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MATHEMATISCHE NACHRICHTEN

Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0,+\infty)$

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Abstract

We consider on the space ℓ_∞ polynomials that are invariant regarding permutations of the sequence variable or regarding finite permutations. Accordingly, they are trivial or factor through c_0. The analogous study, with analogous results, is carried out on $L_\infty[0,+\infty)$, replacing the permutations of $\mathbf{N}$ by measurable bijections of $[0,+\infty)$ that preserve the Lebesgue measure.

KEYWORDS

analytic function on Banach spaces, essentially bounded functions, symmetric polynomial

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46E50, 46G20

1 | INTRODUCTION

We continue the study of the analytic functions on complex Banach spaces that are invariant under the action of a certain set of operators acting on the given space. Such invariant functions have been vaguely called "symmetric" in the mathematical literature.

The topic of "symmetric" functions in infinite dimensions can be traced back to [9] where the case of the Hilbert space ℓ_2 was considered. Since then research on the matter either in sequence spaces, spaces of integrable functions or continuous functions has been done. Sometimes symmetry is so restrictive that the only analytic symmetric functions are the constant ones like for c_0, while at other times there are algebraically independent sequences that generate all symmetric polynomials, as it happens with ℓ_p, $1 \leq p < \infty$, ([7]), that also separate the points in the base space ([1]). In all these cases, "symmetric" means invariant under permutations of the variable sequence.

When turning to function spaces like $L_p([0,1])$, $p \geq 1$, a different notion of symmetry has been used: Invariance under bijections of $[0,1]$ that preserve the Lebesgue measure. There it turns out that on $L_p([0,1])$, $p < \infty$, there is finite algebraic basis of the "symmetric" polynomials ([7] and [2]). A completely different situation occurs in $L_\infty([0,1])$, where an algebraic basis is provided by the sequence of the integrals of the power functions ([6]). Other aspects of the theme have been treated in these references as well as in [3], [4] and [5].

Here we focus on ℓ_∞ and $L_\infty[0,+\infty)$. In the case of ℓ_∞ we deal with either the set of operators arising from all permutations of $\mathbf{N}$ or the subset of those arising from finite permutations of $\mathbf{N}$, while for $L_\infty[0,+\infty)$ we replace the permutations of $\mathbf{N}$ by measurable bijections of $[0,+\infty)$ that preserve the Lebesgue measure and by the subset of those that are eventually the identity, respectively. Accordingly, we have both the *symmetric* and the *finitely symmetric* cases. These are quite different since all symmetric polynomials on ℓ_∞ and $L_\infty[0,+\infty)$ are trivial, while the algebra of all finitely symmetric analytic functions on ℓ_∞ turns to be identified with the algebra of analytic (not necessarily symmetric) functions on the quotient space ℓ_∞/c_0. Realize in passing how different the situation in $L_\infty[0,1]$ and $L_\infty[0,+\infty)$ is. In Section 5 we study the algebra of all finitely symmetric analytic functions on $L_\infty[0,+\infty)$ that can be described in an analogous way to that of ℓ_∞, see Corollary 6.7. Our results stress the presumable fact that on a given Banach space, different meanings attributed to "symmetry" lead to drastically distinct results.

Nevertheless, this is not always so: there is no difference for symmetric and finitely symmetric polynomials on c_0, since they are all trivial.

2 | PRELIMINARIES

A mapping $P : X \to Y$, where X and Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, is called an n-homogeneous polynomial if there exists an n-linear symmetric mapping $A_P : X^n \to Y$ such that

$$P(x) = A_P(\underbrace{x, \dots, x}_{n})$$

for every $x \in X$. Here "symmetric" means that $A_P\big(x_{\tau(1)}, \dots, x_{\tau(n)}\big) = A_P(x_1, \dots, x_n)$ for every permutation $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$. The mapping A_P is called the n-linear symmetric mapping associated with P.

It is known (see e.g. [8], Theorem 1.10) that A_P can be recovered from P by means of the so-called Polarization Formula:

$$A_P(x_1, \dots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n P(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n). \tag{2.1}$$

We shall use the Polynomial Formula (see [8], Theorem 1.8)

$$P(x_1 + \dots + x_k) = \sum_{n_1 + \dots + n_k = n} \frac{n!}{n_1! \dots n_k!} A_P(\underbrace{x_1, \dots, x_1}_{n_1}, \underbrace{x_2, \dots, x_2}_{n_2}, \dots, \underbrace{x_k, \dots, x_k}_{n_k}) \tag{2.2}$$

and its corollary, the Binomial Formula (see [8], Corollary 1.9)

$$P(x + y) = \sum_{m=0}^{n} \binom{n}{m} A_P(\underbrace{x, \dots, x}_{n-m}, \underbrace{y, \dots, y}_{m}). \tag{2.3}$$

Lemma 2.1. *Let $J : X \to X$ be a linear operator and let P be an n-homogeneous polynomial. Then $P(Jx) = P(x)$ for every $x \in X$ if and only if $A_P(Jx_1, \dots, Jx_n) = A_P(x_1, \dots, x_n)$ for every $x_1, \dots, x_n \in X$.*

As usual, $H_b(X)$ denotes the Fréchet space of holomorphic functions of bounded type on X, that is the space of holomorphic functions on X that are bounded on bounded sets in X endowed with the topology of uniform convergence on bounded sets.

3 | SYMMETRIC POLYNOMIALS ON ℓ_∞

A function f on ℓ_∞ is called symmetric if for every bijection $\sigma : \mathbf{N} \to \mathbf{N}$ and every $x \in \ell_\infty$

$$f(x \circ \sigma) = f(x).$$

For $E \subset \mathbf{N}$ let us denote $\mathbf{1}_E$ the sequence $(x(1), \dots, x(m), \dots)$ such that

$$x(m) = \begin{cases} 1, & \text{if } m \in E, \\ 0, & \text{if } m \in \mathbf{N} \setminus E. \end{cases}$$

For an infinite set $E \subset \mathbf{N}$ we denote υ_E an increasing bijection from $\mathbf{N}$ to E.

Proposition 3.1. *Let $\varphi : \ell_\infty \to \mathbf{C}$ be a symmetric* (not necessarily linear) *function such that*

$$\varphi\big(\mathbf{1}_{E_1 \cup E_2}\big) = \varphi\big(\mathbf{1}_{E_1}\big) + \varphi\big(\mathbf{1}_{E_2}\big) \tag{3.1}$$

for every disjoint sets $E_1, E_2 \subset \mathbf{N}$. Then $\varphi(\mathbf{1}_E) = 0$ for every $E \subset \mathbf{N}$.

Proof. Let F and F_1 be infinite subsets of $\mathbf{N}$ such that $\mathbf{N}\setminus F$ and $\mathbf{N}\setminus F_1$ are also infinite. Let us show that $\varphi(\mathbf{1}_F)=\varphi(\mathbf{1}_{F_1})$. Note that the mapping

$$\sigma_{F,F_1}(m)=\begin{cases} v_{F_1}(v_F^{-1}(m)), & \text{if } m\in F,\\ v_{\mathbf{N}\setminus F_1}(v_{\mathbf{N}\setminus F}^{-1}(m)), & \text{if } m\in \mathbf{N}\setminus F,\end{cases}$$

is a bijection from $\mathbf{N}$ to $\mathbf{N}$ such that $\sigma_{F,F_1}(F)=F_1$ and $\sigma_{F,F_1}(\mathbf{N}\setminus F)=\mathbf{N}\setminus F_1$. Therefore $\mathbf{1}_F=\mathbf{1}_{F_1}\circ\sigma_{F,F_1}$. By the symmetry of φ,

$$\varphi(\mathbf{1}_F)=\varphi(\mathbf{1}_{F_1}). \tag{3.2}$$

Let A be an infinite subset of $\mathbf{N}$ such that $\mathbf{N}\setminus A$ is also infinite. We check that $\varphi(\mathbf{1}_A)=0$. Let A_1 and A_2 be disjoint infinite subsets of A such that $A=A_1\cup A_2$. Then, by (3.2),

$$\varphi(\mathbf{1}_A)=\varphi(\mathbf{1}_{A_1})=\varphi(\mathbf{1}_{A_2}).$$

On the other hand, by (3.1),

$$\varphi(\mathbf{1}_A)=\varphi(\mathbf{1}_{A_1})+\varphi(\mathbf{1}_{A_2}).$$

Therefore

$$\varphi(\mathbf{1}_A)=0. \tag{3.3}$$

Let B be an arbitrary infinite subset of $\mathbf{N}$. Let us see that $\varphi(\mathbf{1}_B)=0$. Let B_1 and B_2 be disjoint infinite subsets of B such that $B=B_1\cup B_2$. Then $\mathbf{N}\setminus B_1$ and $\mathbf{N}\setminus B_2$ are infinite. Therefore, by (3.3), $\varphi(\mathbf{1}_{B_1})=0$ and $\varphi(\mathbf{1}_{B_2})=0$. By (3.1),

$$\varphi(\mathbf{1}_B)=\varphi(\mathbf{1}_{B_1})+\varphi(\mathbf{1}_{B_2}).$$

Thus,

$$\varphi(\mathbf{1}_B)=0. \tag{3.4}$$

Let C be a finite subset of $\mathbf{N}$. Then, by (3.1),

$$\varphi(\mathbf{1}_{\mathbf{N}})=\varphi(\mathbf{1}_C)+\varphi(\mathbf{1}_{\mathbf{N}\setminus C}).$$

Since $\mathbf{N}$ and $\mathbf{N}\setminus C$ are both infinite, by (3.4), $\varphi(\mathbf{1}_{\mathbf{N}})=0$ and $\varphi(\mathbf{1}_{\mathbf{N}\setminus C})=0$. Therefore, $\varphi(\mathbf{1}_C)=0$. □

Theorem 3.2. *Let $P:\ell_\infty\to\mathbf{C}$ be a symmetric continuous n-homogeneous polynomial. Then $P=0$.*

Proof. We proceed by induction on n. In the case $n=1$ the polynomial P is a symmetric continuous linear functional. Let

$$x=\sum_{j=1}^{N}a_j\mathbf{1}_{B_j}, \tag{3.5}$$

where $N\in\mathbf{N}$, $a_1,\dots,a_N\in\mathbf{C}$ and $B_1,\dots,B_N$ are disjoint subsets of $\mathbf{N}$. By the linearity of P,

$$P(x)=\sum_{j=1}^{N}a_jP(\mathbf{1}_{B_j}).$$

By Proposition 3.1, $P(\mathbf{1}_{B_j})=0$. Therefore $P(x)=0$. Note that the set of sequences of the form (3.5) is dense in ℓ_∞. Therefore, by the continuity of P, $P(y)=0$ for every $y\in\ell_\infty$.

Assume that the statement of the theorem holds for every $k\in\{1,\dots,n-1\}$. We prove it for n. Let $A_P:(\ell_\infty)^n\to\mathbf{C}$ be the continuous n-linear symmetric form associated with P. By Lemma 2.1, where $J:x\mapsto x\circ\sigma$,

$$A_P(x_1\circ\sigma,\dots,x_n\circ\sigma)=A_P(x_1,\dots,x_n) \tag{3.6}$$

for every $x_1,\dots,x_n\in\ell_\infty$ and for every bijection $\sigma:\mathbf{N}\to\mathbf{N}$.

Lemma 3.3. *Let $F_1, \dots, F_l$ be disjoint subsets of* $\mathbf{N}$*, where $2 \le l \le n$. Then*

$$A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_l}, \dots, \mathbf{1}_{F_l}}_{k_l}\big) = 0,$$

where $k_1, \dots, k_l \in \mathbf{N}$ such that $k_1 + \dots + k_l = n$.

Proof. Without loss of generality we can assume that the set $\Omega = \mathbf{N} \setminus \bigcup_{s=1}^{l-1} F_s$ is infinite. Let $w : \Omega \to \mathbf{N}$ be a bijection. Let

$$\widehat{y}(m) = \begin{cases} y(w(m)), & \text{if } m \in \Omega, \\ 0, & \text{if } m \in \mathbf{N} \setminus \Omega, \end{cases}$$

for $y \in \ell_\infty$. Let us define a mapping $Q : \ell_\infty \to \mathbf{C}$ by

$$Q : y \mapsto A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_{l-1}}, \dots, \mathbf{1}_{F_{l-1}}}_{k_{l-1}}, \underbrace{\widehat{y}, \dots, \widehat{y}}_{k_l}\big).$$

Note that Q is a continuous k_l-homogeneous polynomial. Let us show that Q is symmetric. Let $\sigma : \mathbf{N} \to \mathbf{N}$ be a bijection. Note that

$$\widehat{y \circ \sigma} = \widehat{y} \circ \widetilde{\sigma},$$

where $\widetilde{\sigma} : \mathbf{N} \to \mathbf{N}$ is defined by

$$\widetilde{\sigma}(m) = \begin{cases} w^{-1}(\sigma(w(m))), & \text{if } m \in \Omega, \\ m, & m \in \mathbf{N} \setminus \Omega. \end{cases}$$

Evidently, $\widetilde{\sigma}$ is a bijection. Since $\widetilde{\sigma}(m) = m$ for $m \in \mathbf{N} \setminus \Omega$, it follows that $\mathbf{1}_{F_s} \circ \widetilde{\sigma} = \mathbf{1}_{F_s}$ for $s \in \{1, \dots, l-1\}$. Therefore

$$\begin{aligned} Q(y \circ \sigma) &= A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_{l-1}}, \dots, \mathbf{1}_{F_{l-1}}}_{k_{l-1}}, \underbrace{\widehat{y \circ \sigma}, \dots, \widehat{y \circ \sigma}}_{k_l}\big) \\ &= A_P\big(\underbrace{\mathbf{1}_{F_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{F_1} \circ \widetilde{\sigma}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_{l-1}} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{F_{l-1}} \circ \widetilde{\sigma}}_{k_{l-1}}, \underbrace{\widehat{y} \circ \widetilde{\sigma}, \dots, \widehat{y} \circ \widetilde{\sigma}}_{k_l}\big). \end{aligned}$$

By (3.6),

$$\begin{aligned} &A_P\big(\underbrace{\mathbf{1}_{F_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{F_1} \circ \widetilde{\sigma}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_{l-1}} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{F_{l-1}} \circ \widetilde{\sigma}}_{k_{l-1}}, \underbrace{\widehat{y} \circ \widetilde{\sigma}, \dots, \widehat{y} \circ \widetilde{\sigma}}_{k_l}\big) \\ &= A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_{l-1}}, \dots, \mathbf{1}_{F_{l-1}}}_{k_{l-1}}, \underbrace{\widehat{y}, \dots, \widehat{y}}_{k_l}\big) = Q(y). \end{aligned}$$

Hence, $Q(y \circ \sigma) = Q(y)$. Thus, Q is a continuous k_l-homogeneous symmetric polynomial. Since $k_l < n$, it follows that $Q = 0$ by the induction hypothesis. Let $H = w(F_l)$. Then $\widehat{\mathbf{1}_H} = \mathbf{1}_{F_l}$. Therefore

$$Q(\mathbf{1}_H) = A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_l}, \dots, \mathbf{1}_{F_l}}_{k_l}\big).$$

Thus,

$$A_P\big(\underbrace{\mathbf{1}_{F_1}, \dots, \mathbf{1}_{F_1}}_{k_1}, \dots, \underbrace{\mathbf{1}_{F_l}, \dots, \mathbf{1}_{F_l}}_{k_l}\big) = 0.$$

□

Let E_1 and E_2 be disjoint subsets of $\mathbf{N}$. By the Binomial formula (2.3),

$$P(\mathbf{1}_{E_1\cup E_2}) = P(\mathbf{1}_{E_1}) + \sum_{j=1}^{n-1} \frac{n!}{j!(n-j)!} A_P\big(\underbrace{\mathbf{1}_{E_1},\dots,\mathbf{1}_{E_1}}_{n-j},\underbrace{\mathbf{1}_{E_2},\dots,\mathbf{1}_{E_2}}_{j}\big) + P(\mathbf{1}_{E_2}).$$

By Lemma 3.3,

$$A_P\big(\underbrace{\mathbf{1}_{E_1},\dots,\mathbf{1}_{E_1}}_{n-j},\underbrace{\mathbf{1}_{E_2},\dots,\mathbf{1}_{E_2}}_{j}\big) = 0.$$

Therefore $P(\mathbf{1}_{E_1\cup E_2}) = P(\mathbf{1}_{E_1}) + P(\mathbf{1}_{E_2})$. Thus, by Proposition 3.1,

$$P(\mathbf{1}_E) = 0 \tag{3.7}$$

for every $E \subset \mathbf{N}$.

For x of the form (3.5), by the Polynomial formula (2.2),

$$P(x) = a_1^n P(\mathbf{1}_{B_1}) + \dots + a_N^n P(\mathbf{1}_{B_n}) + \sum_{k_1+\dots+k_l=n,\ l\ge 2} \frac{n!}{k_1!\dots k_l!} A_P\big(\underbrace{\mathbf{1}_{B_1},\dots,\mathbf{1}_{B_1}}_{k_1},\dots,\underbrace{\mathbf{1}_{B_l},\dots,\mathbf{1}_{B_l}}_{k_l}\big).$$

Therefore, by (3.7) and by Lemma 3.3, $P(x) = 0$. Since the set of sequences of the form (3.5) is dense in ℓ_∞ and P is continuous, it follows that $P(y) = 0$ for every $y \in \ell_\infty$. □

4 | FINITELY SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_∞

Definition 4.1. Let $\sigma : \mathbf{N} \to \mathbf{N}$ be a bijection. We call σ a *finite bijection* if there is $a \in \mathbf{N}$ such that the restriction of σ to $\{a, a+1, \dots\}$ is the identity map. A function f on ℓ_∞ is called *finitely symmetric* if

$$f(x\circ\sigma) = f(x)$$

for every finite bijection $\sigma : \mathbf{N} \to \mathbf{N}$ and for every $x \in \ell_\infty$.

Note that there are a lot of finitely symmetric analytic functions on ℓ_∞. For example, if $\mathcal{U}$ is a free ultrafilter on $\mathbf{N}$ and g an entire function on $\mathbf{C}$, then

$$g(x) = \lim_{\mathcal{U}} g(x_n), \qquad x = (x_n) \in \ell_\infty,$$

is a finitely symmetric entire function of bounded type on ℓ_∞. Also every Banach limit is a finitely symmetric linear functional on ℓ_∞. Let us denote by $\mathcal{P}_{fs}(\ell_\infty)$ the algebra of all finitely symmetric polynomials and $H_{bfs}(\ell_\infty)$ the algebra of all finitely symmetric entire functions of bounded type on ℓ_∞.

Proposition 4.2. *Let $f \in H_{bfs}(\ell_\infty)$. Then the restriction of f to c_0 is a constant function.*

Proof. The restriction of f to c_0 is finitely symmetric. Since c_0 is separable and f is continuous, it follows (see [7, Section 1]) that the restriction of f to c_0 is symmetric. But it is well-known that there are no nontrivial symmetric analytic functions on c_0. See [7, Theorem 1.1]. □

Theorem 4.3. *An entire function $f \in H_b(\ell_\infty)$ is finitely symmetric if and only if it factors through c_0, that is, there is $\widetilde{f} \in H_b(\ell_\infty/c_0)$ such that $f = \widetilde{f}\circ Q$, where Q is the quotient map from ℓ_∞ to ℓ_∞/c_0.*

Proof. For every finite permutation $\sigma : \mathbf{N} \to \mathbf{N}$ and $x \in \ell_\infty$ we have $x - x\circ\sigma \in c_0$ and so $Q(x) = Q(x\circ\sigma)$, hence $\widetilde{f}\circ Q(x) = \widetilde{f}\circ Q(x\circ\sigma)$.

In order to prove the reverse statement, it is enough to show that

$$P(x+y) = P(x)$$

for every continuous finitely symmetric n-homogeneous polynomial $P : \ell_\infty \to \mathbf{C}$ and for every $x \in \ell_\infty$ and $y \in c_0$.

Let A_P be the continuous n-linear symmetric form associated with P. By the Binomial formula (2.3),

$$P(x+y) = P(x) + P(y) + \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} A_P\big(\underbrace{y,\dots,y}_{k},\underbrace{x,\dots,x}_{n-k}\big).$$

By Proposition 4.2, $P(y)=0$. We will prove that $A_P(\underbrace{y,\dots,y}_{k},\underbrace{x,\dots,x}_{n-k}) = 0$ for $k \in \{1,\dots,n-1\}$.

First we assume that y has a finite support, that is, $y \in c_{00}$. Let $K = \max\{j \in \mathbf{N} : y(j) \neq 0\}$ and $\Omega_0 = \{1,\dots,K\}$. Let $\Omega_1,\dots,\Omega_n$ be some disjoint infinite sets such that $\mathbf{N}\setminus\Omega_0 = \bigcup_{j=1}^n \Omega_j$. For $j \in \{0,\dots,n\}$ let us define the sequences

$$x_j(m) = \begin{cases} x(m), & \text{if } m \in \Omega_j,\\ 0, & \text{if } m \in \mathbf{N}\setminus\Omega_j. \end{cases}$$

Since $x = \sum_{j=0}^n x_j$, it follows that

$$A_P\big(\underbrace{y,\dots,y}_{k},\underbrace{x,\dots,x}_{n-k}\big) = \sum_{j_1=0}^{n} \cdots \sum_{j_{n-k}=0}^{n} A_P\big(\underbrace{y,\dots,y}_{k}, x_{j_1},\dots,x_{j_{n-k}}\big).$$

Let $j_1,\dots,j_{n-k} \in \{0,\dots,n\}$. Let us prove that $A_P(\underbrace{y,\dots,y}_{k}, x_{j_1},\dots,x_{j_{n-k}}) = 0$. Without loss of generality we can assume that

$j_1 = \cdots = j_{k_0} = 0, j_{k_0+1} = \cdots = j_{k_0+k_1} = 1,\dots, j_{k_0+\ldots+k_{l-1}+1} = \cdots = j_{k_0+\cdots+k_l} = l$, where $l \in \{0,\dots,n-k\}, k_0 \geq 0, k_1,\dots, k_l \geq 1$ (in the case $l \geq 1$) and $k_0 + k_1 + \cdots + k_l = n-k$. Let $w' : \Omega_{l+1} \to \mathbf{N}\setminus\Omega_0$ be an increasing bijection. We define a bijection $w : \Omega_0 \cup \Omega_{l+1} \to \mathbf{N}$ by

$$w(m) = \begin{cases} m, & \text{if } m \in \Omega_0,\\ w'(m), & \text{if } m \in \Omega_{l+1}. \end{cases}$$

For $z \in \ell_\infty$ let

$$\widehat{z}(m) = \begin{cases} (z\circ w)(m), & \text{if } m \in \Omega_0 \cup \Omega_{l+1},\\ 0, & \text{if } m \in \mathbf{N}\setminus\big(\Omega_0\cup\Omega_{l+1}\big). \end{cases}$$

Let $B : (\ell_\infty)^{k+k_0} \to \mathbf{C}$,

$$B : (z_1,\dots,z_{k+k_0}) \mapsto A_P(\widehat{z_1},\dots,\widehat{z_{k+k_0}},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l}).$$

Evidently, B is a continuous symmetric $(k+k_0)$-linear form. For each finite bijection $\sigma : \mathbf{N}\to\mathbf{N}$, we construct a finite bijection $\widetilde{\sigma} : \mathbf{N}\to\mathbf{N}$ according to

$$\widetilde{\sigma}(m) = \begin{cases} \big(w^{-1}\circ\sigma\circ w\big)(m), & \text{if } m \in \Omega_0\cup\Omega_{l+1},\\ m, & \text{if } m \in \mathbf{N}\setminus\big(\Omega_0\cup\Omega_{l+1}\big). \end{cases}$$

It can be checked that $\widehat{z\circ\sigma} = \widehat{z}\circ\widetilde{\sigma}$ and $x_j\circ\widetilde{\sigma} = x_j$ for $z \in \ell_\infty$, and $j \in \{1,\dots,l\}$. Therefore, for every $z_1,\dots,z_{k+k_0} \in \ell_\infty$ and for every finite bijection $\sigma : \mathbf{N}\to\mathbf{N}$

$$\begin{aligned} B(z_1\circ\sigma,\dots,z_{k+k_0}\circ\sigma) &= A_P\big(\widehat{z_1}\circ\widetilde{\sigma},\dots,\widehat{z_{k+k_0}}\circ\widetilde{\sigma},\underbrace{x_1\circ\widetilde{\sigma},\dots,x_1\circ\widetilde{\sigma}}_{k_1},\dots,\underbrace{x_l\circ\widetilde{\sigma},\dots,x_l\circ\widetilde{\sigma}}_{k_l}\big)\\ &= A_P\big(\widehat{z_1},\dots,\widehat{z_{k+k_0}},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l}\big) = B\big(z_1,\dots,z_{k+k_0}\big). \end{aligned}$$

Thus, the restriction of B to the diagonal is a continuous finitely symmetric $(k+k_0)$-homogeneous polynomial. By Proposition 4.2, $B(z,\dots,z)=0$ for every $z\in c_{00}$. By the Polarization formula (2.1), $B(z_1,\dots,z_{k+k_0})=0$ for every $z_1,\dots,z_{k+k_0}\in c_{00}$. Since y and x_0 belong to c_{00}, it follows that $B(\underbrace{y,\dots,y}_{k},\underbrace{x_0,\dots,x_0}_{k_0})=0$, i.e.

$$A_P(\underbrace{\hat{y},\dots,\hat{y}}_{k},\underbrace{\hat{x}_0,\dots,\hat{x}_0}_{k_0},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l})=0.$$

Note that $\hat{y}=y$ and $\hat{x}_0=x_0$. Therefore,

$$A_P(\underbrace{y,\dots,y}_{k},\underbrace{x_0,\dots,x_0}_{k_0},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l})=0.$$

Hence, $P(x+y)=P(x)$ for every $x\in\ell_\infty$ and $y\in c_{00}$. Since each element in c_0 can be approximated by elements with finite support and by the continuity of P we have $P(x+y)=P(x)$ for every $y\in c_0$, $x\in\ell_\infty$. ☐

Let $M_b(\ell_\infty)$, $M_{bfs}(\ell_\infty)$ and $M_b(\ell_\infty/c_0)$ be the spectrum of $H_b(\ell_\infty)$, $H_{bfs}(\ell_\infty)$ and $H_b(\ell_\infty/c_0)$, respectively. Recall that for a given Fréchet algebra $\mathcal{A}$, the spectrum, $M(\mathcal{A})$, is the set of all continuous scalar-valued homomorphisms defined on $\mathcal{A}$.

Corollary 4.4. *The algebra of finitely symmetric entire functions of bounded type on ℓ_∞ is isomorphic to $H_b(\ell_\infty/c_0)$. Moreover, the mapping $\nu\in M_b(\ell_\infty)\mapsto \nu\circ Q^t\in M_b(\ell_\infty/c_0)$ is onto.*

Proof. The mapping $Q^t: H_b(\ell_\infty/c_0)\to H_{bfs}(\ell_\infty)$, given by $Q^t(f)=f\circ Q$ is an algebra isomorphism.

To prove the second statement, notice that the group G of all finite permutations on $\mathbf{N}$ is the union of the finite subgroups $G_a\subset G$ of permutations that coincide with the identity on $[a,+\infty)\cap\mathbf{N}$. So, the assumptions of [2, Theorem 2.5 and Corollary 2.7] are fulfilled and consequently, the mapping $M_b(\ell_\infty)\xrightarrow{\rho} M_{bfs}(\ell_\infty)$ defined by taking the restriction to $H_{bfs}(\ell_\infty)$ is onto. Hence, given $\mu\in M_b(\ell_\infty/c_0)\approx M_{bfs}(\ell_\infty)$, there is $\nu\in M_b(\ell_\infty)$ such that $\mu=\nu_{|H_{bfs}(\ell_\infty)}=\nu\circ Q^t$. ☐

Remark 4.5. Since there is in ℓ_∞/c_0 a (necessarily) complemented copy of ℓ_∞, (see for instance [10]) with projection, say, π, every $f\in H_b(\ell_\infty)$ gives rise to $f\circ\pi$, a finitely symmetric analytic function on ℓ_∞.

5 | SYMMETRIC POLYNOMIALS ON $L_\infty[0,+\infty)$

Let Ω be a Lebesgue measurable subset of $[0,+\infty)$. Let $L_\infty(\Omega)$ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on Ω with norm

$$\|x\|_\infty=\operatorname*{ess\,sup}_{t\in\Omega}|x(t)|.$$

Let Ξ_Ω be the set of all measurable bijections of Ω that preserve the measure.

A function $F: L_\infty(\Omega)\to\mathbf{C}$ is called *symmetric* if for every $x\in L_\infty(\Omega)$ and every $\sigma\in\Xi_\Omega$

$$F(x\circ\sigma)=F(x).$$

Let us denote $\mathcal{P}_s({}^nL_\infty(\Omega))$ the Banach space of all continuous n-homogeneous symmetric polynomials on $L_\infty(\Omega)$. We shall prove that if $\mu(\Omega)=+\infty$, then $\mathcal{P}_s({}^nL_\infty(\Omega))=\{0\}$ for every $n\in\mathbf{N}$. First, we prove some auxiliary results.

Let $D=\bigcup_{k=1}^\infty[\alpha_k,\beta_k)$, where $0\le\alpha_1<\beta_1\le\alpha_2<\beta_2\le\cdots$, such that $\mu(D)=+\infty$. We define the mapping $\delta_D:[0,+\infty)\to D$ in the following way. For $t\in[0,+\infty)$ there exists $m\in\mathbf{N}$ such that $\sum_{k=1}^{m-1}(\beta_k-\alpha_k)\le t<\sum_{k=1}^{m}(\beta_k-\alpha_k)$. We set

$$\delta_D(t)=\alpha_m+t-\sum_{k=1}^{m-1}(\beta_k-\alpha_k). \tag{5.1}$$

It is easy to check that δ_D is a measure preserving bijection.

Let us denote $\Delta_1=\bigcup_{k=1}^\infty[2k-2,2k-1)$ and $\Delta_2=\bigcup_{k=1}^\infty[2k-1,2k)$.

For every $E \subset [0,+\infty)$ let

$$\mathbf{1}_E(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\mathbf{1}_E = \mathbf{1}_{\sigma(E)} \circ \sigma$$

a.e. on $[0,+\infty)$ for every measurable set $E \subset [0,+\infty)$ and for every $\sigma \in \Xi_{[0,+\infty)}$.

Proposition 5.1. *For every measurable set $E \subset [0,+\infty)$ there exists $\sigma_E \in \Xi_{[0,+\infty)}$ such that*

$$\mathbf{1}_E = \mathbf{1}_\Delta \circ \sigma_E$$

a.e. on $[0,+\infty)$, where

$$\Delta = \begin{cases} [0,\mu(E)), & \text{if } \mu(E) < +\infty, \\ [\mu([0,+\infty) \setminus E), +\infty), & \text{if } \mu(E) = +\infty \textit{ and } \mu([0,+\infty)\setminus E) < +\infty, \\ \Delta_1, & \text{if } \mu(E) = +\infty \textit{ and } \mu([0,+\infty)\setminus E) = +\infty. \end{cases} \tag{5.2}$$

Proof. By [6, Proposition 2.2], for every $n \in \mathbf{N}$ there exists $\sigma_n \in \Xi_{[n-1,n]}$ such that

$$\mathbf{1}_{E\cap[n-1,n]} = \mathbf{1}_{[n-1,n-1+a_n)} \circ \sigma_n$$

a.e. on $[n-1,n]$, where $a_n = \mu(E \cap [n-1,n])$. Let $\sigma' : [0,+\infty) \to [0,+\infty)$, $\sigma'(t) = \sigma_n(t)$ for $t \in [n-1,n]$, where $n \in \mathbf{N}$. Then $\sigma' \in \Xi_{[0,+\infty)}$ and

$$\mathbf{1}_E = \mathbf{1}_{\bigcup_{n=1}^\infty [n-1,n-1+a_n)} \circ \sigma' \tag{5.3}$$

a.e. on $[0,+\infty)$.

Let $b_n = \sum_{k=1}^n a_k$, $b_0 = 0$, $c_n = \sum_{k=1}^n (1-a_k)$ and $c_0 = 0$. We define a mapping $\sigma'' : [0,+\infty) \to [0,+\infty)$ in the following way. If $\mu(E) < +\infty$, then we set

$$\sigma''(t) = \begin{cases} b_{n-1} + t - (n-1), & \text{if } t \in [n-1, n-1+a_n), n \in \mathbf{N}, \\ \mu(E) + c_{n-1} + t - (n-1+a_n), & \text{if } t \in [n-1+a_n, n), n \in \mathbf{N}. \end{cases}$$

If $\mu(E) = +\infty$ and $\mu([0,+\infty) \setminus E) < +\infty$, then we set

$$\sigma''(t) = \begin{cases} \mu([0,+\infty)\setminus E) + b_{n-1} + t - (n-1), & \text{if } t \in [n-1, n-1+a_n), n \in \mathbf{N}, \\ c_{n-1} + t - (n-1+a_n), & \text{if } t \in [n-1+a_n, n), n \in \mathbf{N}. \end{cases}$$

If $\mu(E) = +\infty$ and $\mu([0,+\infty) \setminus E) = +\infty$, then we set

$$\sigma''(t) = \begin{cases} \delta_{\Delta_1}(b_{n-1} + t - (n-1)), & \text{if } t \in [n-1, n-1+a_n), n \in \mathbf{N}, \\ \delta_{\Delta_2}(c_{n-1} + t - (n-1+a_n)), & \text{if } t \in [n-1+a_n, n), n \in \mathbf{N}, \end{cases}$$

where δ_{Δ_1} and δ_{Δ_2} are defined by (5.1). In each case

$$\sigma''\left(\bigcup_{n=1}^\infty [n-1, n-1+a_n)\right) = \Delta,$$

where Δ is defined by (5.2). Therefore,

$$\mathbf{1}_{\bigcup_{n=1}^\infty [n-1,n-1+a_n)} = \mathbf{1}_\Delta \circ \sigma'' \tag{5.4}$$

a.e. on $[0,+\infty)$.

By (5.3) and (5.4),

$$\mathbf{1}_E = \mathbf{1}_\Delta \circ \sigma_E$$

a.e. on $[0,+\infty)$, where $\sigma_E = \sigma'' \circ \sigma'$. $\square$

Proposition 5.2. *For every measurable set $E \subset [0,+\infty)$ and for every continuous symmetric polynomial $P : L_\infty[0,+\infty) \to \mathbf{C}$,*

$$P(\mathbf{1}_E) = 0.$$

Proof. For $\alpha > 0$ let S_α be the subspace of $L_\infty[0,+\infty)$ of all functions of the form

$$x = \sum_{n=1}^{\infty} z_n \mathbf{1}_{[\alpha(n-1),\alpha n)},$$

where $(z_1,\dots,z_n,\dots) \in \ell_\infty$. The space S_α is isometrically isomorphic to ℓ_∞. Therefore, the restriction of P to S_α is equal to zero. Let E be a measurable subset of $[0,+\infty)$. By Proposition 5.1, there exists $\sigma_E \in \Xi_{[0,+\infty)}$ such that

$$\mathbf{1}_E = \mathbf{1}_\Delta \circ \sigma_E$$

a.e. on $[0,+\infty)$, where Δ is defined by (5.2). Since P is symmetric,

$$P(\mathbf{1}_E) = P(\mathbf{1}_\Delta).$$

If $\mu(E) < +\infty$, then $\mathbf{1}_\Delta = \mathbf{1}_{[0,\mu(E))} \in S_{\mu(E)}$. If $\mu(E) = +\infty$ and $\mu([0,+\infty) \setminus E) < +\infty$, then $\mathbf{1}_\Delta = \mathbf{1}_{[\mu([0,+\infty)\setminus E),+\infty)} \in S_{\mu([0,+\infty)\setminus E)}$. If $\mu(E) = +\infty$ and $\mu([0,+\infty) \setminus E) = +\infty$, then $\mathbf{1}_\Delta = \mathbf{1}_{\Delta_1} \in S_1$. Therefore, in each case $P(\mathbf{1}_\Delta) = 0$. $\square$

Proposition 5.3. *Let Ω be a measurable subset of $[0,+\infty)$ such that $\mu(\Omega) = +\infty$. Then the space $\mathcal{P}_s({}^nL_\infty(\Omega))$ is isometrically isomorphic to the space $\mathcal{P}_s({}^nL_\infty[0,+\infty))$.*

Proof. By Proposition 5.1, there exists $\sigma_\Omega \in \Xi_{[0,+\infty)}$ such that $\mathbf{1}_\Omega = \mathbf{1}_\Delta \circ \sigma_\Omega$ a.e. on $[0,+\infty)$, where

$$\Delta = \begin{cases} [\mu([0,+\infty) \setminus \Omega),+\infty), & \text{if } \mu([0,+\infty) \setminus \Omega) < +\infty, \\ \Delta_1, & \text{if } \mu([0,+\infty) \setminus \Omega) = +\infty. \end{cases}$$

Let us define $\gamma : [0,+\infty) \to \Omega$ by $\gamma = \sigma_\Omega^{-1}|_\Delta \circ \delta_\Delta$. The mapping γ is a measure preserving bijection.

Let $\beta : L_\infty(\Omega) \to L_\infty[0,+\infty)$, $\beta : x \mapsto x \circ \gamma$. The mapping β is an isometric isomorphism.

Let $\alpha : \mathcal{P}_s({}^nL_\infty[0,+\infty)) \to \mathcal{P}_s({}^nL_\infty(\Omega))$, $\alpha : P \mapsto P \circ \beta$. Evidently, $\alpha(P)$ is a continuous n-homogeneous polynomial for every $P \in \mathcal{P}_s({}^nL_\infty[0,+\infty))$. Let us prove that $\alpha(P)$ is symmetric. Let $\sigma \in \Xi_\Omega$. By the definition, for $x \in L_\infty(\Omega)$,

$$\alpha(P)(x \circ \sigma) = P(\beta(x \circ \sigma)).$$

Note that

$$\beta(x \circ \sigma) = \beta(x) \circ \upsilon(\sigma),$$

where $\upsilon : \Xi_\Omega \to \Xi_{[0,+\infty)}$, $\upsilon : \sigma \mapsto \gamma^{-1} \circ \sigma \circ \gamma$. Therefore,

$$\alpha(P)(x \circ \sigma) = P(\beta(x) \circ \upsilon(\sigma)).$$

By the symmetry of P, $P(\beta(x) \circ \upsilon(\sigma)) = P(\beta(x))$. Thus, $\alpha(P)$ is symmetric. Similarly it can be checked that $\alpha^{-1}(Q)$ is a continuous n-homogeneous symmetric polynomial on $L_\infty[0,+\infty)$ for every $Q \in \mathcal{P}_s({}^nL_\infty(\Omega))$. Since β is an isometric isomorphism, it follows that α is an isometric isomorphism too. $\square$

We obtain the following statement from Propositions 5.2 and 5.3.

Corollary 5.4. *Let Ω be a measurable subset of $[0,+\infty)$ such that $\mu(\Omega) = +\infty$. Then for every measurable set $E \subset \Omega$ and for every continuous symmetric polynomial $P : L_\infty(\Omega) \to \mathbf{C}$,*

$$P(\mathbf{1}_E) = 0.$$

Theorem 5.5. *Let P be a continuous n-homogeneous symmetric polynomial on $L_\infty[0,+\infty)$. Then $P = 0$.*

Proof. Let $A_P : (L_\infty[0,+\infty))^n \to \mathbf{C}$ be the continuous n-linear symmetric form associated with P. By Lemma 2.1, where $J : x \mapsto x \circ \sigma$,

$$A_P(x_1 \circ \sigma, \dots, x_n \circ \sigma) = A_P(x_1, \dots, x_n) \tag{5.5}$$

for every $x_1, \dots, x_n \in L_\infty[0,+\infty)$ and for every $\sigma \in \Xi_{[0,+\infty)}$.

Let us prove that $P(x) = 0$ for every simple measurable function $x \in L_\infty[0,+\infty)$. Let $x = \sum_{j=1}^m z_j \mathbf{1}_{E_j}$, where $z_1, \dots, z_m \in \mathbf{C}$ and $E_1, \dots, E_m$ are disjoint measurable subsets of $[0,+\infty)$. By the n-linearity of A_P,

$$P(x) = A_P(x, \dots, x) = \sum_{j_1=1}^m \dots \sum_{j_n=1}^m z_{j_1} \dots z_{j_n} A_P\left(\mathbf{1}_{E_{j_1}}, \dots, \mathbf{1}_{E_{j_n}}\right).$$

Let us prove that $A_P\left(\mathbf{1}_{E_{j_1}}, \dots, \mathbf{1}_{E_{j_n}}\right) = 0$ for every $j_1, \dots, j_n \in \{1, \dots, m\}$. Without loss of generality we can assume that

$$j_1 = \dots = j_{k_1} = 1,\ j_{k_1+1} = \dots = j_{k_1+k_2} = 2, \dots, j_{k_1+\dots+k_{l-1}+1} = \dots = j_{k_1+\dots+k_l} = l,$$

where $l \in \{1, \dots, m\}$, $k_1 + \dots + k_l = n$, and that $\mu(\Omega) = +\infty$, where

$$\Omega = \begin{cases} [0,+\infty), & \text{if } l = 1, \\ [0,+\infty) \setminus \bigcup_{s=1}^{l-1} E_s, & \text{if } l > 1. \end{cases}$$

For $y \in L_\infty(\Omega)$ we set

$$\widehat{y}(t) = \begin{cases} y(t), & \text{if } t \in \Omega, \\ 0, & \text{if } t \in [0,+\infty) \setminus \Omega. \end{cases}$$

The mapping $Q : L_\infty(\Omega) \to \mathbf{C}$, defined by

$$Q : y \mapsto A_P\left(\mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_{l-1}}, \dots, \mathbf{1}_{E_{l-1}}, \widehat{y}, \dots, \widehat{y}\right)$$

is a continuous k_l-homogeneous polynomial. Let us show that Q is symmetric. Let $\sigma \in \Xi_\Omega$. We set

$$\widetilde{\sigma}(t) = \begin{cases} \sigma(t), & \text{if } t \in \Omega, \\ t, & \text{if } t \in [0,+\infty) \setminus \Omega. \end{cases}$$

Note that $\widetilde{\sigma} \in \Xi_{[0,+\infty)}$ and $\mathbf{1}_{E_s} \circ \widetilde{\sigma} = \mathbf{1}_{E_s}$ for every $s \in \{1, \dots, l-1\}$ because $E_1, \dots, E_{l-1} \subset [0,+\infty) \setminus \Omega$. Evidently, $\widehat{y \circ \sigma} = \widehat{y} \circ \widetilde{\sigma}$ for every $y \in L_\infty(\Omega)$. Therefore,

$$Q(y \circ \sigma) = A_P\left(\mathbf{1}_{E_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_{l-1}} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_{l-1}} \circ \widetilde{\sigma}, \widehat{y} \circ \widetilde{\sigma}, \dots, \widehat{y} \circ \widetilde{\sigma}\right).$$

By (5.5),

$$\begin{aligned} &A_P\left(\mathbf{1}_{E_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_1} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_{l-1}} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_{l-1}} \circ \widetilde{\sigma}, \widehat{y} \circ \widetilde{\sigma}, \dots, \widehat{y} \circ \widetilde{\sigma}\right) \\ &\quad = A_P\left(\mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_{l-1}}, \dots, \mathbf{1}_{E_{l-1}}, \widehat{y}, \dots, \widehat{y}\right). \end{aligned}$$

Therefore, $Q(y \circ \sigma) = Q(y)$. Thus, Q is symmetric.

Note that $E_l \subset \Omega$. Therefore, by Corollary 5.4, $Q(\mathbf{1}_{E_l}) = 0$, i.e.

$$A_P\left(\mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_l}, \dots, \mathbf{1}_{E_l}\right) = 0.$$

Thus, $P(x) = 0$ for every simple measurable function $x \in L_\infty[0,+\infty)$. Since the set of such functions is dense in $L_\infty[0,+\infty)$, by the continuity of P we have that $P(x) = 0$ for every $x \in L_\infty[0,+\infty)$. □

We obtain the following statement from Proposition 5.3 and Theorem 5.5.

Corollary 5.6. *Let Ω be a measurable subset of $[0,+\infty)$ such that $\mu(\Omega) = +\infty$ and let P be a continuous n-homogeneous symmetric polynomial on $L_\infty(\Omega)$. Then $P = 0$.*

6 | FINITELY SYMMETRIC ANALYTIC FUNCTIONS ON $L_\infty[0,+\infty)$

Definition 6.1. We call $\sigma \in \Xi_{[0,+\infty)}$ a *finite bijection* of $[0,+\infty)$ if there is $a \in [0,+\infty)$ such that the restriction of σ to $[a,+\infty)$ is the identity map a.e. We denote $\Xi^0_{[0,+\infty)}$ the set of all finite bijections in $\Xi_{[0,+\infty)}$.

A function f on $L_\infty[0,+\infty)$ is called *finitely symmetric* if

$$f(x \circ \sigma) = f(x), \qquad \forall \sigma \in \Xi^0_{[0,+\infty)},\ \forall x \in L_\infty[0,+\infty).$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$ and let g be an entire function on $\mathbf{C}$. Then

$$f(x) = \lim_{\mathcal{U}} \int_{[n,n+1]} g(x(t))\,dt, \qquad x \in L_\infty[0,+\infty),$$

is a finitely symmetric entire function of bounded type on $L_\infty[0,+\infty)$.

Lemma 6.2. *If $A, B \subset [0,+\infty)$ are disjoint non-null measurable sets such that $[0,+\infty) \setminus (A \cup B)$ is also non-null, then there is a measurable bijection $w : [0,+\infty) \setminus B \to [0,+\infty)$ such that $w([0,+\infty) \setminus (A \cup B))) \overset{a.e.}{=} [0,+\infty) \setminus A$ and $w(t) = t$ for $t \in A$.*

Proof. Put $C = [0,+\infty) \setminus (A \cup B)$. By considering the homeomorphism $\Lambda(x) = \frac{x}{1+x}$ from $[0,+\infty)$ onto $[0,1)$ we reduce the result to the case of the finite measure space $[0,1)$ and non-null subsets $E := \Lambda(A)$, $F := \Lambda(B)$ and $G := \Lambda(C)$. There we may use [11] to assure that there is a measurable bijection Φ of $[0,1)$ and non-trivial disjoint intervals, I_i, $i = 1,2,3$, decomposing $[0,1)$ such that $0 \in I_1 \overset{a.e.}{=} \Phi(E)$, $I_2 \overset{a.e.}{=} \Phi(F)$ and $I_3 \overset{a.e.}{=} \Phi(G)$. It is well-known that there is a measurable bijection $U : I_3 \to I_2 \cup I_3$. Now, consider the mapping

$$u(t) = \begin{cases} U(t), & \text{if } t \in I_3, \\ t, & \text{if } t \in I_1. \end{cases}$$

In this way u is a measurable bijection from $[0,1) \setminus I_2 = (I_1 \cup I_3)$ to $[0,1)$ such that

$$u\big([0,1) \setminus \big(I_1 \cup I_2\big)\big) = u\big(I_3\big) = I_2 \cup I_3.$$

If $v =: \Phi^{-1} \circ u \circ \Phi$, we get that v is a measurable bijection from $[0,1) \setminus F$ to $[0,1)$ such that

$$v\big([0,1) \setminus (E \cup F)\big) \overset{a.e.}{=} [0,1) \setminus E.$$

Finally $w := \Lambda^{-1} \circ v \circ \Lambda$ satisfies the statement. □

Let M_{00} be the space of all functions of the form

$$x = \sum_{j=1}^{N} a_j \mathbf{1}_{E_j}, \tag{6.1}$$

where $N \in \mathbf{N}$, $a_1, \dots, a_N \in \mathbf{C}$ and $E_1, \dots, E_N$ be disjoint bounded non-null measurable subsets of $[0,+\infty)$. Let M_0 be the completion of M_{00} in $L_\infty[0,+\infty)$.

Proposition 6.3. *Let $P : L_\infty[0,+\infty) \to \mathbf{C}$ be a continuous finitely symmetric n-homogeneous polynomial. Then $P(x) = 0$ for every $x \in M_0$.*

Proof. By the density of M_{00} in M_0, it suffices to prove the result for $x \in M_{00}$.

Let $a > 0$ and $y_m = 1_{[0,am)}$ for $m \in \mathbf{N}$. Note that the sequence $\{y_m\}_{m=1}^\infty$ is bounded. By the continuity of P, the sequence $\{P(y_m)\}_{m=1}^\infty$ is bounded too. Since P is finitely symmetric, it follows that

$$P\big(\mathbf{1}_{[0,a)}\big) = P\left(\mathbf{1}_{[a(k-1),ak)}\right) \text{ for every } k \in \mathbf{N} \text{ and} \tag{6.2}$$

$$P(\mathbf{1}_E) = P\big(\mathbf{1}_{[0,\mu(E))}\big) \text{ for every bounded measurable set } E \subset [0,+\infty). \tag{6.3}$$

We proceed by induction on n. In the case $n = 1$ the polynomial P is a linear functional. Therefore,

$$P(y_m) = \sum_{k=1}^{m} P\big(\mathbf{1}_{[a(k-1),ak)}\big).$$

By (6.2),

$$P(y_m) = mP\big(\mathbf{1}_{[0,a)}\big).$$

The sequence $\{P(y_m)\}_{m=1}^{\infty}$ is bounded if and only if $P\big(\mathbf{1}_{[0,a)}\big) = 0$. Since P is finitely symmetric, it follows from (6.3) that for every x of the form (6.1),

$$P(x) = \sum_{j=1}^{N} a_j P\left(\mathbf{1}_{E_j}\right) = 0.$$

Assume that the statement of the proposition holds for every $k \in \{1, \dots, n-1\}$. We prove it for n. Let A_P be the continuous n-linear symmetric form associated with P. By the n-linearity of A_P,

$$P(x) = \sum_{j_1=1}^{N} \dots \sum_{j_n=1}^{N} a_{j_1} \dots a_{j_n} A_P\left(\mathbf{1}_{E_{j_1}}, \dots, \mathbf{1}_{E_{j_n}}\right)$$

for x of the form (6.1). Let $j_1, \dots, j_n \in \{1, \dots, N\}$ be such that $j_m \neq j_s$ for some $m, s \in \{1, \dots, n\}$. Let us prove that in this case $A_P\big(\mathbf{1}_{E_{j_1}}, \dots, \mathbf{1}_{E_{j_n}}\big) = 0$. Without loss of generality we can assume that $j_1 = \dots = j_{k_1} = 1, j_{k_1+1} = \dots = j_{k_1+k_2} = 2, \dots, j_{k_1+\dots+k_{l-1}+1} = \dots = j_{k_1+\dots+k_l} = l$, where $l \geq 2, k_1, \dots, k_l \geq 1$ and $k_1 + \dots + k_l = n$. Since $\bigcup_{s=1}^{l} E_s$ is a bounded set, there exists $c > 0$ such that $\bigcup_{s=1}^{l} E_s \subset [0, c)$, and we may consider $E_1' = E_1 \cup [c, +\infty)$ so that $B = \bigcup_{s=2}^{l} E_s$ and $A = E_1'$ satisfy the assumptions in Lemma 6.2 to find a measurable bijection

$$w : [0, +\infty) \setminus \bigcup_{s=2}^{l} E_s \to [0, +\infty)$$

such that $w(t) = t$ if $t \in E_1'$ and

$$w\left([0, +\infty) \setminus \left(\bigcup_{s=2}^{l} E_s \cup E_1'\right)\right) \overset{a.e.}{=} [0, +\infty) \setminus E_1'.$$

For $z \in L_\infty[0, +\infty)$ let

$$\hat{z}(t) = \begin{cases} (z \circ w)(t), & \text{if } t \in [0, +\infty) \setminus \bigcup_{s=2}^{l} E_s, \\ 0, & \text{if } t \in \bigcup_{s=2}^{l} E_s. \end{cases}$$

Let $B : (L_\infty[0, +\infty))^{k_1} \to \mathbf{C}$,

$$B : (z_1, \dots, z_{k_1}) \mapsto A_P\big(\hat{z}_1, \dots, \widehat{z_{k_1}}, \underbrace{\mathbf{1}_{E_2}, \dots, \mathbf{1}_{E_2}}_{k_2}, \dots, \underbrace{\mathbf{1}_{E_l}, \dots, \mathbf{1}_{E_l}}_{k_l}\big).$$

Evidently, B is a continuous symmetric k_1-linear form. Let us show that

$$B(z_1 \circ \sigma, \dots, z_{k_1} \circ \sigma) = B(z_1, \dots, z_{k_1}) \text{ for } z_1, \dots, z_{k_1} \in L_\infty[0, +\infty) \text{ and } \sigma \in \Xi^0_{[0,+\infty)}.$$

Indeed: Given $\sigma \in \Xi^0_{[0,+\infty)}$, construct $\widetilde{\sigma}$ according to

$$\widetilde{\sigma}(t) = \begin{cases} \big(w^{-1} \circ \sigma \circ w\big)(t), & \text{if } t \in [0, +\infty) \setminus \bigcup_{s=2}^{l} E_s, \\ t, & \text{if } t \in \bigcup_{s=2}^{l} E_s. \end{cases}$$

Observe that $\widetilde{\sigma} \in \Xi^0_{[0,+\infty)}$ because for $t > c$, we have $t \in E_1'$, so $w(t) = t$. It can be checked that $\widehat{z \circ \sigma} = \widehat{z} \circ \widetilde{\sigma}$ for every $z \in L_\infty[0,+\infty)$ and also that $\mathbf{1}_{E_j} \circ \widetilde{\sigma} = \mathbf{1}_{E_j}$ for every $j \in \{2, \dots, l\}$. Therefore,

$$B\big(z_1 \circ \sigma, \dots, z_{k_1} \circ \sigma\big) = A_P\big(\widehat{z_1} \circ \widetilde{\sigma}, \dots, \widehat{z_{k_1}} \circ \widetilde{\sigma}, \underbrace{\mathbf{1}_{E_2} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_2} \circ \widetilde{\sigma}}_{k_2}, \dots, \underbrace{\mathbf{1}_{E_l} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_l} \circ \widetilde{\sigma}}_{k_l}\big).$$

Since

$$\begin{aligned} &A_P\big(\widehat{z_1} \circ \widetilde{\sigma}, \dots, \widehat{z_{k_1}} \circ \widetilde{\sigma}, \underbrace{\mathbf{1}_{E_2} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_2} \circ \widetilde{\sigma}}_{k_2}, \dots, \underbrace{\mathbf{1}_{E_l} \circ \widetilde{\sigma}, \dots, \mathbf{1}_{E_l} \circ \widetilde{\sigma}}_{k_l}\big) \\ &= A_P\big(\widehat{z_1}, \dots, \widehat{z_{k_1}}, \underbrace{\mathbf{1}_{E_2}, \dots, \mathbf{1}_{E_2}}_{k_2}, \dots, \underbrace{\mathbf{1}_{E_l}, \dots, \mathbf{1}_{E_l}}_{k_l}\big), \end{aligned}$$

it follows that $B\big(z_1 \circ \sigma, \dots, z_{k_1} \circ \sigma\big) = B\big(z_1, \dots, z_{k_1}\big)$. Thus, the restriction of B to the diagonal is a continuous finitely symmetric k_1-homogeneous polynomial. By the induction hypothesis, $B\big(\mathbf{1}_{E_1}, \dots, \mathbf{1}_{E_1}\big) = 0$, i.e.

$$A_P\big(\underbrace{\widehat{\mathbf{1}_{E_1}}, \dots, \widehat{\mathbf{1}_{E_1}}}_{k_1}, \underbrace{\mathbf{1}_{E_2}, \dots, \mathbf{1}_{E_2}}_{k_2}, \dots, \underbrace{\mathbf{1}_{E_l}, \dots, \mathbf{1}_{E_l}}_{k_l}\big) = 0.$$

Notice that $\widehat{\mathbf{1}_{E_1}} = \mathbf{1}_{E_1}$. Hence, for x of the form (6.1),

$$P(x) = \sum_{j=1}^{N} a_j^N P\Big(\mathbf{1}_{E_j}\Big).$$

Therefore, $P(y_m) = P\big(\sum_{k=1}^m \mathbf{1}_{[a(k-1),ak)}\big) = mP\Big(\mathbf{1}_{[0,a)}\Big)$. Since the sequence $\{P(y_m)\}_{m=1}^\infty$ is bounded, we have $P\Big(\mathbf{1}_{[0,a)}\Big) = 0$. According to (6.3), for every bounded measurable set $E \subset [0, +\infty)$, we have $P(\mathbf{1}_E) = P\Big(\mathbf{1}_{[0,\mu(E))}\Big)$ thus, $P(\mathbf{1}_E) = 0$. Hence,

$$P(x) = \sum_{j=1}^{N} a_j^n P\big(\mathbf{1}_{[0,\mu(E_j))}\big) = 0.$$

□

Proposition 6.4. *Let $P : L_\infty[0, +\infty) \to \mathbf{C}$ be a continuous finitely symmetric n-homogeneous polynomial. Then $P(x + y) = P(x)$ for every $x \in L_\infty[0, +\infty)$ and $y \in M_{00}$.*

Proof. Let A_P be the continuous n-linear symmetric form associated with P. By the Binomial formula (2.3),

$$P(x + y) = P(x) + P(y) + \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} A_P\big(\underbrace{y, \dots, y}_{k}, \underbrace{x, \dots, x}_{n-k}\big).$$

By Proposition 6.3, $P(y) = 0$. We prove that $A_P(\underbrace{y, \dots, y}_{k}, \underbrace{x, \dots, x}_{n-k}) = 0$ for $k \in \{1, \dots, n-1\}$.

Let Ω_0 be the support of $y \subset [0, a]$. Let $\Omega_1, \dots, \Omega_n$ be disjoint measurable sets such that $[0, +\infty) \setminus \Omega_0 = \bigcup_{j=1}^n \Omega_j$ and $\mu\big(\Omega_j\big) = +\infty$ for every $j \in \{1, \dots, n\}$, and define for $j \in \{0, \dots, n\}$, the functions

$$x_j(t) = \begin{cases} x(t), & \text{if } t \in \Omega_j, \\ 0, & \text{if } t \in [0, +\infty) \setminus \Omega_j. \end{cases}$$

Since $x = \sum_{j=0}^n x_j$, it follows that

$$A_P\big(\underbrace{y, \dots, y}_{k}, \underbrace{x, \dots, x}_{n-k}\big) = \sum_{j_1=0}^{n} \dots \sum_{j_{n-k}=0}^{n} A_P\big(\underbrace{y, \dots, y}_{k}, x_{j_1}, \dots, x_{j_{n-k}}\big).$$

Let $j_1,\dots,j_{n-k}\in\{0,\dots,n\}$. Let us prove that $A_P(\underbrace{y,\dots,y}_{k},x_{j_1},\dots,x_{j_{n-k}})=0$. Without loss of generality we can assume that $j_1=\dots=j_{k_0}=0$, $j_{k_0+1}=\dots=j_{k_0+k_1}=1$, $\dots$, $j_{k_0+\dots+k_{l-1}+1}=\dots=j_{k_0+\dots+k_l}=l$, where $l\in\{0,\dots,n-k\}, k_0\ge 0$, $k_1,\dots,k_l\ge 1$ (in the case $l\ge 1$) and $k_0+k_1+\dots+k_l=n-k$.

Using Lema 6.2 with $A=\Omega_0\cup(\Omega_{l+1}\cap[a,+\infty))$ and $B=[0,+\infty)\setminus(\Omega_0\cup\Omega_{l+1})$, we are led to a measurable bijection $w:\Omega_0\cup\Omega_{l+1}\to[0,+\infty)$ such that $w(t)=t$ for $t\in\Omega_0\cup(\Omega_{l+1}\cap[a,+\infty))$.

For $z\in L_\infty[0,+\infty)$ let

$$\hat z(t)=\begin{cases}(z\circ w)(t), & \text{if } t\in\Omega_0\cup\Omega_{l+1},\\ 0, & \text{if } t\in[0,+\infty)\setminus(\Omega_0\cup\Omega_{l+1}).\end{cases}$$

Let $B:(L_\infty[0,+\infty))^{k+k_0}\to\mathbf{C}$,

$$B:(z_1,\dots,z_{k+k_0})\mapsto A_P\big(\hat z_1,\dots,\widehat{z_{k+k_0}},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l}\big).$$

Evidently, B is a continuous symmetric $(k+k_0)$-linear form. For each $\sigma\in\Xi^0_{[0,+\infty)}$, we construct $\tilde\sigma\in\Xi_{[0,+\infty)}$ according to

$$\tilde\sigma=\begin{cases}\big(w^{-1}\circ\sigma\circ w\big)(t), & \text{if } t\in\Omega_0\cup\Omega_{l+1},\\ t, & \text{if } t\in[0,+\infty)\setminus(\Omega_0\cup\Omega_{l+1}).\end{cases}$$

Also $\tilde\sigma\in\Xi^0_{[0,+\infty)}$ because for $t>a$, we have $w(t)=t$.

It can be checked that $\widehat{z\circ\sigma}=\hat z\circ\tilde\sigma$ and $x_j\circ\tilde\sigma=x_j$ for $z\in L_\infty[0,+\infty)$, and $j\in\{1,\dots,l\}$. Therefore, for every $z_1,\dots,z_{k+k_0}\in L_\infty[0,+\infty)$ and $\sigma\in\Xi^0_{[0,+\infty)}$

$$\begin{aligned}B\big(z_1\circ\sigma,\dots,z_{k+k_0}\circ\sigma\big)&=A_P\big(\hat z_1\circ\tilde\sigma,\dots,\widehat{z_{k+k_0}}\circ\tilde\sigma,\underbrace{x_1\circ\tilde\sigma,\dots,x_1\circ\tilde\sigma}_{k_1},\dots,\underbrace{x_l\circ\tilde\sigma,\dots,x_l\circ\tilde\sigma}_{k_l}\big)\\&=A_P\big(\hat z_1,\dots,\widehat{z_{k+k_0}},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l}\big)=B\big(z_1,\dots,z_{k+k_0}\big).\end{aligned}$$

Thus, the restriction of B to the diagonal is a continuous finitely symmetric $(k+k_0)$-homogeneous polynomial. By Proposition 6.3, $B(z,\dots,z)=0$ for every $z\in M_{00}$. By the Polarization formula (2.1) $B(z_1,\dots,z_{k+k_0})=0$ for every $z_1,\dots,z_{k+k_0}\in M_{00}$. Since y and x_0 belong to M_{00}, it follows that $B(\underbrace{y,\dots,y}_{k},\underbrace{x_0,\dots,x_0}_{k_0})=0$, i.e.

$$A_P(\underbrace{\hat y,\dots,\hat y}_{k},\underbrace{\hat x_0,\dots,\hat x_0}_{k_0},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l})=0.$$

Note that $\hat y=y$ and $\hat x_0=x_0$. Therefore,

$$A_P(\underbrace{y,\dots,y}_{k},\underbrace{x_0,\dots,x_0}_{k_0},\underbrace{x_1,\dots,x_1}_{k_1},\dots,\underbrace{x_l,\dots,x_l}_{k_l})=0.$$

Hence, $P(x+y)=P(x)$ for every $x\in L_\infty[0,+\infty)$ and $y\in M_{00}$. □

Continuity of P implies the following corollary.

Corollary 6.5. *Let $P:L_\infty[0,+\infty)\to\mathbf{C}$ be a continuous finitely symmetric n-homogeneous polynomial. Then $P(x+y)=P(x)$ for every $x\in L_\infty[0,+\infty)$ and $y\in M_0$.*

Let Q be the quotient map from $L_\infty[0,+\infty)$ to $L_\infty[0,+\infty)/M_0$.

Corollary 6.6. *An entire function $f\in H_b(L_\infty[0,+\infty))$ is finitely symmetric if and only if it factors through M_0, that is, there is $\tilde f\in H_b(L_\infty[0,+\infty)/M_0)$ such that $f=\tilde f\circ Q$.*

Corollary 6.7. *The algebra of finitely symmetric entire functions of bounded type on $L_\infty[0,+\infty)$ is isomorphic to $H_b\big(L_\infty[0,+\infty)/M_0\big)$.*

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