. V. Vinnitskii described [11] zeros for functions $f\in H^p_\sigma(\mathbb{C}_+)$ in terms of the following function

$$S(r)=\sum_{1<|\lambda_n|\le r}\left(\frac{1}{|\lambda_n|}-\frac{|\lambda_n|}{r^2}\right)\frac{\mathrm{Re}\lambda_n}{|\lambda_n|},$$

where $\lambda_n\in\mathbb{C}_+$. We obtain the following statement.

Theorem 3. *If $f\in H^p_\cap(\mathbb{C}_+)$, then $S(r)=o(\ln r)$, $r\to+\infty$.*

Proof. Suppose $f\in H^p_\cap(\mathbb{C}_+)$, then $f\in H^p_\sigma(\mathbb{C}_+)$ for all $\sigma>0$. Use the following version of the Carleman formula [4,6,12]

$$\begin{aligned}S(r)=&\frac{1}{\pi r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln|f(re^{i\varphi})|\cos\varphi d\varphi+\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt\\&-\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)|dh(t)|+O(1).\end{aligned}\tag{1}$$

In [11] it is shown that for each function $f\in H^p_\sigma(\mathbb{C}_+),\sigma>0$, the first term on the right side of the last equality is bounded by an independent of r and σ constant. Hence, this term is bounded for each function of the space $H^p_\cap(\mathbb{C}_+)$. Consider the second addend

$$\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt=\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)(\ln|f(it)|e^{-\sigma|t|}+e^{\sigma|t|})dt$$
$$\le\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)(|f(it)|e^{-\sigma|t|}+\sigma|t|)dt.$$

Since $\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\sigma|t|dt=\frac{1}{\pi}\sigma\ln r$ and $f(iy)e^{-\sigma|y|}\in L^p(\mathbb{R})$, this yields

$$\frac{1}{2\pi}\int\limits_{1<|t|\le r}\left(\frac{1}{t^2}-\frac{1}{r^2}\right)\ln|f(it)|dt\le c_3+\frac{1}{\pi}\sigma\ln r.$$

Therefore
$$S(r) = c_4 + \frac{1}{\pi}\sigma \ln r - \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) |dh(t)|.$$
Then, the last addend is negative, we deduce
$$S(r) \le c_4 + \frac{\sigma}{\pi} \ln r.$$
Since the result is true for on of an arbitrary σ, we obtain the statement of the theorem.

□

Theorem 4. *If $f \in H^p_{\cap}(\mathbb{C}_+)$, then $P(r) = o(\ln r)$, $r \to +\infty$, where*
$$P(r) = \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) |dh(t)|.$$

Proof. Let $f \in H^p_{\cap}(\mathbb{C}_+)$, then $f \in H^p_{\sigma}(\mathbb{C}_+)$ for everyone $\sigma > 0$. Using (1), we get $P(r) = K(r) - S(r) + O(1), \quad r \to +\infty$, where
$$K(r) = \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|f(it)| dt.$$
Since
$$\begin{aligned} K(r) &= \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|f(it)|e^{-\sigma|t|} dt + \frac{1}{2\pi} \int\limits_{1<|t|\le r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \sigma|t| dt \\ &\le c_3 + \frac{1}{\pi}\sigma \ln r \quad \text{for all} \quad \sigma > 0, \end{aligned}$$
we deduce $K(r) = o(\ln r)$ as $r \to +\infty$. From Theorem 3 we get the following $S(r) = o(\ln r)$, $r \to +\infty$. Thus $P(r) = o(\ln r), r \to +\infty$. □

Theorem 5. *Let (λ_n) be an arbitrary sequence in $\mathbb{C}_+$. Then $S(r) = o(\ln r)$, $r \to +\infty$, if and only if $S_0(r) = o(\ln r), r \to +\infty$, where*
$$S_0(r) = \sum_{1<|\lambda_n|\le r} \frac{Re\lambda_n}{|\lambda_n|^2}.$$

Proof. It is clear that
$$S_0(r) - S(r) = \sum_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{r^2} \le \sum_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{|\lambda_n| r} = \frac{s(r)}{r},$$
where $s(r) = \sum\limits_{1<|\lambda_n|\le r} \frac{\mathrm{Re}\lambda_n}{|\lambda_n|}$.

In [10] B. V. Vinnitskii proved that
$$S(2r) \ge \frac{3s(r)}{4r}.$$
It follows that
$$S_0(r) - S(r) \le \frac{4rS(2r)}{3r} = \frac{4}{3} S(2r).$$
Since $S(r) = o(\ln r)$, we have $S(2r) = o(\ln r)$. Hence, $S_0(r) - S(r) = o(\ln r), \ r \to +\infty$.

The converse implication is trivial. □

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Нехай $H^p_\sigma(\mathbb{C}_+)$, $1 \le p < +\infty$, $0 \le \sigma < +\infty$, – простір функцій, аналітичних у півплощині $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$, для яких

$$\|f\| := \sup_{\varphi \in (-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

Отримано деякі властивості і опис нулів для функцій з простору $\bigcap\limits_{\sigma>0} H^p_\sigma(\mathbb{C}_+)$.

Ключові слова і фрази: нулі функцій, ваговий простір Гарді, кутові граничні значення.

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DMYTRYSHYN R.I.

A MULTIDIMENSIONAL GENERALIZATION OF THE RUTISHAUSER *QD*-ALGORITHM

In this paper the regular multidimensional C-fraction with independent variables, which is a generalization of regular C-fraction, is considered. An algorithm of calculation of the coefficients of the regular multidimensional C-fraction with independent variables correspondence to a given formal multiple power series is constructed. Necessary and sufficient conditions of the existence of this algorithm are established. The above mentioned algorithm is a multidimensional generalization of the Rutishauser *qd*-algorithm.

Key words and phrases: regular multidimensional C-fraction with independent variables, correspondence, multiple power series, algorithm.

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INTRODUCTION

In constructing the branched continued fractions for a given formal multiple power series the concept of correspondence is used. Some general theory of correspondence for functions of one variables is developed in [15, pp. 148–160] (see also [11, pp. 241–274]) and some aspects of it for functions of several variables are considered in [7], [6, pp. 107–109]. As a result, different types of functional fractions are constructed in [1–6, 8–10, 12–14, 16].

In the present paper we construct and investigate an algorithm for the expansion of a given formal multiple power series into a corresponding regular multidimensional C-fraction with independent variables, which is a generalization of the regular C-fraction [15, p. 128-129]. It is a further expansion of the results obtained in [2].

1 CORRESPONDENCE

Let $\mathcal{L}$ be set of all formal multiple power series of the form

$$L(\mathbf{z}) = \sum_{|m(N)|\geq 0} c_{m(N)}\mathbf{z}^{m(N)}, \tag{1}$$

where $m(N) = m_1, m_2, \ldots, m_N$ is multiindex, $m_i \in \mathbb{Z}_+$, $1 \leq i \leq N$, $0(N) = 0, 0, \ldots, 0$, $|m(N)| = m_1 + m_2 + \cdots + m_N$, $c_{m(N)} \in \mathbb{C}$, $\mathbf{z}^{m(N)} = z_1^{m_1} z_2^{m_2} \cdot \ldots \cdot z_N^{m_N}$, $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^N$. Obviously, this set forms a ring with unity respect to the operations addition and multiplication of series. We define the mapping $\lambda : \mathcal{L} \to \mathbb{N}_0 \cup \{\infty\}$ as follows: $\lambda(L(\mathbf{z})) = \infty$, if $L(\mathbf{z}) \equiv 0$;

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$\lambda(L(\mathbf{z})) = n$, if $L(\mathbf{z}) \not\equiv 0$, where n is the smallest degree of homogeneous polynomial for which $c_{m(N)} \neq 0$, that is $n = |m(N)|$. We consider the sequence of rational functions

$$f_n(\mathbf{z}) = \frac{P_{m_n}(\mathbf{z})}{Q_{l_n}(\mathbf{z})}, \quad n \geq 1,$$

where $P_{m_n}(\mathbf{z})$, $Q_{l_n}(\mathbf{z})$ are polynomials of degrees m_n and l_n respectively, $\mathbf{z} \in \mathbb{C}^N$, moreover, $Q_{l_n}(0, 0, \ldots, 0) \neq 0$.

The sequence $\{f_n(\mathbf{z})\}$ corresponds to series (1) at $\mathbf{z} = (0, 0, \ldots, 0)$, if

$$\lim_{n \to +\infty} \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z}))) = +\infty,$$

where $L(f_n(\mathbf{z}))$ is expansion of function $f_n(\mathbf{z})$ into Taylor series at $\mathbf{z} = (0, 0, \ldots, 0)$. The order of correspondence of $f_n(\mathbf{z})$ is defined by the formula $\nu_n = \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z})))$. This means that the expansion $f_n(\mathbf{z})$ into formal multiple power series coincides with $L(\mathbf{z})$ for all homogeneous polynomials to the degree $(\nu_n - 1)$ inclusively.

Let us introduce the following set of multiindices

$$\mathcal{J} = \{m(N) : \ m(N) = m_1, m_2, \ldots, m_N, \ m_p \in \mathbb{Z}_+, \ 1 \leq p \leq N\}.$$

And now, let us define arithmetical operations on the set $\mathcal{J}$ componentwise. If

$$r(N) = r_1, r_2, \ldots, r_N \in \mathcal{J}, \quad s(N) = s_1, s_2, \ldots, s_N \in \mathcal{J}, \quad k \in \mathbb{Z}_+,$$

then

$$r(N) + s(N) = r_1 + s_1, r_2 + s_2, \ldots, r_N + s_N, \quad kr(N) = kr_1, kr_2, \ldots, kr_N.$$

We consider the regular multidimensional C-fraction with independent variables

$$\frac{a_0}{1 + \operatorname*{D}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)} z_{i_k}}{1}} = \cfrac{a_0}{1 + \sum_{i_1=1}^{N} \cfrac{a_{i(1)} z_{i_1}}{1 + \sum_{i_2=1}^{i_1} \cfrac{a_{i(2)} z_{i_2}}{1 + \ddots}}}, \tag{2}$$

where $i(k) = i_1, i_2, \ldots, i_k$ is multiindex, $a_0 \neq 0$, $a_{i(k)} \neq 0$, $k \geq 1$, $1 \leq i_n \leq i_{n-1}$, $1 \leq n \leq k$, $i_0 = N$, $\mathbf{z} \in \mathbb{C}^N$.

Let $e_0 = 0, 0, \ldots, 0$, $e_r = \delta_{r,1}, \delta_{r,2}, \ldots, \delta_{r,N}$ be a multiindex, $\delta_{r,s}$ be a Kronecker symbol, $1 \leq r, s \leq N$. Let us introduce the following sets of multiindices

$$\mathcal{I} = \{i(k) : \ i(k) = i_1, i_2, \ldots, i_k, \ 1 \leq i_p \leq i_{p-1}, \ 1 \leq p \leq k, \ k \geq 1, \ i_0 = N\},$$
$$\mathcal{I}^* = \{\mathbf{i}^N_{i(k)} : \ \mathbf{i}^N_{i(k)} = e_{i_1} + e_{i_2} + \cdots + e_{i_k}, \ i(k) \in \mathcal{I}\}$$

and the mapping $\varphi : \mathcal{I} \to \mathcal{I}^*$, such that $\varphi(i(k)) = \mathbf{i}^N_{i(k)}$ for all $i(k) \in \mathcal{I}$ (we can show that the mapping φ is bijective).

Let $a_0 = b_0$, $a_{i(k)} = b_{\mathbf{i}^N_{i(k)}}$, $i(k) \in \mathcal{I}$, $\mathbf{i}^N_{i(k)} \in \mathcal{I}^*$. Then we write fraction (2) in the form

$$b_0 \left(1 + \operatorname*{D}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{b_{\mathbf{i}^N_{i(k)}} z_{i_k}}{1}\right)^{-1}, \tag{3}$$

where $b_0 \neq 0$, $b_{\mathbf{i}^N_{i(k)}} \neq 0$, $\mathbf{i}^N_{i(k)} \in \mathcal{I}^*$, $\mathbf{z} \in \mathbb{C}^N$.

Let

$$g_n(\mathbf{z}) = b_0 \left(1 + \mathop{\mathrm{D}}_{k=1}^{n-1} \sum_{i_k=1}^{i_{k-1}} \frac{b_{\mathbf{i}^N_{i(k)}} z_{i_k}}{1} \right)^{-1}$$

be the nth approximant of regular multidimensional C-fraction with independent variables (3), $n \geq 1$.

The correspondence of fraction (3) to series (1) means that the sequence of approximants $\{g_n(\mathbf{z})\}$ corresponds to $L(\mathbf{z})$.

2 ALGORITHM

We shall construct and investigate the algorithm for the expansion of the formal multiple power series (1) into the corresponding regular multidimensional C-fraction with independent variables (3).

Let $c_{0(N)} \neq 0$ and

$$R_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} \frac{c_{m(N)}}{c_{0(N)}} \mathbf{z}^{m(N)}.$$

Next, let

$$R'_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} c^{(e_0)}_{m(N)} \mathbf{z}^{m(N)} \tag{4}$$

be reciprocal to series $R_{e_0}(\mathbf{z})$. The coefficient of FMPS (4) are uniquely determined by recurrent formulas

$$c^{(e_0)}_{m(N)} = - \sum_{|r(N)|=1}^{|m(N)|} c^{(e_0)}_{m(N)-r(N)} \frac{c_{r(N)}}{c_{0(N)}}, \quad m_j \geq 0,\ 1 \leq j \leq N,\ |m(N)| \geq 1, \tag{5}$$

where $c^{(e_0)}_{0(N)} = 1$, moreover, $c^{(e_0)}_{m(N)} = 0$, if here exist an index j, $1 \leq j \leq N$, such that $n_j < 0$.

By condition $c^{(e_0)}_{e_j} \neq 0$, $2 \leq j \leq N$, we write the series (4) in the form

$$R'_{e_0}(\mathbf{z}) = P_{e_1}(z_1) + \sum_{j=2}^{N} c^{(e_0)}_{e_j} z_j R_{e_j}(\mathbf{z}),$$

where

$$P_{e_1}(z_1) = \sum_{\substack{m_1=0 \\ m_j=0,\, 2 \leq j \leq N}}^{\infty} c^{(e_0)}_{m(N)} z_1^{m_1}, \quad R_{e_j}(\mathbf{z}) = \sum_{\substack{|r(N)| \geq 0 \\ r_i=0,\, j+1 \leq i \leq N}} \frac{c^{(e_0)}_{e_j+r(N)}}{c^{(e_0)}_{e_j}} \mathbf{z}^{r(N)}.$$

Then $L(\mathbf{z})$ can be written

$$L(\mathbf{z}) = \frac{c_{0(N)}}{P_{e_1}(z_1) + \sum\limits_{j=2}^{N} c^{(e_0)}_{e_j} z_j R_{e_j}(\mathbf{z})}.$$

Let

$$\sum_{\substack{m_1=0 \\ m_j=0,\, 2 \leq j \leq N}}^{\infty} c_{m(N)} z_1^{m_1}$$

be a normal series (for the notion of normality of formal power series, see [15, pp. 185-190]). Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table for $h = 0$:

$$
\begin{array}{ccccccc}
 & q^{(0)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & & & & \\
e^{(1)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & & & \\
 & q^{(1)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & q^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & & \\
e^{(2)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(1)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & e^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & \\
 & q^{(2)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & q^{(1)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & \vdots & \ddots & \\
e^{(3)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(2)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & \vdots & & & \\
\vdots & q^{(3)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & \vdots & & & & \\
 & \vdots & & & & &
\end{array}
\tag{6}
$$

the entries of which are defined by the initial conditions

$$
e^{(n)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} = 0, \quad q^{(n)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} = \frac{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{m(N)+e_{i_1}+e_{j_h}}}{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{m(N)+e_{j_h}}}, \quad |m(N)| = m_{i_1} = n, \ \ n \ge 0, \tag{7}
$$

moreover,

$$
q^{(n)}_{\mathbf{i}^N_{i(1)}} = \frac{c_{m(N)+e_{i_1}}}{c_{m(N)}}, \quad |m(N)| = m_{i_1} = n, \ n \ge 0,
$$

and the rhombus rule

$$
\begin{gathered}
e^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} + q^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} = q^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} + e^{(n+1)}_{\mathbf{i}^N_{i(r-1)}+\mathbf{j}^N_{j(h)}}, \quad r \ge 1, \ n \ge 0, \\
e^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} q^{(n)}_{\mathbf{i}^N_{i(r+1)}+\mathbf{j}^N_{j(h)}} = q^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} e^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}}, \quad r \ge 1, \ n \ge 0,
\end{gathered}
\tag{8}
$$

The procedure of calculation of the elements of table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) is called the Rutishauser qd-algorithm [15, p. 227].

We put $b_{\mathbf{i}^N_{i(2k-1)}} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}}$, $b_{\mathbf{i}^N_{i(2k)}} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$
\sum_{\substack{m_1=0 \\ m_j=0,\, 2\le j\le N}}^{\infty} \frac{c_{m(N)}}{c_{0(N)}} z_1^{m_1} \sim \left(1 + \mathop{\mathrm{D}}_{\substack{k=1 \\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} \right)^{-1}.
$$

Here the symbol "$\sim$" means the correspondence between the series and the fraction. Moreover, according to Lemma 3 [4] we have

$$
P_{e_1}(z_1) \sim 1 + \mathop{\mathrm{D}}_{\substack{k=1 \\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}} z_1}{1},
$$

since the series $P_{e_1}(z_1)$ is reciprocal to series

$$\sum_{\substack{m_1=0\\ m_j=0,\, 2\le j\le N}}^{\infty} \frac{c_{m(N)}}{c_{0(N)}} z_1^{m_1}.$$

Thus we can write

$$L(\mathbf{z}) \sim \frac{c_{0(N)}}{1+\mathop{\mathrm{D}}\limits_{\substack{k=1\\ i_p=1,\, 1\le p\le k}}^{\infty} \dfrac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} + \sum\limits_{j=2}^{N} c_{e_j}^{(e_0)} z_j R_{e_j}(\mathbf{z})}.$$

Let l be an arbitrary natural number, moreover, $2 \le l \le N$. Next, let

$$\sum_{\substack{m_l=0\\ m_j=0,\, j\ne l,\, 1\le j\le N}}^{\infty} c_{m(N)} z_l^{m_l}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $h = 0$.

We put $b'_{\mathbf{i}^N_{i(2k-1)}} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}}$, $b'_{\mathbf{i}^N_{i(2k)}} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_l=0\\ m_j=0,\, j\ne l,\, 1\le j\le N}}^{\infty} \frac{c_{m(N)} z_l^{m_l}}{c_{0(N)}} \sim \left(1+\mathop{\mathrm{D}}\limits_{\substack{k=1\\ i_p=l,\, 1\le p\le k}}^{\infty} \frac{b'_{\mathbf{i}^N_{i(k)}} z_l}{1}\right)^{-1}.$$

Since

$$c^{(e_0)}_{m(N)} = -\frac{c_{m(N)}}{c_{0(N)}} = b'_{\mathbf{i}^N_{i(1)}}, \quad m_l = 1,\ m_j = 0,\ j \ne l,\ 1 \le j \le N,\ i_1 = l,$$

then we put $b_{\mathbf{i}^N_{i(1)}} = b'_{\mathbf{i}^N_{i(1)}}$, $i_1 = l$.

Thus we can write

$$L(\mathbf{z}) \sim \frac{c_{0(N)}}{1+\mathop{\mathrm{D}}\limits_{\substack{k=1\\ i_p=1,\, 1\le p\le k}}^{\infty} \dfrac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} + \sum\limits_{j_1=2}^{N} b_{\mathbf{j}^N_{j(1)}} z_{j_1} R_{\mathbf{j}^N_{j(1)}}(\mathbf{z})}.$$

Again, let l be an arbitrary natural number, moreover, $2 \le l \le N$. Next, let

$$R'_{e_l}(\mathbf{z}) = \sum_{\substack{|m(N)|\ge 0\\ m_i=0,\, l+1\le i\le N}} c^{(e_l)}_{m(N)} \mathbf{z}^{m(N)} \tag{9}$$

be reciprocal to series $R_{e_l}(\mathbf{z})$. The coefficients of series (9) are uniquely determined by recurrent formulas for $m_i = 0,\ j_h + 1 \le i \le N,\ |m(N)| \ge 1$, and $\mathbf{j}^N_{j(h)} = e_l$

$$c^{(\mathbf{j}^N_{j(h)})}_{m(N)} = -\sum_{|r(N)|=1}^{|m(N)|} c^{(\mathbf{j}^N_{j(h)})}_{m(N)-r(N)} \frac{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{r(N)+e_{j_h}}}{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{e_{j_h}}}, \tag{10}$$

where $c^{(\mathbf{j}^N_{j(h)})}_{0(N)} = 1$, moreover, $c^{(\mathbf{j}^N_{j(h)})}_{n(N)} = 0$, if here exist an index p, $1 \le p \le N$, such that $n_p < 0$.

By condition $c^{(e_l)}_{e_j} \ne 0$, $2 \le j \le l$, we write the series (9) in the form

$$R'_{e_l}(\mathbf{z}) = P_{e_l+e_1}(z_1) + \sum_{j=2}^{l} c^{(e_l)}_{e_j} z_j R_{e_l+e_j}(\mathbf{z}),$$

where

$$P_{e_l+e_1}(z_1) = \sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} c^{(e_l)}_{m(N)} z_1^{m_1}, \quad R_{e_l+e_j}(\mathbf{z}) = \sum_{\substack{|r(N)|\ge 0\\ r_i=0,\ j+1\le i\le N}} \frac{c^{(e_l)}_{e_j+r(N)}}{c^{(e_l)}_{e_j}} \mathbf{z}^{r(N)}.$$

Then $R_{e_l}(\mathbf{z})$ can be written as follows

$$R_{e_l}(\mathbf{z}) = \left(P_{e_l+e_1}(z_1) + \sum_{j=2}^{l} c^{(e_l)}_{e_j} z_j R_{e_l+e_j}(\mathbf{z}) \right)^{-1}.$$

Let

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} c^{(e_0)}_{m(N)+e_l} z_1^{m_1}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $\mathbf{j}^N_{j(h)} = e_l$.

We put $b_{\mathbf{i}^N_{i(2k-1)}+e_l} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}+e_l}$, $b_{\mathbf{i}^N_{i(2k)}+e_l} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}+e_l}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_1^{m_1} \sim \left(1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=1,\ 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+e_l} z_1}{1} \right)^{-1}.$$

Since the series $P_{e_l+e_1}(z_1)$ is reciprocal to series

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_1^{m_1},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_1}(z_1) \sim 1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+e_l} z_1}{1}.$$

Let t be an arbitrary natural number, moreover, $2 \le t \le l-1$. Next, let

$$\sum_{\substack{m_t=0\\ m_j=0,\, j\ne t,\, 1\le j\le N}}^{\infty} c^{(e_0)}_{m(N)+e_l} z_t^{m_t}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $i_p = t$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $\mathbf{j}^N_{j(h)} = e_l$.

We put $b'_{\mathbf{i}^N_{i(2k-1)}+e_l} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}+e_l}$, $b'_{\mathbf{i}^N_{i(2k)}+e_l} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}+e_l}$, $i_p = t$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_t=0\\ m_j=0,\, j\ne t,\, 1\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_t^{m_t} \sim \left(1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=t,\, 1\le p\le k}}^{\infty} \frac{b'_{\mathbf{i}^N_{i(k)}+e_l} z_t}{1}\right)^{-1}.$$

Since the series $P_{e_l+e_r}(z_t)$ is reciprocal to series

$$\sum_{\substack{m_t=0\\ m_j=0,\, j\ne t,\, 1\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_t^{m_t},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_r}(z_t) \sim 1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=t,\, 1\le p\le k}}^{\infty} \frac{b'_{\mathbf{i}^N_{i(k)}+e_l} z_t}{1}.$$

Since

$$c^{(e_l)}_{e_t} = -\frac{c^{(e_0)}_{e_l+e_t}}{c^{(e_0)}_{e_l}} = -\frac{c_{0(N)} c_{e_l+e_t} - c^2_{e_l}}{c_{e_l} c_{0(N)}} = b'_{e_l+e_t}, \quad c^{(e_l)}_{e_l} = -\frac{c^{(e_0)}_{2e_l}}{c^{(e_0)}_{e_l}} = -\frac{c_{0(N)} c_{2e_l} - c^2_{e_l}}{c_{e_l} c_{0(N)}} = b'_{2e_l},$$

than we put $b_{e_l+e_t} = b'_{e_l+e_t}$, $b_{2e_l} = b'_{2e_l}$.

Thus we can write

$$L(\mathbf{z}) \sim \cfrac{c_{0(N)}}{Q_{\mathbf{j}^N_{j(0)}}(z_1) + \sum\limits_{j_1=2}^{N} \cfrac{b_{\mathbf{j}^N_{j(1)}} z_{j_1}}{Q_{\mathbf{j}^N_{j(1)}}(z_1) + \sum\limits_{j_2=2}^{j_1} b_{\mathbf{j}^N_{j(2)}} z_{j_2} R_{\mathbf{j}^N_{j(2)}}(\mathbf{z})}},$$

where

$$Q_{\mathbf{j}^N_{j(h)}}(z_1) = 1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=1,\,1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}} z_1}{1}, \quad h \ge 0,$$

moreover, $j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, if $h \ge 1$.

Next, computing the coefficients

$$c^{(\mathbf{j}^N_{j(h)})}_{m(N)}, \quad m_i = 0,\ j_h + 1 \le i \le N,\ |m(N)| \ge 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*,$$

by recurrent formulas (10) and continuing process of iteration under the conditions that the series

$$\sum_{\substack{m_l=0\\ m_i=0,\, i\neq l,\, 1\le i\le N}}^{\infty} c_{m(N)} z_l^{m_l}, \qquad \sum_{\substack{m_t=0\\ m_i=0,\, i\neq t,\, 1\le i\le N}}^{\infty} c^{(e_0)}_{m(N)+e_p} z_t^{m_t}, \qquad \sum_{\substack{m_r=0\\ m_i=0,\, i\neq r,\, 1\le i\le N}}^{\infty} c^{(\mathbf{j}^N_{j(h)})}_{m(N)+e_{j_h}} z_r^{m_r}, \tag{11}$$

where $1 \le l \le N,\ 1 \le t \le p-1,\ 2 \le p \le N,\ 1 \le r \le j_h - 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, are normal, for series (1) we obtain fraction (3), where $c_0 = c_{0(N)}$, $b_{\mathbf{i}^N_{i(2k-1)}+\mathbf{j}^N_{j(h)}} = -q^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $b_{\mathbf{i}^N_{i(2k)}+\mathbf{j}^N_{j(h)}} = -e^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $i_p = n,\ 1 \le p \le k,\ 1 \le n \le j_h - 1,\ k \ge 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$ (the numbers $q^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $e^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $i_p = n,\ 1 \le p \le k,\ 1 \le n \le j_h - 1,\ k \ge 1,\ j_r \neq 1,$ $1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, are the diagonal elements of the qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8)).

Thus, if the coefficients of the formal multiple power series (1) are given, then the recurrent algorithm of calculation of the coefficients of the regular multidimensional C-fraction with independent variables (3) is constructed. This algorithm is a multidimensional generalization of Rutishauser qd-algorithm [15, p. 227]. The correspondence of fraction (3) to series (1) can be proved by a scheme proposed in [5].

Hence, the following theorem holds:

Theorem. *The regular multidimensional C-fraction with independent variables* (3) *corresponds to the given formal multiple power series* (1) *if and only if the formal power series* (11) *are normal.*

We remark that some examples of functions of two variables represented by regular two-dimensional C-fractions with independent variables are given in [2].

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Дмитришин Р.І. *Багатовимірне узагальнення* qd*-алгоритму Рутисхаузера* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 230–238.

Розглядається регулярний багатовимірний C-дріб з нерівнозначними змінними, який є узагальненням регулярного C-дробу. Побудовано алгоритм обчислення коефіцієнтів багатовимірного C-дробу з нерівнозначними змінними, відповідного заданому формальному кратному степеневому ряду, який є узагальненням *qd*-алгоритму Рутисхаузера. Встановлено необхідні та достатні умови існування такого алгоритму.

Ключові слова і фрази: регулярний багатовимірний C-дріб з нерівнозначними змінними, відповідність, кратний степеневий ряд, алгоритм.

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ANALOGUES OF WHITTAKER'S THEOREM FOR LAPLACE-STIELTJES INTEGRALS

Lower estimates on a sequence for the maximum of the integrand of Laplace-Stieltjes integrals are found. Using these estimates we obtained analogues of Whittaker's theorem for entire functions given by lacunary power series.

Key words and phrases: Laplace-Stieltjes integral, maximum of integrand, Whittaker's theorem.

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INTRODUCTION

For an entire function

$$g(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad z = re^{i\theta}, \tag{1}$$

let $M_g(r) = \max\{|g(z)| : |z| = r\}$ and $\varrho = \overline{\lim\limits_{r\to+\infty}} \dfrac{\ln \ln M_g(r)}{\ln r}$, $\lambda = \underline{\lim\limits_{r\to+\infty}} \dfrac{\ln \ln M_g(r)}{\ln r}$ be the order and the lower order of g correspondingly. J.M. Whittaker [1] has proved that $\lambda \le \varrho\beta$, where $\beta = \underline{\lim\limits_{n\to+\infty}} (\ln \lambda_n)/\ln \lambda_{n+1}$. For an analytic in $\{z : |z| < 1\}$ function (1) of the order $\varrho_0 = \overline{\lim\limits_{r\uparrow 1}} \dfrac{\ln \ln M_g(r)}{-\ln(1-r)}$ and the lower order $\lambda_0 = \overline{\lim\limits_{r\uparrow 1}} \dfrac{\ln \ln M_g(r)}{-\ln(1-r)}$ L.R. Sons [2] tried to prove that $\lambda_0 + 1 \le (\varrho_0 + 1)\beta$. In [3] this result is disproved and it is showed that $\lambda_0 \le \varrho_0\beta$, i. e. absolute analogue of Whittaker's theorem is valid. Moreover, in [3] it is obtained analogues of Whittaker's theorem for Dirichlet series $\sum\limits_{n=0}^{\infty} a_n e^{\lambda_n s}$, $s = \sigma + it$, with an arbitrary abscissa of the absolute convergence $\sigma_a = A \in (-\infty, +\infty]$, where $0 = \lambda_0 < \lambda_n \uparrow +\infty$, $n \to \infty$.

Here we investigate similar problems for Laplace-Stieltjes integrals.

1 MAIN RESULTS

Let V be the class of all nonnegative nondecreasing unbounded continuous on the right functions F on $[0, +\infty)$. We say that $F \in V(l)$ if $F \in V$ and $F(x) - F(x-0) \le l < +\infty$ for all $x \ge 0$.

For a nonnegative function f on $[0, +\infty)$ the integral

$$I(\sigma) = \int_0^{\infty} f(x) e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \tag{2}$$

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is called of Laplace-Stieltjes [4]. Integral (1) is a direct generalisation of the ordinary Laplace integral $I(\sigma) = \int_0^\infty f(x)e^{x\sigma}dx$ and of the Dirichlet series $\sum\limits_{n=0}^{\infty} a_n e^{\lambda_n \sigma}$ with nonnegative coefficients a_n and exponents λ_n, $0 \le \lambda_n \uparrow +\infty$, $n \to \infty$, if we choose $F(x) = n(x) = \sum\limits_{\lambda_n \le x} 1$ and $f(\lambda_n) = a_n \ge 0$ for all $n \ge 0$. The maximal therm of this Dirichlet series is defined by formula $\mu(\sigma) = \max\{a_n e^{\lambda_n\sigma} : n \ge 0\}$.

By $\Omega(A)$ we denote the class of all positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, A)$. From now on, we denote by φ the inverse function to Φ', and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. It is clear that the function φ is continuously differentiable and increasing to A on $(0, +\infty)$. The function Ψ is [4–6] continuously differentiable and increasing to A on $(-\infty, A)$.

For $\Phi \in \Omega(A)$ and $0 < a < b < +\infty$ we put

$$G_1(a,b,\Phi) = \frac{ab}{b-a}\int_a^b \frac{\Phi(\varphi(t)}{t^2}dt, \quad G_2(a,b,\Phi) = \Phi\left(\frac{1}{b-a}\int_a^b \varphi(t)dt\right).$$

It is known [5] that $G_1(a,b,\Phi) < G_2(a,b,\Phi)$, and in [3] the following Lemma is proved.

Lemma 1. *Let (x_k) be an increasing to $+\infty$ sequence of positive numbers, $\Phi \in \Omega(A)$ and $\mu_D(\sigma)$ be the maximal term of formal Dirichlet series*

$$D(s) = \sum_{k=1}^{\infty} \exp\{-x_k\Psi(\varphi(x_k)) + sx_k\}, \quad s = \sigma + it.$$

Then

$$\varlimsup_{\sigma\uparrow A} \frac{\ln \mu_D(\sigma)}{\Phi(\sigma)} = 1, \quad \varlimsup_{\sigma\uparrow A} \frac{\ln\ln \mu_D(\sigma)}{\ln \Phi(\sigma)} = 1, \tag{3}$$

$$\varliminf_{\sigma\uparrow A} \frac{\ln \mu_D(\sigma)}{\Phi(\sigma)} = \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \tag{4}$$

and if

$$\ln \mu_D(\sigma) + \left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right)\ln \Phi(\sigma) \ge 0, \quad \sigma \in [\sigma_0,\, A), \tag{5}$$

then

$$\varliminf_{\sigma\uparrow A} \frac{\ln\ln \mu_D(\sigma)}{\ln \Phi(\sigma)} = \varliminf_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{6}$$

It is clear that integral (2) either converges for all $\sigma \in \mathbb{R}$ or diverges for all $\sigma \in \mathbb{R}$ or there exists a number σ_c such that integral (2) converges for $\sigma < \sigma_c$ and diverges for $\sigma > \sigma_c$. In the latter case the number σ_c is called abscissa of the convergence of integral (2). If integral (2) converges for all $\sigma \in \mathbb{R}$ then we put $\sigma_c = +\infty$, and if it diverges for all $\sigma \in \mathbb{R}$ then we put $\sigma_c = -\infty$.

Let

$$\mu(\sigma, I) = \sup\{f(x)e^{x\sigma} : x \ge 0\}, \quad \sigma \in \mathbb{R},$$

be the maximum of the integrand. Then either $\mu(\sigma, I) < +\infty$ for all $\sigma \in \mathbb{R}$ or $\mu(\sigma, I) = +\infty$ for all $\sigma \in \mathbb{R}$ or there exists a number σ_μ such that $\mu(\sigma, I) < +\infty$ for all $\sigma < \sigma_\mu$ and $\mu(\sigma, I) = +\infty$

for for all $\sigma > \sigma_\mu$. By analogy the number σ_μ is called abscissa of maximum of the integrand. It is well known ([4]) that if $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$ then $\sigma_c \geq \sigma_\mu$.

For each Dirichlet series $\sigma_c \leq \sigma_\mu$. In general case this inequality can be not executed. We will say in this connection as in [4] that a nonnegative function f has regular variation in regard to F if there exist $a \geq 0$, $b \geq 0$ and $h > 0$ such that for all $x \geq a$

$$\int\limits_{x-a}^{x+b} f(t)dF(t) \geq hf(x). \tag{7}$$

In [4] it is proved that if $F \in V$ and f has regular variation in regard to F then $\sigma_c \leq \sigma_\mu$. We need also the following lemma.

Lemma 2 ([4]). *Let $\sigma_\mu = A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. In order that $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$, it is necessary and sufficient that $\ln f(x) \leq -x\Psi(\varphi(x))$ for all $x \geq x_0$.*

Let L be the class of all positive continuous functions α increasing to $+\infty$ on $(x_0, +\infty)$, $x_0 \geq -\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, and $\alpha \in L_{si}$ if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$.

Using Lemmas 1 and 2 first we will prove the following theorem.

Theorem 1. *Let $\sigma_\mu = +\infty$, $\Phi \in \Omega(+\infty)$, $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for all $\sigma \geq \sigma_0$ and $X = (x_k)$ be a some sequence of positive numbers increasing to $+\infty$. Suppose that f is a nonincreasing function. Then:*

1) if either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\Phi \in L^0$, or $x_{k+1} - x_k \leq H < +\infty$ for all $k \geq 0$, or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\Phi \in L^0$, then

$$\varliminf_{\sigma \to +\infty} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \leq \varliminf_{k \to \infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}; \tag{8}$$

2) if

$$\ln \sigma + \left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right) \ln \Phi(\sigma) \geq q > -\infty, \quad \sigma \geq \sigma_0, \tag{9}$$

and either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\ln \Phi \in L^0$, or $\ln f(x_k) \leq a \ln f(x_{k+1})$, $0 < a < 1$, and $\ln \Phi \in L_{si}$, or $x_{k+1} - x_k \leq H < +\infty$ for all $k \geq 0$, or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\Phi \in L^0$ or $x_{k+1} \leq Ax_k$ for all $k \geq 0$ and $\ln \Phi \in L_{si}$ then

$$\varliminf_{\sigma \to +\infty} \frac{\ln \ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \leq \varliminf_{k \to \infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{10}$$

Proof. We remark that in view of the condition $\sigma_\mu = +\infty$ we have $f(x) \to 0$ as $x \to +\infty$ and $\sigma = o(\ln \mu(\sigma, I))$ as $\sigma \to +\infty$. Now, we put $x_0 = 0$ and $\mu(\sigma, I; X) = \max\{f(x_k)e^{\sigma x_k} : k \geq 0\}$. Clearly,

$$\ln \mu(\sigma, I) = \sup_{x \geq 0}(\ln f(x) + \sigma x) \geq \sup_{k \geq 0}(\ln f(x_k) + \sigma x_k) = \ln \mu(\sigma, I, X). \tag{11}$$

Therefore, $\ln\,\mu(\sigma, I; X) \le \Phi(\sigma)$ for all $\sigma \ge \sigma_0$ and by Lemma 2 $\ln\,f(x_k) \le -x_k\Psi(\varphi(x_k))$ for all $k \ge k_0$. Hence it follows that $\ln\,\mu(\sigma, I; X) \le \ln\,\mu_D(r)$ for $\sigma \ge \sigma_0$. Therefore, by Lemma 1 from (4) we obtain

$$\varliminf_{\sigma\to+\infty} \frac{\ln\,\mu(\sigma, I; X)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{12}$$

On the other hand for $\sigma > 0$

$$\ln\,\mu(\sigma, I) = \max_{k\ge0} \sup_{x_k\le x<x_{k+1}} (\ln\,f(x) + x\sigma) \le \max_{k\ge0}(\ln\,f(x_k) + x_{k+1}\sigma). \tag{13}$$

If $\ln\,f(x_k) = (1 + o(1))\ln\,f(x_{k+1})$ as $k \to \infty$ then for every $\varepsilon > 0$ we have $\ln\,f(x_k) \le (\ln\,f(x_{k+1}))/(1 + \varepsilon)$ for all $k \ge k_0 = k_0(\varepsilon)$. Therefore,

$$\begin{aligned}
&\max_{k>0}(\ln\,f(x_k) + x_{k+1}\sigma)\\
&\quad= \max\left\{\max_{k\le k_0}(\ln\,f(x_k) + x_{k+1}\sigma), \max_{k\ge k_0}\left(\frac{\ln\,f(x_k)}{\ln\,f(x_{k+1})}\ln\,f(x_{k+1}) + x_{k+1}\sigma\right)\right\}\\
&\quad\le \max\left\{O(\sigma), \max_{k\ge k_0}\left(\frac{\ln\,f(x_{k+1})}{1+\varepsilon} + x_{k+1}\sigma\right)\right\}\\
&\quad\le \frac{1}{1+\varepsilon}\max_{k\ge0}(\ln\,f(x_{k+1}) + x_{k+1}\sigma(1+\varepsilon)) + O(\sigma), \quad \sigma \to +\infty.
\end{aligned}$$

Hence and from (13) it follows that $\ln\,\mu(\sigma, I) \le \ln\,\mu\,(\sigma(1+\varepsilon), I; X)$ for $\sigma \ge \sigma_0^*$. Thus,

$$\begin{aligned}
\varliminf_{r\to+\infty} \frac{\ln\,\mu(\sigma, I)}{\Phi(\sigma)} &\le \varliminf_{\sigma\to+\infty} \frac{\ln\,\mu(\sigma(1+\varepsilon), I; X)}{\Phi(\sigma)}\\
&\le \varliminf_{r\to+\infty} \frac{\ln\,\mu(\sigma, I; X)}{\Phi(\sigma)} \varlimsup_{\sigma\to+\infty} \frac{\Phi(\sigma(1+\varepsilon))}{\Phi(\sigma)} \le A(\varepsilon) \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)},
\end{aligned} \tag{14}$$

where $A(\varepsilon) = \varlimsup\limits_{r\to+\infty} \dfrac{\Phi(\sigma(1+\varepsilon))}{\Phi(\sigma)}$. For $\Phi \in L^0$ in [7] is proved that $A(\varepsilon) \searrow 1$ as $\varepsilon \downarrow 0$. Therefore, (14) implies (8).

If $x_{k+1} = (1 + o(1))x_k$ as $k \to \infty$ then for arbitrary $\varepsilon > 0$ from (13) it follows that

$$\ln\,\mu(\sigma, I) \le \ln\,\mu\,(\sigma(1+\varepsilon), I; X) + O(\sigma), \quad \sigma_0^*(\varepsilon) \le \sigma \to +\infty,$$

whence in view of the condition $\Phi \in L^0$ as above we obtain (8).

If $\ln\,f(x_k) - \ln\,f(x_{k+1}) = O(1)$ as $k \to \infty$ then from (13) we have

$$\ln\,\mu(\sigma, I) \le \max_{k\ge0}(\ln\,f(x_{k+1}) + x_k\sigma + \ln\,f(x_k) - \ln\,f(x_{k+1})) \le \ln\,\mu(\sigma, I; X) + \text{const}, \tag{15}$$

that is in view of (12)

$$\varliminf_{r\to+\infty} \frac{\ln\,\mu(\sigma, I)}{\Phi(\sigma)} \le \varliminf_{\sigma\to+\infty} \frac{\ln\,\mu(\sigma, I; X)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{16}$$

Finally, if $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ then from (13) follows that

$$\ln\,\mu(\sigma, I) \le \max_{k\ge0}(\ln\,f(x_k) + x_k\sigma + \sigma(x_{k+1} - x_k)) \le \ln\,\mu(\sigma, I; X) + H\sigma, \tag{17}$$

that is in view of (12) we obtain again (16). The first part of Theorem 1 is proved.

Now we will prove the second part. Since $\ln\sigma = o(\ln\mu(\sigma, I))$ as $\sigma \to +\infty$, condition (9) follows from (5).

If either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ then from either (16), or (17) in view of (12) and Lemma 1 we obtain

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\mu(\sigma, I)}{\ln\Phi(\sigma)} \le \varliminf_{\sigma\to+\infty} \frac{\ln\ln\mu(\sigma, I; X)}{\ln\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}.$$

If either $\ln f(x_k) \le (1+o(1))\ln f(x_{k+1})$ or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ as $x \to +\infty$ then as above from (13) we have $\ln\ln\mu(\sigma, I) \le \ln\ln\mu\,(\sigma(1+\varepsilon), I; X)$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma_0(\varepsilon)$, whence (10) follows in view of the condition $\ln\Phi \in L^0$.

If $\ln f(x_k) \le a\ln f(x_{k+1})$, $0 < a < 1$, then from (13) we have

$$\ln\mu(\sigma, I) \le a\max_{k\ge0}(\ln f(x_{k+1}) + x_{k+1}\sigma/a) = a\ln\mu(\sigma/a, I; X);$$

and since $\ln\Phi \in L_{si}$, we obtain

$$\varliminf_{\sigma\to+\infty} \frac{\ln\ln\mu(\sigma, I)}{\ln\Phi(\sigma)} \le \varliminf_{r\to+\infty} \frac{\ln\ln\mu(\sigma/a, I; X)}{\ln\Phi(\sigma/a)} \varlimsup_{r\to+\infty} \frac{\ln\Phi(\sigma/a)}{\ln\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}.$$

If $x_{k+1} \le Ax_k$ for all $k \ge 0$ then $\ln\mu(\sigma, I) \le \ln\mu\,(A\sigma, I; X) + O(\sigma)$ as $\sigma \to +\infty$, whence in view of the condition $\ln\Phi \in L_{si}$ we obtain (10). The proof of Theorem 1 is complete. □

Now we consider the case $\sigma_\mu = 0$. Let $\hat{L}$ be the class of all positive continuous on $(\sigma_0, 0)$, $\sigma_0 \ge -\infty$, functions β, increasing to $+\infty$. We say that $\beta \in \hat{L}^0$ if $\beta \in \hat{L}$ and $\beta((1+o(1))\sigma) = (1+o(1))\beta(\sigma)$ as $\sigma \uparrow 0$, and $\beta \in \hat{L}_{si}$ if $\beta(c\sigma) = (1+o(1))\beta(\sigma)$ as $\sigma \uparrow 0$ for each $c \in (0, +\infty)$.

Lemma 3. *Let $\beta \in \hat{L}$ and $B(\delta) = \varlimsup_{\sigma\uparrow0} \dfrac{\beta(\sigma/(1+\delta))}{\beta(\sigma)}$ $(\delta > 0)$. In order that $\beta \in \hat{L}^0$, it is necessary and sufficient that $B(\delta) \to 1$ as $\delta \downarrow 0$.*

Proof. Suppose that $\beta \in \hat{L}^0$ but $B(\delta) \not\to 1$ as $\delta \downarrow 0$. Since the function $B(\delta)$ is nondecreasing, there exists $\lim\limits_{\delta\downarrow0} B(\delta) = b^* > 1$, that is $B(\delta) \ge b^* > 1$. We choose an arbitrary sequence $(\delta_n) \downarrow 0$. For every δ_n there exists a sequence $(\sigma_{n,k}) \uparrow 0$ such that $\beta((1+\delta_n)\sigma_{n,k}) \ge b\beta(\sigma_{n,k})$, $1 < b < b^*$. We put $\sigma_1 = \sigma_{1,1}$ and $\sigma_n = \min\{\sigma_{n,k} \ge \sigma_{n-1} : k \ge n-1\}$ and construct a function $\gamma(\sigma) \to 0$, $\sigma \uparrow 0$, such that $\gamma(\sigma_n) = \delta_n$. Then $\beta(\sigma_n/(1+\gamma(\sigma_n))) = \beta(\sigma_n/(1+\delta_n)) \ge b\beta(\sigma_n)$. In view of definition of $\hat{L}^0$ it is impossible.

On the contrary, let $B(\delta) \to 1$ as $\delta \downarrow 0$ but $\beta \notin \hat{L}^0$. Then there exists a function $\gamma(\sigma) \to 0$, $\sigma \uparrow 0$, and sequence $(\sigma_n) \uparrow 0$, $n \to \infty$, such that $\lim\limits_{n\to\infty} \beta(\sigma_n/(1+\gamma(\sigma_n))/\beta(\sigma_n) = a \ne 1$. Clearly, $a < 1$ provided $\gamma(\sigma_n) < 0$ and $a > 1$ provided $\gamma(\sigma_n) > 0$. We examine, for example, the second case. Let $\delta > 0$ be an arbitrary number. Then $\gamma(\sigma_n) < \delta$ for $n \ge n_0$ and

$$B(\delta) = \varlimsup_{\sigma\uparrow0} \frac{\beta(\sigma/(1+\delta))}{\beta(\sigma)} \ge \varlimsup_{n\to\infty} \frac{\beta(\sigma_n/(1+\delta))}{\beta(\sigma_n)} \ge \varlimsup_{n\to\infty} \frac{\beta(\sigma_n/(1+\gamma(\sigma_n)))}{\beta(\sigma_n)} = a > 1,$$

which is impossible. Lemma 3 is proved. □

Theorem 2. *Let $\sigma_\mu = 0$, $\Phi \in \Omega(0)$, $\ln\mu(\sigma, I) \le \Phi(\sigma)$ for all $\sigma \ge \sigma_0$ and $X = (x_k)$ be some sequence $X = (x_k)$ of positive numbers increasing to $+\infty$. Suppose that $f(x) \nearrow +\infty$ as $x \to +\infty$. Then:*

1) if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ *or* $x_{k+1} - x_k \le H < +\infty$ *for all* $k \ge 0$*, or* $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ *as* $k \to \infty$ *and* $\Phi \in \hat{L}^0$*, or* $x_{k+1} = (1+o(1))x_k$ *as* $k \to \infty$ *and* $\Phi \in \hat{L}^0$*, or* $x_{k+1} \le Ax_k$ *for* $k \ge 0$ *and* $\Phi \in \hat{L}_{si}$ *then*

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \le \varlimsup_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}, \tag{18}$$

2) if

$$\left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right) \ln \Phi(\sigma) \ge q > -\infty, \quad \sigma \in [\sigma_0, 0), \tag{19}$$

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \le \varlimsup_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{20}$$

Proof. As above let $\mu(\sigma, I; X) = \max\{f(x_k)e^{\sigma x_k} : k \ge 0\}$. Clearly, (11) holds. Therefore, $\ln \mu(\sigma, I; X) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$ and by Lemma 2 $\ln f(x_k) \le -x_k\Psi(\varphi(x_k))$ for all $k \ge k_0$, that is $\ln \mu(\sigma, I; X) \le \ln \mu_D(r)$ for $\sigma \ge \sigma_0$. Therefore, by Lemma 1

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I; X)}{\Phi(\sigma)} \le \varlimsup_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{21}$$

On the other hand for $\sigma < 0$ now we have

$$\ln \mu(\sigma, I) = \max_{k\ge 0} \sup_{x_k \le x < x_{k+1}} (\ln f(x) + x\sigma) \le \max_{k\ge 0} (\ln f(x_{k+1}) + x_k\sigma). \tag{22}$$

Therefore, if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ hence we obtain either $\ln \mu(\sigma, I) \le \ln \mu(\sigma, I; X) + H$ or $\ln \mu(\sigma, I) \le \ln \mu(\sigma, I; X) + H\sigma$, whence

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \le \varlimsup_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I; X)}{\Phi(\sigma)}. \tag{23}$$

Inequalities (21) and (23) imply (18).

If either $x_{k+1} = (1+o(1))x_k$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ then from (23) as in the proof of Theorem 1 for every $\varepsilon > 0$ we have correspondingly $\ln \mu(\sigma, I) \le \ln \mu(\sigma/(1+\varepsilon), I; X)$ and $\ln \mu(\sigma, I) \le (1+\varepsilon) \ln \mu(\sigma/(1+\varepsilon), I; X)$ for $\sigma \in [\sigma_0(\varepsilon), 0)$, whence in view of condition $\ln \Phi \in \hat{L}^0$, of Lemma 3 and of the arbitrariness of ε we obtain (23) and, thus, (18) holds.

Finally, if $x_{k+1} \le Ax_k$ for $k \ge 0$ then $\ln \mu(\sigma, I) \le \ln \mu(\sigma/A, I; X)$, whence in view of condition $\Phi \in \hat{L}_{si}$ we obtain again (23). The first part of Theorem 2 is proved.

For the proof of the second part we remark that from the condition $f(x) \nearrow +\infty$ as $x \to +\infty$ it follows that $\ln \mu(\sigma, I) \uparrow +\infty$ as $\sigma \uparrow 0$. Therefore, (19) implies (5). We remark also that if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\ln \Phi \in \hat{L}^0$ or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\ln \Phi \in \hat{L}^0$ or $x_{k+1} \le Ax_k$ for $k \ge 0$ and $\ln \Phi \in \hat{L}_{si}$ then from the inequalities obtained above we get (20). If $\ln f(x_{k+1}) \le A \ln f(x_k)$ for $k \ge 0$ then from (21) we obtain the inequality $\ln \mu(\sigma, I) \le A \ln \mu(\sigma/A, I; X)$, whence in view of the condition $\ln \Phi \in \hat{L}_{si}$ inequality (20) follows. The proof of Theorem 2 is complete. ☐

2 ANALOGUES OF WHITTAKER'S THEOREM

Examing the other scale of growth from Theorems 1 and 2 gives us a possible to get the series of results for Laplace-Stieltjes integrals. Here we will be stopped only for two cases which more frequent at meet in mathematical works. The most used characteristics of growth for integrals (2) with $\sigma_c = +\infty$ (by analogy with Dirichlet series) are R-order $\varrho_R[I]$, lower R-order $\lambda_R[I]$ and (if $\varrho_R[I] \in (0, +\infty)$) R-type $T_R[I]$, lower R-type $t_R[I]$, which are defined by formulas

$$\varrho_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \ln I(\sigma)}{\sigma}, \quad \lambda_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \ln I(\sigma)}{\sigma},$$
$$T_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\exp\{\sigma\varrho_R[I]\}}, \quad t_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\exp\{\sigma\varrho_R[I]\}}.$$

We will show that in this formulas $\ln I(\sigma)$ can be replaced by $\ln \mu(\sigma, I)$ and will use the following Lemmas for this purpose.

Lemma 4 ([4, 8]). *Let $F \in V$, f has regular variation in regard to F and either $\sigma_\mu = +\infty$ or $\sigma_\mu = 0$ and $\varlimsup_{x\to+\infty} f(x) = +\infty$. Then $\ln \mu(\sigma, I) \le (1 + o(1)) \ln I(\sigma)$ as $\sigma \uparrow \sigma_\mu$.*

Lemma 5 ([4, 9]). *Let $F \in V$, $\sigma_\mu = +\infty$ and $\varlimsup_{x\to+\infty} (\ln F(x))/x = \tau < +\infty$. Then $I(\sigma) \le \mu(\sigma + \tau + \varepsilon, I)$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma(\varepsilon)$.*

It is easy to check that these lemmas imply the following statement.

Proposition 1. *Let $F \in V$, f has regular variation in regard to F and $\sigma_\mu = +\infty$. If $\ln F(x) = O(x)$ as $x \to +\infty$ then*

$$\varrho_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \ln \mu(\sigma, I)}{\sigma}, \quad \lambda_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \ln \mu(\sigma, I)}{\sigma}, \tag{24}$$

and if $\ln F(x) = o(x)$ as $x \to +\infty$ then

$$T_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \mu(\sigma, I)}{\exp\{\sigma\varrho_R[I]\}}, \quad t_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \mu(\sigma, I)}{\exp\{\sigma\varrho_R[I]\}}. \tag{25}$$

Using Theorem 1 and Proposition 1 we prove the following theorem.

Theorem 3. *Let $F \in V$, $\sigma_\mu = +\infty$ and $X = (x_k)$ be some sequence of positive numbers increasing to $+\infty$. Suppose that f is a nonincreasing function and has regular variation in regard to F.*

If $\ln F(x) = O(x)$ as $x \to +\infty$ and $\ln f(x_k) = (1 + o(1)) \ln f(x_{k+1})$ as $k \to \infty$ then

$$\lambda_R[I] \le \beta\varrho_R[I], \quad \beta = \varliminf_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}}. \tag{26}$$

If $\ln F(x) = o(x)$ as $x \to +\infty$ and $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ then

$$t_R[I] \le T_R[I] \frac{\gamma}{1-\gamma} \exp\left\{1 + \frac{\gamma \ln \gamma}{1-\gamma}\right\} \ln \frac{1}{\gamma}, \quad \gamma = \varliminf_{k\to\infty} \frac{x_k}{x_{k+1}}. \tag{27}$$

Proof. From (24) and (25) for every ε and all $\sigma \geq \sigma_0(\varepsilon)$ we have accordingly $\ln \mu(\sigma, I) \leq \exp\{(\varrho_R[I]+\varepsilon)\sigma\}$ and $\ln \mu(\sigma, I) \leq (T_R[I]+\varepsilon)\exp\{\varrho_R[I]\sigma\}$. We choose $\Phi \in \Omega(+\infty)$ such that $\Phi(\sigma) = Te^{\varrho\sigma}$ for $\sigma \geq \sigma_0(\varepsilon)$, where either $\varrho = \varrho_R[I]+\varepsilon$ and $T = 1$ or $\varrho = \varrho_R[I]$ and $T = T_R[I]+\varepsilon$. Then $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for $\sigma \geq \sigma_0(\varepsilon)$, $\ln \Phi \in L^0$ and it is well known ([4, 10]) that

$$G_1(x_k, x_{k+1}, \Phi) = \frac{1}{\varrho}\frac{x_k x_{k+1}}{x_{k+1}-x_k}\ln\frac{x_{k+1}}{x_k}$$

and

$$G_2(x_k, x_{k+1}, \Phi) = \frac{1}{e\varrho}\exp\left\{\frac{x_{k+1}\ln x_{k+1} - x_k \ln x_k}{x_{k+1}-x_k}\right\}.$$

Since $\Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma)^2 = 1$, condition (9) holds and by Theorem 1 we have

$$\lambda_R[I] \leq \varrho \varliminf_{k\to\infty}\frac{(x_{k+1}-x_k)\ln\left(\dfrac{x_k x_{k+1}}{x_{k+1}-x_k}\ln\dfrac{x_{k+1}}{x_k}\right)}{x_{k+1}\ln x_{k+1} - x_k\ln x_k} \tag{28}$$

provided $\ln f(x_k) = (1+o(1))\ln f(x_{k+1})$ as $k \to \infty$, and

$$t_R[I] \leq eT \varliminf_{k\to\infty}\frac{\dfrac{x_k x_{k+1}}{x_{k+1}-x_k}\ln\dfrac{x_{k+1}}{x_k}}{\exp\left\{\dfrac{x_{k+1}\ln x_{k+1} - x_k\ln x_k}{x_{k+1}-x_k}\right\}} \tag{29}$$

provided $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$.

We suppose that $\beta < 1$. Then there exist a number $\beta^* \in (\beta, 1)$ and an increasing sequence (k_j) of positive integers such that $\ln x_{k_j} \leq \beta^* \ln x_{k_j+1}$, that is $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$. Therefore, from (28) we obtain

$$\lambda_R[I] \leq \varrho\varliminf_{j\to\infty}\frac{(x_{k_j+1}-x_{k_j})\ln\left(\dfrac{x_{k_j}x_{k_j+1}}{x_{k_j+1}-x_{k_j}}\ln\dfrac{x_{k_j+1}}{x_{k_j}}\right)}{x_{k_j+1}\ln x_{k_j+1} - x_{k_j}\ln x_{k_j}}$$
$$\leq \varrho\varliminf_{j\to\infty}\frac{\ln x_{k_j} + o(1) + \ln\ln x_{k_j+1}}{\ln x_{k_j+1}} \leq \varrho\beta^*,$$

whence in view of the arbitrariness of β^* and ε we obtain inequality (26) follows.

Further, if $\gamma \in (0,1)$, then $x_{k_j} = (1+o(1))\gamma x_{k_j+1}$ as $j \to \infty$ for some increasing sequence (k_j) of positive integers and from (29) we obtain

$$t_R[I] \leq eT\varliminf_{j\to\infty}\frac{x_{k_j}x_{k_j+1}\ln(x_{k_j+1}/x_{k_j})}{(x_{k_j+1}-x_{k_j})\exp\left\{\dfrac{x_{k_j+1}\ln x_{k_j+1} - x_{k_j}\ln x_{k_j}}{x_{k_j+1}-x_{k_j}}\right\}}$$
$$= eT\varliminf_{j\to\infty}\frac{\gamma x_{k_j+1}\ln(1/\gamma)}{(1-\gamma)\exp\{\ln x_{k_j+1} - (\gamma\ln\gamma)/(1-\gamma)\}} = T\frac{\gamma}{1-\gamma}\ln\frac{1}{\gamma}\exp\left\{1+\frac{\gamma\ln\gamma}{1-\gamma}\right\},$$

whence in view of the arbitrariness of ε we get (27). Since $\dfrac{\gamma}{1-\gamma}\ln\dfrac{1}{\gamma}\exp\left\{1+\dfrac{\gamma\ln\gamma}{1-\gamma}\right\} \to 1$ as $\gamma \to 1$, then inequality (27) is obvious if $\gamma = 1$. Finally, if $\gamma = 0$, then $\ln x_{k_j} = o(\ln x_{k_j+1})$ as $j \to \infty$ for some increasing sequence (k_j) of positive integers and from (29) we obtain

$$t_R[I] \leq eT\varliminf_{j\to\infty}\frac{x_{k_j}(\ln x_{k_j+1} - \ln x_{k_j})}{\exp\{\ln x_{k_j+1} + o(1)\}} = eT\varliminf_{j\to\infty}\frac{x_{k_j}}{x_{k_j+1}}\ln\frac{x_{k_j+1}}{x_{k_j}} = 0,$$

i.e. inequality (27) holds. The proof of Theorem 3 is complete. □

Now we consider the case $\sigma_\mu = 0$. The order $\varrho_0[I]$, the lower order $\lambda_0[I]$ and (if $0 < \varrho_0[I] < +\infty$) the type $T_0[I]$ and the lower type $t_0[I]$ are defined by formulas

$$\varrho_0[I] = \varlimsup_{\sigma\uparrow 0} \frac{\ln\ln I(\sigma)}{\ln(1/|\sigma|)}, \quad \lambda_0[\varphi] = \varliminf_{\sigma\uparrow 0} \frac{\ln\ln I(\sigma)}{\ln(1/|\sigma|)},$$
$$T_0[I] = \varlimsup_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln I(\sigma), \quad t_0[I] = \varliminf_{\sigma\uparrow 0} |\sigma|^{\varrho_*[I]} \ln I(\sigma).$$

We will show that in this formulas $\ln I(\sigma)$ can be replaced by $\ln \mu(\sigma, I)$ and will use for this purpose the following lemmas.

Lemma 6 ([4,9]). *Let $F \in V$, $\sigma_\mu = 0$ and $\ln F(x) \le h \ln f(x)$ for $x \ge x_0$. Then for every $\varepsilon > 0$ and all $\sigma \in [\sigma_0(\varepsilon), 0)$*

$$\ln I(\sigma) \le (1+h+\varepsilon) \ln \mu\left(\frac{\sigma}{1+h+\varepsilon}, I\right) + K, \quad K = K(\varepsilon) = const.$$

Lemma 7 ([4,9]). *Let $F \in V$, $\sigma_\mu = 0$ and $\ln F(x) = o(x\gamma(x))$ as $x \to +\infty$, where γ is a positive continuous and decreasing to 0 function on $[0, +\infty)$ such that $x\gamma(x) \uparrow +\infty$ as $x \to +\infty$. Then for every $\varepsilon > 0$ and all $\sigma \in [\sigma_0(\varepsilon), 0)$*

$$\ln I(\sigma) \le \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \frac{\varepsilon|\sigma|}{1+\varepsilon} \gamma^{-1}\left(\frac{|\sigma|}{\varepsilon(1+\varepsilon)^2}\right).$$

Lemmas 4, 6 and 7 imply the following statement.

Proposition 2. *Let $F \in V$, $\sigma_\mu = +\infty$, f has regular variation in regard to F and $f(x) \nearrow +\infty$ as $x \to +\infty$. If either $\ln F(x) = O(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ then*

$$\varrho_0[I] = \varlimsup_{\sigma\uparrow 0} \frac{\ln\ln \mu(\sigma, I)}{\ln(1/|\sigma|)}, \quad \lambda_0[\varphi] = \varliminf_{\sigma\uparrow 0} \frac{\ln\ln \mu(\sigma, I)}{\ln(1/|\sigma|)}, \tag{30}$$

and if either $\ln F(x) = o(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ then

$$T_0[I] = \varlimsup_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln \mu(\sigma, I), \quad t_0[I] = \varliminf_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln \mu(\sigma, I)). \tag{31}$$

Proof. If $\ln F(x)) = O(\ln f(x))$ (accordingly $\ln F(x) = o(\ln f(x))$) as $x \to +\infty$ then formulas (30) (accordingly (31)) easy follows from Lemmas 4 and 6.

If we choose function γ such that $\gamma(x) = x^{\delta-1}$ for $x \ge x_0$, where $\delta \in (0,1)$ is an arbitrary numbers, then γ satisfies the conditions of Lemma 7. Therefore, if $\ln F(x) = o(x^\delta)$ as $x \to +\infty$ then

$$\ln I(\sigma) \le \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \frac{\varepsilon|\sigma|}{1+\varepsilon}\left(\frac{\varepsilon(1+\varepsilon)^2}{|\sigma|}\right)^{1-\delta}$$
$$= \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \varepsilon^{2-\delta}(1+\varepsilon)^{1-2\delta}|\sigma|^\delta = \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + o(1), \quad \sigma \uparrow 0,$$

whence the formulas (30) and (31) follow. It remained to notice that the condition $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ implies the condition $\ln F(x) = o(x^\delta)$ as $x \to +\infty$ for $\delta \in (0,1)$. Proposition 2 is proved. □

Using Theorem 2 and Proposition 2 we prove the following theorem.

Theorem 4. *Let $F \in V$, $\sigma_\mu = 0$ and $X = (x_k)$ be some sequence of positive numbers increasing to $+\infty$. Suppose that f has regular variation in regard to F and $f(x) \nearrow +\infty$ as $x \to +\infty$.*

If either $\ln F(x) = O(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ and $\ln f(x_{k+1}) = O(\ln f(x_k))$ as $k \to \infty$ then

$$\lambda_0[I] \le \beta\varrho_0[I], \quad \beta = \varliminf_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}}. \tag{32}$$

If either $\ln F(x) = o(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ and $\ln f(x_{k+1}) = (1+o(1))\ln f(x_k)$ as $k \to \infty$ then

$$t_0[I] \le T_0[I]A(\gamma), \quad \gamma = \varliminf_{k\to\infty} \frac{x_k}{x_{k+1}}, \tag{33}$$

where

$$A(\gamma) =: \frac{\gamma^{\varrho/(\varrho+1)}(1-\gamma^{1/(\varrho+1)})(1-\gamma^{\varrho/(\varrho+1)})^{\varrho}}{(1-\gamma)^{\varrho+1}}.$$

Proof. If $\varrho_0[I] < +\infty$ ($T_0[I] < +\infty$) then $\ln \mu(\sigma, I) \le \Phi(\sigma) = \dfrac{T}{|\sigma|^\varrho}$ for all $\sigma \in [\sigma_0(\varepsilon), 0)$, where either $\varrho = \varrho_0[I] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_0[I]$ and $T = T_0[I] + \varepsilon$. Clearly, $\Phi \in \hat{L}^0$ and $\ln\Phi \in \hat{L}_{si}$. It is known [4, p. 40] that for this function

$$G_1(x_k, x_{k+1}, \Phi) = \frac{T(\varrho+1)}{(T\varrho)^{\varrho/(\varrho+1)}} \frac{x_k x_{k+1}}{x_{k+1}-x_k}\left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}}\right)$$

and

$$G_2(x_k, x_{k+1}, \Phi) = T\left(\frac{(\varrho+1)(T\varrho)^{1/(\varrho+1)}}{\varrho}\frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k}\right)^{-\varrho}.$$

We remark that

$$\left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right)\ln\Phi(\sigma) = \frac{1}{\varrho}\ln\frac{T}{|\sigma|^\varrho} \uparrow +\infty, \quad \sigma \uparrow 0,$$

that is (19) holds.

Therefore, if $\ln f(x_{k+1}) = O(\ln f(x_k))$ as $k \to \infty$ then by Theorem 2 in view of arbitrariness of ε

$$\lambda_0[I] \le \varrho_0[I] \varliminf_{k\to\infty} \frac{\ln\left(\dfrac{x_k x_{k+1}}{x_{k+1}-x_k}\left(\dfrac{1}{x_k^{1/(\varrho+1)}} - \dfrac{1}{x_{k+1}^{1/(\varrho+1)}}\right)\right)}{\ln\left(\dfrac{x_{k+1}-x_k}{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}\right)^{\varrho}} \tag{34}$$

and if $\ln f(x_{k+1}) = (1+o(1))\ln f(x_k)$ as $k \to \infty$ then

$$t_0[I] \le T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} \varliminf_{k\to\infty} \frac{x_k x_{k+1}}{x_{k+1}-x_k}\left(\frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}}\right)\left(\frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k}\right)^{\varrho}. \tag{35}$$

We suppose that $\beta < 1$. Then there exists a number $\beta^* \in (\beta, 1)$ and an increasing sequence (k_j) of positive integers such that $\ln x_{k_j} \le \beta^* \ln x_{k_j+1}$, that is $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$. Therefore, from (34) we obtain

$$\lambda_0[I] \le \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln\left(\dfrac{x_{k_j}x_{k_j+1}}{x_{k_j+1}-x_{k_j}}\left(\dfrac{1}{x_{k_j}^{1/(\varrho+1)}} - \dfrac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\right)}{\ln\left(\dfrac{x_{k_j+1}-x_{k_j}}{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}}\right)^{\varrho}}$$

$$= \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln x_{k_j}^{\varrho/(\varrho+1)}}{\varrho \ln x_{k_j+1}^{1/(\varrho+1)}} = \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln x_{k_j}}{\ln x_{k_j+1}} \le \varrho_0[I]\beta^*,$$

i.e. in view of arbitrariness of β^* we obtain the inequality $\lambda_0[I] \le \beta\varrho_0[I]$. For $\beta = 1$ this inequality is trivial.

Now we suppose that $\gamma \in (0, 1)$. Then there exists an increasing sequence (k_j) of positive integers such that $x_{k_j} = (1+o(1))\gamma\, x_{k_j+1}$ as $j \to \infty$. Therefore, from (35) we obtain

$$t_0[I] \le T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho} \varliminf_{j\to\infty} \frac{x_{k_j}x_{k_i+1}}{x_{k_j+1}-x_{k_j}}\left(\frac{1}{x_{k_j}^{1/(\varrho+1)}} - \frac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\left(\frac{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}}{x_{k_j+1}-x_{k_j}}\right)^{\varrho}$$

$$\le T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho}\frac{\gamma}{\gamma-1}\left(\frac{1}{\gamma^{1/(\varrho+1)}}-1\right)\frac{(1-\gamma^{\varrho/(\varrho+1)})^\varrho}{(1-\gamma)^\varrho} = T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^\varrho}A(\gamma).$$

It is easy to show that $A(\gamma) \to \frac{\varrho^\varrho}{(\varrho+1)^{\varrho+1}}$ as $\gamma \to 1$ that (2) is transformed in obvious inequality $t_0[\varphi] \le T_0[\varphi]$ as $\gamma \to 1$. If $\gamma = 0$ then $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$ and from (2) we obtain easy that $t_0[I] = 0$, because $A(0) = 0$. The proof of Theorem 4 is complete. □

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Для максимуму підінтегрального виразу інтегралу Лапласа-Стілтьєса знайдено нижні оцінки на деякій послідовності. Використовуючи ці оцінки, отримано аналоги тереми Уіттекера для цілих функцій, зображених лакунарними степеневими рядами.

Ключові слова і фрази: інтеграл Лапласа-Стілтьєса, максимум підінтегрального виразу, теорема Уіттекера.

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COINCIDENCE POINT THEOREMS FOR $\varphi - \psi-$CONTRACTION MAPPINGS IN METRIC SPACES INVOLVING A GRAPH

Some new coupled coincidence and coupled common fixed point theorems for $\varphi - \psi-$contraction mappings are established. We have also an application to some integral system to support the results.

Key words and phrases: coupled coincidence point, coupled fixed point, edge preserving, directed graph.

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INTRODUCTION AND PRELIMINARIES

In 2009, *Lakshmikantham* and *Ćirić* [2] introduced a generalization of monotonicity that called mixed g-monotone property. The authors established some coupled coincidence and coupled fixed point results related the mappings have mixed g-monotone property in the partially ordered metric space.

Definition 1 ([2]). *An element $(x, y) \in X^2$ is said to be a coupled coincidence point of a mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2 ([2]). *An element $(x, y) \in X^2$ is said to be a coupled common fixed point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.*

Definition 3 ([2]). *Let X be a nonempty set and $F : X^2 \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.*

Now, we furnish the following class of auxiliary functions which will be used densely in the sequel.

Definition 4 ([11]). *Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$, which satisfy following:*

(φ_1) *φ is continuous and non-decreasing;*

(φ_2) *$\varphi(t) = 0$ iff $t = 0$;*

(φ_3) *$\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$ and Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$, which satisfy ($\psi_1$);*

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(ψ_1) *ψ is continuous function with the condition $\varphi(t) > \psi(t)$ for all $t > 0$.*

By (φ_1), (φ_2) and (ψ_1) we have that $\psi(0) = 0$.

Next, we give the following coupled fixed point theorems as the main results of Işık and Türkoğlu [11].

Theorem 1 ([11]). *Let $(X, \leq, d)$ be a complete partially ordered metric space. Suppose that $F : X^2 \to X$ is a mapping having the mixed monotone property on X. Assume there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x,y), F(u,v))) \leq 2^{-1} \times \psi(d(x,u) + d(y,v)) \tag{1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.
Suppose that either

(a) F is continuous or;

(b) X has the following properties:

1) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

2) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n.

If there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Then F has a coupled fixed point.

The existence of fixed points of contraction mappings in metric space endowed with graph has been initiated by Jachymski [4]. Fixed point theorems for single valued and multivalued operators in such metric spaces have been studied by some authors since 2007 (see [5]— [10] and so on).

Let (X, d) be a metric space, Δ be a diagonal of X^2, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. Furthermore, denote by G^{-1} the graph obtained from G by reversing the direction of the edges in G. Hence, $E(G^{-1}) = \{(x,y) \in X^2 : (y,x) \in E(G)\}$.

Definition 5 ([4]). *A function $g : X \to X$ is $G-$continuous if*

(a) for all x, $x_* \in X$ and any sequence $(n_i)_{i \in N}$ of positive integers, $(x_{n_i}) \to x_*$ and $(x_{n_i}, x_{n_i+1}) \in E(G)$, for $n \in N$, implies $g(x_{n_i}) \to gx_*$;

(b) for all y, $y_* \in X$ and any sequence $(n_i)_{i \in N}$ of positive integers, $(y_{n_i}) \to y_*$ and $(y_{n_i}, y_{n_i+1}) \in E(G^{-1})$, for $n \in N$, implies $g(y_{n_i}) \to gy_*$.

Definition 6 ([9]). *Let (X, d) be a complete metric space, G be a directed graph and $F : X \times X \to X$ be a mapping. Then*

(i) F is called $G-$continuous if for all (x, y), $(x_*, y_*) \in X^2$ and for any sequence $(n_i)_{i \in N}$ of positive integers such that $(x_{n_i}) \to x_*$, $(y_{n_i}) \to y_*$ as $i \to \infty$ and $(x_{n_i}, x_{n_i+1}) \in E(G)$, $(y_{n_i}, y_{n_i+1}) \in E(G^{-1})$, for $n \in N$, implies $F(x_{n_i}, y_{n_i}) \to F(x_*, y_*)$ and $F(y_{n_i}, x_{n_i}) \to F(y_*, x_*)$ as $i \to \infty$;

(ii) (X, d, G) *has property A if (a) for any sequence* $(x_n)_{n\in N}$ *in* X *with* $(x_n) \to x_*$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$*, then* $(x_n, x_*) \in E(G)$*; (b) for any sequence* $(y_n)_{n\in N}$ *in* X *with* $(y_n) \to y_*$ *as* $n \to \infty$ *and* $(y_n, y_{n+1}) \in E\left(G^{-1}\right)$ *for* $n \in \mathbb{N}$*, then* $(y_n, y_*) \in E\left(G^{-1}\right)$.

Consider the set $CCoinFix\,(Fg)$ of all coupled coincidence points of mappings $F : X^2 \to X$, $g : X \to X$ and the set $\left(X^2\right)_{Fg}$ as follows:

$$CCoinFix\,(Fg) = \left\{(x, y) \in X^2 : gx = F(x, y) \text{ and } gy = F(y, x)\right\} \text{ and}$$
$$\left(X^2\right)_{Fg} = \left\{(x, y) \in X^2 : (gx, F(x, y)) \in E(G) \text{ and } (gy, F(y, x)) \in E\left(G^{-1}\right)\right\}.$$

In 2016, Eshi et al. [12] introduced the concept of $G - g-$contraction mapping as follows.

Definition 7 ([12])**.** $F : X^2 \to X$ *is called* $G - g-$*contraction if:*

(i) g *is edge preserving, i.e.,*$(gx, gu) \in E(G)$ *and* $(gy, gv) \in E\left(G^{-1}\right) \Rightarrow (g(gx), g(gu)) \in E(G)$ *and* $(g(gy), g(gv)) \in E\left(G^{-1}\right)$*;*

(ii) F *is* $g-$*edge preserving, i.e.,*$(gx, gu) \in E(G)$ *and* $(gy, gv) \in E\left(G^{-1}\right) \Rightarrow (F(x, y), F(u, v)) \in E(G)$ *and* $(F(y, x), F(v, u)) \in E\left(G^{-1}\right)$*;*

(iii) for all x, y, u, $v \in X$ *such that,* $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E\left(G^{-1}\right)$, $d(F(x, y), F(u, v)) \leq \frac{k}{2}\left[(gx, gu) + (gy, gv)\right]$*, where* $k \in \left[0, \frac{1}{2}\right)$ *is called the contraction constant of* F.

Proposition 1 ([12])**.** *If* $F : X^2 \to X$ *is* $g-$*edge preserving and* $F\left(X^2\right) \subseteq g(X)$*. Also, let* $(x_n)_{n\in N}$, $(y_n)_{n\in N}$, $(u_n)_{n\in N}$ *and* $(v_n)_{n\in N}$ *be sequences in metric space* (X, d) *endowed with a directed graph* G*. Then*

(a) $(gx, gu) \in E(G)$ *and* $(gy, gv) \in E\left(G^{-1}\right) \Rightarrow (F(x_n, y_n), F(u_n, v_n)) \in E(G)$ *and* $(F(y_n, x_n), F(v_n, u_n)) \in E\left(G^{-1}\right)$ *for all* $n \in N$*;*

(b) $(x, y) \in \left(X^2\right)_{Fg} \Rightarrow (F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \in E(G)$ *and* $(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \in E\left(G^{-1}\right)$ *for all* $n \in N$*;*

(c) $(x, y) \in \left(X^2\right)_{Fg} \Rightarrow (F(x_n, y_n), F(y_n, x_n)) \in \left(X^2\right)_{Fg}$ *for all* $n \in N$.

In this paper, we prove coupled coincidence and coupled common fixed point theorems for contaction mappings in metric spaces endowed with a directed graph. Our results extend and improve the results obtained by Eshi et al. in [12], Işık and Türkoğlu in [11], Chifu and Petrusel in [9] so on. Moreover, we have an application to some integral system to support the results.

1 Main results

Definition 8. *Let* (X, d) *be a complete metric space endowed with a directed graph* G*. The mappings* $F : X^2 \to X$, $g : X \to X$ *are called a* $\varphi - \psi-$*contraction if:*

1) g *is edge preserving,* F *is* $g-$*edge preserving;*

2) there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ satisfying $(gx, gu) \in E(G)$ and $(gy, gv) \in E(G^{-1})$,

$$\varphi(d(F(x,y), F(u,v))) \leq 2^{-1} \times \psi(d(gx, gu) + d(gy, gv)). \tag{2}$$

Lemma 1. *Let (X, d) be complete metric space endowed with a directed graph G, and let $F : X^2 \to X$, $g : X \to X$ be a $\varphi - \psi-$contraction and $F(X^2) \subseteq g(X)$. Also, let (x_n), (y_n) be sequences in X. If for each $(x, y) \in (X^2)_{Fg}$, then*

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty.$$

Proof. Let x_0, $y_0 \in X$. Since $F(X^2) \subseteq g(X)$, we can constitute x_1, $y_1 \in X$ such that $F(x_0, y_0) = gx_1$ and $F(y_0, x_0) = gy_1$. Again, we can constitute x_2, $y_2 \in X$ such that $F(x_1, y_1) = gx_2$ and $F(y_1, x_1) = gy_2$. Continuing this procedure above we obtain sequences (x_n) and (y_n) in X such that

$$gx_n = F(x_{n-1}, y_{n-1}) \text{ and } gy_n = F(y_{n-1}, x_{n-1}) \tag{3}$$

for all $n \geq 1$, $x = x_0$ and $y = y_0$. Let $(x_0, y_0) \in (X^2)_{Fg}$ such that $(gx_0, F(x_0, y_0)) = (gx_0, gx_1) \in E(G)$ and $(gy_0, F(y_0, x_0)) = (gy_0, gy_1) \in E(G^{-1})$. Then, by Proposition 1 (b), we get $(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \in E(G)$ and $(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \in E(G^{-1})$. Thus we have that $(gx_n, gx_{n+1}) \in E(G)$ and $(gy_n, gy_{n+1}) \in E(G^{-1})$ for all $n \in N$. Using the $\varphi - \psi-$contaction (2) and (3), we have that

$$\begin{aligned} \varphi(d(gx_{n+1}, gx_n)) &= \varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\leq 2^{-1} \times \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \text{ and} \end{aligned} \tag{4}$$

$$\begin{aligned} \varphi(d(gy_{n+1}, gy_n)) &= \varphi(d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \\ &\leq 2^{-1} \times \psi(d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})) \end{aligned} \tag{5}$$

for all $n \in N$. From (4) and (5) we get

$$\varphi(d(gx_{n+1}, gx_n)) + \varphi(d(gy_{n+1}, gy_n)) \leq \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})). \tag{6}$$

From (φ_3), we obtain that

$$\varphi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \leq \psi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})).$$

Regarding the properties φ and ψ, we conclude that

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}).$$

It follows that $\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)$ is decreasing. Then $\lim_{n\to\infty} \rho_n = \rho$ for some $\rho \geq 0$. Taking the limit as $n \to \infty$ in (6), we have $\varphi(\rho) \leq \psi(\rho)$. From the properties φ and ψ, we obtain that $\rho = 0$, and thus

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty.$$

□

Theorem 2. *Let (X,d) be complete metric space endowed with a directed graph G, and let $F : X^2 \to X$, $g : X \to X$ be a $\varphi - \psi-$contraction and $F\left(X^2\right) \subseteq g\left(X\right)$. Let g be $G-$continuous and commutes with F. Suppose that:*

(i) F is $G-$continuous, or

(ii) the tripled (X,d,G) has a property A.

Then $CCoinFix\left(Fg\right) \neq \varnothing$ iff $\left(X^2\right)_{Fg} \neq \varnothing$.

Proof. Let $CCoinFix\left(Fg\right) \neq \varnothing$. Then there exists $(x_*,y_*) \in CCoinFix\left(Fg\right)$ such that $(gx_*, F(x_*,y_*)) = (gx_*, gx_*) \in \Delta \subset E(G)$ and $(gy_*, F(y_*,x_*)) = (gy_*, gy_*) \in \Delta \subset E\left(G^{-1}\right)$. It follows that $(x_*,y_*) \in \left(X^2\right)_{Fg}$, so that $\left(X^2\right)_{Fg} \neq \varnothing$.

Now, suppose that $\left(X^2\right)_{Fg} \neq \varnothing$. Then there exists $(x_0,y_0) \in \left(X^2\right)_{Fg}$, e.g., $(gx_0, F(x_0,y_0)) \in E(G)$, $(gy_0, F(y_0,x_0)) \in E\left(G^{-1}\right)$. Then, by Proposition 1 (b), we get $(F(x_{n-1},y_{n-1}), F(x_n,y_n)) \in E(G)$ and $(F(y_{n-1},x_{n-1}), F(y_n,x_n)) \in E\left(G^{-1}\right)$. Thus we have that

$$(gx_n, gx_{n+1}) \in E(G) \text{ and } (gy_n, gy_{n+1}) \in E\left(G^{-1}\right) \tag{7}$$

for all $n \in N$. By Lemma 1, we have

$$\rho_n := d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \to 0 \text{ as } n \to \infty. \tag{8}$$

Next, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. If possible, assume that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\left\{gx_{n(k)}\right\}$, $\left\{gx_{m(k)}\right\}$ of $\{gx_n\}$ and $\left\{gy_{n(k)}\right\}$, $\left\{gy_{m(k)}\right\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\gamma_k := d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \geq \varepsilon. \tag{9}$$

Farther, corresponding to $m(k)$, we can choose $n(k)$ in the manner that it is the smallest integer for which (9) holds. Then,

$$d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right) < \varepsilon. \tag{10}$$

Using (9), (10), and triangular inequality, we obtain

$$\varepsilon \leq \gamma_k < \varepsilon + d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gy_{n(k)}, gy_{n(k)-1}\right). \tag{11}$$

Letting $k \to \infty$ in (11) and by (8), we have

$$\gamma_k := d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \to \varepsilon \text{ as } k \to \infty. \tag{12}$$

From the triangle inequality, we get

$$\begin{aligned}\gamma_k &= d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right) \\ &\leq d\left(gx_{n(k)+1}, gx_{m(k)+1}\right) + d\left(gy_{n(k)+1}, gy_{m(k)+1}\right) + \rho_{n(k)} + \rho_{m(k)}.\end{aligned}$$

From property φ, we have

$$\begin{aligned}
\varphi\left(\gamma_k\right) &\leq \varphi\left(d\left(gx_{n(k)+1}, gx_{m(k)+1}\right)\right)+\varphi\left(d\left(gy_{n(k)+1}, gy_{m(k)+1}\right)\right)+\varphi\left(\rho_{n(k)}+\rho_{m(k)}\right) \\
&\leq \varphi\left(d\left(F\left(x_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}\right)\right)\right) \\
&+\varphi\left(d\left(F\left(y_{n(k)}, x_{n(k)}\right), F\left(y_{m(k)}, x_{m(k)}\right)\right)\right)+\varphi\left(\rho_{n(k)}+\rho_{m(k)}\right) \\
&\leq 2^{-1} \times \psi\left(d\left(gx_{n(k)}, gx_{m(k)}\right)+d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \\
&+2^{-1} \times \psi\left(d\left(gy_{n(k)}, gy_{m(k)}\right)+d\left(gx_{n(k)}, gx_{m(k)}\right)\right)+\varphi\left(\rho_{n(k)}+\rho_{m(k)}\right) \\
&\leq \psi\left(\gamma_k\right)+\varphi\left(\rho_{n(k)}+\rho_{m(k)}\right).
\end{aligned} \tag{13}$$

Taking $k \to \infty$ in (13) and from (8) and (12), we obtain a following contradiction:

$$\varphi(\varepsilon) \leq \psi(\varepsilon)+\varphi(0)=\psi(\varepsilon).$$

Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. As (X, d) is complete, there exists $x_*, y_* \in X$ such that

$$gx_n \to x_* \text{ and } gy_n \to y_* \text{ as } n \to \infty. \tag{14}$$

Since g be $G-$continuous, we have

$$g\left(gx_n\right) \to gx_* \text{ and } g\left(gy_n\right) \to gy_* \text{ as } n \to \infty.$$

Moreover as F and g are commutative

$$g\left(gx_{n+1}\right) = g\left(F\left(x_n, y_n\right)\right)=F\left(gx_n, gy_n\right), \tag{15}$$

$$g\left(gy_{n+1}\right) = g\left(F\left(y_n, x_n\right)\right)=F\left(gy_n, gx_n\right). \tag{16}$$

We now prove that

$$F\left(x_*, y_*\right)=gx_* \text{ and } F\left(y_*, x_*\right)=gy_*.$$

Suppose assumption (i) holds. From (15) and (16), we have

$$\begin{aligned}
gx_* &= \lim_{n \to \infty} g\left(gx_{n+1}\right)=\lim_{n \to \infty} F\left(gx_n, gy_n\right)=F\left(x_*, y_*\right), \\
gy_* &= \lim_{n \to \infty} g\left(gy_{n+1}\right)=\lim_{n \to \infty} F\left(gy_n, gx_n\right)=F\left(y_*, x_*\right);
\end{aligned}$$

that is, $\left(x_*, y_*\right)$ is a coincidence point of F and g.

Suppose now assumption (ii) holds. From (7) and (14), using property A, we get $\left(gx_n, x_*\right) \in E(G)$ and $\left(gy_n, y_*\right) \in E\left(G^{-1}\right)$ for each $n \in \mathbb{N}$. By (2), we get

$$\begin{aligned}
&\varphi\left(d\left(gx_*, F\left(x_*, y_*\right)\right)+d\left(gy_*, F\left(y_*, x_*\right)\right)\right) \\
&\quad\leq \varphi\left(d\left(gx_*, gx_{n+1}\right)+d\left(gx_{n+1}, F\left(x_*, y_*\right)\right)+d\left(gy_*, gy_{n+1}\right)+d\left(gy_{n+1}, F\left(y_*, x_*\right)\right)\right) \\
&\quad\leq \varphi\left(d\left(gx_*, gx_{n+1}\right)\right)+\varphi\left(d\left(F\left(x_n, y_n\right), F\left(x_*, y_*\right)\right)\right) \\
&\quad+\varphi\left(d\left(gy_*, gy_{n+1}\right)\right)+\varphi\left(d\left(F\left(y_n, x_n\right), F\left(y_*, x_*\right)\right)\right) \\
&\quad\leq \psi\left(d\left(gx_n, gx_*\right)+d\left(gy_n, gy_*\right)\right)+\varphi\left(d\left(gx_*, gx_{n+1}\right)\right)+\varphi\left(d\left(gy_*, gy_{n+1}\right)\right).
\end{aligned}$$

Letting $n \to \infty$, we obtain $\varphi\left(d\left(gx_*, F\left(x_*, y_*\right)\right)+d\left(gy_*, F\left(y_*, x_*\right)\right)\right)=0$. From properties φ, we have $d\left(gx_*, F\left(x_*, y_*\right)\right)+d\left(gy_*, F\left(y_*, x_*\right)\right)=0$. Hence, $gx_*=F\left(x_*, y_*\right)$ and $gy_*=F\left(y_*, x_*\right)$. □

Definition 9. *Let (X,d) be a complete metric space endowed with a directed graph G. The mappings $F : X^2 \to X$, $g : X \to X$ are called a $\psi-$contraction if:*

(i) g is edge preserving, F is $g-$edge preserving;

(ii) there exists $\psi \in \Psi$ such that for all x, y, u, $v \in X$ satisfying $(gx,gu) \in E(G)$ and $(gy,gv) \in E\left(G^{-1}\right)$,

$$d\left(F\left(x,y\right),F\left(u,v\right)\right) \leq 2^{-1} \times \psi\left(d\left(gx,gu\right)+d\left(gy,gv\right)\right).$$

Theorem 3. *Let (X,d) be complete metric space endowed with a directed graph G, and let $F : X^2 \to X$, $g : X \to X$ be a $\psi-$contraction and $F\left(X^2\right) \subseteq g\left(X\right)$. Let g be $G-$continuous and commutes with F. Suppose that:*

(i) F is $G-$continuous, or

(ii) the tripled (X,d,G) has a property A.

Then $CCoinFix\left(Fg\right) \neq \varnothing$ iff $\left(X^2\right)_{Fg} \neq \varnothing$.

Proof. Taking $\varphi\left(t\right) = t$, along the lines of the proof of Theorem 2, we have the requested results. By virtue of the analogy, we skip the details of the proof. □

If we choose the functions $\varphi\left(t\right) = t$ and $\psi\left(t\right) = kt$, for $t \in [0,\infty)$ and $k \in \left[0,\frac{1}{2}\right)$ in Theorem 2, we have the following corollary.

Corollary 1 ([12]). *Let (X,d) be complete metric space endowed with a directed graph G, and let $F : X^2 \to X$ be a $G-g-$contraction with contraction constant $k \in \left[0,\frac{1}{2}\right)$ and $F\left(X^2\right) \subseteq g\left(X\right)$. Let g be $G-$continuous and commutes with F. Suppose that (i) F is $G-$continuous, or (ii) the tripled (X,d,G) has a property A. Then $CCoinFix\left(Fg\right) \neq \varnothing$ iff $\left(X^2\right)_{Fg} \neq \varnothing$.*

Remark 1. *In the case where $(X,\preccurlyeq)$ is partially ordered complete metric space, taking $E\left(G\right) = \{(x,y) \in X \times X : x \preccurlyeq y\}$, the functions $\varphi\left(t\right) = t$ and $\psi\left(t\right) = kt$, for $t \in [0,\infty)$ and $k \in [0,1)$, Theorem 2 generalize and improve the results obtained by Bhaskar and Lakshmikantham ([1], Theorem 2.1) and Chifu and Petrusel ([9], Theorem 2.1). If we take the function $\psi\left(t\right) = \varphi\left(t\right) - \psi_1\left(t\right)$, for $t \in [0,\infty)$, where $\psi_1 \in \Psi$, Theorem 2 generalize the results given by Luong and Thuan ([3], Theorem 2.1). In Theorem 2, let g be the identity mapping. Then it is easy to see that our conclusions enhance the results achieved by Işık and Türkoğlu [11].*

Theorem 4. *In addition to Theorem 2, suppose that for any two elements $(x,y), (x_*,y_*) \in X^2$, there exists $(p,r) \in X^2$ such that*

$$\left(F\left(x,y\right),F\left(p,r\right)\right) \in E\left(G\right),\ \left(F\left(y,x\right),F\left(r,p\right)\right) \in E\left(G^{-1}\right) \text{ and}$$
$$\left(F\left(x_*,y_*\right),F\left(p,r\right)\right) \in E\left(G\right),\ \left(F\left(y_*,x_*\right),F\left(r,p\right)\right) \in E\left(G^{-1}\right).$$

Then, F and g have a unique coupled common fixed point.

Proof. By Theorem 2, we have $CCoinFix\,(Fg) \neq \varnothing$. Suppose $(x, y), (x_*, y_*)$ are coupled fixed points of F, e.g.,

$$gx = F(x, y),\ gy = F(y, x) \text{ and } gx_* = F(x_*, y_*),\ gy_* = F(y_*, x_*). \tag{17}$$

Consider sequences $\{p_n\}$ and $\{r_n\}$ as follows

$$p_0 = p, r_0 = r, p_{n+1} = F(p_n, r_n) \text{ and } r_{n+1} = F(r_n, p_n) \text{ for all } n \geq 0.$$

From assumption, we get

$$(F(x, y), F(p, r)) = (gx, gp_1) \in E(G),\ (F(y, x), F(r, p)) = (gy, gr_1) \in E\left(G^{-1}\right) \text{ and}$$

$$\begin{aligned}&(F(x_*, y_*), F(p, r)) = (gx_*, gp_1) \in E(G),\\&(F(y_*, x_*), F(r, p)) = (gy_*, gr_1) \in E\left(G^{-1}\right).\end{aligned}$$

Since F is $g-$edge preserving, we have

$$(F(x, y), F(p_1, r_1)) = (gx, gp_2) \in E(G),\ (F(y, x), F(r_1, p_1)) = (gy, gr_2) \in E\left(G^{-1}\right),$$

$$\begin{aligned}&(F(x_*, y_*), F(p_1, r_1)) = (gx_*, gp_2) \in E(G),\\&(F(y_*, x_*), F(r_1, p_1)) = (gy_*, gr_2) \in E\left(G^{-1}\right).\end{aligned}$$

Continuing this procedure above, we obtain

$$(gx, gp_n) \in E(G),\ (gy, gr_n) \in E\left(G^{-1}\right) \text{ and}$$

$$(gx_*, gp_n) \in E(G),\ (gy_*, gr_n) \in E\left(G^{-1}\right).$$

By (2), we have

$$\begin{aligned}&\varphi(d(gx_*, p_{n+1})) + \varphi(d(r_{n+1}, gy_*))\\&\quad = \varphi(d(F(x_*, y_*), F(p_n, r_n))) + \varphi(d(F(r_n, p_n), F(y_*, x_*)))\\&\quad \leq 2^{-1} \times \psi(d(gx_*, gp_n) + d(gy_*, gr_n)) + 2^{-1} \times \psi(d(gr_n, gy_*) + d(gp_n, gx_*)).\end{aligned}$$

By the property of φ, we have

$$\varphi(d(gx_*, gp_{n+1}) + d(gr_{n+1}, gy_*)) \leq \psi(d(gx_*, gp_n) + d(gy_*, gr_n)). \tag{18}$$

By (φ_1) and (ψ_1), we have

$$d(gx_*, gp_{n+1}) + d(gr_{n+1}, gy_*) \leq d(gx_*, gp_n) + d(gy_*, gr_n).$$

Therefore, the sequence $\{f_n\}$ defined by $f_n = d(gx_*, gp_n) + d(gy_*, gr_n)$, is a nonnegative decreasing sequence, and consequently, there exists some $f \geq 0$ such that

$$d(gx_*, gp_n) + d(gy_*, gr_n) \to f \text{ as } n \to \infty.$$

Suppose that $f > 0$. Then taking limit as $n \to \infty$ in (18) and using the continuity of φ and ψ, we get

$$\varphi(f) \leq \psi(f)$$

which implies, from the properties of φ and ψ, that $\psi(f) = 0$ and eventually, $f = 0$. Hence

$$d(gx_*, gp_n) + d(gy_*, gr_n) \to 0 \text{ as } n \to \infty,$$

which implies

$$\lim_{n\to\infty} d(gx_*, gp_n) = 0 = \lim_{n\to\infty} d(gy_*, gr_n).$$

Similarly

$$\lim_{n\to\infty} d(gx, gp_n) = 0 = \lim_{n\to\infty} d(gy, gr_n).$$

By the triangular inequality we obtain

$$d(gx_*, gx) \le d(gx_*, gp_n) + d(gp_n, gx), \quad d(gy_*, gy) \le d(gy_*, gr_n) + d(gr_n, gy), \tag{19}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (19), we obtain that $d(gx_*, gx) = 0 = d(gy_*, gy)$. Hence, we get

$$gx_* = gx \text{ and } gy_* = gy. \tag{20}$$

Let $gx_* = gx = t$ and $gy_* = gy = s$.

Owing to commutativity of F and g, by (17), we have

$$g(gx_*) = g(F(x_*, y_*)) = F(gx_*, gy_*) \Rightarrow gt = F(t, s) \text{ and}$$

$$g(gy_*) = g(F(y_*, x_*)) = F(gy_*, gx_*) \Rightarrow gs = F(s, t).$$

Hence, (t, s) is a coupled coincidence point. Thus, by repeating previous argument for (x_*, y_*) and (t, s),

$$gx_* = gt \Rightarrow t = gt \text{ and } gy_* = gs \Rightarrow s = gs.$$

Therefore, $t = gt = F(t, s)$ and $s = gs = F(s, t)$. Hence, (t, s) is a coupled common fixed point of F and g.

To show the uniqueness, suppose that (k, l) is another coupled common fixed point of F and g. Hence,

$$k = gk = F(k, l) \text{ and } l = gl = F(l, k). \tag{21}$$

By (20), we have

$$gk = gt = t \text{ and } gl = gs = s. \tag{22}$$

Thus, from (21) and (22), we get $k = t$ and $l = s$. Then, $k = gk = gt = t$ and $l = gl = gs = s$. □

2 APPLICATION

We consider the following integral system:

$$\begin{aligned} x(t) &= h(t) + \lambda \int_{-t}^{t} A(t, s, x(s), y(s))\, ds, \\ y(t) &= h(t) + \lambda \int_{-t}^{t} A(t, s, y(s), x(s))\, ds, \end{aligned} \tag{23}$$

for $t \in [-T, T]$, $T > 0$, $\lambda \in R$.

Recall that the Bielecki-type norm on $X := C\left(\left[-T,T\right],R^{n}\right)$,

$$\|x\|_{B}=\max_{t\in[-T,T]}\left|x\left(t\right)e^{-\tau(t-T)}\right| \text{ for all } x\in X,$$

where $\tau>0$, is arbitrarily chosen. Consider $\|x-y\|_{B}=\max_{t\in[-T,T]}\left|x\left(t\right)-y\left(t\right)\right|e^{-\tau(t-T)}$ for all $x,y\in X$.

Define the graph G with partial order relation by

$$x,y\in X,\ x\leq y\Leftrightarrow x\left(t\right)\leq y\left(t\right)\quad\text{for any } t\in I.$$

Thus $\left(X,\|x\|_{B}\right)$ is complete metric space endowed with a directed graph G.

If we take into consideration $E\left(G\right):=\left\{\left(x,y\right)\in X^{2}:x\leq y\right\}$, then $\Delta\left(X^{2}\right)\subseteq E\left(G\right)$. On the other hand $E\left(G^{-1}\right):=\left\{\left(x,y\right)\in X^{2}:y\leq x\right\}$. Furthermore, $\left(X,\|x\|_{B},G\right)$ has property A.

Then $\left(X^{2}\right)_{Fg}=\left\{\left(x,y\right)\in X^{2}:gx\leq F\left(x,y\right)\text{ and } F\left(y,x\right)\leq gy\right\}$. We consider the following conditions:

1. $A:\left[-T,T\right]\times\left[-T,T\right]\times R^{n}\times R^{n}\rightarrow R^{n}$ and $h:\left[-T,T\right]\rightarrow R^{n}$ are continuous;

2. for all $x,y,u,v\in R^{n}$ with $x\leq u$, $v\leq y$ we have $A\left(t,s,x,y\right)\leq A\left(t,s,u,v\right)$ for all t, $s\in\left[-T,T\right]$;

3. for all $t,s\in\left[-T,T\right]$ and for all $x,y,u,v\in R^{n}$

$$\left|A\left(t,s,x,y\right)-A\left(t,s,u,v\right)\right|\leq\psi\left(\left|x-u\right|+\left|y-v\right|\right),$$

   where $\psi\in\Psi$ such that $\psi\left(\alpha t\right)\leq\alpha\psi\left(t\right)$ for all $t\in\left[-T,T\right]$ and for all $\alpha\geq0$;

4. there exists $\left(x_{0},y_{0}\right)\in X^{2}$ such that

$$\begin{aligned} x_{0}\left(t\right) &\leq h\left(t\right)+\lambda\int_{-t}^{t}A\left(t,s,x_{0}\left(s\right),y_{0}\left(s\right)\right)ds,\\ y_{0}\left(t\right) &\geq h\left(t\right)+\lambda\int_{-t}^{t}A\left(t,s,y_{0}\left(s\right),x_{0}\left(s\right)\right)ds, \end{aligned}$$

   where $t\in\left[-T,T\right]$.

Theorem 5. *Suppose that conditions (1)—(4) are satisfied. Then there exists at least one solution of (23).*

Proof. Let $F:X^{2}\rightarrow X$ and $g:X\rightarrow X$ be defined as

$$\begin{aligned} &F\left(x,y\right)\left(t\right)=h\left(t\right)+\lambda\int_{-t}^{t}A\left(t,s,x\left(s\right),y\left(s\right)\right)ds,\ t\in\left[-T,T\right],\\ &g\left(x\right)\left(t\right)=x\left(t\right). \end{aligned}$$

Then (23) can be indicated as

$$gx=F\left(x,y\right)\text{ and } gy=F\left(y,x\right). \tag{24}$$

By (24), the solution of this system is a coupled coincidence point of the mappings F and g, if we prove the assumptions in Theorem 3.

Let $x, y, u, v \in X$ be such that $gx \leq gu$ and $gv \leq gy$,

$$\begin{aligned} F(x,y)(t) &= h(t) + \lambda \int_{-t}^{t} A(t,s,x(s),y(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,g(x)(s),g(y)(s))\,ds \\ &\leq h(t) + \lambda \int_{-t}^{t} A(t,s,g(u)(s),g(v)(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,u(s),v(s))\,ds = F(u,v)(t) \end{aligned}$$

for all $t \in [-T,T]$. Therefore $(F(x,y), F(u,v)) \in E(G)$.

$$\begin{aligned} F(v,u)(t) &= h(t) + \lambda \int_{-t}^{t} A(t,s,v(s),u(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,g(v)(s),g(u)(s))\,ds \\ &\leq h(t) + \lambda \int_{-t}^{t} A(t,s,g(y)(s),g(x)(s))\,ds \\ &= h(t) + \lambda \int_{-t}^{t} A(t,s,y(s),x(s))\,ds = F(y,x)(t) \end{aligned}$$

for all $t \in [-T,T]$. Therefore $(F(y,x), F(v,u)) \in E\left(G^{-1}\right)$. Then, F is $g-$edge preserving.

We shall show that F is $\psi-$contraction. We have

$$\begin{aligned} & |F(x,y)(t) - F(u,v)(t)| \\ & \leq |\lambda| \int_{-t}^{t} |A(t,s,x(s),y(s)) - A(t,s,u(s),v(s))|\,ds \\ & \leq |\lambda| \int_{-t}^{t} \psi(|x(s)-u(s)| + |y(s)-v(s)|) \left(e^{-\tau(t-T)} e^{\tau(t-T)}\right) \\ & \leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B)\, e^{\tau(t-T)} \end{aligned}$$

for all $t \in [-T,T]$; therefore,

$$|F(x,y)(t) - F(u,v)(t)|\, e^{-\tau(t-T)} \leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B). \tag{25}$$

Applying maximum in (25), we have

$$\|F(x,y) - F(u,v)\|_B \leq \frac{|\lambda|}{\tau} \psi(\|x-u\|_B + \|y-v\|_B).$$

If we take τ such that $\frac{|\lambda|}{\tau} = \frac{1}{2} \Leftrightarrow |\lambda| = \frac{\tau}{2}$, then F is $\psi-$contraction.

From assumption (4) show that there exists $(x_0, y_0) \in X^2$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$, which implies that $\left(X^2\right)_{Fg} \neq \varnothing$. Also, F and g are commutative.

On the other hand, by virtue of (1) and of the fact that $(X, \|x\|_B, G)$ has property A we get that (i) or (ii) from Theorem 3 is fulfilled. Hence, there exists a coupled coincidence point $(x_*, y_*) \in X^2$ of the mapping F and g, which is the solution of the integral system (23). □

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Йолакан Е., Кізілтанк Г., Кір М. *Теореми про точки співпадіння для $\varphi - \psi -$скоротних відображень в метричних просторах еволюції графів* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 251–262.

У статті отримано деякі нові теореми про зв'язні точки співпадання та зв'язні фіксовані точки для $\varphi - \psi -$скоротних відображень. Також були отримані застосування отриманих результатів у дослідженні інтегральних систем.

Ключові слова і фрази: зв'язна точка співпадання, зв'язна фіксована точка, вершина збереження, напрямлений граф.

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REPRESENTATION OF SPECTRA OF ALGEBRAS OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS OF BOUNDED TYPE

The paper contains a description of a symmetric convolution of the algebra of block-symmetric analytic functions of bounded type on ℓ_1-sum of the space $\mathbb{C}^2$. We show that the specrum of such algebra does not coincide of point evaluation functionals and we describe characters of the algebra as functions of exponential type with plane zeros.

Key words and phrases: algebraic basis, block-symmetric polynomials, block-symmetric analytic functions, spectrum, symmetric intertwining, symmetric convolution.

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INTRODUCTION

In resent years there is an increasing interest to investigations of invariants of the permutation group S_∞ of positive integers. This group can be represented on a Banach space X with symmetric basis as a group of operators of perturbation of basis vectors. The action of this group has a natural extension to the action on the algebra $H_b(X)$ of analytic functions of bounded type on X. Invariants of this representation of S_∞ are so-called symmetric analytic functions of bounded type on X. The algebras of symmetric analytic functions $H_{bs}(X)$ were investigated by many authors ([1, 2, 9]). In particular, it is known that $H_{bs}(\ell_p)$ admits an algebraic basis for $1 \leq p < \infty$.

On the other hand, there are more representations of S_∞ in Banach spaces. For example, if $\mathcal{X}$ is a directs sum of infinite many of "blocks" which consists of linear subspaces isomorphic each to other, then S_∞ may to act as a group of permutations of the "blocks". For this case we have invariants — the algebra of block-symmetric analytic functions. Note that this algebra is much more complicated and in the general case has no algebraic basis (see e.g. [6, 12]). In the case $\dim \mathcal{X} < \infty$, block-invariant polynomials were investigated in the classical theory of invariants [5, 11].

1 MAIN RESULTS

Let

$$\mathcal{X}^2 = \oplus_{\ell_1}\mathbb{C}^2 = \ell_1 \otimes \mathbb{C}^2$$

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be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X}^2$ can be represented as a sequence $u = \begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \ldots \right)$, where $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathbb{C}^2$, with the norm $\|u\| = \sum\limits_{k=1}^{\infty} (|x_k| + |y_k|)$. Also, we will use notation $u(x,y)$, where $x, y \in \ell_1$, $x = \sum\limits_{k=1}^{\infty} x_k e_k$, $y = \sum\limits_{k=1}^{\infty} y_k e_k$. Here e_k is the standard symmetric basis in ℓ_1.

A polynomial P on the space $\mathcal{X}^2$ is called *block-symmetric* (or *vector-symmetric*) if:

$$P\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right) = P\left(\begin{pmatrix} x_{\sigma(1)} \\ y_{\sigma(1)} \end{pmatrix}, \ldots, \begin{pmatrix} x_{\sigma(m)} \\ y_{\sigma(m)} \end{pmatrix}, \ldots \right),$$

for every permutation σ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{C}^2$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In [7] it was shown that the following vectors form an algebraic bases of "power" block-symmetric polynomials of $\mathcal{P}_{vs}(\mathcal{X}^2)$:

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p}, \tag{1}$$

where $0 \le p \le n$, $(x_i, y_i) \in \mathbb{C}^2$, $i \ge 1$. Also, there is a basis of "elementary" block-symmetric polynomials:

$$R^{p,n-p}(x,y) = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_{n-p} \\ i_k \neq j_l}}^{\infty} x_{i_1} \ldots x_{i_p} y_{j_1} \ldots y_{j_{n-p}}, \tag{2}$$

where $0 \le p \le n$, $n \ge 1$ and $(x_i, y_i) \in \mathbb{C}^2$.

In the finite case, generating elements of algebra of block-symmetric polynomials on the space $\mathcal{X}_m^2 = \oplus_{\ell_1}^m \mathbb{C}^2$ are algebraic dependent. In [12] was proved the following theorem.

Theorem 1. *For every nonsymmertic polynomial ξ of a system of generating elements of $\mathcal{P}_{vs}(\mathcal{X}_m^2)$ there exist symmetric polynomials a_k in this system such that*

$$\xi^{m!} - a_1 \xi^{m!-1} + \cdots + (-1)^{m!-1} a_{m!-1} \xi^1 + (-1)^{m!} a_{m!} = 0.$$

Let σ be some permutation on the set of natural numbers $\mathbb{N}$. We denote by T_σ the linear operator on $\mathcal{X}^2$ associated with σ by the formula

$$T_\sigma\Big(\sum_{k=1}^{\infty} x_k e_k, \sum_{k=1}^{\infty} y_k e_k \Big) = \Big(\sum_{k=1}^{\infty} x_{\sigma(k)} e_k, \sum_{k=1}^{\infty} y_{\sigma(k)} e_k \Big).$$

For any $(x,y), (z,t) \in \mathcal{X}^2$ we denote $(x,y) \sim (z,t)$ if there exists a permutation σ on $\mathbb{N}$ such that $(x,y) = T_\sigma(z,t)$.

Theorem 2. *Let $(x,y), (z,t) \in \mathcal{X}^2$ and $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, where $0 \le p \le i$ for every $i \ge 1$. Then $(x,y) \sim (z,t)$.*

Proof. Let $G(x)$ be a symmetric polynomial of degree n in the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ on ℓ_1. We set $P(x,y) = G(x+jy)$, where $0 \le j \le n$, $(x,y) \in \mathcal{X}^2$. Obviously, $P(x,y)$ is a block-symmetric polynomial. In [13] it was proved that the block-symmetric polynomial $P(x,y)$ will be represented as an algebraic combination of $F_k(x+jy)$, where $F_n(x) = \sum_{k=1}^{\infty} x_k^n$. So for the polynomial $P(x,y)$ according to [1, Theorem 1.3] we obtain that $x + jy = T_\sigma(z+jt)$. On the other hand, we can denote by $T_\sigma(x) = T_\sigma(x,0)$, $T_\sigma(y) = T_\sigma(0,y)$ and we obtain that $x + jy = T_\sigma((z,0) + j(0,t)) = T_\sigma(z) + jT_\sigma(t)$.

For us it is enough to consider $j = 1,2$. We obtain two equalities

$$x + y = T_\sigma(z) + T_\sigma(t), \quad x + 2y = T_\sigma(z) + 2T_\sigma(t),$$

which imply $x = T_\sigma(z)$, $y = T_\sigma(t)$. That is, $(x,jy) = T_\sigma(z,t)$.

Since $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, $0 \le p \le i$ for every $i \ge 1$ it follows that $F_i(x+jy) = F_i(z+jt)$ and so $(x,y) \sim (z,t)$. $\square$

Let $H_{bvs}(\mathcal{X}^2)$ be the algebra of block-symmetric analytic functions of bounded type (that is, bounded on bounded subsets) on $\mathcal{X}^2$. This algebra is generated by polynomials $H^{1,0}, \dots, H^{p,n-p}, \dots, H^{0,n}, \dots$, where $n \ge 1, 0 \le p \le n$. Let us denote by $M_{bvs}(\mathcal{X}^2)$ the specrum of algebra $H_{bvs}(\mathcal{X}^2)$.

For given $(x,y), (z,t) \in \mathcal{X}^2$,

$$(x,y) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \dots \right)$$

and

$$(z,t) = \left(\begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \dots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \dots \right),$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$, we put

$$(x,y) \bullet (z,t) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \dots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \dots \right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{3}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the *intertwining* and the operator $\mathcal{T}_{(z,t)}$ is the *intertwining operator*. Some elementary properties of $\mathcal{T}_{(z,t)}$ was proved in [6].

Let $\mathbb{C}\{t_1, t_2\}$ be the space of all pover series over $\mathbb{C}^2$. We denote by $\mathcal{R}$ and $\mathcal{H}$ the following maps from $M_{bvs}(\mathcal{X}^2)$ into $\mathbb{C}\{t_1, t_2\}$

$$\mathcal{R}(\varphi) = \sum_{\substack{n=0 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(R^{p,n-p}),$$

and

$$\mathcal{H}(\varphi) = \sum_{\substack{n=1 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(H^{p,n-p}).$$

Note

$$\mathcal{R}((x,y)\bullet(z,t)) = \mathcal{R}(x,y)\mathcal{R}(z,t),$$

and

$$\mathcal{H}((x,y)\bullet(z,t)) = \mathcal{H}(x,y) + \mathcal{H}(z,t),$$

where $(x,y),(z,t) \in \mathcal{X}^2$. We will prove these equalities in Theorem 4 for more general situation.

Following [3] we define the symmetric convolution.

Definition 1. *For any $f \in H_{bvs}(\mathcal{X}^2)$ and $\theta \in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\theta \star f$ is defined by*

$$(\theta \star f)(x,y) = \theta[\mathcal{T}_{(x,y)}(f)].$$

Definition 2. *For any $\varphi, \theta \in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\varphi \star \theta$ is defined by*

$$(\varphi \star \theta)(f) = \varphi(\theta \star f) = \varphi((z,t) \mapsto \theta(\mathcal{T}_{(z,t)} f)).$$

Theorem 3. *For any $\varphi, \theta \in M_{bvs}(\mathcal{X}^2)$ the symmetric convolution is commutative, associative and*

$$(\varphi \star \theta)(H^{p,n-p}) = \varphi(H^{p,n-p}) + \theta(H^{p,n-p}), \tag{4}$$

where $0 \leq p \leq n$.

Proof. First we will prove the equality (4). Indead, for polynomials $H^{p,n-p}$ we have

$$\begin{aligned}(\theta \star H^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x,y)}(H^{p,n-p})) \\ &= \theta(H^{p,n-p}(x,y) + H^{p,n-p}) = H^{p,n-p}(x,y) + \theta(H^{p,n-p}).\end{aligned}$$

Therefore,

$$\begin{aligned}(\varphi \star \theta)(H^{p,n-p}) &= \varphi(H^{p,n-p}(x,y) + \theta(H^{p,n-p})) \\ &= \varphi(H^{p,n-p}) + \theta(H^{p,n-p}).\end{aligned}$$

From this equality it follows the associativity and commutativity of $\varphi \star \theta \in M_{bvs}(\mathcal{X}^2)$. $\square$

Similarly to Lemma 3.1 and Proposition 8.2 in [4] (see also [12]) it is possible to show that

$$||R^{p,n-p}|| \leq \frac{2}{p!(n-p)!}$$

and $\mathcal{R}(\varphi)(t)$ is a function of exponential type for every fixed $\varphi \in M_{bvs}(\mathcal{X}^2)$.

Theorem 4. *The following identities hold*

1. $\mathcal{H}(\varphi \star \theta) = \mathcal{H}(\varphi) + \mathcal{H}(\theta)$,

2. $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$.

Proof. The first statement it follows from Theorem 3. To prove the second statement we observe that

$$R^{p,n-p}((x,y)\bullet(z,t)) = \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}(z,t).$$

Thus

$$\begin{aligned}(\theta\star R^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x^1,x^2)}(R^{p,n-p}))\\ &= \theta\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}(\varphi\star\theta)\Big(R^{p,n-p}\Big) &= \varphi\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x^1,x^2)\theta\Big(R^{p-r,n-p-(i-r)}\Big)\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} \varphi\Big(R^{r,i-r}\Big)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

On the other hand

$$\begin{aligned}\mathcal{R}(\varphi)\mathcal{R}(\theta) &= \sum_{\substack{i=0\\0\le k\le i}}^{\infty} t_1^k t_2^{i-k}\varphi(R^{k,i-k})\sum_{\substack{m=0\\0\le r\le m}}^{\infty} t_1^r t_2^{m-r}\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty}\sum_{\substack{k+r=p\\i+m=n}} t_1^p t_2^{n-p}\varphi(R^{k,i-k})\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p}\sum_{\substack{k+r=p\\i+m=n}}\varphi(R^{k,i-k})\theta(R^{r,m-r}) = \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p}(\varphi\star\theta)\Big(R^{p,n-p}\Big)\\ &= \mathcal{R}(\varphi\star\theta).\end{aligned}$$

$\square$

Lemma 1. *If $\varphi=\delta_{(x,y)}$, then for every $(x,y)\in\mathcal{X}^2$:*

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2) = \prod_{i=1}^{\infty}(1+x_it_1+y_it_2) = \sum_{n=0}^{\infty} G_n(xt_1+yt_2),$$

where $(x_i,y_i)\in\mathbb{C}^2$, $i\ge 1$ and $G_n(xt_1+yt_2) = \sum\limits_{k_1<k_2<\ldots<k_n}(x_{k_1}t_1+y_{k_1}t_2)\ldots(x_{k_n}t_1+y_{k_n}t_2)$ and $G_0=1$.

Proof. For every $(x,y) \in \mathcal{X}^2$, the product

$$\prod_{i=1}^{\infty}(1 + x_i t_1 + y_i t_2)$$

is absolutely convergent if the series $\sum\limits_{i=1}^{\infty}(x_i t_1 + y_i t_2)$ is absolutely convergent. Since

$$\begin{aligned}\sum_{i=1}^{\infty}|x_i t_1 + y_i t_2| &\le \sum_{i=1}^{\infty}(|x_i||t_1| + |y_i||t_2|) = |t_1|\sum_{i=1}^{\infty}|x_i| + |t_2|\sum_{i=1}^{\infty}|y_i| \\ &\le \max\{|t_1|,|t_2|\}\Big(\sum_{i=1}^{\infty}|x_i| + \sum_{i=1}^{\infty}|y_i|\Big) \\ &\le \max\{|t_1|,|t_2|\}\sqrt{2}\Big(\sum_{i=1}^{\infty}\left(|x_i|^2 + |y_i|^2\right)^{1/2}\Big) < \infty,\end{aligned}$$

we obtain that $\prod\limits_{i=1}^{\infty}(1 + x_i t_1 + y_i t_2)$ is absolutely convergent, and so the product is convergent as well. Since for every $1 \le m < \infty$ will be performed the equality

$$\sum_{\substack{n=0\\0\le p\le n}}^{m} t_1^p t_2^{n-p}\delta_{(x,y)}(R^{p,n-p}) = \prod_{i=1}^{m}(1 + x_i t_1 + y_i t_2)$$

and series and product are convergent, we obtain that

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2) = \prod_{i=1}^{\infty}(1 + x_i t_1 + y_i t_2).$$

It is known from Combinatorics [8] that $\sum\limits_{n=0}^{\infty} t^n G_n(x) = \prod\limits_{i=1}^{\infty}(1 + x_i t_1)$ for every $x \in c_{00}$, where $G_n(x) = \sum\limits_{k_1<\ldots<k_n}^{\infty} x_{k_1}\ldots x_{k_n}$ is the basis of elementary symmetric polynomials of algebra $\mathcal{H}_{bs}(\ell_1)$. Since it is true for every $x \in \ell_1$,

$$\begin{aligned}\sum_{n=0}^{\infty} G_n(xt_1 + yt_2) &= \sum_{n=0}^{\infty}(t_1 t_2)^n G_n\Big(\frac{x}{t_2} + \frac{y}{t_1}\Big) = \prod_{i=1}^{\infty}\Big(1 + \Big(\frac{x_i}{t_2} + \frac{y_i}{t_1}\Big)t_1 t_2\Big) \\ &= \prod_{i=1}^{\infty}(1 + x_i t_1 + y_i t_2).\end{aligned}$$

□

Now we show that the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\mathcal{X}^2$ does not coincide of point evaluation functionals.

Example 1. *Let k, l are same fixed nonzero complex numbers. Now we consider the sequence of elements*

$$\begin{aligned}e_1(k,l) &= \left(\begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right), \\ e_2(k,l) &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right), \\ &\ldots\ldots\ldots\ldots \\ e_n(k,l) &= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} k \\ l \end{pmatrix}, \ldots \right), \\ &\ldots\ldots\ldots\ldots\end{aligned}$$

in $\mathcal{X}^2$ and for every n put

$$v_n(k,l) = \frac{1}{n}(e_1(k,l) + e_2(k,l) + \ldots + e_n(k,l)) \in \mathcal{X}^2.$$

Then $\delta_{v_n(k,l)}(H^{0,1}) \to l$, $\delta_{v_n(k,l)}(H^{1,0}) \to k$, $\delta_{v_n(k,l)}(H^{p,i-p}) \to 0$ as $n \to \infty$ for every $1 \le k \le i$, where $1 \le p \le i$. By the reletive compactness of bounded subset of $M_{bvs}(\mathcal{X}^2)$ there is an accumulation point $\varphi_{(k,l)}$ of the sequence $\delta_{v_n(k,l)}$, such that $\varphi_{(k,l)}(H^{0,1}) = l$, $\varphi_{(k,l)}(H^{1,0}) = k$, $\varphi_{(k,l)}(H^{p,i-p}) = 0$ for all $1 \le i \le m$, where $1 \le p \le i$. From Theorem 2 it follows that there is no poit $(x,y) \in \mathcal{X}^2$, such that $\delta_{(x,y)} = \varphi_{(k,l)}$. Indeed, if such a point exists, then $(x,y) \sim (0,0)$. Therefore $\delta_{v_n(k,l)}(H^{0,1}) = \delta_{v_n(k,l)}(H^{1,0}) = 0$, but we have that $\delta_{v_n(k,l)}(H^{0,1}) = l$, $\delta_{v_n(k,l)}(H^{1,0}) = k$.

Example 2. *Let $\varphi_{(k,l)}$ be as in Example 1. We know that $\mathcal{H}(\varphi_{(k,l)}) = k + l$. To find $\mathcal{R}(\varphi_{(k,l)})$ note that*

$$R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{n^p n^{s-p}} \binom{n}{s} \binom{s}{p},$$

hence

$$\varphi(R^{p,s-p}) = \lim_{n\to\infty} R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{p!(s-p)!}$$

and so

$$\begin{aligned}\mathcal{R}(\varphi_{(k,l)})(t_1,t_2) &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} t_1^p t_2^{s-p} \varphi(R^{p,s-p}) \\ &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} \frac{(kt_1)^p (lt_2)^{s-p}}{p!(s-p)!} = e^{kt_1+lt_2}.\end{aligned}$$

Theorem 5. *The invertible elements of semigroup $(M_{bvs}(\mathcal{X}^2), \star)$ are functionals only of the form $\varphi_{(k,l)} = \mathcal{R}(\varphi_{(k,l)})(t_1,t_2) = e^{kt_1+lt_2}$.*

Proof. Since by Theorem 4 $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$, $\varphi_{(-k,-l)}$ is inverse to $\varphi_{(k,l)}$. In the other hand, if φ is invertible and $\psi = \varphi^{-1}$, then $\mathcal{R}(\psi) = \frac{1}{\mathcal{R}(\varphi)(t_1,t_2)}$ is an entire function of exponential type and so has no zeros. So we have that $\mathcal{R}(\varphi)(t_1,t_2) = e^{kt_1+lt_2}$ for some complex numbers k,l. □

Corollary 1. *Let Φ be a homomorphism on the subspace of block-symmetric polynomials in $H_{bvs}(\mathcal{X}^2)$ to itself such that $\Phi(H^{p,k-p}) = -H^{p,k-p}$ for every p,k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to continuous homomorphism $\tilde{\Phi}$ of $H_{bvs}(\mathcal{X}^2)$. Then for $(x,y) \in \mathcal{X}^2$

$$H^{p,k-p}(x,y) + \Phi(H^{p,k-p})(x,y) = 0 \tag{5}$$

for all p,k. Note that this equality is true for

$$(x_0,y_0) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right).$$

Let us denote $\psi = \delta_{(x_0,y_0)} \circ \tilde{\Phi}$. From the continuity of homomorphism $\tilde{\Phi}$ we have, that $\psi \in M_{bvs}(\mathcal{X}^2)$. From equality (5) we have, that $\delta_{(x,y)} \star \psi = \delta_{(0,0)}$, $\psi = \delta^{-1}_{(x_0,y_0)}$. According to the Theorem 5 $\delta_{(x_0,y_0)}$ is not invertible. □

Let $f(z)$ be an entire function of many variable. We will say that $f(z)$, where $z \in \mathbb{C}^n$, has "plane" zeros if the set of zeros is

$$Z_f = \Big\{z \in \mathbb{C}^n : f(z) = 0\Big\} = \bigcup_{k=1}^{\infty} H_k,$$

where $H_k = \{z : \langle z, a^k |a^k|^{-2}\rangle = 1\}$ is hyperplane in $\mathbb{C}^n$. Here $a^k \in \mathbb{C}^n$ are feets of perpendiculars dropped from the origin onto zeros hyperplanes H_k of the function $f(z)$ (see [10]).

Theorem 6. *Let φ be a character such that $\mathcal{R}(\varphi)$ is a polynomial. Then $R(\varphi)$ have a plane zeros, that is $KerR(\varphi)$ consists of one-codimensional linear subspaces.*

Proof. Let us denote $\Lambda_{t_1t_2}(G_n) = G_n(xt_1 + yt_2)$. Now we consider the equation $\sum\limits_{n=0}^{m} \lambda^n \varphi(\Lambda_{t_1t_2}(G_n)) = 0$ with m solutions z_k, $1 \le k \le m$. Hence $\prod\limits_{i=1}^{m}(1 + z_k\lambda) = 0$. Obviously, every solution z_k can be represented as $z_k = x_k t_1 + y_k t_1$, where x_k, y_k are indeterminants and t_1, t_2 are some complex numbers. If we take $t_1 = 1, t_2 = 0$ and $t_1 = 2, t_2 = 1$, then can fined x_k, y_k. So we have the system of $2m$ equation and $2m$ indeterminants x_k, y_k, $1 \le k \le m$. The solutions of that system are $x_k = z_k$, $y_k = -z_k$, $1 \le k \le m$. Hence x_k, y_k can be clearly define. If $\lambda = 1$, then we obtain the equality

$$\mathcal{R}(\varphi)(t_1, t_2) = \sum_{n=0}^{m} \varphi(\Lambda_{t_1t_2}(G_n)) = \prod_{i=1}^{m}(1 + x_i t_1 + y_i t_2) = 0.$$

Hence φ has plane zeros. □

According to the analog of Hadamard's Theorem [10] the function $\mathcal{R}(\varphi)(t_1, t_2)$ with plane zeros is of the form

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{n} \Big(1 - \big(t_1 \tfrac{\overline{a}_1^k}{|a^k|^2} + t_2 \tfrac{\overline{a}_2^k}{|a^k|^2}\big)\Big),$$

where $\{(a_1^k, a_2^k)\}$ are the zeros of $\mathcal{R}(\varphi)(t_1, t_2)$, $P(t_1, t_2)$ is analytic polynomial and we have

$$\sum_{k=1}^{n} \tfrac{1}{|a_k|} < \infty.$$

According to the Lemma 1

$$\mathcal{R}(\delta_{(x,y)})(t_1, t_2) = \prod_{i=1}^{m}(1 + x_i t_1 + y_i t_2),$$

and so the zeros of $\mathcal{R}(\delta_{(x,y)})(t_1, t_2)$ are

$$a_1^k = -\frac{\overline{x}_k}{|x_k|^2 + |y_k|^2}, \qquad a_2^k = -\frac{\overline{y}_k}{|x_k|^2 + |y_k|^2}.$$

On the other hand, if $f(t_1, t_2)$ is the function of the exponential type with plane zeros, then it can be represented as

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{\infty} \Big(1 - \big(t_1 \tfrac{\overline{a}_1^k}{|a^k|^2} + t_2 \tfrac{\overline{a}_2^k}{|a^k|^2}\big)\Big),$$

if

$$\sum_{k=1}^{\infty} \tfrac{1}{|a_k|} < \infty.$$

So for $\varphi \in M_{bvs}(\mathcal{X}^2)$, which we can represanted as $\varphi = \varphi_{(k,l)} \star \delta_{(x,y)}$, where $(x, y) \in \mathcal{X}^2$, $(x_k, y_k) = -\big(\tfrac{\overline{a}_1^k}{|a_k|^2}, \tfrac{\overline{a}_2^k}{|a_k|^2}\big)$ and $\varphi_{(k,l)}$ was defined in Example 1, we have that

$$\mathcal{R}(\varphi)(t_1, t_2) = f(t_1, t_2).$$

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Кравців В.В., Загороднюк А.В. *Представлення спектра алгебр блочно-симетричних аналітичних функцій обмеженого типу* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 263–271.

У статті описано симетричну згортку характерів алгебри блочно-симетричних аналітичних фнкцій обмеженого типу на ℓ_1-сумі простору $\mathbb{C}^2$. Авторами показано, що спектр такої алгебри не не збігається з множиною класів еквівалентності функціоналів значенні в точках, описано характери такої алгебри, як функції експоненціального типу з "плоскими" нулями.

Ключові слова і фрази: алгебраїчний базис, блочно-симетичні поліноми, блочно-симетричні аналітичні фукції, спектр, симетричний зсув, симетрична згортка.

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A WORPITZKY BOUNDARY THEOREM FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

For a branched continued fraction of a special form we propose the limit value set for the Worpitzky-like theorem when the element set of the branched continued fraction is replaced by its boundary.

Key words and phrases: element set, value set, branched continued fraction of special form.

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INTRODUCTION

A lot of convergence criteria for continued fractions are characterized by convergence domains. Such domains are indicated in the complex plane, that if elements a_k, b_k of a continued fraction belong to these domains then the continued fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots}}} = \mathop{\mathrm{D}}_{k=1}^{\infty}\frac{a_k}{b_k}$$

converges. At first convergence domains for continued fractions we can find in papers of Worpitzky (1865), Pringsheim (1899) and Van Vleck (1901) [8].

Despite of the fact that a well known convergence theorem for continued fractions was proposed by J. Worpitzky in 1865, its new proofs, generalizations and applications are actual even at present [3, 6, 8].

H. Waadeland [10] formulated the Worpitzky theorem in a slightly more general form than classical one [8], using conditions on the coefficients of the continued fraction proposed by F. Paydon and H. Wall [9].

Theorem 1. *Let $\rho \in (0, 1/2]$ be any positive number, and let all elements of a continued fraction*

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\ddots}}} = \mathop{\mathrm{D}}_{i=1}^{\infty}\frac{a_i}{1}, \tag{1}$$

$a_i,\ i = 1, 2, \ldots,$ be complex numbers, bounded by

$$|a_i| \le \rho(1-\rho), \quad i = 1, 2, \ldots. \tag{2}$$

Then the continued fraction (1) converges and its values are contained in the disk $|w| \le \rho$.

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For the continued fraction (1) Haakon Waadeland raised the question: What happens to the set of values of the continued fraction (1) when the condition (2) in the Worpitzky theorem would be replaced by $|a_i| = \rho(1-\rho),\ i = 1,2,\ldots$? Answering on his question H.Waadeland proved [10], that the set of all possible values of the continued fraction (1) is the annulus

$$\rho \cdot \frac{1-\rho}{1+\rho} \leq |w| \leq \rho.$$

In the classical case of the theorem ($\rho = 1/2$), i.e. $|a_i| = 1/4, \quad i = 1,2,\ldots,$ the annulus is $1/6 \leq |w| \leq 1/2$.

The same question one can put for multidimensional generalizations of the continued fraction, such as for example,

a branched continued fraction (BCF) [3]

$$1+\sum_{i_1=1}^{N}\cfrac{a_{i_1}z_{i_1}}{1+\sum\limits_{i_2=1}^{N}\cfrac{a_{i_1i_2}z_{i_2}}{1+\sum\limits_{i_3=1}^{N}\cfrac{a_{i_1i_2i_3}z_{i_3}}{1+_{\ddots}}}} = 1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}z_{i_k}}{1}, \tag{3}$$

where $a_{i_1i_2\ldots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1i_2\ldots i_k$ be multiindex;

a branched continued fraction with independent variables [1]

$$\cfrac{a_{00}}{1+\sum\limits_{i_1=1}^{N}\cfrac{a_{i_1}z_{i_1}}{1+\sum\limits_{i_2=1}^{i_1}\cfrac{a_{i_1i_2}z_{i_2}}{1+\sum\limits_{i_3=1}^{i_2}\cfrac{a_{i_1i_2i_3}z_{i_3}}{1+_{\ddots}}}}} = \cfrac{a_{00}}{1+\mathop{\mathrm{D}}\limits_{k=1}^{\infty}\sum\limits_{i_k=1}^{i_{k-1}}\cfrac{a_{i(k)}z_{i_k}}{1}}, \tag{4}$$

where $a_{i_1i_2\ldots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1i_2\ldots i_k$ be multiindex $1 \leq i_k \leq i_{k-1},\ k = 1,2,\ldots,\ i_0 = N$;

or a two-dimensional continued fraction (TDCF) [6]

$$\mathop{D}_{i=0}^{\infty}\frac{a_{i,i}z_1z_2}{\Phi_i}, \quad \Phi_i = 1+\mathop{D}_{j=1}^{\infty}\frac{a_{i+j,i}z_1}{1}+\mathop{D}_{j=1}^{\infty}\frac{a_{i,i+j}z_2}{1}, \tag{5}$$

where $a_{i,j}, i = 0,1,\ldots, j = 1,2,\ldots,$ be complex numbers, z_1, z_2 be complex variables.

It was found this question for the branched continued fraction (3) with $z_1 = z_2 = \ldots = z_N = 1$ is answered by the following theorem [11].

Theorem 2. *Let $\rho \in (0,1/2]$ and $N \geq 2$ be an integer. In the family of branched continued fractions*

$$1+\sum_{i_1=1}^{N}\cfrac{a_{i_1}}{1+\sum\limits_{i_2=1}^{N}\cfrac{a_{i_1i_2}}{1+\sum\limits_{i_3=1}^{N}\cfrac{a_{i_1i_2i_3}}{1+_{\ddots}}}} = 1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{1}, \tag{6}$$

where $a_{i_1i_2\ldots i_k}$ be complex numbers, $i(k) = i_1i_2\ldots i_k$ be multiindex, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right| = \dfrac{\rho(1-\rho)}{N}$, then the set of possible branched continued fraction values is the closed disk $|w| \leq \rho$.

Thus, in this case the set of possible BCF values is unchanged when all elements of (6) are restricted to the boundary of the disk.

For TDCF (5) with $z_1 = z_2 = 1$ the answer is proposed by the following theorem [7].

Theorem 3. *Let ρ be a real number in $(0,1/2]$, and let F_ρ be the family of two-dimensional continued fractions*

$$\mathop{D}_{i=0}^{\infty} \frac{a_{i,i}}{\Phi_i}, \quad \Phi_i = 1 + \mathop{D}_{j=1}^{\infty} \frac{a_{i+j,i}}{1} + \mathop{D}_{j=1}^{\infty} \frac{a_{i,i+j}}{1}, \tag{7}$$

where $a_{i,j}$, $i = 0,1,\ldots, j = 1,2,\ldots$, be complex numbers that satisfy conditions $|a_{i,j}| = \frac{1}{2}\rho(1-\rho)$, $i,j \geq 1$.

Then the set of all possible values f of the TDCF (7) in F_ρ is the annulus A_ρ, given by

$$R \cdot \frac{\rho(1-\rho)}{4R-\rho(1-\rho)} \leq |f| \leq R, \quad R = \frac{1}{2}\left(\sqrt{1-2\rho(1-\rho)} + \sqrt{1-4\rho(1-\rho)}\right).$$

In the case $\rho = 1/2$ the annulus is $\left(8+\sqrt{2}\right)/124 \leq |f| \leq 1/2\sqrt{2}$.

In the present paper the answer will be done for the branched continued fraction with independent variables (4) with $z_1 = z_2 = \ldots = z_N = 1$ (named the branched continued fraction of the special form [2, 5, 4]).

1 THE WORPITZKY-LIKE THEOREMS FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

Since the beginning we prove the Worpitsky-like theorem in a slightly more general form than it was done in [1].

Theorem 4. *Let $\rho \in (0,1/2]$ and $N \geq 2$ be an integer. In the BCF of the special form*

$$\cfrac{a_{00}}{1 + \mathop{D}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1}}, \tag{8}$$

where $a_{i_1 i_2 \ldots i_k}$ be complex numbers, $i(k) = i_1 i_2 \ldots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}$, $k = 1,2,\ldots$, $i_0 = N$, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right| \leq \alpha_{i_{k-1}} = \dfrac{\rho(1-\rho)}{i_{k-1}}$, $|a_{00}| \leq \rho(1-\rho)$.

Then the BCF of the special form (8) converges, and its values are contained in the disk $|w| \leq \rho$.

Proof. It is not difficult to show that a periodic continued fraction

$$\cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \vdots}}} \tag{9}$$

is the majorant fraction for the BCF of special form (8).

It means that approximants of these fractions satisfy the relation:

$$|f_n - f_m| \leq M \cdot |g_n - g_m|,$$

where f_n, g_n are the nth approximants of the BCF of the special form (8) and continued fraction (9) respectively, M is a certain constant, m, n are natural numbers.

For the difference between the nth and mth approximants of the BCF of the special form (8) the following relation is true [1]:

$$f_n - f_m = (-1)^m \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \dots \sum_{i_m=1}^{i_{m-1}} \frac{a_{00} \cdot \prod_{k=1}^{m} a_{i(k)}}{\prod_{k=0}^{m} Q_{i(k)}^{(n-1)} \prod_{k=0}^{m-1} Q_{i(k)}^{(m-1)}}, \ n > m \geq 1, \tag{10}$$

where

$$Q_{i(s)}^{(s)} = 1, \quad Q_{i(k)}^{(s)} = 1 + \sum_{i_{k+1}=1}^{i(k)} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}}, \ k = \overline{1, s-1}, s \geq 2,$$

$$Q^{(s)} = Q_{i(0)}^{(s)} = 1 + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{Q_{i(1)}^{(s)}}, \ s \geq 1, \quad f_n = \frac{a_{00}}{Q_{i(0)}^{(n-1)}}.$$

Using the method of complete mathematical induction it is easy to prove that

$$\left| Q_{i(k)}^{(s)} \right| \geq h_{s-k}, \tag{11}$$

where h_m is the m th approximant of the continued fraction

$$1 - \cfrac{\rho(1-\rho}{1 - \cfrac{\rho(1-\rho}{1 - \genfrac{}{}{0pt}{}{}{\vdots}}}$$

for all possible index sets.

Let us write the difference formula for approximants of the continued fraction (9)

$$g_n - g_m = \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod_{i=0}^{m} h_{n-i-1} \prod_{i=0}^{m-1} h_{m-i-1}}. \tag{12}$$

From (11) follows that all $Q_{i(k)}^{(s)} \neq 0$. Hence, taking into account (10) and (12) we have

$$\begin{aligned} |f_n - f_m| &\leq \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \dots \sum_{i_m=1}^{i_{m-1}} \frac{|a_{00}| \cdot \prod_{k=1}^{m} \left|a_{i(k)}\right|}{\prod_{k=0}^{m} \left|Q_{i(k)}^{(n-1)}\right| \prod_{k=0}^{m-1} \left|Q_{i(k)}^{(m-1)}\right|} \\ &\leq \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod_{k=0}^{m} h_{n-k-1} \prod_{k=0}^{m-1} h_{m-k-1}} = g_n - g_m. \end{aligned}$$

The continued fraction (9) converges, and therefore the BCF of the special form (8) is also convergent.

Let us write the m th approximant of (8) in the form

$$z = \frac{a_{00}}{1 + \sum\limits_{i_1=1}^{N} \frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}}} = \frac{a_{00}}{(1+w)}.$$

From the conditions of the theorem on the fraction coefficients and inequalities (11) one can write

$$|w| = \left| \sum_{i_1=1}^{N} \frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}} \right| \leq \frac{\rho(1-\rho)}{h_{m-2}} = g_{m-1}.$$

Putting $g_n = P_n/Q_n$, where P_n is the nth numerator and Q_n is the nth denominator of the approximant g_n it is easy to find by induction that

$$Q_n = \sum_{i=0}^{n} \rho^i (1-\rho)^{n-i}.$$

If Q is the value of the infinite fraction (9), and $Q_n > 0,\ n = 1, 2, \ldots$, then we get

$$g_n - g_{n-1} = \frac{(\rho(1-\rho))^n}{Q_n Q_{n-1}} \geq 0,$$

i.e., the sequence $\{g_n\}$ grows monotonically. Hence,$|w| \leq Q$. Since $Q = \rho(1-\rho) \cdot (1-Q)^{-1}$, and taking into account that $Q = 0$, if $\rho = 0$, the solution of this quadratic equation with respect to Q gives $Q = \rho$.

Therefore, $|w| \leq \rho$, and $|z| \leq \rho$. $\square$

Now we obtain the boundary version of this theorem.

Theorem 5. *Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the family of branched continued fractions of the special form F_ρ*

$$\frac{a_{00}}{1 + \mathop{D}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1}}, \tag{13}$$

where $a_{i_1 i_2 \ldots i_k}$ be complex numbers, $i(k) = i_1 i_2 \ldots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}$, $k = 1, 2, \ldots$, $i_0 = N$, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right| = \dfrac{\rho(1-\rho)}{i_{k-1}}$, $|a_{00}| = \rho(1-\rho)$, the set of all possible branched continued fractions of the special form values is the annulus A_ρ, given by

$$\rho \cdot \frac{1-\rho}{1+\rho} \leq |w| \leq \rho.$$

Proof. Let f_0 be a possible value of the BCF of the special form (13). Then all values f with $|f| = |f_0|$ are possible BCF of the special form values in F_ρ. Hence the set of values of such fraction must be a disk or an annulus, in both cases centered at the origin. From the Worpitzky-like theorem (Theorem 4) follows that this disk or annulus must be contained in the disk $|f| \leq \rho$.

We shall first prove that the set of all values must be contained in A_ρ. Any BCF of the special form in F_ρ can be written in the form

$$f = \frac{\rho(1-\rho)e^{i\theta}}{1+\omega},\ \theta \in [0, 2\pi),\ \omega = \sum_{i_1=1}^{N} \frac{a_{i(1)}}{1 + \mathop{D}\limits_{k=1}^{\infty} \sum\limits_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1}}.$$

Since $\dfrac{a_{i(1)} \cdot N}{1 + \overset{\infty}{\underset{k=1}{D}} \sum\limits_{i_{k+1}=1}^{i_k} \dfrac{a_{i(k+1)}}{1}} \in F_\rho$ we have, using the previous Theorem 4

$$\left| a_{i(1)} \cdot N \cdot \left(1 + \overset{\infty}{\underset{k=1}{D}} \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \rho.$$

It means that

$$\left| a_{i(1)} \cdot \left(1 + \overset{\infty}{\underset{k=1}{D}} \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \frac{\rho}{N},$$

and $|\omega| \leq \rho$. Since $|\omega| \leq \rho$ it follows that for any value f of a BCF of the special form in F_ρ we have $|f| \geq \rho \cdot \dfrac{1-\rho}{1+\rho}$.

That is sharp, follows from the fact that

$$\rho = \cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \ddots}},$$

and that the right-hand side is in F_ρ.

We next prove that A_ρ is contained in the set of values of BCFs of the special form in F_ρ with independent variables $|\omega| \leq \rho$.

By the mapping $\xi = 1/1 + \omega$ the circle $\omega = \rho$ is mapped onto the circle

$$\left| \xi - \frac{1}{1-\rho^2} \right| = \frac{\rho}{1-\rho^2}.$$

Then, by $\xi \to \rho(1-\rho)e^{i\theta}\xi$, for all $\theta \in [0, 2\pi)$ we get all points in the annulus A_ρ.

Hence, A_ρ is contained in the set of BCF with independent variables values for F_ρ. $\square$

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Для гіллястого ланцюгового дробу спеціального вигляду запропоновано межову множину значень у теоремі типу Ворпіцького, коли множина елементів гіллястого ланцюгового дробу замінена її межею.

Ключові слова і фрази: множина елементів, множина значень, гіллястий ланцюговий дріб спеціального вигляду.

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POINTWISE STABILIZATION OF THE POISSON INTEGRAL FOR THE DIFFUSION TYPE EQUATIONS WITH INERTIA

In this paper we consider the pointwise stabilization of the Poisson integral for the diffusion type equations with inertia in the case of finite number of parabolic degeneracy groups. We establish necessary and sufficient conditions of this stabilization for a class of bounded measurable initial functions.

Key words and phrases: Poisson integral, Kolmogorov equation, diffusion type equation with inertia, stabilization, degenerate parabolic equation, surface level, average on border.

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INTRODUCTION

In this paper we consider pointwise stabilization of the Poisson integral for diffusion type equations with inertia which have finite number groups of variables with diffusion degeneration.

Stabilization problems for solutions of the Cauchy problem for parabolic equations were studied by S.D. Eidelman and V.P. Repnikov [1, 2]. Necessary and sufficient conditions of pointwise stabilization of the Poisson integral for the Kolmogorov equation were obtained by S.D. Eidelman, V.P. Repnikov and G.P. Malytska [3, 4]. Generalization of these results in the case of three degeneration groups can be found in the work [5].

1 NOTATIONS AND PROBLEM STATEMENT

Let $x := (x_{11}, x_{12}, \dots, x_{1n_1}; \dots; x_{k1}, x_{k2}, \dots, x_{kn_k}; \dots; x_{p1}, x_{p2}, \dots, x_{pn_p}; x_{p+1,1}, \dots, x_{m1})$, $n_1 \geq n_2 \geq \dots \geq n_p > 1$, $n_k \in \mathbb{N}$, $k = \overline{1,p}$, $p \in \mathbb{N}$, $m \geq p$, $\sum\limits_{k=1}^{p} n_k + m - p = n$, $x \in \mathbb{R}^n$. Consider the Cauchy problem

$$\partial_t u\,(t,x) - \sum_{k=1}^{p}\sum_{j=1}^{n_k} x_{kj}\partial_{x_{k\,j+1}} u\,(t,x) = \sum_{v=1}^{m} \partial^2_{x_v^2} u\,(t,x)\,, \tag{1}$$

$$u\,(t,x)\,|_{t=\tau} = u_0\,(x)\,,\ \ 0 \leq \tau < t \leq T < +\infty,\ x \in \mathbb{R}^n, \tag{2}$$

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where $u_0(x)$ is a Lebesgue measurable and bounded function in $\mathbb{R}^n$. The fundamental matrix of solutions $G(t-\tau,\, x,\, \xi)$ with $t>\tau, x\in\mathbb{R}^n, \xi\in\mathbb{R}^n$ of the Cauchy problem (1), (2) was found in [6]. Hence,

$$G(t-\tau,\, x,\, \xi) = (2\sqrt{\pi})^{-n}(t-\tau)^{-\mu}\prod_{\nu=1}^{p}\prod_{k=1}^{n_\nu} k(k+1)\dots(2k-2)(2k-1)^{-\frac{1}{2}}\, e^{-\rho(t,x;\tau,\xi)}, \tag{3}$$

where

$$\begin{aligned}\rho(t,x;\tau,\xi) = {} & \sum_{\nu=1}^{m}|x_{\nu 1}-\xi_{\nu 1}|^2 4^{-1}(t-\tau)^{-1}\sum_{\nu=1}^{p}\sum_{k=2}^{n_\nu}(k-1)^2k^2\dots(2k-3)^2(2k-1)\\ & (t-\tau)^{-(2k-1)}\Big|\sum_{j=0}^{k-1}\tfrac{x_{\nu k-j}(t-\tau)^j}{j!}-\xi_{\nu k}-\Big(\sum_{j=0}^{k-2}\tfrac{x_{\nu k-1-j}(t-\tau)^j}{j!}-\xi_{\nu k-1}\Big)(t-\tau)2^{-1}+\dots\\ & +(-1)^{k-l}\tfrac{2l(2l+1)\dots(2l+(k-l)-2)(2l+2(k-l)-1)}{k\dots(2k-1)}\tfrac{(t-\tau)^{(k-l)}}{(k-l)!}\big(\sum_{j=0}^{l-1}\tfrac{x_{\nu l-j}(t-\tau)^j}{j!}-\xi_{\nu l}\big)+\dots\\ & +\tfrac{(-1)^{k-1}(t-\tau)^{(k-1)}}{k\dots(2k-2)}(x_{\nu 1}-\xi_{\nu 1})\Big|^2,\ \mu=\tfrac{m}{2}+\tfrac{\sum_{k=1}^{p}(n_k-1)^2}{2}.\end{aligned}$$

Here $\rho(t,x;0,\xi)=r^2$ is the family of surfaces of the fundamental solutions of the problem (1), (2). Let us denote by $F_{r,t}^{x,0}$ a figure which is bounded by the ellipsoid

$$\rho(t,x;0,\xi)=r^2, \tag{4}$$

where ξ is a variable. Let v_n be the volume of the figure which is bounded by the surface $\rho_1(\alpha)\equiv 1$, where

$$\rho_1(\alpha)=\sum_{\nu=1}^{m}\alpha_{\nu 1}^2+\sum_{\nu=1}^{m}\sum_{k=2}^{n_\nu}(\alpha_{\nu k}-(2k-3)^{1/2}(2k-1)^{1/2}(k-1)^{-1}\alpha_{\nu k-1}).$$

Let $M_t^x(r)$ is the average of $u_0(x)$ with respect to $F_{r,t}^x$ which is bounded by surfaces (4).

Definition 1. *Function $u_0(x)$ has threshold average $M^x(r)$ on bodies $F_{r,t}^x$ if there exists the following limit $\lim\limits_{t\to\infty}M_t^x(r)=M^x(r)$.*

2 POINTWISE STABILIZATION OF THE POISSON INTEGRAL OF THE CAUCHY PROBLEM (1), (2)

Theorem 1. *If $u_0(x)$ has a threshold average on ellipsoids $F_{r,t}^{x,0}$, which almost for all r is equal to $M^x(r)$, then the Poisson integral of the equation* (1) *stabilizes (as $t\to\infty$) to the number*

$$\iota=(2\pi)^{-n/2}v_n\int_0^{+\infty}r^{n+1}e^{-r^2}M^x(r)dr.$$

Proof. Consider the Poisson integral of the equation (1)

$$u(t,x)=\int_{\mathbb{R}^n}G(t,x;0,\xi)u_0(\xi)d\xi. \tag{5}$$

Let us make the following change of variables

$$\begin{cases} x_{\nu 1} - \xi_{\nu 1} = -2t^{1/2}\alpha_{\nu 1},\ \nu = \overline{1,m}, \\ (k-1)k\dots(2k-3)(2k-1)^{1/2}t^{-\frac{2k-1}{2}}\Big[\sum\limits_{j=0}^{k-1}\frac{x_{\nu k-j}(t-\tau)^j}{j!} - \xi_{\nu k} \\ -\Big(\sum\limits_{j=0}^{k-2}\frac{x_{\nu k-1-j}(t-\tau)^j}{j!} - \xi_{\nu k-1}\Big)\frac{t-\tau}{2} + \dots \\ +\frac{(-1)^{k-l}(t-\tau)^{(k-l)}}{(k-l)!}\frac{2l(2l+1)\dots(2l+(k-l)-2)(2l+2(k-l)-1)}{k\dots(2k-1)}\Big(\sum\limits_{j=0}^{l-1}\frac{x_{\nu l-j}(t-\tau)^j}{j!} - \xi_{\nu l}\Big) \\ +\dots+(-1)^{k-2}(t-\tau)^{(k-2)}\frac{x_{\nu 2}-\xi_{\nu 2}+(t-\tau)x_{\nu 1}}{2(k+1)\dots(2k-3)} + \frac{(-1)^{k-1}(t-\tau)^{(k-1)}}{k\dots(2k-2)}(x_{\nu 1}-\xi_{\nu 1})\Big] \\ = -(\alpha_{\nu k} - (2k-3)^{1/2}(2k-1)^{1/2}(k-1)^{-1}\alpha_{\nu k-1} + \dots + (-1)^l\alpha_{k-l}l \\ (2l+1)\dots(2l+(k-l)-2)(2k+2(k-l)-1)(2k-1)^{-1/2}((2(k-l)-1)!)^{-1} \\ (k-l)^{-1}(2(k-l)-1)^{1/2} + \dots + (-1)^{k-1}2\alpha_{\nu 1}(2k-1)^{1/2}), \quad \nu = \overline{1,p},\ k = \overline{1,n_\nu}. \end{cases} \tag{6}$$

Then equation (5) takes the form

$$u(t,x) = \pi^{-x/2}\int\limits_{R^x} exp\{-\rho_1(\alpha)\}u_0(\xi(\alpha,x,t))d\alpha, \tag{7}$$

where $u_0(\xi(\alpha,x,t))$ is the value of $u_0(\xi)$, and $\xi(\alpha,x,t)$ is determined by the system (6). Let us consider positively defined quadratic form

$$\rho_1(\alpha) = \sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu} c_{\nu kj}\alpha_{\nu k}\alpha_{\nu j},$$

and respective family of disjoint ellipsoids

$$\sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu} c_{\nu kj}\alpha_{\nu k}\alpha_{\nu j} = r^2.$$

In the integral (7) we consider new integration variables

$$\begin{cases} \alpha_{11} = r\Phi(\Psi)\cos\Psi_1, \\ \alpha_{21} = r\Phi(\Psi)\sin\Psi_1\cos\Psi_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_{m1} = r\Phi(\Psi)\sin\Psi_1\sin\Psi_2\dots\sin\Psi_{n-1}, \end{cases} \tag{8}$$

where $0 \le r < +\infty, \Psi = (\Psi_1\dots\Psi_{n-1}), 0 \le \Psi_j \le \pi, j = \overline{1,n-2}, 0 \le \Psi_{n-1} \le 2\pi$. The function $\Phi(\Psi)$ is defined by the equality

$$\Phi^2(\Psi)\sum_{\nu=1}^{m}\sum_{k,j=1}^{n_\nu} c_{\nu kj}\alpha'_{\nu k}\alpha'_{\nu j} = 1,$$

where $\alpha_1^{'}1 = \cos\Psi_1$, $\alpha_1^{'}2 = \sin\Psi_1\cos\Psi_2, \dots, \alpha_{m1}^{'} = \sin\Psi_1\sin\Psi_2\dots\sin\Psi_{n-2}\cos\Psi_{n-1}$. Note that $J = r^{n-1}J_1$ is the Jacobian of the transformation (8), where

$$J_1 = \Phi^n(\Psi)\sin^{n-2}\Psi_1\sin^{n-3}\Psi_2\dots\sin\Psi_{n-1}.$$

Let us denote $u_0(t,r,\Psi,x) := u_0(\xi(\alpha,x,t))$, where α is defined by (8). Then we obtain

$$u(t,x) = \pi^{-n/2}\int_0^{+\infty} r^{n-1}e^{-r^2}dr\int_{\Sigma_1} u_0(t,r,\Psi,x)Jd\Psi = \pi^{-n/2}\int_0^{+\infty} e^{-r^2}\frac{\partial}{\partial r}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,x)J_1 d\Psi dr$$
$$= 2\pi^{-n/2}\int_0^{+\infty} re^{-r^2}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,x)Jd\Psi dr,$$

where Σ_1 is the unit sphere in $\mathbb{R}^n$, J is the Jacobian of the transformation (8). Therefore for $M_t^x(r)$ we have

$$u(t,x) = 2\pi^{-n/2}v_n\int_0^{+\infty} r^{n+1}e^{-r^2}(r^n v_n)^{-1}\int_0^r \rho^{n-1}d\rho\int_{\Sigma_1} u_0(t,r,\Psi,n)Jd\Psi dr$$
$$= 2\pi^{-n/2}v_n\int_0^{+\infty} r^{n+1}e^{-r^2}M_t^x(r)dr.$$

It remains to pass to the limit in the above integral as $t\to\infty$. It can be done according to the Lebesgue theorem because there exists a threshold average. From boundedness of $u_0(x)$ immediately follows uniform boundedness of $M_t^x(r)$ by t.

Note that it is sufficient to show the existence of threshold average in some fixed point x_1 that leads to existence of threshold average in any point x and to stabilization at every compact. □

Theorem 2. *Let $u_0(x)\ge 0$. For stabilization of the Poisson integral (5) to zero it is necessary and sufficient that $u_0(x)$ has a threshold average $M^x(r)$, which almost everywhere is equal to zero.*

Proof. The sufficiency follows from Theorem 1. Let us show that from stabilization of the integral (5) it follows the existence of a zero threshold average on $F_{r,t}^x$:

$$M_t^x(r) = \frac{1}{mesF_{r,t}^x}\int_{F_{r,t}^x} u_0(\xi)d\xi \le ct^{-N_1/3}\int_{\mathbb{R}^N} exp\{-\rho(t^{1/3},x,0,\xi)\}u_0(\xi)d\xi = c_1 u(t^{1/3},x), \quad (9)$$

where $N_1 = \frac{m-p}{2} + \sum\limits_{k=1}^{p} n_k^2$. In the inequality (9) $mesF_{r,t}^x$ replaced by volume of the parallelepiped

$$\begin{cases} |\xi_{\nu 1} - x_{\nu 1}| \le t^{1/6}, \ \nu = \overline{1,m}, \\ |\xi_{\nu k} - x_{\nu k}| \le t^{\frac{2k-1}{6}}, \nu = \overline{1,p}, \ k = \overline{2,n_p}. \end{cases}$$

Since $u(t,x)\to 0$ as $t\to\infty$, then from (9) it follows that $M_{t,r}^x \to 0$ as $t\to\infty$ for any r. □

3 Conclusion

If there exists a threshold average of a measurable bounded initial function, then theorems about pointwise stabilization of the Poisson integral for diffusion type equations with inertia also take place for systems of Kolmogorov equations with constant coefficients [7, 8]. Stabilization of the Poisson integral of the equation (1) is related to the stability problem of derivative prices on financial markets [9, 10, 11].

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В роботі розглянуто поточкову стабілізацію інтеграла Пуассона для рівнянь типу дифузії з інерцією у випадку скінченної кількості груп виродження параболічності, встановлено необхідні і достатні умови такої стабілізації у класі обмежених вимірних початкових функцій.

Ключові слова і фрази: інтеграл Пуассона, рівняння Колмогорова, рівняння типу дифузії з інерцією, стабілізація, вироджене параболічне рівняння, поверхні рівня, граничне середнє.

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ON GENERALIZED COMPLEX SPACE FORMS SATISFYING CERTAIN CURVATURE CONDITIONS

We study Ricci soliton (g, V, λ) of generalized complex space forms when the Riemannian, Bochner and W_2 curvature tensors satisfy certain curvature conditions like semi-symmetric, Einstein semi-symmetric, Ricci pseudo symmetric and Ricci generalized pseudo symmetric. In this study it is shown that shrinking, steady and expansion of the generalized complex space forms depend on the solenoidal property of vector V. Also we prove that generalized complex space form with conservative Bochner curvature tensor is constant scalar curvature.

Key words and phrases: generalized complex space forms, Ricci soliton, Einstein manifold, Einstein semi-symmetric, pseudo symmetric.

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1 INTRODUCTION

A Kähler manifold with constant holomorphic sectional curvature is a complex space form and it has a specific form of its curvature tensor. More generally an almost Hermition manifold M is called a generalized complex space form $M(f_1, f_2)$ if its Riemannian curvature tensor R satisfies,

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ - g(Y,JZ)JX + 2g(X,JY)JZ\}, \end{aligned} \tag{1}$$

for all $X, Y, Z \in TM$, where f_1 and f_2 are smooth functions on M [21]. In [21], an important obstruction for such a space was presented by Tricerri and Vanhecke: if M is connected, dim ≥ 6 and f_2 is not identically zero, then M is a complex-space-form (in particular, f_1 and f_2 must be constant). Olszak [16] proved the existence of generalized complex space form. The authors Alegre and Carriazo studied structures on generalized Sasakian space forms [1]. The authors De [7], Kim [12], Atceken [13], Nagaraja [14], et. al., have contributed to the study of Sasakian space forms in which they put different symmetric conditions on projctive curvature tensor etc.

A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel [5], i.e. $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold the notion of semi-symmetric manifold was defined by

$$(R(X,Y)\cdot R)(U,V,W) = 0, \quad X,Y,U,V,W \in \chi(M)$$

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and it is studied by many authors [15,17]. A complete intrinsic classification of these was given by Szabo [20].

For a $(0,k)$-tensor field T on M, $k \geq 1$, and a symmetric $(0,2)$ tensor fields g and S on M, we define the $(0,k+2)$ tensor fields $R \cdot T$, $Q(A,T)$ and $Q(B,T)$ by

$$\begin{aligned}
&(R \cdot T)(X_1, \ldots, X_k, X, Y) = \\
&\quad - T(R(X,Y)X_1, X_2, \ldots, X_k - \cdots - T(X_1, X_2, \ldots, X_{k-1}, R(X,Y)X_k), \\
&Q(g,T)(X_1, \ldots, X_k, X, Y) = \\
&\quad - T((X \wedge_g Y)X_1, X_2, \ldots, X_k - \ldots - T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_g Y)X_k), \\
&Q(S,T)(X_1, \ldots, X_k, X, Y) = \\
&\quad - T((X \wedge_S Y)X_1, X_2, \ldots, X_k - \cdots - T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_S Y)X_k),
\end{aligned}$$

where $(X \wedge_g Y)$ and $(X \wedge_S Y)$ are the endomorphism given by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y, \quad (X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y.$$

A Riemannian manifold is said to be pseudo symmetric (in the sense of Deszcz [6,9]) if

$$R \cdot R = L_R Q(g,R)$$

holds on the set $U_R = \{x \in M \mid R - \frac{r}{n(n-1)} G \neq 0 \text{ at } x\}$, where G is the $(0,4)$–tensor defined by $G(X_1, X_2, X_3, X_4)$=$g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some function on U_R.

A Riemannian manifold is said to be Ricci generalized pseudo symmetric (in the sense of Deszcz [6,9]) if

$$R \cdot R = L_R Q(S,R)$$

holds on the set $U_R = \{x \in M : Q(S,R) \neq 0 \text{ at } x\}$, and L_R is some function on U_R. A Riemannian manifold is said to be Bochner Ricci generalized pseudo symmetric if

$$R \cdot B = L_B Q(S,B)$$

holds on the set $U_B = \{x \in M : B \neq 0 \text{ at } x\}$, and L_B is some function on U_B and B is the Bochner curvature tensor. If $L_B = 0$ on U_B, then a Bochner Ricci generalized pseudo symmetric manifold is Bochner semisymmetric. But L_B need not be zero, in general and hence there exists Bochner Ricci generalized pseudo symmetric manifolds which are not Bochner semisymmetric manifolds. Thus the class of Bochner Ricci generalized pseudo symmetric manifolds is a natural extension of the class of Bochner semisymmetric manifolds.

Also we need the notion of Ricci solitons. It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M,g). A Ricci soliton is a triple (g,V,λ) with g a Riemannian metric such that

$$L_V g + 2S + 2\lambda g = 0, \tag{2}$$

where V is the potential vector field, λ a real scalar, S is Ricci tensor of M and L_V denotes the Lie derivative operator along V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively [10].

In the context of generalized complex space forms, the authors Bharathi and Bagewadi [3], Bagewadi and Praveena [2, 19] extended the study to W_2 curvature, H-projective, Bochner and pseudoprojective curvature tensors. Motivated by these ideas, in this paper, we extend the study of Ricci soliton in which curvature tensor on generalized complex space forms satisfy several semi-symmetric and pseudo-symmetric conditions. The paper is organized as follows. In the section 2 we give definitions, notions and basic results for generalized complex space forms. In sections 3 and 4 we study Bochner semi-symmetric and Einstein semi-symmetric on generalized complex space forms. In sections 5 and 6 we find the characterizations of generalized complex space forms satisfying the pseudo-symmetric conditions like $R \cdot B = L_B Q(S, B)$. and $B \cdot W_2 = L_1 Q(g, W_2)$. Finally we obtain generalized complex space form with conservative Bochner curvature tensor is of constant scalar curvature.

2 PRELIMINARIES

Let M be a complex n-dimensional Kähler manifold, with a complex structure J and a positive–definite metric g which satisfies the following conditions [4]

$$J^2 = -I, \quad g(JX, JY) = g(X, Y) \quad \text{and} \quad \nabla J = 0,$$

where ∇ means covariant derivation according to the Levi-civita connection. The scalar curvature $r = \Sigma S(e_i, e_i)$, therefore

$$(\nabla_X S)(e_i, e_i) = \nabla_X r = dr(X).$$

Let Q be the Ricci operator defined by $g(QX, Y) = S(X, Y)$. Then

$$(\nabla_Z S)(X, Y) = g((\nabla_Z Q)(X), Y).$$

Taking $Y = Z = e_i$ and taking summation over i in the above equation we get

$$\begin{aligned}(\nabla_{e_i} S)(X, e_i) &= g((\nabla_{e_i} Q)(X), e_i),\\ (divQ)(X) &= tr(Z \to (\nabla_Z Q)(X)) = \sum g((\nabla_{e_i} Q)(X), e_i).\end{aligned}$$

But it is known [8, 18] that $(divQ)(X) = \frac{1}{2}dr(X)$. Hence $(\nabla_{e_i} S)(X, e_i) = \frac{1}{2}dr(X)$ and $(\nabla_{e_i} S)(JX, e_i) = \frac{1}{2}dr(JX)$. It is known [11] that in a Kähler manifold the Ricci tensor S satisfies

$$(divR)(X, Y)Z = (\nabla_Z S)(X, Y) - (\nabla_X S)(Z, Y) = (\nabla_{JY} S)(JX, Z). \tag{3}$$

Using equation (1) we have

$$S(X, Y) = \{(n-1)f_1 + 3f_2\}g(X, Y), \tag{4}$$

$$QX = [(n-1)f_1 + 3f_2]X, \tag{5}$$

$$r = n[(n-1)f_1 + 3f_2], \tag{6}$$

where S is the Ricci tensor, Q is the Ricci operator and r is scalar curvature of the space form $M(f_1, f_2)$.

Given a complex n-dimensional Kähler manifold M, the Bochner curvature tensor and W_2 curvature tensor are given by [11]

$$\begin{aligned}B(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2n+4}[&g(Y,Z)S(X,U) - S(X,Z)g(Y,U)\\ &+ g(JY,Z)S(JX,U) - S(JX,Z)g(JY,U) + S(Y,Z)g(X,U)\\ &- g(X,Z)S(Y,U) + S(JY,Z)g(JX,U) - g(JX,Z)S(JY,U)\\ &- 2S(Y,JX)g(JZ,U) - 2S(JZ,U)g(JX,Y)]\\ &+ \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(JY,Z)g(JX,U)\\ &- g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)],\end{aligned} \tag{7}$$

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{8}$$

Definition 1. *The Einstein Tensor denoted by E is defined by*

$$E(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y), \tag{9}$$

where S is a Ricci tensor and r is the scalar curvature.

Definition 2 ([9,20]). *A n-dimensional generalized complex space form is said to be:*

1) *Bochner-Semi-symmetric if it satisfies*

$$(R(X,Y)\cdot B)(U,V,W) = 0 \text{ for all } X,Y \in \chi(M);$$

2) *Einstein-Semi-symmetric if it satisfies*

$$(R(X,Y)\cdot E)(U,V,W) = 0 \text{ for all } X,Y \in \chi(M).$$

3 Bochner semi-symetric generalized complex space forms

Let generalized complex space form $M(f_1, f_2)$ be Bochner semi-symmetric and by definition it satisfies the equation $R \cdot B = 0$, i.e. for any tangent vectors X, Y, U, Z and W, this implies

$$(R(X,Y)\cdot B)(U,Z,W) = 0.$$

Therefore

$$R(X,Y)B(U,Z)W - B(R(X,Y)U,Z)W - B(U,R(X,Y)Z)W - B(U,Z)R(X,Y)W = 0.$$

Taking inner product with T we have,

$$\begin{aligned}g(R(X,Y)B(U,Z)W,T) - g(B(R(X,Y)U,Z)W,T) - g(B(U,R(X,Y)Z)W,T)\\ - g(B(U,Z)R(X,Y)W,T) = 0.\end{aligned} \tag{10}$$

Using equations (1) and (7) in (10) and putting $X = Z = e_i$, further again putting $Y = T = e_i$ to the simplified equation, where e_i is an an orthonarmal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \le i \le n$, we get

$$f_2\left\{\frac{2n-8}{2n+4}S(U,W) - \frac{5n+2}{(2n+4)(2n+2)}rg(U,W)\right\} = 0.$$

If $f_2 \neq 0$, then

$$\left\{\frac{2n-8}{2n+4}S(U,W) - \frac{5n+2}{(2n+4)(2n+2)}rg(U,W)\right\} = 0.$$

This implies,

$$S(U,W) = \frac{5n+2}{(2n-8)(2n+2)}rg(U,W). \tag{11}$$

That is $M(f_1, f_2)$ is an Einstein manifold. Hence we can state the following result.

Theorem 1. *A generalized complex space form $M(f_1, f_2)$ is an Einstein manifold provided by $f_2 \neq 0$ if Bochner curvature tensor satisfies $R \cdot B = 0$.*

Using equation (11) in (2), we get

$$(L_V g)(U,W) + 2\left[\frac{5n+2}{(2n-8)(2n+2)}\right] rg(U,W) + 2\lambda g(U,W) = 0, \tag{12}$$

setting $U = W = e_i$ in (12) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$divV + \frac{5n+2}{(2n-8)(2n+2)}rn + \lambda n = 0. \tag{13}$$

If V is solenoidal then $divV = 0$. Therefore the equation (13) can be reduced to

$$\lambda = -\frac{5n+2}{(2n-8)(2n+2)}r.$$

Thus, we can state the following.

Corollary 1. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner semi-symmetric. If V is solenoidal then it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

4 EINSTEIN SEMI-SYMMETRIC GENERALIZED COMPLEX SPACE FORM

Let R and E satisfy the equation $R \cdot E = 0$ in $M(f_1, f_2)$. Then this equation leads to

$$(R(X,Y) \cdot E(U,W)) = 0,$$

where X, Y, U and W are any tangent vectors. The above equation can be expressed as

$$E(R(X,Y)U,W) + E(U,R(X,Y)W) = 0. \tag{14}$$

In view of (9) equation (14) becomes

$$S(R(X,Y)U,W) - \frac{r}{2}g(R(X,Y)U,W) + S(U,R(X,Y)W) - \frac{r}{2}g(U,R(X,Y)W) = 0. \tag{15}$$

Using equation (1) in (15) and by replacing $X = U = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \leq i \leq n$, we get

$$f_1[-nS(Y,W) + rg(Y,W)] = 0.$$

If $f_1 \neq 0$, then

$$S(Y,W) = \frac{r}{n}g(Y,W). \tag{16}$$

Then we can state the following.

Theorem 2. *If generalized complex space form is Einstein semi-symmetric then it is an Einstein manifold provided* $f_1 \neq 0$.

Using equation (16) in (2), we get

$$(L_V g)(Y, W) + 2\frac{r}{n} g(Y, W) + 2\lambda g(Y, W) = 0. \tag{17}$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $Y = W = e_i$ in (17) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$(L_V g)(e_i, e_i) + 2\frac{r}{n} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0.$$

This implies

$$divV + r + \lambda n = 0. \tag{18}$$

If V is solenoidal then $divV = 0$. Therefore the equation (18) can be reduced to

$$\lambda = -\frac{r}{n}.$$

Thus we can state the following.

Corollary 2. *Let* (g, V, λ) *be a Ricci soliton in a generalized complex space form satisfying Einstein semi-symmetric condition. Then* V *is solenoidal if and only if it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

5 Bochner Ricci-generalized pseudo-symmetric generalized complex space forms

Let us consider the Ricci-generalized Bochner pseudosymmetric generalized complex space form $M(f_1, f_2)$. Then we have

$$(R(X, Y) \cdot B)(U, Z, W) = L_B((X \Lambda_S Y \cdot B)(U, Z, W).$$

This implies

$$\begin{aligned} &R(X, Y)B(U, Z)W - B(R(X, Y)U, Z)W - B(U, R(X, Y)Z)W - B(U, Z)R(X, Y)W \\ &= L_B[(X \Lambda_S Y)B(U, Z)W - B((X \Lambda_S Y)U, Z)W - B(U, (X \Lambda_S Y)Z)W - B(U, W)(X \Lambda_S Y)W]. \end{aligned}$$

Taking inner product with T we have,

$$\begin{aligned} &g(R(X, Y)B(U, Z)W, T) - g(B(R(X, Y)U, Z)W, T) - g(B(U, R(X, Y)Z)W, T) \\ &\quad - g(B(U, Z)R(X, Y)W, T) = L_B[g((X \Lambda_S Y)B(U, Z)W, T) - g(B((X \Lambda_S Y)U, Z)W, T) \\ &\quad - g(B(U, (X \Lambda_S Y)Z)W, T) - g(B(U, Z)(X \Lambda_S Y)W, T)]. \end{aligned} \tag{19}$$

Using equations (7), (4) and (5) in (19) and substituting $X = Z = e_i$, further again substituting $Y = T = e_i$ in the resulting equation, where $\{e_i\}$, $i, 1 \leq i \leq n$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned} &f_2 \left\{ \frac{2n-8}{2n+4} S(U, W) - \frac{5n+2}{(2n+4)(2n+2)} rg(U, W) \right\} \\ &\quad = L_B \left[\frac{4((n-1)f_1 + 3f_2 - 1) - n(r+1)}{2n+4} S(U, W) + \frac{r(n+2) - (n+4)}{(2n+2)(2n+4)} rg(U, W) \right]. \end{aligned}$$

This implies that

$$\left[\frac{f_2(2n-8) - L_B(4((n-1)f_1 + 3f_2 - 1) - n(r+1))}{2n+4}\right] S(U,W)$$
$$- \left[\frac{f_2(5n+2) + L_B(r(n+2) - (n+4))}{(2n+4)(2n+2)}\right] rg(U,W) = 0.$$

The above equation implies

$$[\alpha S(U,W) - \beta r g(U,W)] = 0,$$

where $\alpha = \left[\frac{f_2(2n-8)-L_B(4((n-1)f_1+3f_2-1)-n(r+1))}{2n+4}\right]$ and $\beta = \left[\frac{f_2(5n+2)+L_B(r(n+2)-(n+4))}{(2n+4)(2n+2)}\right]$. This implies

$$S(U,W) = \frac{\beta r}{\alpha} g(U,W). \tag{20}$$

Theorem 3. *A Bochner Ricci-generalized pseudo-symmetric generalized complex space form is an Einstein manifold.*

Using equation (20) in (2), we get

$$(L_V g)(U,W) + 2\frac{\beta r}{\alpha} g(U,W) + 2\lambda g(U,W) = 0. \tag{21}$$

Contraction of (21) over U and W gives

$$(L_V g)(e_i,e_i) + 2\frac{\beta r}{\alpha} g(e_i,e_i) + 2\lambda g(e_i,e_i) = 0.$$

This implies

$$divV + \frac{\beta r}{\alpha} n + \lambda n = 0. \tag{22}$$

If V is solenoidal then $divV = 0$. Therefore the equation (22) can be reduced to

$$\lambda = -\frac{\beta r}{\alpha}.$$

Thus we can state the following.

Corollary 3. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner Ricci-Generalized pseudo-symmetric generalized complex space forms. Then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.*

6 GENERALIZED COMPLEX SPACE FORM SATISFYING $B \cdot W_2 = L_1 Q(g, W_2)$

We assume that $B \cdot W_2 = L_1 Q(g, W_2)$ hold on $M(f_1, f_2)$, then we have

$$(B(X,Y) \cdot W_2)(U,V,Z) = L_1[((X \wedge Y) \cdot W_2)(U,V)Z].$$

This implies,

$$B(X,Y)W_2(U,V)Z - W_2(B(X,Y)U,V)Z - W_2(U,B(X,Y)V)Z - W_2(U,V)B(X,Y)Z$$
$$= L_1[(X\Lambda_g Y)W_2(U,V)Z - W_2((X\Lambda_g Y)U,V)Z - W_2(U,(X\Lambda_g Y)V)Z - W_2(U,V)(X\Lambda_g Y)Z.$$

Taking inner product with T we have,

$$\begin{aligned} & g(B(X,Y)W_2(U,V)Z,T) - g(W_2(B(X,Y)U,V)Z,T) - g(W_2(U,B(X,Y)V)Z,T) \\ & - g(W_2(U,V)B(X,Y)Z,T) = L_B[g((X\Lambda_g Y)W_2(U,V)Z,T) - g(W_2((X\Lambda_g Y)U,V)Z,T) \\ & - g(W_2(U,(X\Lambda_g Y)V)Z,T - g(W_2(U,V)(X\Lambda_g Y)Z,T)]. \end{aligned} \tag{23}$$

Applying equations (1), (7) and (8) in (23) and putting $X = V = e_i$, further again putting $Y = T = e_i$ in the resulting equation and taking summation over $i, 1 \leq i \leq n$, we get

$$\frac{\gamma}{(n-1)}S(U,Z) + \frac{\delta}{(n-1)}rg(U,Z) = L_1[\frac{1}{n-1}[nS(U,Z) - rg(U,Z)]] \tag{24}$$

where

$$\gamma = \frac{(2n+2)[(6n^3 - 8n^2 - 39n - 22)f_2 - 2(n^3 + 4n^2 + 7n - 18)(n+1)f_1] + rn(2n+4)}{(2n+2)(2n+4)},$$
$$\delta = \frac{-f_1(2n+2)(4n+2) + 6f_2(2n+2)(n+1) + r}{(2n+2)(2n+4)}.$$

Equation (24) implies

$$[\gamma S(U,W) + \delta rg(U,W)] = L_1[nS(U,Z) - rg(U,Z)].$$

The above equation implies

$$S(U,W) = Arg(U,W), \tag{25}$$

where $A = \frac{(L_1+\delta)}{L_1 n - \gamma}$. Thus we can state.

Theorem 4. *A n-dimensional generalized complex space form satisfying $B \cdot W_2 = L_1 Q(g, W_2)$ is an Einstein manifold.*

Using equation (25) in (2), we get

$$(L_V g)(U,W) + 2Arg(U,W) + 2\lambda g(U,W) = 0. \tag{26}$$

Taking $U = W = e_i$ and summing over $i = 1, 2, ..., n$ in (26) we obtain

$$(L_V g)(e_i,e_i) + 2Arg(e_i,e_i) + 2\lambda g(e_i,e_i) = 0.$$

This implies

$$divV + Arn + \lambda n = 0. \tag{27}$$

If V is solenoidal then $divV = 0$. Therefore the equation (27) can be reduced to

$$\lambda = -Ar. \tag{28}$$

Thus we can state the following.

Corollary 4. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying $B \cdot W_2 = L_1 Q(g, W_2)$. Then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.*

7 GENERALIZED COMPLEX SPACE FORM WITH $divB = 0$

Assume that the Bochner curvature tensor of a generalized complex space form is conservative that is $divB = 0$. Using equations (4) and (5) in (7), then we obtain

$$\begin{aligned} B(X,Y,Z) = R(X,Y,Z) - 2\frac{[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ], \end{aligned} \tag{29}$$

Differentiating (29) covariantly, contracting and our assumption yields.

$$\begin{aligned} 0 = (divR)(X,Y)Z - 2\frac{d[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X \\ -g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ], \end{aligned} \tag{30}$$

Using equation (3) in (30) we obtain

$$\begin{aligned} 0 = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - 2\frac{d[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X - g(X,Z)Y \\ + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned} \tag{31}$$

Taking $[(n-1)f_1+3f_2] = constant = k_1 \neq 0$ in equation (31) we obtain

$$\begin{aligned} 0 = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y \\ + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned} \tag{32}$$

Again using equation (3) in (32) we get

$$\begin{aligned} 0 = (\nabla_{JZ} S)(JY,X) + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned}$$

Replace Z by JZ in the above equation we get

$$\begin{aligned} (\nabla_Z S)(JY,X) = \frac{dr}{(2n+2)(2n+4)}[g(Y,JZ)X - g(X,JZ)Y + g(Y,Z)JX \\ - g(X,Z)JY + 2g(JX,Y)Z]. \end{aligned} \tag{33}$$

Contraction of (33) over Y and Z after simplification we get $dr(JX) = 0$. If $dr(JX) = 0$ then $dr(X) = 0$ so r is constant. Using $r = constant$ in (32) we get

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

We can state the following.

Theorem 5. *A n-dimensional generalized complex space form with conservative Bochner curvature tensor is constant scalar curvature provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

Theorem 6 ([8])**.** *Let M be a Kaehler manifold of dimension* $n \geq 4$. *Then div R=0 and div C=0 are equivalent.*

Using above Theorem we can state the following.

Theorem 7. *Let M be a generalized complex space form of dimension* $n \geq 4$. *Then div R=0, div C=0 and div B=0 are equivalent provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

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Правіна М.М., Багеваді Ц.С. *Про узагальнені форми в комплексному просторі, які задовільняють певні умови кривини* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 284–294.

Ми вивчаємо солітон Річчі (g, V, λ) на узагальнених формах в комплексному просторі при умовах, що тензори з кривиною Рімана, Бохнера і W_2 задовільняють певні умови кривини, а саме напівсиметричності, Ейнштейнової напівсиметричності, псевдосиметричності Річч та узагальненої псевдосиметричності Річчі. У роботі показано, що стиснення, випрямлення і розширення узагальнених форм в сомплексному просторі залежить від соленоїдальних властивостей вектора V. Також доведено, що узагальнена форма у комплексному просторі з звичайним тензором кривизни Бохнера має сталу скалярну кривизну.

Ключові слова і фрази: узагальнені форми у комплексному просторі, многовид Ейнштейна, напівсиметричність Ейнштейна, псевдосиметричність.

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PREVYSOKOVA N.V.

FAMILY OF WAVELET FUNCTIONS ON THE GALOIS FUNCTION BASE

We construct a family of wavelet systems on the Galois function base. We research and prove properties of systems of the built family.

Key words and phrases: wavelet, Galois function, scaling function, wavelet function.

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INTRODUCTION

The main methods of solving problems of a digital signal processing are spectral analysis, synthesis, filtering, coding and compressing based on discrete orthogonal transforms and wavelet transforms [1–5]. The signal is presented in the form of a function of time. Wavelet transforms may be considered as time-frequency representations or decompositions of a signal. A signal decomposition can be done by the basis built from a single wavelet function using wavelet scale changes and shifts [1–4, 6]. Each function of the basis describes some frequency of the signal and its location in the time domain.

An important step of a wavelet analysis is the choice of transform basis which depends on the processing tasks and on the signal. The problem of choice of a basis and the wavelet transform based on it is rather relevant and it is being researched subject.

The paper [4] systematizes basises of wavelet functions and wavelet transforms, but the problem of choice of a wavelet is solved only partially [1–4, 6]. For descrete analysis the wavelets of Daubechies, Haar, Meyer, Coifman, symlets, biorthogonal wavelets and wavelet-packet Walsh functions are used [1–4, 6].

To solve practical problems orthogonal or symmetric wavelets with compact carrier that ensure efficient transform algorithm can be chosen. But wavelets that simultaneously satisfy all of this properties are unknown. The only symmetric orthogonal wavelets with compact support are Haar wavelets but they do not satisfy the given processing qualities in many problems. To ensure symmetry multivalued biorthogonal wavelets are used. Daubechies wavelets are much smoother than Haar wavelets but they are multivalued and do not have analytical expression that complicates the process of their forming and calculation transforming.

From the recursively ordered Walsh system the Galois functions are generated [5], the latter take only two values (± 1) and the sequence of values is in full correlation. These features can provide simple algorithms for information processing in the basis based on Galois functions [5], but the researches of the Galois functions properties in various spaces and the possibility of its application for wavelet transform have not been done yet.

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Thus performing of a time-frequency analysis and processing of a broad class of one-dimensional signals with finite and limited energy, mathematical models of which are functions in space $L_2([0,T))$, necessitated the construction of wavelet basis on the base of Galois functions in this space and research their properties.

The goal of this article is the construction of a family of wavelet systems based on mother or generating Galois functions and proving properties of the constructed systems to create basises for discrete wavelet transform in the space $L_2([0,T))$.

The article provides the results of building of wavelet systems based on Galois functions, of synthesed scaling functions for built wavelets systems and proves the required properties of wavelets basises in the space $L_2([0,T))$.

1 DEFINING WAVELET SYSTEM ON GALOIS FUNCTIONS BASE

For the purpose of constructing of a system of wavelet functions for discrete transforms of signals presented by functions $f \in L_2([0,T))$ as a mother wavelet the first function $Gal_{n,0}(\theta)$ of a recursively ordered Galois system, which is defined in [5] is used.

The Galois functions system [5, p. 46] with the recursive ordering [5, p. 36] $\{Gal_{n,i}(\theta)\}$, $\theta \in [0,M)$ is defined according to the generating vector of Galois field $GF(2^n)$ from a recursive sequence or a recursive orderly system of Walsh functions [5, p. 36], where $M = 2^n$, $M \leq T$, $n = 1,2,3,\ldots$ is a degree of irreducible polynomial Galois fields $GF(2^n)$; $i = 0,1,\ldots 2^n - 1$. Examples of creating recursive sequences are shown in the following text.

Example 1. *Vector of coefficients $(p_0, p_1, p_2) = (1,1,1)$ corresponds to irreducible polynomial $x^2 + x + 1$, which generates Galois field $GF(2^2)$. Non-zero elements of vector determine the rule $p_{i+2} = p_i \oplus p_{i+1}$ for the formation of a recursive sequence. Initial vector with unitary elements $(v_0, v_1) = (1,1)$ is chosen as a primary vector. From the primary vector according to this rule $v_{i+2} = v_i \oplus v_{i+1}$ there are defined the elements of a recursive sequence wich are repeated with period $2^n - 1$. Fragment of $n-1$ zero elements of the sequence is supplemented by one zero. Elements of supplemented sequence are denoted as g_i:*

$$\{0, v_{i+2}, v_i, v_{i+1}\} = \{g_0, g_1, g_2, g_3\} = \{0,0,1,1\},$$

where $\oplus$ denotes the addition modulo two.

Example 2. *Vector of coefficients $(p_0, p_1, p_2, p_3) = (1,1,0,1)$ corresponds to irreducible polynomial $x^3 + x^2 + 1$, which generates Galois field $GF(2^3)$. This vector also determines the rule $p_{i+3} = p_i \oplus p_{i+1}$ for the formation of a recursive sequence. From the initial vector $(v_0, v_1, v_2) = (1,1,1)$ according to the rule $v_{i+3} = v_i \oplus v_{i+1}$ there are defined the elements of a recursive sequence, supplemented by zero and submitted the following fragment:*

$$\{0, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_i, v_{i+1}, v_{i+2}\} = \{g_0, g_1, g_2, \ldots, g_7\} = \{0,0,0,1,0,1,1,1\}.$$

Example 3. *Vector of coefficients $(p_0, p_1) = (1,1)$ corresponds to irreducible polynomial $x+1$, which generates Galois field $GF(2^1)$. This vector also determines the rule $p_{i+1} = p_i$ for the formation of a recursive sequence. From the initial vector $(v_0) = (1)$ according to this rule $v_{i+1} = v_i \oplus 1$ there are defined the elements of a recursive sequence, supplemented by zero and submitted the following fragment: $\{0, v_i\} = \{g_0, g_1\} = \{0,1\}$.*

Elements of the fragment of a recursive sequence supplemented by zero are signed as $\{g_0, g_1, g_2, \ldots, g_{2^n-1}\}$.

The value of the first function $Gal_{n,0}(\theta)$ of a recursively ordered Galois system $\{Gal_{n,i}(\theta)\}$ of order n at the points $\theta = \theta_j = j$ in the interval $\theta \in [0, M)$ is obtained from an element of a recursive sequences fragment via transform

$$Gal_{n,0}(\theta_j) = 1 - 2g_j, \tag{1}$$

where $j = 0, 1, \ldots 2^n - 1$, g_j — elements of a fragment of a recursive sequence.

In the intervals $\theta \in [j, j+1)$ functions $Gal_{n,0}(\theta)$ are continuous constants and take values

$$Gal_{n,0}(\theta) = Gal_{n,0}(\theta_j). \tag{2}$$

Since $g_j = 1$ or $g_j = 0$, therefore according to (1) and (2) functions $Gal_{n,0}(\theta) = \pm 1$.

Each next function of Galois system $\{Gal_{n,i}(\theta)\}$ is received from the previous unit cyclic shift either left or right by $\theta = 1$ [5], so the first function can create two different systems. For each irreducible polynomial of Galois field $GF(2^n)$ or generating vector several systems Galois functions can be built.

These functions $Gal_{n,0}(\theta)$ are defined as mother wavelets for systems of order n

$$Gal_n(\theta) = Gal_{n,0}(\theta).$$

Mother wavelet $Gal_n(\theta)$ is defined in the interval $[0, M)$, outside this interval the function $Gal_n(\theta) = 0$.

The norm of function $Gal_n(\theta)$ equals $\|Gal_n(\theta)\| = (\int\limits_0^M Gal_n^2(\theta)\, d\theta)^{\frac{1}{2}}$. Wavelet-functions must have unitary norm $\|Gal_n(\theta)\| = 1$, that is why function values are $Gal_n(\theta) = \pm\sqrt{\frac{1}{2^n}}$.

The graphics mother Galois wavelets $Gal_1(\theta)$, $Gal_2(\theta)$, $Gal_3(\theta)$, $Gal_4(\theta)$ are shown in fig. 1 — fig. 4 accordingly.

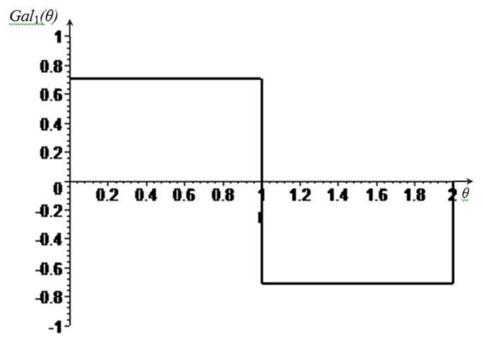

Figure 1: Galois wavelet, $n = 1$.

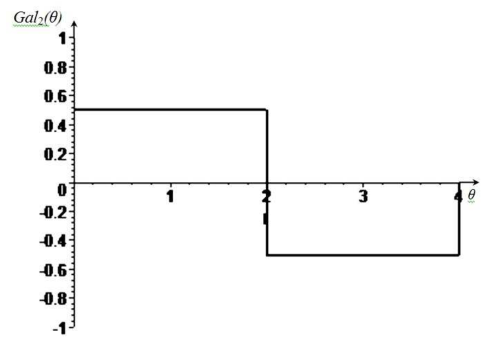

Figure 2: Galois wavelet, $n = 2$.

On the basis of each mother function $Gal_n(\theta)$ with the help of scale and parralel shift a system of wavelet-function is formed and defined as

$$Gal_{n,m,k}(t) = 2^{\frac{m-1}{2}} Gal_n(2^{m-1}t - Nk), \tag{3}$$

where $t = \frac{N}{M}\theta$; $N = 2^p$ is the quantity of functions in the system; $p = 1, 2, 3, \ldots$; $m = 0, 1, \ldots, \log_2 N + 1$; $k = 0, 1, \ldots, N \cdot 2^{-m}$.

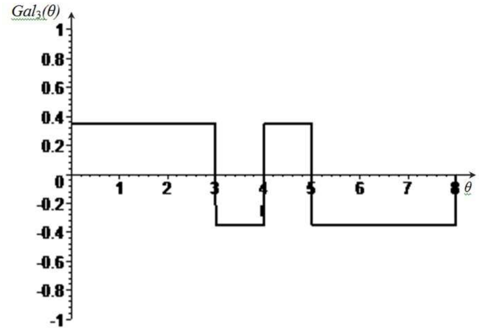

Figure 3: Galois wavelet, $n = 3$.

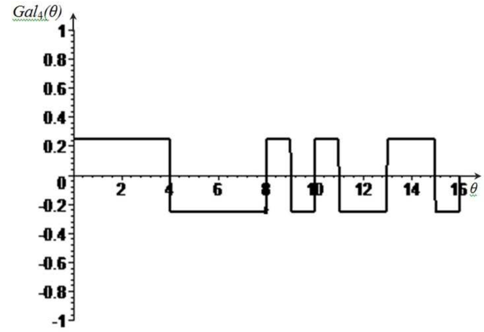

Figure 4: Galois wavelet, $n = 4$.

Non-normalized functions are $Gal_{n,m,k}(t) = \pm 1$ for $t \in [0, T)$, $T = N$ and $Gal_{n,m,k}(t) = 0$ for other t.

Normalized functions $Gal_{n,m,k}(t) = \pm\sqrt{\frac{2^{m-1}}{N}}$ are piecewise constants in intervals $t \in \left[\frac{q}{l}, \frac{q+1}{l}\right)$, where $q = 0, 1, \ldots, lN - 1$; $l = 2^{n-1}$.

The graphics of eight wavelet functions $\{Gal_{2,m,k}(t)\}$ built by the formula (3) from mother Galois wavelet $Gal_2(\theta)$ are shown in fig. 5.

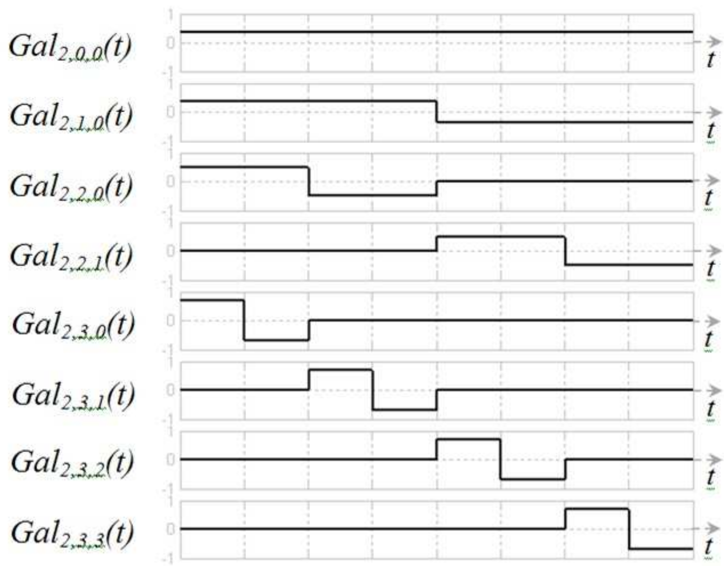

Figure 5: Graphics of wavelet functions of two-order system with mother Galois wavelet.

The graphics of eight wavelet functions $\{Gal_{3,m,k}(t)\}$, built by the formula (3) from mother Galois wavelet $Gal_3(\theta)$ are shown in fig. 6.

The set $\{Gal_{n,m,k}(t)\}$ of systems, based on mother wavelets for different values of $n = 1, 2, 3, \ldots$ forms a family of wavelet functions on the Galois functions basis.

From the result of construction of wavelet functions according (1) — (2) and fig. 1 — fig. 2 we can conclude that mother wavelets $Gal_1(\theta)$ i $Gal_2(\theta)$ of systems by orders $n = 1$ and $n = 2$ are Haar wavelets and the system wavelet functions built on their basis (fig. 5) is an orthogonal Haar system.

It is known that Haar system or Haar wavelet functions is the orthonormal basis [1–6] in the space $L_2([0, T))$, that is why in this paper proving properties and synthesis of scaling functions will be done for cases $n \geq 3$.

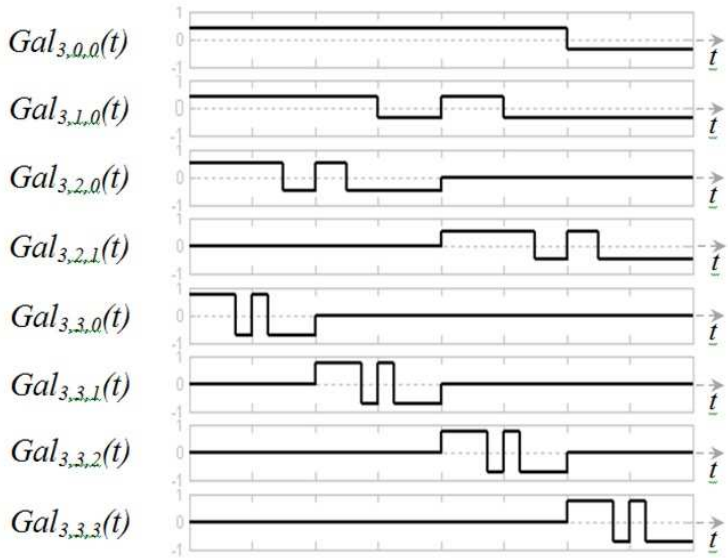

Figure 6: Graphics of wavelet functions of third-order system with mother Galois wavelet, $n = 3$.

2 SYNTHESIS OF SCALING FUNCTION

To execute the multiresolution decomposition [6, p. 86] or multiresolution analysis and to record wavelet transform in the filter form the scaling functions are used.

Scaling functions must form the basis, in which mother wavelet decomposes [1,3,4,6].

To build scaling functions for Galois wavelets a well known method of construction of scaling functions for Haar systems [3,6] is used.

For mother wavelet $Gal_n(\theta)$ the scaling function $\varphi(\theta)$ is defined as

$$\varphi(\theta) = \begin{cases} 1, \theta \in [0,1), \\ 0, \theta \in [1, M). \end{cases}$$

In the space $L_2(R)$ there is build the system of functions $\varphi_{0,b}(\theta)$, $b \in Z$, received from $\varphi(\theta)$ by shifts on integer number b

$$\varphi_{0,b}(\theta) = \varphi(\theta - b).$$

Space in $L_2(R)$, being generated by linear combinations of shift functions, is a closure of linear span of system $\varphi_{0,b}(\theta)$, signed V_0. Obviously, the system $\varphi_{0,b}(\theta)$ forms an orthonormal basis of space V_0.

On the next step a system of functions $\varphi_{1,b}(\theta)$ is created by scaling and shifting of function $\varphi_{0,b}(\theta)$

$$\varphi_{1,b}(\theta) = \sqrt{2}\varphi(2\theta - b).$$

System $\varphi_{1,b}(\theta)$ creates an orthonormal basis in space V_1, which is the closure of the linear span of the system $\varphi_{1,b}(\theta)$.

Function $\varphi(\theta) \in V_0$ is a linear combination of elements of space V_1

$$\begin{aligned} \varphi(\theta) &= \varphi(2\theta) + \varphi(2\theta - 1), \\ \varphi(\theta) &= \frac{1}{\sqrt{2}}\varphi_{1,0}(2\theta) + \frac{1}{\sqrt{2}}\varphi_{1,1}(2\theta - 1). \end{aligned} \tag{4}$$

On the next step there is built a space V_2, generated by functions

$$\varphi_{2,b}(\theta) = 2\varphi(2^2\theta - b).$$

For constructed spaces V_0, V_1, V_0 insertion $V_0 \subset V_1 \subset V_2$ is right. The procedure of construction of functions system is extended for any $k \in Z$. It results in a constructed orthonormal functions system

$$\varphi_{k,b}(\theta) = \sqrt{2^k}\varphi(2^k\theta - b).$$

There are the following inclusion of spaces $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k$.

According to the definition [6, p. 76] the function $\varphi(\theta) \in L_2(R)$ is called a scaling function if it can be presented in the following form

$$\varphi(\theta) = \sqrt{2}\sum_{s\in Z} h_s\varphi(2\theta - s),$$

where numbers h_s satisfy the condition $\sum\limits_{s\in Z} |h_s|^2 < \infty$.

Decomposition (4) proves performing of the scaling function definition for $\varphi(\theta)$.

Mother wavelet $Gal_n(\theta)$ is decomposed into the functions system $\{\varphi(2\theta)\}$

$$Gal_n(\theta) = \sqrt{2}\sum_{s=0}^{2^{n+1}-1} h_s\varphi(2\theta - s),$$

where the coefficients h_s are called filters.

Example 4. *Non-normalized wavelet function $Gal_3(\theta)$ is decomposed in the system of scaling functions $\varphi(2\theta)$ by the following way*

$$\begin{aligned} Gal_3(\theta) = {} & 1\cdot\varphi(2\theta) + 1\cdot\varphi(2\theta-1) + 1\cdot\varphi(2\theta-2) + 1\cdot\varphi(2\theta-3) + 1\cdot\varphi(2\theta-4) \\ & + 1\cdot\varphi(2\theta-5) + (-1)\cdot\varphi(2\theta-6) + (-1)\cdot\varphi(2\theta-7) + 1\cdot\varphi(2\theta-8) + 1\cdot\varphi(2\theta-9) \\ & + (-1)\cdot\varphi(2\theta-10) + (-1)\cdot\varphi(2\theta-11) + (-1)\cdot\varphi(2\theta-12) + (-1)\cdot\varphi(2\theta-13) \\ & + (-1)\cdot\varphi(2\theta-14) + (-1)\cdot\varphi(2\theta-15). \end{aligned}$$

The corresponding filters are $h_0 = 1, h_1 = 1, h_2 = 1, h_3 = 1, h_4 = 1, h_5 = 1, h_6 = -1, h_7 = -1, h_8 = 1, h_9 = 1, h_{10} = -1, h_{11} = -1, h_{12} = -1, h_{13} = -1, h_{14} = -1, h_{15} = -1$.

3 PROPERTIES OF WAVELET SYSTEMS BASED ON GALOIS FUNCTIONS IN $L_2([0,T))$

The wavelets system (3) based on Galois functions may be used as a basis for wavelet transforms if the following properties of wavelet basises are performed [3, 4, 6]:

1) it has a compact carrier (a finite time interval);

2) it has at least one zero moment;

3) a basis is orthogonal or it is a Riesz basis.

These properties for systems of wavelets with mother Galois functions are proved by the following propositions.

1) The existence of a compact carrier of a wavelet.

It is known that the function $f(t)$ has a compact carrier if $f(t) = 0$ for $t < a$ or $t > b$, where $-\infty < a < b < \infty$ [3, p. 15]. Wavelets with a compact carrier have a finite number of nonzero coefficients of expansion.

Proposition 1. *Mother wavelet $Gal_n(\theta)$ of Galois wavelet system has a compact carrier.*

Proof. According to the definition (1)—(2) function $Gal_n(\theta)$ in interval $[0, M)$ is piecewise constant, it has non-zero values $Gal_n(\theta) = \pm\sqrt{\frac{1}{2^n}}$ and outside the interval its value equals zero, therefore it has a compact carrier. □

2) The existence of one zero moment.

According to the definition [6, p. 129], function $f(t) \in L_2(R)$ has L zero moment if equality is satisfied

$$\int\limits_{-\infty}^{\infty} t^r f(t)\, dt = 0 \tag{5}$$

for all integers $r = 0, 1, \dots, L-1$. If the mother-wavelet has successive moments equal to zero the wavelet coefficients decrease quickly.

Proposition 2. *The mother wavelet $Gal_n(\theta)$ of the Galois wavelet system has one zero moment*

$$\int\limits_{-\infty}^{\infty} Gal_n(\theta)\, d\theta = 0.$$

Proof. According to the property of Galois function [5] it is

$$\int\limits_{0}^{M} Gal_n(\theta)\, d\theta = 0,$$

and outside the interval $[0, M)$ value of function is zero.

Sums of lengths the intervals where $Gal_n(\theta) = \sqrt{\frac{1}{2^n}}$ and $Gal_n(\theta) = -\sqrt{\frac{1}{2^n}}$ are equal. Therefore, according to the definition (5) functions $Gal_n(\theta)$ have a zero moment and satisfy the basic requirements for wavelet functions. However, there is only one zero moment because the direct checking shows that

$$\int\limits_{-\infty}^{\infty} \theta Gal_n(\theta)\, d\theta \neq 0.$$

□

3) Orthogonality of system or Riesz basis. Built systems $\{Gal_{1,m,k}(t)\}$ and $\{Gal_{2,m,k}(t)\}$ coincide with the orthogonal Haar system. Built systems $\{Gal_{n,m,k}(t)\}$ for $n = 3, 4, \dots$ are nonorthogonal. We know that the demand for orthogonality of wavelets system may be weakened, but it is necessary for the system to form the Riesz basis [2–4, 6].

According to the definition [6, p. 111] system $\varphi_v(t)$ in Hilbert space H called Riesz basis if there are such positive constants A i B that for any element $f(t) \in H$ the following inequality is performed

$$A\|f(t)\|^2 \leqslant \sum_{v=1}^{\infty} |\langle f(t), \varphi_v(t)\rangle|^2 \leqslant B\|f(t)\|^2. \tag{6}$$

Proposition 3. *System* $\{Gal_{n,m,k}(t)\}$ *is the Riesz basis in space* $L_2([0,T])$.

Proof. To prove that the properties (6) of wavelet systems with mother Galois functions form Riesz basis, it must be established that there are such constants A i B, $0 < A \leq B < \infty$ for which the inequality is performed

$$A\|f(t)\|^2 \leqslant \sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \leqslant B\|f(t)\|^2, \tag{7}$$

where $\|f(t)\|^2 = \int\limits_0^N f^2(t)\,dt$, $v = 1,2,\ldots,N$ is serial number of the wavelet in the system $\{Gal_{n,m,k}(t)\}$.

Numbers m and k in the system $\{Gal_{n,m,k}(t)\}$ with the triple numeration are connected with the serial number v of the wavelet by the formula $v = 2^{m-1} + k + 1$.

Since the number of functions in the proposed system is finite and equals N, the sum in the middle of inequality (7) contains a finite number of components

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 = \sum_{v=1}^{N} \left| \int\limits_0^T f(t)\cdot Gal_v(t)\,dt \right|^2.$$

With Bunyakovsky inequality $\left(\int\limits_a^b x(t)\cdot y(t)\,dt\right)^2 \leq \int\limits_a^b x^2(t)\,dt \int\limits_a^b y^2(t)\,dt$ for any $x(t)$, $y(t)$ an assessment of the latter expression and following transforms there are performed

$$\begin{aligned}\sum_{v=1}^{N}\left(\int\limits_0^T f(t)\cdot Gal_v(t)\,dt\right)^2 &\leq \sum_{v=1}^{N}\left(\int\limits_0^T f^2(t)\,dt\cdot\int\limits_0^T Gal_v^2(t)\,dt\right)\\ &= \|f(t)\|^2\cdot\sum_{v=1}^{N}\|Gal_v(t)\|^2.\end{aligned}$$

Functions $\{Gal_v(t)\}$ are normalized and the norm is $\|Gal_v(t)\| = 1$. Selection of the first and the last expressions in latest inequality sets the ratio

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \leq \|f(t)\|^2\cdot\sum_{v=1}^{N}\|Gal_v(t)\|^2 = \|f(t)\|^2\cdot N.$$

So there exists the constant $N > 0$ and the right side of inequality (7) is proved. On the other hand, we must prove that there exists a constant $A > 0$ and there performs the inequality

$$A\|f(t)\|^2 \leq \sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \text{ or} \tag{8}$$

$$A \leq \frac{\sum\limits_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2}{\|f(t)\|^2}. \tag{9}$$

Since function $f(t)$ is bounded and it is designated as $p \leq f(t) \leq P$, then the following inequalities are executed $\int\limits_{q}^{q+1} f(t)\,dt \geq \int\limits_{q}^{q+1} p\,dt$ and $\int\limits_{q}^{q+1} (-f(t))\,dt \geq \int\limits_{q}^{q+1} (-P)\,dt$.

Since normalized functions $Gal_v(t) = \pm\sqrt{\frac{2^{m-1}}{N}}$ are piecewise constants in intervals $t \in \left[\frac{q}{l}, \frac{q+1}{l}\right)$, $q = 0, 1, \ldots, lN-1$ and each function $Gal_v(t) = Gal_{n,m,k}(t) \neq 0$ is not zero-value in the interval $t \in \left[\frac{k}{2^{m-log_2 N-1}}, \frac{k+1}{2^{m-log_2 N-1}}\right)$, then

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 = \sum_{v=1}^{N} \left|\int\limits_{0}^{T} f(t) Gal_v(t)\,dt\right|^2 = \sum_{v=1}^{N} \left|\int\limits_{\frac{k}{2^{m-log_2 N-1}}}^{\frac{k+1}{2^{m-log_2 N-1}}} f(t) Gal_{n,m,k}(t)\,dt\right|^2.$$

Assume designation $I_1 = \cup\left[\frac{q_s}{l}, \frac{q_s+1}{l}\right)$ — for combining intervals, where values of functions are $Gal_v(t) = \sqrt{\frac{2^{m-1}}{N}}$, and $I_2 = \cup\left[\frac{q_r}{l}, \frac{q_r+1}{l}\right)$ — for combining intervals, where values of functions are $Gal_v(t) = -\sqrt{\frac{2^{m-1}}{N}}$, $s = 0, 1, \ldots, lN-1$, $r = 0, 1, \ldots, lN-1$.

$$\begin{aligned}
&\sum_{v=1}^{N} \left|\int\limits_{\frac{k}{2^{m-log_2 N-1}}}^{\frac{k+1}{2^{m-log_2 N-1}}} f(t) Gal_{n,m,k}(t)\,dt\right|^2 = \sum_{v=1}^{N} \left|\int\limits_{I_1} \sqrt{\frac{2^{m-1}}{N}} f(t)\,dt + \int\limits_{I_2} \sqrt{\frac{2^{m-1}}{N}} (-f(t))\,dt\right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left|\left(\int\limits_{I_1} f(t)\,dt + \int\limits_{I_2} (-f(t))\,dt\right)\right|^2 \geq \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left|\left(\int\limits_{I_1} p\,dt + \int\limits_{I_2} (-P)\,dt\right)\right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left|\left(p\int\limits_{I_1} dt + (-P)\int\limits_{I_2} dt\right)\right|^2 = \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left|\left(p\frac{N}{2^m} + (-P)\frac{N}{2^m}\right)\right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1} N^2}{N 2^{2m}} (p-P)^2 = 2^{-m-1} N^2 (p-P)^2.
\end{aligned}$$

The function $f^2(t)$ is bounded. It is assumed that $f^2(t) \leq S$, $S \in R$, then the inequality is executed

$$\int\limits_{0}^{T} f^2(t)\,dt \leq \int\limits_{0}^{T} S\,dt = S \cdot N.$$

Substituting the last result in (9) allows to reach the following conclusion: when choosing $A \leq \frac{N(p-P)^2}{2^{m+1}S}$ inequalities (8) i (7) are performed. The statement is proved. □

According to proven propositions 1 — 3 mother Galois wavelet functions have a compact carrier, one vanishing zero moment, wavelet function systems $Gal_{n,m,k}(t)$ for different n form Riesz basises, that satisfy the necessary conditions for wavelet basises in space $L_2([0, T))$.

4 CONCLUSIONS

Thus, it was proved that the first functions of recursively ordered Galois systems are mother or generating wavelets. There was synthesized the orthogonal scaling functions system in which mother wavelets decompose.

On the basis of mother wavelets of different orders n there were built wavelet functions systems. The set of built systems is a family of wavelet functions that are generated by Galois functions.

The article also proves necessary conditions (properties) of wavelet system for the built system. It is proved that each system of family is the Riesz basis. The proved conditions enable using wavelet systems with generating functions Galois as basises of discrete wavelet transforms in the space $L_2([0, T))$. A significant advantage of implementation of these transforms compared to others is that all the basic functions are piecewise constant and take only two values.

Transforms in built basises can be used for analysis and processing of one-dimensional signals with finite energy.

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DIVISOR PROBLEM IN SPECIAL SETS OF GAUSSIAN INTEGERS

Let A_1 and A_2 be fixed sets of gaussian integers. We denote by $\tau_{A_1,A_2}(\omega)$ the number of representations of ω in form $\omega = \alpha\beta$, where $\alpha \in A_1, \beta \in A_2$. We construct the asymptotical formula for summatory function $\tau_{A_1,A_2}(\omega)$ in case, when ω lie in the arithmetic progression, A_1 is a fixed sector of complex plane, $A_2 = \mathbb{Z}[i]$.

Key words and phrases: Gaussian numbers, divisor problem, asymptotic formula, arithmetic progression.

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INTRODUCTION

Let A_1 and A_2 be fixed infinite sets of natural numbers. We let $\tau_{A_1,A_2}(n)$ denote the number of representations of n in form $n = m_1m_2$, where $m_1 \in A_1, m_2 \in A_2$. To investigate average order of function $\tau_{A_1,A_2}(n)$, it is usual to consider the summatory function

$$\sum_{n\leqslant x} \tau_{A_1,A_2}(n),$$

where x is a large real variable. For $A_1 = A_2 = \mathbb{N}$, this is the classical Dirichlet divisor problem about the number of lattice points (u, v) under the hyperbola $uv \leqslant x,\ u, v \geqslant 1$. Historical review results on the divisor problem can be found in the monograph of Krätzel [4]. The best estimate to-date is due to Huxley [3]

$$\sum_{n\leqslant x} \tau_{\mathbb{N},\mathbb{N}}(n) = x\log x + (2\gamma - 1) + O(x^{\frac{131}{416}} (\log x)^{\frac{26947}{8320}}).$$

In articles [5–9] the authors discussed special cases of sets of natural numbers A_1, A_2.

The similar problem was considered over the ring of the Gaussian integers $\mathbb{Z}[i]$ in the work of Varbanets and Zarzycki [9] in case, when

$$A_1 = \mathbb{Z}[i], \quad A_2 = \{\alpha \in \mathbb{Z}[i] : \alpha \equiv \alpha_0 \pmod{\gamma}\}, \quad \alpha_0, \gamma \in \mathbb{Z}[i].$$

The following asymptotic formula was obtained

$$\sum_{\substack{\omega=\alpha\beta \\ \alpha\equiv\alpha_0 \pmod{\gamma} \\ N(\alpha\beta)\leqslant x}} 1 = \frac{\pi^2 x\log x}{N(\gamma)} + c(\alpha_0, \gamma)\frac{x}{N(\gamma)} + O\left(\left(\frac{x}{N(\gamma)}\right)^{\frac{1}{2}+\varepsilon}\right) + O\left(\left(\frac{x}{N(\alpha_1)}\right)^{\theta}\right) + O(x^{\varepsilon}),$$

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where $\theta < \frac{1}{3}$, α_1 is a number of form $\alpha_0 + \beta\gamma$, $\beta \in \{0, \pm 1, \pm i\}$ with the smallest norm, the constant $c(\alpha_0, \gamma)$ is computable and depends on α_0 and γ.

In the present paper, we investigate the distribution of values of the divisor function not only in an arithmetic progression, but in narrow sectorial region also. By the $\tau_S(\omega)$ we denote the function $\tau_{A_1,A_2}(\omega)$ in case, when $A_1 = \mathbb{Z}[i]$, $A_2 = S(\varphi)$ is a fixed sector of complex plane

$$S(\varphi) := \{\alpha \in \mathbb{Z}[i] : \varphi_1 < \arg\alpha \leqslant \varphi_2,\ \varphi = \varphi_2 - \varphi_1\}.$$

The main point of this paper is to construct an asymptotic formula for sum

$$T(x, \gamma, \omega_0, S(\varphi)) = \sum_{\substack{\omega \equiv \omega_0 \pmod{\gamma}, \\ N(\omega) \leq x}} \tau_S(\omega),$$

in particular to investigate the ranges of γ and x for which this formula is nontrivial. Applying the method of Vinogradov we get the asymptotic formula in case, when the norm of a difference of progression grows.

In this paper we denote by $\mathbb{Z}[i]$ the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \,|\, a, b \in \mathbb{Z}\}.$$

For $\alpha \in \mathbb{Z}[i]$ we put $Sp(\alpha) = \alpha + \overline{\alpha} = \operatorname{Re}\alpha$, $N(\alpha) = \alpha \cdot \overline{\alpha}$, where $\overline{\alpha}$ denotes a complex conjugate with α. $Sp(\alpha)$ and $N(\alpha)$ we name a trace and a norm (respectively) of α from $\mathbb{Z}[i]$. Moreover, $\exp(x) := e^x$, $\quad e_q(z) := e^{2\pi i \frac{z}{q}}$ for $q \in \mathbb{N}$. The Vinogradov's symbol as in $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; ε is an arbitrary small positive number that is not necessarily the same at each occurence; the constants implied by the O (or $\ll$) — notation depend at most on ε. $\zeta(s)$ is the Riemann zeta-function; $L(s, \chi_4)$ is the Dirichlet L-function with the non-principal character modulo 4. $\mathfrak{B} := \{0, \pm 1, \pm i\}$. $\overline{\varphi}(\alpha) = N(\alpha)\prod_{p|\alpha}(1 - N(p)^{-1})$ denotes the Euler function in $\mathbb{Z}[i]$.

1 Preliminaries

We begin this section with few background definitions and facts. Note that every non-zero Gaussian number has associated element in each quadrant of the complex plane. Therefore without loss of generality, we assume $0 \leqslant \varphi_1 < \varphi_2 \leqslant \frac{\pi}{2}$. Let $\chi(\varphi)$ be a characteristic function of sector S. We will follow the idea of Vinogradov [1]. We first mention some classical results.

Lemma 1 ([1]). *Suppose r is an integer, $r > 0$, $\Omega > 0$, $0 < \Delta < \frac{1}{2}\Omega$, φ_1, φ_2 are real numbers, $\Delta \leqslant \varphi_2 - \varphi_1 \leqslant \Omega - 2\Delta$. Then there exists a periodic function $f(\varphi) = f(\varphi; \varphi_1, \varphi_2)$ with period Ω such that:*

1. *$f(\varphi) = 1$ in the interval $[\varphi_1, \varphi_2]$; $0 \leqslant f(\varphi) \leqslant 1$ in the intervals $[\varphi_1 - \Delta, \varphi_1]$ and $[\varphi_2, \varphi_2 + \Delta]$;*
2. *$f(\varphi) = 0$ in the interval $[\varphi_2 + \Delta, \varphi_2 + \Omega - \Delta]$;*
3. *$f(\varphi)$ can be expanded into Fourier series of the form*

$$f(\varphi) = \sum_{m=-\infty}^{\infty} a_m \exp\left(2\pi i \frac{m\varphi}{\Omega}\right),$$

where $a_0 = \frac{1}{\Omega}(\varphi_2 - \varphi_1 + \Delta)$, $|a_m| \leqslant \begin{cases} \Omega^{-1}(\varphi_2 - \varphi_1 + \Delta), \\ 2(\pi|m|)^{-1}, \\ 2(\pi|m|)^{-1}(r\Omega(\pi|m|\Delta)^{-1})^r. \end{cases}$

Remark 1. *There exist numbers θ_i, $|\theta_i| \leqslant 1$, $i = 1, 2$, such that*

$$\chi(\varphi) = f(\varphi; \varphi_1, \varphi_2) + \theta_1 f(\varphi; \varphi_1 - \Delta, \varphi_1) + \theta_2 f(\varphi; \varphi_2, \varphi_2 + \Delta). \tag{1}$$

Let δ, $\delta_0 \in \mathbb{Q}[i]$ and $m \in \mathbb{Z}$. Let for $\operatorname{Re} s > 1$ we define the Hecke Z-function with the shift

$$Z_m(s; \delta, \delta_0) = \sum_{\substack{\omega \in \mathbb{Z}[i] \\ \omega \neq -\delta}} \frac{\exp\left(4mi \arg\left(\omega + \delta\right)\right)}{N(\omega + \delta)} \exp\left(2\pi i \operatorname{Re}(\delta_0 \omega)\right).$$

Lemma 2. *$Z_m(s; \delta, \delta_0)$ is an entire function if $m \neq 0$ and $\delta_0 \notin \mathbb{Z}[i]$. For $m = 0$ and $\delta_0 \in \mathbb{Z}[i]$ Hecke Z-function $Z_0(s; \delta, \delta_0)$ is a holomorphic function in the whole complex plane except at $s = 1$, where it has a simple pole with residue π. It satisfies the functional equation*

$$\pi^{-s}\Gamma(2|m| + s)Z_m(s; \delta, \delta_0) = \pi^{-(1-s)}\Gamma(2|m| + 1 - s)Z_m(1 - s; -\overline{\delta_0}, \delta) \exp\left(-2\pi i \operatorname{Re}(\delta_0 \delta)\right) \tag{2}$$

in all cases.

For the proof in the case $\delta = \delta_0 = 0$ see [2]. The proof in other cases similar.

Lemma 3 ([9]). *Let δ be a Gaussian rational, $N(\delta) < 1$. Then $Z_0(s; \delta, 0)$ has the following Laurent expansion*

$$Z_0(s; \delta, 0) = \frac{\pi}{s - 1} + a_0(\delta) + a_1(\delta)(s - 1) + \ldots,$$

where

$$a_0(\delta) = \begin{cases} \pi\gamma + 4L'(s, \chi_4), & \text{if } \delta \in \mathbb{Z}[i], \\ \pi\gamma + 4L'(s, \chi_4) + \sum\limits_{\beta \in \mathfrak{B}} \left(N(\delta + \beta)\right)^{-1} + b_0(\gamma), & \text{if } 0 < N(\delta) < 1; \end{cases}$$

γ is the Euler's constant, $b_0(\gamma) = -4 + O\left(N^{\frac{1}{2}}(\delta)\right)$.

By the Stirling's formula for Gamma-function to the terms of the second order $O(t^{-2})$ we have for $|t| > 1$, $\sigma > 0$

$$\begin{aligned} \Gamma(\sigma + it) &= \sqrt{2\pi} t^{\sigma - \frac{1}{2}} \\ &\times \exp\left(i\left(t\log t - t + \frac{\pi}{2}\left(\sigma - \frac{1}{2}\right) + \left(\sigma - \sigma^2 - \frac{1}{6}\right)(2t)^{-1} + O(t^{-2})\right)\right) \exp\left(-\frac{\pi|t|}{2}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\Gamma(2|m| + 1 - s)}{\Gamma(2|m| + s)} &= \exp\left(it(2 - \log(4m^2 + t^2) + \frac{|2m| + 1}{4m^2 + t^2} + \frac{(2|m| + 1)^2}{(4m^2 + t^2)^2})\right) \\ &\times \frac{1}{2}(4m^2 + t^2)^{1 - 2\sigma} \exp\left(\sigma - \frac{1}{2} + \frac{t^2}{16}(4m^2 + 2|m| + t^2)^{-1}\right) \\ &\times \left(1 + O(m^2 + t^2)^{-\frac{1}{2}}\right). \end{aligned} \tag{3}$$

Applying the estimations for $|t| \geqslant 2$, $\sigma = 1$

$$Z^*_m(s;\delta,\delta_0) := Z_m(s;\delta,\delta_0) - \sum_{\omega\in\mathfrak{B}} \frac{e^{4mi\arg(\omega+\delta)}}{N(\omega+\delta)^s} e^{2\pi i\,\mathrm{Re}(\delta_0\omega)} \ll \log^4(t^2+m^2)$$

and functional equation (2), from (3) and Phragmen–Lindelöf theorem in the strip $-1 \leqslant \mathrm{Re}(s) \leqslant 1$ we infer

$$Z^*_m(s;\delta,\delta_0) \ll (m^2+t^2)^{\frac{1-\sigma}{2}} \left(\log(m^2+t^2)\right)^{\frac{1-\sigma}{2}}, \quad |m| \geqslant 1. \tag{4}$$

Let $\alpha,\beta,\gamma \in \mathbb{Z}[i]$. We define the Kloosterman sum for the ring of Gaussian integers

$$K(\alpha,\beta;\gamma) = \sum_{\substack{\xi,\xi'(\mathrm{mod}\gamma)\\ \xi\cdot\xi'\equiv 1(\gamma)}} e^{\pi i Sp\left(\frac{\alpha\xi+\beta\xi'}{\gamma}\right)}.$$

Lemma 4. *Let $\alpha,\beta,\gamma \in \mathbb{Z}[i], \gamma \neq 0$. Then the estimate*

$$K(\alpha,\beta;\gamma) \ll (N(\gamma)N((\alpha,\beta,\gamma)))^{\frac{1}{2}}\tau(\gamma)$$

holds. Moreover,

$$K(\alpha,\beta;\gamma) = \sum_{\delta|(\alpha,\beta,\gamma} N(\delta)K\left(1,\frac{\alpha\beta}{\delta^2};\frac{\gamma}{\delta}\right). \tag{5}$$

Proof. This lemma follows from multiplicative property of $K(\alpha,\beta;\gamma)$ on γ and the Bombieri estimate of an exponential sum on the algebraic curve over the finite field. The formula (5) is a generalized Kuznetsov's identity for Kloosterman sums. □

2 THE MAIN RESULTS

Lemma 5. *Let $\gamma,\omega_0 \in \mathbb{Z}[i], N(\gamma) > 1$, $(\omega_0,\gamma) = \beta, N(\beta) < N(\gamma)$. Then for every $\varepsilon > 0$, $N(\gamma) \ll x^{\frac{2}{3}-\varepsilon}$ we have*

$$T_0(x,\gamma,\omega_0) = c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)} + c_1(\gamma,\omega_0)\frac{x}{N(\gamma)} + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right),$$

where $c_0(\gamma,\omega_0), c_1(\gamma,\omega_0)$ are computable constants

$$c_0(\gamma,\omega_0) = \pi^2 N(\beta)\overline{\varphi}\left(\frac{\gamma}{\beta}\right)N^{-1}(\gamma)\tau(\beta), \tag{6}$$

$$c_1(\gamma,\omega_0) = \pi^2\sum_{\delta|\gamma}\left[2E - 1 + 2\frac{L'(1,\chi_4)}{L(1,\chi_4)} + \sum_{p|\gamma/\delta}{}^{*}\frac{\log N(p)}{N(p)-1}\right]\prod_{p|\gamma/\delta}{}^{*}(1-N(p)^{-1}). \tag{7}$$

Proof. Without loss of generality we will consider a case $(\omega_0,\gamma) = 1$. For $\mathrm{Re}\,s > 1$ we denote

$$F(s) := \sum_{\substack{\omega\in\mathbb{Z}[i]\\ \omega\equiv\omega_0(\gamma)}} \frac{\tau(\omega)}{N(\omega)^s}, \quad F^*(s) := F(s) - \sum_{\beta\in\mathfrak{B}} \frac{\tau(\omega_0+\beta\gamma)}{N(\omega_0+\beta\gamma)^s}.$$

It is clear, that

$$F(s) = N^{-2s}(\gamma) \sum_{\substack{\alpha_i \in (\mathrm{mod}\gamma) \\ \alpha_1\alpha_2 \equiv \omega_0(\gamma)}} Z_0\left(s, \frac{\alpha_1}{\gamma}, 0\right) Z_0\left(s, \frac{\alpha_2}{\gamma}, 0\right).$$

By using the Abel's Lemma about partial summation of Dirichlets' series, we have for $c = 1+\varepsilon$, $1 < T \le x$, where $\varepsilon > 0$ is arbitrary small

$$T_0(x,\gamma,\omega_0) = \frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT} F^*(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{TN(\gamma)}\right). \tag{8}$$

From Lemma 4 we have the functional equation

$$F(s) = \frac{\pi^{2(2s-1)}}{N^{2s}(\gamma)}\frac{\Gamma^2(1-s)}{\Gamma^2(s)}\Psi(1-s),$$

where

$$\Psi(1-s) = \sum_{\omega}\frac{1}{N(\omega)^s}\sum_{\alpha\beta=\omega}\Phi(\alpha,\beta;\gamma), \quad \Phi(\alpha,\beta;\gamma) = \sum_{\substack{\alpha_1,\alpha_2(\mathrm{mod}\gamma) \\ \alpha_1\alpha_2\equiv\omega_0(\gamma)}} e^{\pi i Sp\left(\frac{\alpha\alpha_1+\beta\alpha_2}{\gamma}\right)}.$$

We consider the function $F^*(s)$ in the strip $-\frac{1}{4} \le \mathrm{Re}\, s \le 1+\varepsilon$. It is obviously that $F^*(1+\varepsilon+it) \ll N(\gamma)^{-1-\varepsilon}$. On the line $\mathrm{Re}\, s = -\frac{1}{4}$ we apply the functional equation for $Z_0(s;\delta,0)$, (3), Lemma 4 and then obtain $F^*(1+\varepsilon+it) \ll N(\gamma)^{1/2+\varepsilon}(|t|+3)^3$.

Applying the Phragmen-Lindelöf theorem in the strip $-\frac{1}{4} \leqslant \mathrm{Re}(s) \leqslant 1+\varepsilon$ we infer for $|t| \le T$

$$F^*(-\varepsilon+it) \ll N(\gamma)^{1/5+\varepsilon}T^{12/5+\varepsilon}.$$

To deal with integral in (8) we shift the line of integration to $\mathrm{Re}\, s = -\varepsilon$. By the Theorem of residues we obtain

$$\begin{aligned} T_0(x,\gamma,\omega_0) &= \underset{s=0}{res}\left(F^*(s))\frac{x^s}{s}\right) + \underset{s=1}{res}\left(F^*(s))\frac{x^s}{s}\right) + \frac{1}{2\pi i}\int\limits_{-\varepsilon-iT}^{-\varepsilon+iT} F^*(s)\frac{x^s}{s}ds \\ &+ O\left(x^{\varepsilon}\right) + O\left(x^{-\varepsilon}N(\gamma)^{1/5+\varepsilon}T^{7/5+\varepsilon}\right) + O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right). \end{aligned} \tag{9}$$

Further, applying Lemma 2 we get

$$\begin{aligned} \underset{s=1}{res}\left(F^*(s))\frac{x^s}{s}\right) &= \frac{\pi^2 x\log x}{N(\gamma)}\prod_{p|\gamma}{}^*(1-N(p)^{-1}) \\ &+ \frac{\pi^2 x}{N(\gamma)}\prod_{p|\gamma}{}^*(1-N(p)^{-1})\left[-1+2\left(E+\frac{L'(1,\chi_4)}{L(1,\chi_4)}+\sum_{p|\gamma}{}^*(\frac{\log N(p)}{N(p)-1})\right)\right], \end{aligned} \tag{10}$$

where sign $\prod^*$ means that the product conducts by all the non-associated prime Gaussian numbers. Moreover, $F(0) = 0$ if $N(\gamma) > 1$.

$$\underset{s=0}{res}\left(F^*(s)\frac{x^s}{s}\right) = \underset{s=0}{res}\left(-\sum_{\beta\in\mathfrak{B}}\frac{\tau(\omega_0+\beta\gamma)}{N(\omega_0+\beta\gamma)^s}\frac{x^s}{s}\right) \ll N(\gamma)^{\varepsilon}. \tag{11}$$

Observe that by Lemma 4

$$\sum_{\alpha\beta=\omega}|\Phi(\alpha,\beta;\gamma)| = \sum_{\alpha\beta=\omega}|K(\alpha,\beta\omega_0;\gamma)| \ll N(\gamma)^{1/2}N((\omega,\gamma))^{1/2}\tau(\gamma)\tau(\omega).$$

Now by the termwise integration and applying the Stirling formula for gamma function and the method of stationary phase we get

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{-\varepsilon-iT}^{-\varepsilon+iT}F^*(s)\frac{x^s}{s}ds = &\sum_{\substack{\omega\\ 0<N(\omega)\le Y}}\frac{\pi^2}{N(\omega)}\sum_{\alpha\beta=\omega}\Phi(\alpha,\beta;\gamma)\frac{y^{3/8}}{4\sqrt{2/\pi}}e\left(-\frac{1}{8}-\frac{1}{2\pi}y^{1/4}\right)\\ &\times(1+O(y^{-1/8}))+O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right)+O\left(x^{\varepsilon}\right)\\ &+O\left(\sum_{\substack{\omega\\ N(\omega)>Y}}y^{-\varepsilon}T^{1+4\varepsilon}N(\gamma)^{1/2+\varepsilon}N((\omega,\gamma))^{1/2}\tau(\omega)N(\omega)^{-1}\right)\end{aligned}\tag{12}$$

where $Y\le X=\left(\frac{4}{\pi}\right)^4\frac{T^4N^2(\gamma)}{x}$, $y=\frac{\pi^4xN(\omega)}{N^2(\gamma)}$. Thus, by combining (8)—(12) and taking $T=x^{1/2}N(\gamma)^{-3/4}$, $Y=x^{1/3}$ we obtain the assertion of Lemma 5. □

Theorem 1. *Let $\gamma,\omega_0\in\mathbb{Z}[i], N(\gamma)>1$, $(\omega_0,\gamma)=\beta, N(\beta)<N(\gamma)$. Then for every $\varepsilon>0$, $x\ge N^{\frac{3}{2}}(\gamma)$ and $\varphi_2-\varphi_1\gg\frac{N^{\frac{3}{4}}(\gamma)}{x^{\frac{1}{2}-\varepsilon}}$, the following formula holds*

$$\begin{aligned}T(x,\gamma,\omega_0,S(\varphi)) = &\frac{2(\varphi_2-\varphi_1)}{\pi}\left(c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)}\right.\\ &\left.+\ (c_1(\gamma,\omega_0)+A_0(\varphi))\frac{x}{N(\gamma)}\right)+O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right),\end{aligned}$$

where $c_0(\gamma,\omega_0), c_1(\gamma,\omega_0), A_0(\varphi)$ are computable constants, which defined in (6), (7). The constant in symbol "O" depends only on ε.

Proof. Let $m\neq 0$. Denote

$$c_m(\omega)=\sum_{\alpha\beta=\omega}e^{4mi\arg\alpha}.$$

For $\operatorname{Re}s>1$ we have

$$\begin{aligned}F_m(s) &= \sum_{\substack{\omega\in\mathbb{Z}[i]\\ \omega\equiv\omega_0(\gamma)}}\frac{c_m(\omega)}{N(\omega)^s}=\frac{e^{4mi\arg\gamma}}{N^{2s}(\gamma)}\sum_{\substack{\alpha_i\in(\mathrm{mod}\gamma)\\ \alpha_1\alpha_2\equiv\omega_0(\gamma)}}Z_m\left(s,\frac{\alpha_1}{\gamma},0\right)Z_0\left(s,\frac{\alpha_2}{\gamma},0\right),\\ F_m^*(s) &= F_m(s)-\sum_{\substack{\beta\in\mathfrak{B}\\ \alpha\beta=\omega_0+\beta\gamma}}\frac{e^{4mi\arg\alpha}}{N(\omega_0+\beta\gamma)^s}.\end{aligned}$$

Thus, repeating the arguments of the proof of Lemma 5, we obtain for $m\neq 0$

$$\begin{aligned}T_m(x,\gamma,\omega_0) = &\sum_{\substack{\omega\in\mathbb{Z}[i]\\ \omega\equiv\omega_0(\gamma)}}c_m(\omega)=\frac{\pi xe^{4mi\arg\gamma}}{N^2(\gamma)}\sum_{\alpha_1\in(\mathrm{mod}\gamma)}{}'Z_m\left(1,\frac{\alpha_1}{\gamma},0\right)\\ &+O\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon}\right)+O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right),\end{aligned}\tag{13}$$

where sign $'$ in summation $\sum$ denotes that α_1 runs reduced residue system modulo γ.

By the Lemma 1 and (1) we have

$$\begin{aligned} T(x,\gamma,\omega_0,S(\varphi)) &= \sum_{\substack{\omega\equiv\omega_0 \pmod{\gamma},\\ N(\omega)\leq x}} \tau_S(\omega) = \sum_{\substack{\omega\equiv\omega_0 \pmod{\gamma},\\ N(w)\leqslant x}} \sum_{\alpha|\omega} \chi_s(\arg\alpha) \\ &= \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} (f(\arg\alpha;\varphi_1,\varphi_2)+\theta_1 f(\arg\alpha;\varphi_1-\Delta,\varphi_1) \\ &+\theta_2 f(\arg\alpha;\varphi_2,\varphi_2+\Delta)) := \sum\nolimits_0 +\theta_1\sum\nolimits_1 +\theta_2\sum\nolimits_2, \end{aligned} \tag{14}$$

where f is the function from Lemma 1, associated, respectively, with segments $[\varphi_1,\ \varphi_2]$, $[\varphi_1-\Delta,\ \varphi_1]$, $[\varphi_2,\ \varphi_2+\Delta]$. The sums $\sum_0$, $\sum_1$, $\sum_2$ can be investigated similarly, so we consider the case $\sum_0$. We have

$$\begin{aligned} \sum\nolimits_0 &= \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} f(\arg\alpha;\varphi_1,\varphi_2) = \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} \sum_{m=-\infty}^{+\infty} a_m \exp(4mi\arg\alpha) \\ &= \sum_{m=-\infty}^{+\infty} a_m \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} \exp(4mi\arg\alpha) = a_0 T_0(x,\gamma,\omega_0) + \sum_{|m|\geqslant 1} a_m T_m(x,\gamma,\omega_0), \end{aligned} \tag{15}$$

where $a_0 = \frac{1}{\Omega}(\varphi_2-\varphi_1+\Delta)$, $\Omega = \frac{\pi}{2}$, the exact value of Δ will be defined later. The sum over m we split into two parts: $1 \leq |m| \leqslant \Delta^{-1}$, $|m| > \Delta^{-1}$. For $|m| \leqslant \Delta^{-1}$ we use the estimation $|a_m| \leq (2\pi|m|)^{-1}$, when $|m| > \Delta^{-1}$ we apply $|a_m| \leq 2(\pi|m|)^{-1}(r\Omega(\pi|m|\Delta)^{-1})^r$, $r=2$. Substituting these estimates into (15), using the Lemma 5, (4) and (13) we obtain

$$\begin{aligned} \sum\nolimits_0 &= \frac{2}{\pi}(\varphi_2-\varphi_1+\Delta)T_0(x,\gamma,\omega_0) + \frac{\pi x}{N^2(\gamma)} \sum_{\alpha_1\in(\mathrm{mod}\gamma)}{}' \sum_{|m|\geqslant 1} a_m e^{4mi\arg\gamma} Z_m\left(1,\frac{\alpha_1}{\gamma},0\right) \\ &+ O\left(\sum_{1\leq|m|\leq\Delta^{-1}} m^{-1}\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon} + \frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right)\right) \\ &+ O\left(\sum_{|m|>\Delta^{-1}} m^{-3}\Delta^{-2}\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon} + \frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right)\right) \\ &= \frac{1}{2\pi}(\varphi_2-\varphi_1+\Delta)T_0(x,\gamma,\omega_0) + (\varphi_2-\varphi_1)\frac{x}{N(\gamma)}A_0(\varphi_2-\varphi_1,\Delta) \\ &+ O\left(N(\gamma)^{1/5+\varepsilon}\Delta^{-2-\varepsilon}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right), \end{aligned}$$

where $A_0(\varphi_2-\varphi_1,\Delta) = A_0(\varphi)+O(\Delta)$ limited for $\varphi_2-\varphi_1\to 0$ and $\Delta\to 0$. Let $\Delta^{-1} = \frac{x^{\frac{1}{4}}}{N^{\frac{9}{40}}(\gamma)}$. In such case we have

$$\begin{aligned} \sum\nolimits_0 = \frac{2(\varphi_2-\varphi_1)}{\pi} &\left(c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)}\right. \\ &\left.+\ (c_1(\gamma,\omega_0)+A_0(\varphi))\frac{x}{N(\gamma)}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right). \end{aligned} \tag{16}$$

The sums $\sum_1$, $\sum_2$ have similar representations, but we write Δ instead $\varphi_2 - \varphi_1$. $A_0(\varphi)$ can be obtained using Lemma 1 for the case $r = 1$. The assertion of the Theorem 1 follows from (14), (16). The proof is completed. $\square$

In the same way the asymptotic formula for summary function of $\tau_{A_1,A_2}(\omega)$ can be proved, where $A_1 = S(\varphi)$, $A_2 = \{\alpha \in \mathbb{Z}[i] : \alpha \equiv \alpha_0 (\mathrm{mod}\ \gamma)\}$.

The asymptotic formula for the $T_0(x, \gamma, \omega_0)$ can be used for investigation of number of solutions in Gaussian integers of the equation $\alpha_1\alpha_2 - \alpha_3\alpha_4 = \beta$, $N(\alpha_1\alpha_2) \le x$.

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О.В.Савастру *Проблема дільників на спеціальних множинах цілих гаусових чисел* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 305–312.

Нехай A_1 та A_2 — це задані множини цілих гаусових чисел. Через $\tau_{A_1,A_2}(\omega)$ позначимо кількість уявлень ω у вигляді $\omega = \alpha\beta$, де $\alpha \in A_1, \beta \in A_2$. Побудована асимптотична формула для суматорної функції, яка відповідає функції $\tau_{A_1,A_2}(\omega)$, у випадку, коли ω належить арифметичній прогресії, A_1 — сектор роствору φ у комплексній площині, $A_2 = \mathbb{Z}[i]$.

Ключові слова і фрази: гаусові числа, проблема дільників, асимптотична формула, арифметична прогресія.

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APPLICATION OF THE FUNCTIONAL CALCULUS TO SOLVING OF INFINITE DIMENSIONAL HEAT EQUATION

In this paper we study infinite dimensional heat equation associated with the Gross Laplacian. Using the functional calculus method, we obtain the solution of appropriate Cauchy problem in the space of polynomial ultradifferentiable functions. The semigroup approach is considered as well.

Key words and phrases: infinite dimensional heat equation, Gross Laplacian, space of polynomial ultradifferentiable functions, space of polynomial ultradistributions.

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INTRODUCTION

The mathematical framework of white noise analysis, which was founded in works of Yu. Berezansky, Yu. Samoilenko [1] and T. Hida [5], is based on an infinite dimensional analogue of the Schwartz distribution theory.

In 1967 L. Gross [4] introduced Laplacian Δ_G on an abstract Wiener space as a natural infinite dimensional analogue of the finite dimensional Laplacian and studied potential theory associated with Δ_G. Within the white noise framework, the Gross Laplacian has been formulated by Kuo in [8] as a continuous linear operator acting on test white noise functions. The Gross Laplacian and appropriate Cauchy problem have been studied for example in [2, 9].

The aim of this work is to use the functional calculus constructed in [12] in order to solve the infinite dimensional heat equation associated with the Gross Laplacian.

1 PRELIMINARIES

1.1 Spaces of functions

Denote $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\partial^k := \partial^k/\partial t^k$. Fix any real $\beta > 1$. An infinitely differentiable function φ is called an ultradifferentiable function of the Gevrey class (see [7]) if for each segment $[\mu,\nu] \subset \mathbb{R}$ there exist constants $h > 0$ and $C > 0$ such that the inequality $\sup_{t\in[\mu,\nu]} |\partial^k \varphi(t)| \le Ch^k k^{k\beta}$ holds for all $k \in \mathbb{Z}_+$. For a fixed $h > 0$ let us consider the subspace

$$\mathcal{G}^h_\beta[\mu,\nu] := \Big\{\varphi \in C^\infty : \operatorname{supp}\varphi \subset [\mu,\nu], \|\varphi\|_{\mathcal{G}^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{t\in[\mu,\nu]} \frac{|\partial^k\varphi(t)|}{h^k k^{k\beta}} < \infty\Big\}.$$

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Each subspace $\mathcal{G}^h_\beta[\mu,\nu]$ is a Banach space (see [7]) and all maps $\mathcal{G}^h_\beta[\mu,\nu] \hookrightarrow \mathcal{G}^l_\beta[\mu,\nu]$ with $h < l$ are compact inclusions. Consider the space

$$\mathcal{G}_\beta := \bigcup_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad \mathcal{G}_\beta \simeq \limind_{\mu<\nu,\, h>0} \mathcal{G}^h_\beta[\mu,\nu],$$

of Gevrey ultradifferentiable functions with compact supports and endow it with topology of inductive limit with respect to above mentioned compact inclusions. Let $\mathcal{G}'_\beta$ be its dual space of Roumieu ultradistributions.

Let $h > 0$ be any positive real and $\mu,\nu \in \mathbb{R}$ be any reals such that $\mu < \nu$. In the space of entire functions of exponential type we consider the subspace $E^h_\beta[\mu,\nu]$ of functions with the finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+} \sup_{z\in\mathbb{C}} \frac{|z^k \psi(z) e^{-H_{[\mu,\nu]}(\eta)}|}{h^k k^{k\beta}}, \qquad \text{where} \quad H_{[\mu,\nu]}(\eta) := \sup_{t\in[\mu,\nu]} t\eta.$$

Each space $E^h_\beta[\mu,\nu]$ is a Banach one, and all maps $E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu']$ with $[\mu,\nu] \subset [\mu',\nu']$, $h < h'$, are compact inclusions. Consider the space

$$E_\beta := \bigcup_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu], \qquad E_\beta \simeq \limind_{\mu<\nu,\, h>0} E^h_\beta[\mu,\nu],$$

and endow it with the topology of inductive limit with respect to above mentioned compact inclusions.

Consider the Fourier-Laplace transformation

$$\widehat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}} e^{-itz} \varphi(t)\, dt, \qquad \varphi \in \mathcal{G}_\beta,\ z \in \mathbb{C}.$$

Let $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$ be the adjoint mapping. It is known [13], that $F(\mathcal{G}_\beta) = E_\beta$ and $F'(E'_\beta) = \mathcal{G}'_\beta$.

1.2 Polynomial ultradifferentiable functions and polynomial ultradistributions

For any locally convex space $\mathcal{X}$, let $\mathcal{X}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, be the symmetric nth tensor degree of $\mathcal{X}$, completed in the projective tensor topology. For any $x \in \mathcal{X}$ we denote $x^{\otimes n} := \underbrace{x \otimes \cdots \otimes x}_{n} \in \mathcal{X}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$. Set $\mathcal{X}^{\widehat{\otimes} 0} := \mathbb{C}$, $x^{\otimes 0} := 1 \in \mathbb{C}$.

To define the locally convex space $\mathcal{P}({}^n\mathcal{G}'_\beta)$ of n-homogeneous polynomials on $\mathcal{G}'_\beta$ we use the canonical topological linear isomorphism $\mathcal{P}({}^n\mathcal{G}'_\beta) \simeq (\mathcal{G}'^{\widehat{\otimes} n}_\beta)'$, described in [3]. We equip $\mathcal{P}({}^n\mathcal{G}'_\beta)$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{G}'_\beta$. Set $\mathcal{P}({}^0\mathcal{G}'_\beta) := \mathbb{C}$. The space $\mathcal{P}(\mathcal{G}'_\beta)$ of all continuous polynomials on $\mathcal{G}'_\beta$ is defined to be the complex linear span of all $\mathcal{P}({}^n\mathcal{G}'_\beta)$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Let $\mathcal{P}'(\mathcal{G}'_\beta)$ mean the strong dual of $\mathcal{P}(\mathcal{G}'_\beta)$. Elements of the spaces $\mathcal{P}(\mathcal{G}'_\beta)$ and $\mathcal{P}'(\mathcal{G}'_\beta)$ we call the polynomial test ultradifferentiable functions and polynomial ultradistributions, respectively.

Denote

$$\Gamma(\mathcal{G}_\beta) := \bigoplus_{n\in\mathbb{Z}_+} {}_{fin}\, \mathcal{G}^{\widehat{\otimes} n}_\beta \subset \bigoplus_{n\in\mathbb{Z}_+} \mathcal{G}^{\widehat{\otimes} n}_\beta \qquad \text{and} \qquad \Gamma(\mathcal{G}'_\beta) := \bigtimes_{n\in\mathbb{Z}_+} \mathcal{G}'^{\widehat{\otimes} n}_\beta.$$

Note, that we consider only the case when the elements of direct sum consist of finite but not fixed number of addends. It is well known [11, 4.4], that $\langle\, \Gamma(\mathcal{G}'_\beta),\, \Gamma(\mathcal{G}_\beta)\,\rangle$ is a dual pair with respect to the bilinear form

$$\langle\, \boldsymbol{f},\, \boldsymbol{p}\,\rangle = \Big\langle \mathop{\times}_{n\in\mathbb{Z}_+} f_n,\ \bigoplus_{n\in\mathbb{Z}_+} p_n \Big\rangle = \sum_{n\in\mathbb{Z}_+} \langle\, f_n,\, p_n\,\rangle, \quad \boldsymbol{p}\in\Gamma(\mathcal{G}_\beta), \quad \boldsymbol{f}\in\Gamma(\mathcal{G}'_\beta), \tag{1}$$

where $p_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$ and $f_n \in \mathcal{G}_\beta'^{\widehat{\otimes} n} \simeq (\mathcal{G}_\beta^{\widehat{\otimes} n})'$.

By analogy we can construct the dual pairs $\langle\, \Gamma(E'_\beta),\, \Gamma(E_\beta)\,\rangle$ and $\langle\, \mathcal{P}'(E'_\beta),\, \mathcal{P}(E'_\beta)\,\rangle$.

We have the following assertion (see also [10, Proposition 2.1]).

Proposition 1.1. *There exist the linear topological isomorphisms*

$$\Upsilon : \mathcal{P}'(\mathcal{G}'_\beta) \longrightarrow \Gamma(\mathcal{G}'_\beta), \qquad \Psi : \mathcal{P}'(E'_\beta) \longrightarrow \Gamma(E'_\beta).$$

Using the Proposition 1.1 and tensor structure of the space $\Gamma(\mathcal{G}'_\beta)$, we extend the map F'^{-1} onto $\Gamma(\mathcal{G}'_\beta)$. First, for elements of total subset of the space $\mathcal{G}_\beta'^{\widehat{\otimes} n}$ we define the operator $\mathcal{F}'^{\otimes n}$: $f^{\otimes n} \longmapsto \widehat{f}^{\otimes n}$, $\mathcal{F}'^{\otimes 0} := I_{\mathbb{C}}$, where $\widehat{f}^{\otimes n} := \left(F'^{-1} f\right)^{\otimes n}$. Next, we extend the map $\mathcal{F}'^{\otimes n}$ onto whole space $\mathcal{G}_\beta'^{\widehat{\otimes} n}$ by linearity and continuity. As a result we obtain the map $\mathcal{F}'^{\otimes n} \in \mathscr{L}\big(\mathcal{G}_\beta'^{\widehat{\otimes} n}, E_\beta'^{\widehat{\otimes} n}\big)$ And finally, we define the mapping $\mathcal{F}'^{\otimes}$ by the formula

$$\mathcal{F}'^{\otimes} := \big(\mathcal{F}'^{\otimes n}\big) : \Gamma(\mathcal{G}'_\beta) \ni \boldsymbol{f} = \big(f_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{f}} := \big(\widehat{f}_n\big) \in \Gamma(E'_\beta),$$

where $f_n \in \mathcal{G}_\beta'^{\widehat{\otimes} n}$, $\widehat{f}_n := \mathcal{F}'^{\otimes n} f_n \in E_\beta'^{\widehat{\otimes} n}$.

The following commutative diagram

$$\begin{tikzcd}
\mathcal{P}'(\mathcal{G}'_\beta) \arrow[r, "\mathcal{F}'^{\otimes}_{\mathcal{P}}"] \arrow[d, "\Upsilon"'] & \mathcal{P}'(E'_\beta) \\
\Gamma(\mathcal{G}'_\beta) \arrow[r, "\mathcal{F}'^{\otimes}"] & \Gamma(E'_\beta) \arrow[u, "\Psi^{-1}"']
\end{tikzcd} \tag{2}$$

uniquely defines the operator $\mathcal{F}'^{\otimes}_{\mathcal{P}} \in \mathscr{L}\big(\mathcal{P}'(\mathcal{G}'_\beta), \mathcal{P}'(E'_\beta)\big)$.

2 CONVOLUTION OF POLYNOMIAL ULTRADISTRIBUTIONS

Let $g \in \mathcal{G}'_\beta$. Define the shift operator on the space $\mathcal{P}(\mathcal{G}'_\beta)$ with the formula

$$\mathscr{T}_g P(f) := P(f+g), \qquad P \in \mathcal{P}(\mathcal{G}'_\beta), \quad f \in \mathcal{G}'_\beta.$$

It is easy to see, that $\mathscr{T}_g$ is a linear continuous operator from the space $\mathcal{P}(\mathcal{G}'_\beta)$ into itself.

Let the symbol $\circledcirc_k$ denotes the (right) k-contraction [6] of symmetric tensor product, i.e., $g^{\otimes k} \circledcirc_k \varphi^{\otimes s} := \langle g, \varphi\rangle^k \varphi^{\otimes(s-k)}$, $k \le s$, $g \in \mathcal{G}'_\beta$, $\varphi \in \mathcal{G}_\beta$.

Let us show, that for any $g \in \mathcal{G}'_\beta$ the shift operator $\mathscr{T}_g$ acts as follows $P = \sum_n \langle \cdot^{\otimes n}, p_n\rangle \mapsto \mathscr{T}_g P = \sum_n \langle \cdot^{\otimes n}, q_n\rangle$, where $p_n, q_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$, $n = 0, 1, \ldots, m$, $m = \deg P$, and the elements q_n can be obtained by the formula

$$q_n = \sum_{k=0}^{m-n} \frac{(n+k)!}{n!k!} g^{\otimes k} \circledcirc_k p_{n+k}.$$

Without loss of generality we can prove this for polynomials of view $P_{\varphi,m}=\sum_{k=0}^m\langle\ \cdot\ ^{\otimes k},\varphi^{\otimes k}\rangle$, where $(1,\varphi,\varphi^{\otimes 2},\dots,\varphi^{\otimes m},0,\dots)\in\Gamma(\mathcal{G}_\beta)$, $\varphi\in\mathcal{G}_\beta$, $m\in\mathbb{Z}_+$.

Indeed,

$$\begin{aligned}
\mathcal{T}_g P_{\varphi,m}(f)=P_{\varphi,m}(f+g)&=\sum_{k=0}^m\langle(f+g)^{\otimes k},\varphi^{\otimes k}\rangle=\sum_{k=0}^m\sum_{n=0}^k C_k^n\langle f^{\otimes n}\widehat{\otimes}g^{\otimes(k-n)},\varphi^{\otimes k}\rangle\\
&=\sum_{n=0}^m\sum_{k=n}^m C_k^n\langle f^{\otimes n}\widehat{\otimes}g^{\otimes(k-n)},\varphi^{\otimes k}\rangle=\sum_{n=0}^m\sum_{k=0}^{m-n} C_{n+k}^n\langle f^{\otimes n}\widehat{\otimes}g^{\otimes k},\varphi^{\otimes(n+k)}\rangle\\
&=\sum_{n=0}^m\sum_{k=0}^{m-n} C_{n+k}^n\big\langle f^{\otimes n},\langle g,\varphi\rangle^k\varphi^{\otimes n}\big\rangle=\sum_{n=0}^m\Big\langle f^{\otimes n},\sum_{k=0}^{m-n}C_{n+k}^n\langle g,\varphi\rangle^k\varphi^{\otimes n}\Big\rangle\\
&=\sum_{n=0}^m\Big\langle f^{\otimes n},\sum_{k=0}^{m-n}C_{n+k}^n g^{\otimes k}\circledcirc_k\varphi^{\otimes(n+k)}\Big\rangle.
\end{aligned}$$

Let us define the convolution of a polynomial ultradistribution $U\in\mathcal{P}'(\mathcal{G}'_\beta)$ and a test function $P\in\mathcal{P}(\mathcal{G}'_\beta)$ with the formula $(U*P)(g):=\langle U,\mathcal{T}_gP\rangle$, $g\in\mathcal{G}'_\beta$, where in the right side there is the pairing of the dual pair $\big\langle\mathcal{P}'(\mathcal{G}'_\beta),\mathcal{P}(\mathcal{G}'_\beta)\big\rangle$ (see Proposition 1.1 and formula (1)).

If $U\in\mathcal{P}'(\mathcal{G}'_\beta)$ and $P\in\mathcal{P}(\mathcal{G}'_\beta)$ are represented in the form $U=\bigtimes_{n\in\mathbb{Z}_+}\langle u_n,\cdot^{\otimes n}\rangle$ and $P=\sum_{n=0}^m\langle\cdot^{\otimes n},p_n\rangle$ respectively, then the convolution may be written in the explicit form

$$\begin{aligned}
(U*P)(g)&=\sum_{n=0}^m\Big\langle u_n,\sum_{k=0}^{m-n}C_{n+k}^n g^{\otimes k}\circledcirc_k p_{n+k}\Big\rangle=\sum_{n=0}^m\sum_{k=0}^{m-n}C_{n+k}^n\langle u_n\widehat{\otimes}g^{\otimes k},p_{n+k}\rangle\\
&=\sum_{k=0}^m\sum_{n=0}^{m-k}C_{n+k}^n\langle g^{\otimes k},u_n\circledcirc_n p_{n+k}\rangle=\sum_{k=0}^m\Big\langle g^{\otimes k},\sum_{n=0}^{m-k}C_{n+k}^n u_n\circledcirc_n p_{n+k}\Big\rangle.
\end{aligned}\tag{3}$$

It is clear, that $q_k=\sum_{n=0}^{m-k}C_{n+k}^n u_n\circledcirc_n p_{n+k}$ belongs to the space $\mathcal{G}_\beta^{\widehat{\otimes}k}$ for each $k=0,1,\dots,m$. It follows, that the convolution $U*P$ is a polynomial from the space $\mathcal{P}(\mathcal{G}'_\beta)$.

For any polynomial ultradistribution $U\in\mathcal{P}'(\mathcal{G}'_\beta)$ the mapping C_U, defined with the formula $C_U:\mathcal{P}(\mathcal{G}'_\beta)\ni P\mapsto U*P\in\mathcal{P}(\mathcal{G}'_\beta)$, is said to be the convolution operator, associated with U.

Let us show, that the composition of two convolution operators C_V and C_U, associated with any $V,U\in\mathcal{P}'(\mathcal{G}'_\beta)$, is a convolution operator, associated with some polynomial ultradistribution, which we denote by $V*U$. Let $V,U\in\mathcal{P}'(\mathcal{G}'_\beta)$ and $P\in\mathcal{P}(\mathcal{G}'_\beta)$ are represented in the form $V=\bigtimes_{n\in\mathbb{Z}_+}\langle g^{\otimes n},\cdot^{\otimes n}\rangle$, $U=\bigtimes_{n\in\mathbb{Z}_+}\langle f^{\otimes n},\cdot^{\otimes n}\rangle$ and $P=\sum_{n=0}^m\langle\cdot^{\otimes n},\varphi^{\otimes n}\rangle$ respectively, where $f,g\in\mathcal{G}'_\beta$, $\varphi\in\mathcal{G}_\beta$.

Using formula (3), we obtain the following equalities.

$$\begin{aligned}
(C_V\circ C_U)(P)=V*(U*P)&=\sum_{n=0}^m\Big\langle\ \cdot\ ^{\otimes n},\sum_{j=0}^{m-n}C_{n+j}^j g^{\otimes j}\circledcirc_j q_{n+j}\Big\rangle\\
&=\sum_{n=0}^m\Big\langle\ \cdot\ ^{\otimes n},\sum_{j=0}^{m-n}C_{n+j}^j g^{\otimes j}\circledcirc_j\Big(\sum_{k=0}^{m-n-j}C_{n+j+k}^k f^{\otimes k}\circledcirc_k\varphi^{\otimes(n+j+k)}\Big)\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{m} \Big\langle \,\cdot\,^{\otimes n}, \sum_{j=0}^{m-n} \sum_{k=0}^{m-n-j} C_{n+j}^{j} C_{n+j+k}^{k} \langle g, \varphi\rangle^{j} \langle f, \varphi\rangle^{k} \varphi^{\otimes n} \Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle \,\cdot\,^{\otimes n}, \sum_{j+k=0}^{m-n} \frac{(n+j+k)!}{n!j!k!} (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{j+k} \varphi^{\otimes (n+j+k)} \Big\rangle \\
&= \sum_{n=0}^{m} \Big\langle \,\cdot\,^{\otimes n}, \sum_{s=0}^{m-n} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{s} \varphi^{\otimes (n+s)} \Big\rangle \\
&= \sum_{n=0}^{m} \sum_{s=0}^{m-n} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} \big\langle \,\cdot\,^{\otimes n}, (g^{\otimes j} \widehat{\otimes} f^{\otimes k}) \circledcirc_{s} \varphi^{\otimes (n+s)} \big\rangle \\
&= \sum_{s=0}^{m} \sum_{n=0}^{m-s} \frac{(n+s)!}{n!s!} \sum_{j+k=s} \frac{s!}{j!k!} \big\langle \,\cdot\,^{\otimes n} \widehat{\otimes} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \varphi^{\otimes (n+s)} \big\rangle \\
&= \sum_{s=0}^{m} \Big\langle \sum_{j+k=s} \frac{s!}{j!k!} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \sum_{n=0}^{m-s} C_{n+s}^{s} (\,\cdot\,^{\otimes n}) \circledcirc_{n} \varphi^{\otimes (n+s)} \Big\rangle.
\end{aligned}
$$

It follows, that the composition $C_V \circ C_U$ is the convolution operator, associated with

$$
V * U = \bigtimes_{n \in \mathbb{Z}_+} \Big\langle \sum_{j+k=n} \frac{n!}{j!k!} g^{\otimes j} \widehat{\otimes} f^{\otimes k}, \,\cdot\,^{\otimes n} \Big\rangle \in \mathcal{P}'(\mathcal{G}'_\beta). \tag{4}
$$

For any polynomial ultradistribution $U \in \mathcal{P}'(\mathcal{G}'_\beta)$ let us define the formal series

$$
e^{*U} := \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} U^{*n}, \qquad \text{where} \quad U^{*n} := \underbrace{U * \cdots * U}_{n}. \tag{5}
$$

Note, that each partial sum of this series belongs to the space $\mathcal{P}'(\mathcal{G}'_\beta)$.

3 Heat equation associated with the Gross Laplacian

Let $\{U_t : t \in J\}$ be a family of elements from the space $\mathcal{P}'(\mathcal{G}'_\beta)$, let J be an arbitrary interval $[0, \alpha]$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Let us assume, that the function $t \longmapsto U_t$ is a continuous map from J into $\mathcal{P}'(\mathcal{G}'_\beta)$. Then the function $t \longmapsto \mathcal{F}'^{\otimes}_{\mathcal{P}} U_t$ is continuous map from J into $\mathcal{P}'(E'_\beta)$, where the mapping $\mathcal{F}'^{\otimes}_{\mathcal{P}}$ is defined with formula (2). Therefore, for each $t \in J$ the set $\{\mathcal{F}'^{\otimes}_{\mathcal{P}} U_s : s \in [0, t]\}$ is a compact subset in $\mathcal{P}'(E'_\beta)$. In particular, it is bounded. It follows, that the element

$$
\int_0^t \mathcal{F}'^{\otimes}_{\mathcal{P}} U_s \, ds,
$$

belongs to the space $\mathcal{P}'(E'_\beta)$ for each $t \in J$. Hence, in the space $\mathcal{P}'(\mathcal{G}'_\beta)$ there exists a unique element, which we denote $\int_0^t U_s \, ds$, such that

$$
\mathcal{F}'^{\otimes}_{\mathcal{P}} \int_0^t U_s \, ds = \int_0^t \mathcal{F}'^{\otimes}_{\mathcal{P}} U_s \, ds.
$$

Moreover, the map $E_t = \int_0^t U_s \, ds$, $t \in J$, is differentiable and satisfies the equality $\frac{\partial}{\partial t} E_t = U_t$.

Let $\{U_t : t \in J\}$ be any described above family of elements from $\mathcal{P}'(\mathcal{G}'_\beta)$. Let us consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} X_t = U_t * X_t, & t \in J, \\ X_0 = P, & P \in \mathcal{P}(\mathcal{G}'_\beta). \end{cases} \tag{6}$$

Theorem 1. *Cauchy problem* (6) *has a unique solution in* $\mathcal{P}(\mathcal{G}'_\beta)$*, which can be presented in the view*

$$X_t = e^{* \int_0^t U_s\, ds} * P, \qquad t \in J, \tag{7}$$

where $e^{* \int_0^t U_s\, ds}$ *is treated in the sense of the formula* (5).

Proof. Using Picard's iteration, the solution X_t of Cauchy problem (6) is written informally in the form (7). Since the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ has a finite number of addends, a value of $e^{* \int_0^t U_s\, ds} * P$ depends on some partial sum of the series $e^{* \int_0^t U_s\, ds}$. Formula (3) implies that solution (7) belongs to the space $\mathcal{P}(\mathcal{G}'_\beta)$. $\square$

As an application of Theorem 1 we consider the generalized heat equation, associated with the Gross Laplacian.

Let the trace operator τ be defined by

$$\langle \tau, \varphi \widehat{\otimes} \psi \rangle := \int_{\mathbb{R}^d_+} \varphi(t)\psi(t)\, dt, \qquad \varphi, \psi \in \mathcal{G}_\beta.$$

It is clear, that $\tau \in \mathscr{L}(\mathcal{G}_\beta^{\widehat{\otimes} 2}, \mathbb{C}) = (\mathcal{G}_\beta^{\widehat{\otimes} 2})' \simeq \mathcal{G}_\beta'^{\widehat{\otimes} 2}$.

The Gross Laplacian Δ_G by definition (see e.g. [8]) is the following operator

$$\Delta_G : P = \sum_{n=0}^{m} \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle \longmapsto \Delta_G P := \sum_{n=0}^{m-2} (n+2)(n+1) \langle \tau, \varphi^{\otimes 2} \rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{G}_\beta.$$

Theorem 2. *The Gross Laplacian* Δ_G *acts as a convolution operator, i.e.*

$$\frac{1}{2} \Delta_G P = U_\tau * P, \qquad P \in \mathcal{P}(\mathcal{G}'_\beta),$$

where U_τ *is a polynomial ultradistribution from the space* $\mathcal{P}'(\mathcal{G}'_\beta)$*, that corresponds to the element* $(0, 0, \tau, 0, \dots) \in \Gamma(\mathcal{G}'_\beta)$.

Proof. The polynomial ultradistribution U_τ can be written in the form

$$U_\tau = \bigtimes_{n \in \mathbb{Z}_+} \langle u_{\tau,n}, \cdot^{\otimes n} \rangle = \big(0, 0, \langle \tau, \cdot^{\otimes 2} \rangle, 0, \dots\big),$$

where $u_{\tau,n} = \tau$ if $n = 2$ and $u_{\tau,n} = 0$ if $n \neq 2$.

Let the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ be of the form $P = \sum_{n=0}^{m} \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle$, $\varphi \in \mathcal{G}_\beta$. Using equalities (3), we obtain the required result

$$\begin{aligned} U_\tau * P = \sum_{n=0}^{m} \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^{k} u_{\tau,k} \circledcirc_k \varphi^{\otimes (n+k)} \Big\rangle &= \sum_{n=0}^{m-2} \big\langle \cdot^{\otimes n}, C_{n+2}^{2} \tau \circledcirc_2 \varphi^{\otimes (n+2)} \big\rangle \\ &= \sum_{n=0}^{m-2} C_{n+2}^{2} \langle \tau, \varphi^{\otimes 2} \rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n} \rangle = \frac{1}{2} \Delta_G P. \end{aligned}$$

$\square$

Theorem 3. *Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t} X_t = \frac{1}{2}\Delta_G X_t, & t \in J, \\ X_0 = P, & P \in \mathcal{P}(\mathcal{G}'_\beta), \end{cases} \tag{8}$$

for heat equation, associated with the Gross Laplacian, has a unique solution in $\mathcal{P}(\mathcal{G}'_\beta)$ *given by*

$$X_t = e^{*tU_\tau} * P, \qquad t \in J.$$

Proof. Theorem 2 allows us to rewrite the heat equation in the view $\frac{\partial}{\partial t} X_t = U_\tau * X_t$. It follows from Theorem 1 that the Cauchy problem has a unique solution given by

$$X_t = e^{*\int_0^t U_\tau\, ds} * P = e^{*tU_\tau} * P.$$

We can rewrite it in explicit form. Using formula (4), let us find $(tU_\tau)^{*n}$. For $n = 2$ we obtain

$$(tU_\tau) * (tU_\tau) = \bigtimes_{n\in\mathbb{Z}_+} \Big\langle t^2 \sum_{j+k=n} \frac{n!}{j!k!} u_{\tau,j} \widehat{\otimes} u_{\tau,k}, \cdot^{\otimes n} \Big\rangle = \Big(0,0,0,0, \frac{4!}{2!2!} t^2 \langle \tau^{\otimes 2}, \cdot^{\otimes 4}\rangle, 0, \dots\Big),$$

since $u_{\tau,n}$ does not vanish only for $n = 2$. Using the mathematical induction, it is easy to prove that

$$(tU_\tau)^{*n} = \Big(\underbrace{0,\dots,0}_{2n}, \frac{(2n)!}{2^n} t^n \langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle, 0, \dots\Big).$$

It follows

$$\begin{aligned} e^{*tU_\tau} &= \sum_{n\in\mathbb{Z}_+} \frac{1}{n!} (tU_\tau)^{*n} = \sum_{n\in\mathbb{Z}_+} \frac{1}{n!} \Big(\underbrace{0,\dots,0}_{2n}, \frac{(2n)!}{2^n} t^n \langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle, 0, \dots\Big) \\ &= \Big(1, 0, t\langle \tau, \cdot^{\otimes 2}\rangle, 0, 3t^2 \langle \tau^{\otimes 2}, \cdot^{\otimes 4}\rangle, 0, \dots, 0, \underbrace{\frac{(2n)!}{n!} \frac{t^n}{2^n} \langle \tau^{\otimes n}, \cdot^{\otimes 2n}\rangle}_{2n\text{-th place}}, 0, \dots\Big). \end{aligned} \tag{9}$$

It only remains to find the convolution $e^{*tU_\tau} * P$. Let the polynomial $P \in \mathcal{P}(\mathcal{G}'_\beta)$ be written in the form $P = \sum_{n=0}^m \langle \cdot^{\otimes n}, \varphi^{\otimes n}\rangle$, $\varphi \in \mathcal{G}_\beta$. For any $n \in \mathbb{Z}_+$ let us denote $e_{2n} := \frac{(2n)!}{n!} \frac{t^n}{2^n} \tau^{\otimes n}$ and $e_{2n+1} := 0$. Then e^{*tU_τ} can be rewritten as $e^{*tU_\tau} = \bigtimes_{n\in\mathbb{Z}_+} \langle e_n, \cdot^{\otimes n}\rangle$. Therefore, we obtain

$$\begin{aligned} e^{*tU_\tau} * P &= \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{m-n} C_{n+k}^k e_k \circledcirc_k \varphi^{\otimes(n+k)} \Big\rangle = \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} C_{n+2k}^{2k} e_{2k} \circledcirc_{2k} \varphi^{\otimes(n+2k)} \Big\rangle \\ &= \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{(2k)!n!} \frac{(2k)!}{k!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k}\rangle \varphi^{\otimes n} \Big\rangle \\ &= \sum_{n=0}^m \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k}\rangle \langle \cdot^{\otimes n}, \varphi^{\otimes n}\rangle, \end{aligned}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the floor function.

Hence, if the polynomial P from (8) has the form $P = \sum_{n=0}^m \langle \cdot^{\otimes n}, p_n\rangle$, $p_n \in \mathcal{G}_\beta^{\widehat{\otimes} n}$, then the solution of Cauchy problem for heat equation associated with the Gross Laplacian can be expressed as

$$X_t = \sum_{n=0}^m \Big\langle \cdot^{\otimes n}, \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \tau^{\otimes k} \circledcirc_{2k} p_{n+2k} \Big\rangle.$$

$\square$

4 SEMIGROUP GENERATED BY THE GROSS LAPLACIAN

Our next goal is to construct an one-parameter semigroup $\{\mathsf{G}_t : t \geq 0\}$ with the infinitesimal generator $\frac{1}{2}\Delta_G$. This semigroup can be formally expressed as $\mathsf{G}_t = e^{t\frac{1}{2}\Delta_G}$.

Since $\frac{1}{2}\Delta_G P = U_\tau * P$, results of previous section imply

$$\mathsf{G}_t P := \sum_{n=0}^{m} \sum_{k=0}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \, \cdot \, ^{\otimes n}, \varphi^{\otimes n} \rangle, \tag{10}$$

where $P = \sum_{n=0}^{m} \langle \, \cdot \, ^{\otimes n}, \varphi^{\otimes n} \rangle$, $\varphi \in \mathcal{G}_\beta$.

Proposition 4.1. *The mapping $\mathbb{R}_+ \ni t \longmapsto \mathsf{G}_t \in \mathscr{L}(\mathcal{P}(\mathcal{G}'_\beta))$, where G_t is defined by formula* (10)*, is a strongly continuous one-parameter semigroup of continuous linear operators from $\mathcal{P}(\mathcal{G}'_\beta)$ into itself with infinitesimal generator $\frac{1}{2}\Delta_G$.*

Proof. Formula (10) can be rewritten as

$$\mathsf{G}_t P = P + \sum_{n=0}^{m-2} \sum_{k=1}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \, \cdot \, ^{\otimes n}, \varphi^{\otimes n} \rangle, \tag{11}$$

therefore the equality $\mathsf{G}_0 = I_{\mathcal{P}(\mathcal{G}'_\beta)}$ is clear.

Formulas (4), (9) and the following equalities

$$\mathsf{G}_t \mathsf{G}_s = e^{t\frac{1}{2}\Delta_G} e^{s\frac{1}{2}\Delta_G} = e^{*tU_\tau} * e^{*sU_\tau} = e^{*(t+s)U_\tau} = e^{(t+s)\frac{1}{2}\Delta_G} = \mathsf{G}_{t+s}$$

imply the semigroup property $\mathsf{G}_t \mathsf{G}_s = \mathsf{G}_{t+s}$.

To prove the strong continuity of the semigroup, we need to show that for any $P \in \mathcal{P}(\mathcal{G}'_\beta)$ the function $t \longmapsto \mathsf{G}_t P$ is continuous. Using representation (11), we obtain

$$\begin{aligned}
\lim_{t\to 0} \sup_f |\mathsf{G}_t P - P| &= \lim_{t\to 0} \sup_f \Big| \sum_{n=0}^{m} \sum_{k=1}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{t^k}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle f^{\otimes n}, \varphi^{\otimes n} \rangle \Big| \\
&\leq \lim_{t\to 0} \sup_f \sum_{n=0}^{m} \sum_{k=1}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{|t|^k}{2^k} \big| \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \big| \big| \langle f^{\otimes n}, \varphi^{\otimes n} \rangle \big| \\
&= \sum_{n=0}^{m} \sup_f \big| \langle f^{\otimes n}, \varphi^{\otimes n} \rangle \big| \lim_{t\to 0} \sum_{k=1}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{|t|^k}{2^k} \big| \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \big| = 0.
\end{aligned}$$

It remains to show that the Gross Laplacian is the generator of the semigroup G_t. Using representation (11), we can write

$$\begin{aligned}
\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m} \sum_{k=1}^{\left\lfloor \frac{m-n}{2} \right\rfloor} \frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \langle \, \cdot \, ^{\otimes n}, \varphi^{\otimes n} \rangle \\
- \sum_{n=0}^{m-2} \frac{(n+2)(n+1)}{2} \langle \tau, \varphi^{\otimes 2} \rangle \langle \, \cdot \, ^{\otimes n}, \varphi^{\otimes n} \rangle .
\end{aligned}$$

Note, that $\lfloor \frac{m-n}{2} \rfloor = 0$ for $n = m-1$ and for $n = m$. So, we can rewrite the above equality

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-2} \Bigg(\sum_{k=1}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle - \frac{(n+2)(n+1)}{2} \langle \tau, \varphi^{\otimes 2} \rangle \Bigg) \langle \,\cdot\, ^{\otimes n}, \varphi^{\otimes n} \rangle.$$

It is clear that $\frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle = \frac{(n+2)(n+1)}{2} \langle \tau, \varphi^{\otimes 2} \rangle$ with $k = 1$, therefore

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-2} \Bigg(\sum_{k=2}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \Bigg) \langle \,\cdot\, ^{\otimes n}, \varphi^{\otimes n} \rangle.$$

Note, that $\lfloor \frac{m-n}{2} \rfloor = 1$ for $n = m-2$ and for $n = m-3$. So, we obtain

$$\frac{\mathsf{G}_t P - P}{t} - \frac{1}{2}\Delta_G P = \sum_{n=0}^{m-4} \Bigg(\sum_{k=2}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{t^{k-1}}{2^k} \langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle \Bigg) \langle \,\cdot\, ^{\otimes n}, \varphi^{\otimes n} \rangle.$$

From the above formula we can derive the required result

$$\lim_{t\to 0} \sup_f \Big| \frac{\mathsf{G}_t P(f) - P(f)}{t} - \frac{1}{2}\Delta_G P(f) \Big| \leq \sum_{n=0}^{m-4} \sup_f \big| \langle f^{\otimes n}, \varphi^{\otimes n} \rangle \big| \lim_{t\to 0} \sum_{k=2}^{\lfloor \frac{m-n}{2} \rfloor} \frac{(n+2k)!}{k!n!} \frac{|t|^{k-1}}{2^k} |\langle \tau^{\otimes k}, \varphi^{\otimes 2k} \rangle| = 0.$$

□

Corollary 4.1. *Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t} X_t = \frac{1}{2}\Delta_G X_t, & t \in J, \\ X_0 = P, & P \in \mathcal{P}(\mathcal{G}'_\beta), \end{cases}$$

for heat equation associated with the Gross Laplacian has a unique solution in $\mathcal{P}(\mathcal{G}'_\beta)$ *given by*

$$X_t = \mathsf{G}_t P, \qquad t \in J.$$

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Шарин С.В. *Застосування функціонального числення до розв'язання задачі Коші для нескінченновимірного рівняння теплопровідності* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 313–322.

У цій роботі ми вивчаємо нескінченновимірне рівняння теплопровідності, породжене лапласіаном Ґросса. Використовуючи метод функціонального числення, ми отримуємо розв'язок відповідної задачі Коші у просторі поліноміальних ультрадиференційовних функцій. Також розглянуто напівгруповий підхід розв'язання такої задачі.

Ключові слова і фрази: нескінченновимірне рівняння теплопровідності, лапласіан Ґросса, простір поліноміальних ультрадиференційовних функцій, простір поліноміальних ультрарозподілів.

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Віталій Іванович Сущанський

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29 жовтня 2016 року, після тривалої важкої хвороби, у м. Глівіцах (Польща) помер видатний український математик, доктор фізико-математичних наук, професор кафедри алгебри та математичної логіки Київського національного університету імені Тараса Шевченка, професор Інституту математики Сілезької політехніки (Польща) Віталій Іванович Сущанський.

Віталій Іванович народився 14 листопада 1946 року в селі Ходорків Попільнянського району Житомирської області, де і пройшло його дитинство. Після успішного закінчення у 1964 році середньої школи він був зарахований на механіко-математичний факультет Київського університету імені Тараса Шевченка. У цей час завідувачем кафедри алгебри та математичної логіки був її засновник, відомий математик Л.А. Калужнін, якого завжди оточувала здібна студентська молодь, серед якої був і Віталій Сущанський.

У 1969 році Віталій Іванович з відзнакою закінчив механіко-математичний факультет та вступив до аспірантури при кафедрі алгебри та математичної логіки, де займався науково-дослідницькою роботою під керівництвом Л.А. Калужніна. У лютому 1972 року В.І. Сущанський захистив кандидатську дисертацію за темою "Вінцеві добутки елементарних абелевих груп та їх застосування". Докторську дисертацію за спеціальністю "алгебра, теорія чисел та математична логіка" за темою "Вінцеві добутки, ізометрії напівскінченних метрик Бера і резидуально скінченні групи" було захищено у березні 1991 року у Ленінградському відділенні Інституту математики АН СРСР імені В.А. Стеклова.

У 1971–2004 роках Віталій Іванович Сущанський працює на кафедрі алгебри та математичної логіки Київського національного університету імені Тараса Шевченка, пройшовши шлях від асистента до професора та її завідувача (1998–2004). Протягом цього періоду все його життя та енергія були спрямовані на наукову роботу з обдарованими студентами. Серед них В. Некрашевич, А. Олійник, Я. Лавренюк, Ю. Леонов, Є. Бондаренко, Д. Савчук та багато інших. Підготувавши собі достойну зміну, В.І. Сущанський, на запрошення адміністрації Сілезької політехніки (м. Глівіци, Польща), у 2004 році стає звичайним професором Інституту математики, а пізніше керівником відділу алгебри цього навчального закладу.

Віталій Іванович проводив величезну організаційно-наукову роботу. Був постійним членом програмних комітетів та одним з організаторів міжнародних алгебраїчних конференцій в Україні. Ним було організовано семінар з теорії груп у Київському університеті, який досі є одним з найбільш важливих координаційних центрів теорії груп в Україні. Йому належить ідея організувати Київський науковий алгебраїчний семінар "Під кінець року", щоб українські алгебраїсти, що працюють за кордоном, могли на різдвяні канікули приїхати та зробити доповідь про свої наукові досягнення. Впродовж багатьох років Віталій Іванович був членом спеціалізованої вченої ради із захисту докторських дисертацій з алгебри і дискретної математики у Київському національному університеті імені Тараса Шевченка, спочатку як вчений секретар, а після захисту докторської дисертації певний час головою ради, членом редколегій журналів "Алгебра та дискретна математика", "Математичні студії", "Карпатські математичні публікації", головним редактором єдиного в Україні математичного науково-популярного журналу "У світі математики", членом Київського та Польського математичних товариств.

Широта та різносторонність інтересів В.І. Сущанського та його наукова мобільність вражали. Він одночасно проводив спільні алгебраїчні дослідження у різних напрямках як із українськими так і зарубіжними математиками. Ним видано понад 150 наукових робіт майже із сорока зарубіжними авторами, під його керівництвом захищено 27 кандидатських дисертацій в Україні та 5 у Польщі і 5 докторських дисертацій. Віталій Іванович візитував у багатьох закордонних університетах: у Фрайбурзькому університеті (1998, Німеччина), Манітобському університеті (1999, Канада), університеті Бразилії (2000), Упсали (2003, Швеція), Техаському A&M університеті (2006, США). Серед грантів та нагород Віталія Івановича INTAS Award (1994–1998), Research and Conference Grant, J. Soros Foundation (1994), J. Soros Professorship (1995–1996), Research Grant DAAD (1998), нагорода ректора Сілезької політехніки за наукові досягнення.

У працях В.І. Сущанського (разом з професором Л.А. Калужніним) отримали подальший розвиток дослідження будови вінцевих добутків груп, систематично вивчалися операції на групах підстановок. Віталій Іванович застосував вінцеві добутки за нескінченними послідовностями груп підстановок для побудови нових прикладів груп бернсайдівського типу — нескінченних періодичних груп зі скінченним числом твірних, розв'язав за допомогою оригінальних конструкцій відомі проблеми теорії факторизовних груп, разом зі своїми учнями отримав низку важливих результатів про будову груп автоморфізмів дерев, заклав основи теорії груп та напівгруп автоматних перетворень, дослідив класи спряженості в групах автоморфізмів різних типів дерев, описав нормальну будову груп автоморфізмів шарово-однорідних дерев, охарактеризував широкі класи підгруп бернсайдівського типу в групах автоморфізмів однорідного кореневого дерева. У працях з Р.І. Григорчуком і В.В. Некрашевичем розвинув теорію скінченно-автоматних груп перетворень як для випадку синхронних автоматів типу Мілі та Мура, так і для випадку асинхронних автоматів. У спільних роботах з Р.І. Григорчуком, Ю.Г. Леоновим і В.В. Некрашевичем побудував теорію унітрикутних зображень самоподібних груп і пов'язав її з теорією підстановочних динамічних систем. Разом зі своїми учнями А.Бєр, Я.Шашсковським, Ю.Лещенко вивчав групи, що занурюються у групи автоморфізмів дерев та їх силовські підгрупи. Разом з В.В. Некрашевичем і П.Ґавроном описав класи спряженості основних груп автоморфізмів дерев. Разом з А.С. Олійником побудував приклади вільних груп нескінченних унітрикутних матриць. Це далеко неповний перелік наукового доробку Сущанського Віталія Івановича. Не забував Віталій Іванович і про школярів. Ним було написано чимало науково-популярних статей спеціально для журналу "У світі математики".

Світла пам'ять про Віталія Івановича як скромну, високо інтелігентну, чуйну людину, здатну завжди прийти на допомогу, назавжди залишиться в наших серцях.

Артемович О.Д., Безущак О.О., Гаврилків В.М., Григорчук Р.І., Гой Т.П., Дрозд Ю.А., Загороднюк А.В., Зарічний М.М., Заторський Р.А., Кириченко В.В., Малицька Г.П., Никифорчин О.Р., Олійник А.С., Олійник Б.В., Петравчук А.П., Петричкович В.М., Пилипів В.М., Сікора В.С., Скасків О.Б., Філевич П.М., Черевко І.М., Шарин С.В.