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# REPRESENTATION OF SPECTRA OF ALGEBRAS OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS OF BOUNDED TYPE

The paper contains a description of a symmetric convolution of the algebra of block-symmetric analytic functions of bounded type on $\ell_1$-sum of the space $\mathbb{C}^2$. We show that the specrum of such algebra does not coincide of point evaluation functionals and we describe characters of the algebra as functions of exponential type with plane zeros.

*Key words and phrases:* algebraic basis, block-symmetric polynomials, block-symmetric analytic functions, spectrum, symmetric intertwining, symmetric convolution.

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## INTRODUCTION

In resent years there is an increasing interest to investigations of invariants of the permutation group $S_\infty$ of positive integers. This group can be represented on a Banach space $X$ with symmetric basis as a group of operators of perturbation of basis vectors. The action of this group has a natural extension to the action on the algebra $H_b(X)$ of analytic functions of bounded type on $X$. Invariants of this representation of $S_\infty$ are so-called symmetric analytic functions of bounded type on $X$. The algebras of symmetric analytic functions $H_{bs}(X)$ were investigated by many authors ([1, 2, 9]). In particular, it is known that $H_{bs}(\ell_p)$ admits an algebraic basis for $1 \le p < \infty$.

On the other hand, there are more representations of $S_\infty$ in Banach spaces. For example, if $\mathcal{X}$ is a directs sum of infinite many of "blocks" which consists of linear subspaces isomorphic each to other, then $S_\infty$ may to act as a group of permutations of the "blocks". For this case we have invariants — the algebra of block-symmetric analytic functions. Note that this algebra is much more complicated and in the general case has no algebraic basis (see e.g. [6, 12]). In the case $\dim \mathcal{X} < \infty$, block-invariant polynomials were investigated in the classical theory of invariants [5, 11].

## 1 MAIN RESULTS

Let

$$\mathcal{X}^2 = \oplus_{\ell_1}\mathbb{C}^2 = \ell_1 \otimes \mathbb{C}^2$$

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be an infinite $\ell_1$-sum of copies of Banach space $\mathbb{C}^2$. So any element $u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X}^2$ can be represented as a sequence $u = \begin{pmatrix} x \\ y \end{pmatrix} = \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \ldots \right)$, where $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathbb{C}^2$, with the norm $\|u\| = \sum\limits_{k=1}^{\infty} (|x_k| + |y_k|)$. Also, we will use notation $u(x,y)$, where $x, y \in \ell_1$, $x = \sum\limits_{k=1}^{\infty} x_k e_k$, $y = \sum\limits_{k=1}^{\infty} y_k e_k$. Here $e_k$ is the standard symmetric basis in $\ell_1$.

A polynomial $P$ on the space $\mathcal{X}^2$ is called *block-symmetric* (or *vector-symmetric*) if:

$$P\left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right) = P\left( \begin{pmatrix} x_{\sigma(1)} \\ y_{\sigma(1)} \end{pmatrix}, \ldots, \begin{pmatrix} x_{\sigma(m)} \\ y_{\sigma(m)} \end{pmatrix}, \ldots \right),$$

for every permutation $\sigma$ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{C}^2$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In [7] it was shown that the following vectors form an algebraic bases of "power" block-symmetric polynomials of $\mathcal{P}_{vs}(\mathcal{X}^2)$ :

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p}, \tag{1}$$

where $0 \le p \le n$, $(x_i, y_i) \in \mathbb{C}^2$, $i \ge 1$. Also, there is a basis of "elementary" block-symmetric polynomials:

$$R^{p,n-p}(x,y) = \sum_{\substack{i_1 < \ldots < i_p \\ j_1 < \ldots < j_{n-p} \\ i_k \neq j_l}}^{\infty} x_{i_1} \ldots x_{i_p} y_{j_1} \ldots y_{j_{n-p}}, \tag{2}$$

where $0 \le p \le n$, $n \ge 1$ and $(x_i, y_i) \in \mathbb{C}^2$.

In the finite case, generating elements of algebra of block-symmetric polynomials on the space $\mathcal{X}_m^2 = \oplus_{\ell_1}^m \mathbb{C}^2$ are algebraic dependent. In [12] was proved the following theorem.

**Theorem 1.** *For every nonsymmertic polynomial $\xi$ of a system of generating elements of $\mathcal{P}_{vs}(\mathcal{X}_m^2)$ there exist symmetric polynomials $a_k$ in this system such that*

$$\xi^{m!} - a_1 \xi^{m!-1} + \cdots + (-1)^{m!-1} a_{m!-1} \xi^1 + (-1)^{m!} a_{m!} = 0.$$

Let $\sigma$ be some permutation on the set of natural numbers $\mathbb{N}$. We denote by $T_\sigma$ the linear operator on $\mathcal{X}^2$ associated with $\sigma$ by the formula

$$T_\sigma\Big( \sum_{k=1}^{\infty} x_k e_k, \sum_{k=1}^{\infty} y_k e_k \Big) = \Big( \sum_{k=1}^{\infty} x_{\sigma(k)} e_k, \sum_{k=1}^{\infty} y_{\sigma(k)} e_k \Big).$$

For any $(x,y), (z,t) \in \mathcal{X}^2$ we denote $(x,y) \sim (z,t)$ if there exists a permutation $\sigma$ on $\mathbb{N}$ such that $(x,y) = T_\sigma(z,t)$.

**Theorem 2.** *Let $(x,y), (z,t) \in \mathcal{X}^2$ and $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, where $0 \le p \le i$ for every $i \ge 1$. Then $(x,y) \sim (z,t)$.*

*Proof.* Let $G(x)$ be a symmetric polynomial of degree $n$ in the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ on $\ell_1$. We set $P(x,y) = G(x+jy)$, where $0 \le j \le n$, $(x,y) \in \mathcal{X}^2$. Obviously, $P(x,y)$ is a block-symmetric polynomial. In [13] it was proved that the block-symmetric polynomial $P(x,y)$ will be represented as an algebraic combination of $F_k(x+jy)$, where $F_n(x) = \sum\limits_{k=1}^{\infty} x_k^n$. So for the polynomial $P(x,y)$ according to [1, Theorem 1.3] we obtain that $x + jy = T_\sigma(z+jt)$. On the other hand, we can denote by $T_\sigma(x) = T_\sigma(x,0)$, $T_\sigma(y) = T_\sigma(0,y)$ and we obtain that $x + jy = T_\sigma((z,0) + j(0,t)) = T_\sigma(z) + jT_\sigma(t)$.

For us it is enough to consider $j = 1,2$. We obtain two equalities

$$x + y = T_\sigma(z) + T_\sigma(t), \quad x + 2y = T_\sigma(z) + 2T_\sigma(t),$$

which imply $x = T_\sigma(z)$, $y = T_\sigma(t)$. That is, $(x, jy) = T_\sigma(z,t)$.

Since $H^{p,i-p}(x,y) = H^{p,i-p}(z,t)$, $0 \le p \le i$ for every $i \ge 1$ it follows that $F_i(x+jy) = F_i(z+jt)$ and so $(x,y) \sim (z,t)$. $\square$

Let $H_{bvs}(\mathcal{X}^2)$ be the algebra of block-symmetric analytic functions of bounded type (that is, bounded on bounded subsets) on $\mathcal{X}^2$. This algebra is generated by polynomials $H^{1,0}, \ldots, H^{p,n-p}, \ldots, H^{0,n}, \ldots$, where $n \ge 1, 0 \le p \le n$. Let us denote by $M_{bvs}(\mathcal{X}^2)$ the specrum of algebra $H_{bvs}(\mathcal{X}^2)$.

For given $(x,y), (z,t) \in \mathcal{X}^2$,

$$(x,y) = \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right)$$

and

$$(z,t) = \left( \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right),$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$, we put

$$(x,y) \bullet (z,t) = \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{3}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the *intertwining* and the operator $\mathcal{T}_{(z,t)}$ is the *intertwining operator*. Some elementary properties of $\mathcal{T}_{(z,t)}$ was proved in [6].

Let $\mathbb{C}\{t_1, t_2\}$ be the space of all pover series over $\mathbb{C}^2$. We denote by $\mathcal{R}$ and $\mathcal{H}$ the following maps from $M_{bvs}(\mathcal{X}^2)$ into $\mathbb{C}\{t_1, t_2\}$

$$\mathcal{R}(\varphi) = \sum_{\substack{n=0 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(R^{p,n-p}),$$

and

$$\mathcal{H}(\varphi) = \sum_{\substack{n=1 \\ 0 \le p \le n}}^{\infty} t_1^p t_2^{n-p} \varphi(H^{p,n-p}).$$

Note
$$\mathcal{R}((x,y)\bullet(z,t)) = \mathcal{R}(x,y)\mathcal{R}(z,t),$$
and
$$\mathcal{H}((x,y)\bullet(z,t)) = \mathcal{H}(x,y)+\mathcal{H}(z,t),$$
where $(x,y),(z,t)\in\mathcal{X}^2$. We will prove these equalities in Theorem 4 for more general situation.

Following [3] we define the symmetric convolution.

**Definition 1.** *For any $f\in H_{bvs}(\mathcal{X}^2)$ and $\theta\in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\theta\star f$ is defined by*
$$(\theta\star f)(x,y) = \theta[\mathcal{T}_{(x,y)}(f)].$$

**Definition 2.** *For any $\varphi,\theta\in H_{bvs}(\mathcal{X}^2)'$, symmetric convolution $\varphi\star\theta$ is defined by*
$$(\varphi\star\theta)(f) = \varphi(\theta\star f) = \varphi((z,t)\mapsto\theta(\mathcal{T}_{(z,t)}f)).$$

**Theorem 3.** *For any $\varphi,\theta\in M_{bvs}(\mathcal{X}^2)$ the symmetric convolution is commutative, associative and*
$$(\varphi\star\theta)(H^{p,n-p}) = \varphi(H^{p,n-p})+\theta(H^{p,n-p}), \tag{4}$$
*where $0\le p\le n$.*

*Proof.* First we will prove the equality (4). Indead, for polynomials $H^{p,n-p}$ we have
$$\begin{aligned}(\theta\star H^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x,y)}(H^{p,n-p}))\\ &= \theta(H^{p,n-p}(x,y)+H^{p,n-p}) = H^{p,n-p}(x,y)+\theta(H^{p,n-p}).\end{aligned}$$
Therefore,
$$\begin{aligned}(\varphi\star\theta)(H^{p,n-p}) &= \varphi(H^{p,n-p}(x,y)+\theta(H^{p,n-p}))\\ &= \varphi(H^{p,n-p})+\theta(H^{p,n-p}).\end{aligned}$$

From this equality it follows the associativity and commutativity of $\varphi\star\theta\in M_{bvs}(\mathcal{X}^2)$. $\square$

Similarly to Lemma 3.1 and Proposition 8.2 in [4] (see also [12]) it is possible to show that
$$||R^{p,n-p}||\le\frac{2}{p!(n-p)!}$$
and $\mathcal{R}(\varphi)(t)$ is a function of exponential type for every fixed $\varphi\in M_{bvs}(\mathcal{X}^2)$.

**Theorem 4.** *The following identities hold*

1. $\mathcal{H}(\varphi\star\theta) = \mathcal{H}(\varphi)+\mathcal{H}(\theta)$,
2. $\mathcal{R}(\varphi\star\theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$.

*Proof.* The first statement it follows from Theorem 3. To prove the second statement we observe that

$$R^{p,n-p}((x,y)\bullet(z,t)) = \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}(z,t).$$

Thus

$$\begin{aligned}(\theta\star R^{p,n-p})(x,y) &= \theta(\mathcal{T}_{(x^1,x^2)}(R^{p,n-p}))\\ &= \theta\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)R^{p-r,n-p-(i-r)}\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x,y)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}(\varphi\star\theta)\Big(R^{p,n-p}\Big) &= \varphi\Big(\sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} R^{r,i-r}(x^1,x^2)\theta\Big(R^{p-r,n-p-(i-r)}\Big)\Big)\\ &= \sum_{\substack{i=0\\0\le p\le n\\0\le r\le i}}^{n} \varphi\Big(R^{r,i-r}\Big)\theta\Big(R^{p-r,n-p-(i-r)}\Big).\end{aligned}$$

On the other hand

$$\begin{aligned}\mathcal{R}(\varphi)\mathcal{R}(\theta) &= \sum_{\substack{i=0\\0\le k\le i}}^{\infty} t_1^k t_2^{i-k}\varphi(R^{k,i-k}) \sum_{\substack{m=0\\0\le r\le m}}^{\infty} t_1^r t_2^{m-r}\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty}\sum_{\substack{k+r=p\\i+m=n}} t_1^p t_2^{n-p}\varphi(R^{k,i-k})\theta(R^{r,m-r})\\ &= \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p}\sum_{\substack{k+r=p\\i+m=n}}\varphi(R^{k,i-k})\theta(R^{r,m-r}) = \sum_{\substack{n=0\\0\le p\le n}}^{\infty} t_1^p t_2^{n-p}(\varphi\star\theta)\Big(R^{p,n-p}\Big)\\ &= \mathcal{R}(\varphi\star\theta).\end{aligned}$$

□

**Lemma 1.** *If $\varphi=\delta_{(x,y)}$, then for every $(x,y)\in\mathcal{X}^2$ :*

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2) = \prod_{i=1}^{\infty}(1+x_it_1+y_it_2) = \sum_{n=0}^{\infty} G_n(xt_1+yt_2),$$

*where $(x_i,y_i)\in\mathbb{C}^2$, $i\ge1$ and $G_n(xt_1+yt_2) = \sum\limits_{k_1<k_2<\ldots<k_n}^{\infty}(x_{k_1}t_1+y_{k_1}t_2)\ldots(x_{k_n}t_1+y_{k_n}t_2)$ and $G_0=1$.*

*Proof.* For every $(x,y)\in\mathcal{X}^2$, the product

$$\prod_{i=1}^{\infty}(1+x_it_1+y_it_2)$$

is absolutely convergent if the series $\sum\limits_{i=1}^{\infty}(x_it_1+y_it_2)$ is absolutely convergent. Since

$$\begin{aligned}\sum_{i=1}^{\infty}|x_it_1+y_it_2| &\le \sum_{i=1}^{\infty}(|x_i||t_1|+|y_i||t_2|) = |t_1|\sum_{i=1}^{\infty}|x_i|+|t_2|\sum_{i=1}^{\infty}|y_i| \\ &\le \max\{|t_1|,|t_2|\}\Big(\sum_{i=1}^{\infty}|x_i|+\sum_{i=1}^{\infty}|y_i|\Big) \\ &\le \max\{|t_1|,|t_2|\}\sqrt{2}\Big(\sum_{i=1}^{\infty}\big(|x_i|^2+|y_i|^2\big)^{1/2}\Big)<\infty,\end{aligned}$$

we obtain that $\prod\limits_{i=1}^{\infty}(1+x_it_1+y_it_2)$ is absolutely convergent, and so the product is convergent as well. Since for every $1\le m<\infty$ will be performed the equality

$$\sum_{\substack{n=0\\0\le p\le n}}^{m} t_1^pt_2^{n-p}\delta_{(x,y)}(R^{p,n-p})=\prod_{i=1}^{m}(1+x_it_1+y_it_2)$$

and series and product are convergent, we obtain that

$$\mathcal{R}(\delta_{(x,y)})(t_1,t_2)=\prod_{i=1}^{\infty}(1+x_it_1+y_it_2).$$

It is known from Combinatorics [8] that $\sum\limits_{n=0}^{\infty}t^nG_n(x)=\prod\limits_{i=1}^{\infty}(1+x_it_1)$ for every $x\in c_{00}$, where $G_n(x)=\sum\limits_{k_1<\ldots<k_n}^{\infty}x_{k_1}\ldots x_{k_n}$ is the basis of elementary symmetric polynomials of algebra $\mathcal{H}_{bs}(\ell_1)$. Since it is true for every $x\in\ell_1$,

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(xt_1+yt_2) &= \sum_{n=0}^{\infty}(t_1t_2)^nG_n\Big(\frac{x}{t_2}+\frac{y}{t_1}\Big)=\prod_{i=1}^{\infty}\Big(1+\Big(\frac{x_i}{t_2}+\frac{y_i}{t_1}\Big)t_1t_2\Big) \\ &= \prod_{i=1}^{\infty}(1+x_it_1+y_it_2).\end{aligned}$$

□

Now we show that the spectrum of the algebra of block-symmetric analytic functions of bounded type on $\mathcal{X}^2$ does not coincide of point evaluation functionals.

**Example 1.** *Let $k,l$ are same fixed nonzero complex numbers. Now we consider the sequence of elements*

$$e_1(k,l)=\Big(\begin{pmatrix}k\\l\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix},\ \ldots,\ \begin{pmatrix}0\\0\end{pmatrix},\ \ldots\Big),$$
$$e_2(k,l)=\Big(\begin{pmatrix}0\\0\end{pmatrix},\ \begin{pmatrix}k\\l\end{pmatrix},\ \ldots,\ \begin{pmatrix}0\\0\end{pmatrix},\ \ldots\Big),$$
$$\cdots\cdots\cdots\cdots$$
$$e_n(k,l)=\Big(\begin{pmatrix}0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\end{pmatrix},\ \ldots,\ \begin{pmatrix}k\\l\end{pmatrix},\ \ldots\Big),$$
$$\cdots\cdots\cdots\cdots$$

*in $\mathcal{X}^2$ and for every $n$ put*

$$v_n(k,l) = \frac{1}{n}(e_1(k,l) + e_2(k,l) + \ldots + e_n(k,l)) \in \mathcal{X}^2.$$

*Then $\delta_{v_n(k,l)}(H^{0,1}) \to l$, $\delta_{v_n(k,l)}(H^{1,0}) \to k$, $\delta_{v_n(k,l)}(H^{p,i-p}) \to 0$ as $n \to \infty$ for every $1 \le k \le i$, where $1 \le p \le i$. By the reletive compactness of bounded subset of $M_{bvs}(\mathcal{X}^2)$ there is an accumulation point $\varphi_{(k,l)}$ of the sequence $\delta_{v_n(k,l)}$, such that $\varphi_{(k,l)}(H^{0,1}) = l$, $\varphi_{(k,l)}(H^{1,0}) = k$, $\varphi_{(k,l)}(H^{p,i-p}) = 0$ for all $1 \le i \le m$, where $1 \le p \le i$. From Theorem 2 it follows that there is no poit $(x,y) \in \mathcal{X}^2$, such that $\delta_{(x,y)} = \varphi_{(k,l)}$. Indeed, if such a point exists, then $(x,y) \sim (0,0)$. Therefore $\delta_{v_n(k,l)}(H^{0,1}) = \delta_{v_n(k,l)}(H^{1,0}) = 0$, but we have that $\delta_{v_n(k,l)}(H^{0,1}) = l$, $\delta_{v_n(k,l)}(H^{1,0}) = k$.*

**Example 2.** *Let $\varphi_{(k,l)}$ be as in Example 1. We know that $\mathcal{H}(\varphi_{(k,l)}) = k + l$. To find $\mathcal{R}(\varphi_{(k,l)})$ note that*

$$R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{n^p n^{s-p}} \binom{n}{s} \binom{s}{p},$$

*hence*

$$\varphi(R^{p,s-p}) = \lim_{n\to\infty} R^{p,s-p}(v_n(k,l)) = \frac{k^p l^{s-p}}{p!(s-p)!}$$

*and so*

$$\begin{aligned}\mathcal{R}(\varphi_{(k,l)})(t_1,t_2) &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} t_1^p t_2^{s-p} \varphi(R^{p,s-p}) \\ &= \lim_{n\to\infty} \sum_{\substack{s=0\\0\le p\le s}}^{n} \frac{(kt_1)^p (lt_2)^{s-p}}{p!(s-p)!} = e^{kt_1+lt_2}.\end{aligned}$$

**Theorem 5.** *The invertible elements of semigroup $(M_{bvs}(\mathcal{X}^2), \star)$ are functionals only of the form $\varphi_{(k,l)} = \mathcal{R}(\varphi_{(k,l)})(t_1,t_2) = e^{kt_1+lt_2}$.*

*Proof.* Since by Theorem 4 $\mathcal{R}(\varphi \star \theta) = \mathcal{R}(\varphi)\mathcal{R}(\theta)$, $\varphi_{(-k,-l)}$ is inverse to $\varphi_{(k,l)}$. In the other hand, if $\varphi$ is invertible and $\psi = \varphi^{-1}$, then $\mathcal{R}(\psi) = \frac{1}{\mathcal{R}(\varphi)(t_1,t_2)}$ is an entire function of exponential type and so has no zeros. So we have that $\mathcal{R}(\varphi)(t_1,t_2) = e^{kt_1+lt_2}$ for some complex numbers $k,l$. □

**Corollary 1.** *Let $\Phi$ be a homomorphism on the subspace of block-symmetric polynomials in $H_{bvs}(\mathcal{X}^2)$ to itself such that $\Phi(H^{p,k-p}) = -H^{p,k-p}$ for every $p,k$. Then $\Phi$ is discontinuous.*

*Proof.* If $\Phi$ is continuous it may be extended to continuous homomorphism $\tilde{\Phi}$ of $H_{bvs}(\mathcal{X}^2)$. Then for $(x,y) \in \mathcal{X}^2$

$$H^{p,k-p}(x,y) + \Phi(H^{p,k-p})(x,y) = 0 \tag{5}$$

for all $p,k$. Note that this equality is true for

$$(x_0,y_0) = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right).$$

Let us denote $\psi = \delta_{(x_0,y_0)} \circ \tilde{\Phi}$. From the continuity of homomorphism $\tilde{\Phi}$ we have, that $\psi \in M_{bvs}(\mathcal{X}^2)$. From equality (5) we have, that $\delta_{(x,y)} \star \psi = \delta_{(0,0)}$, $\psi = \delta^{-1}_{(x_0,y_0)}$. According to the Theorem 5 $\delta_{(x_0,y_0)}$ is not invertible. □

Let $f(z)$ be an entire function of many variable. We will say that $f(z)$, where $z \in \mathbb{C}^n$, has "plane" zeros if the set of zeros is

$$Z_f = \Big\{ z \in \mathbb{C}^n : f(z) = 0 \Big\} = \bigcup_{k=1}^{\infty} H_k,$$

where $H_k = \{z : \langle z, a^k |a^k|^{-2} \rangle = 1\}$ is hyperplane in $\mathbb{C}^n$. Here $a^k \in \mathbb{C}^n$ are feets of perpendiculars dropped from the origin onto zeros hyperplanes $H_k$ of the function $f(z)$ (see [10]).

**Theorem 6.** *Let $\varphi$ be a character such that $\mathcal{R}(\varphi)$ is a polynomial. Then $R(\varphi)$ have a plane zeros, that is $Ker R(\varphi)$ consists of one-codimensional linear subspaces.*

*Proof.* Let us denote $\Lambda_{t_1 t_2}(G_n) = G_n(xt_1 + yt_2)$. Now we consider the equation $\sum\limits_{n=0}^{m} \lambda^n \varphi(\Lambda_{t_1 t_2}(G_n)) = 0$ with $m$ solutions $z_k$, $1 \le k \le m$. Hence $\prod\limits_{i=1}^{m} (1 + z_k \lambda) = 0$. Obviously, every solution $z_k$ can be represented as $z_k = x_k t_1 + y_k t_1$, where $x_k, y_k$ are indeterminants and $t_1, t_2$ are some complex numbers. If we take $t_1 = 1, t_2 = 0$ and $t_1 = 2, t_2 = 1$, then can fined $x_k, y_k$. So we have the system of $2m$ equation and $2m$ indeterminants $x_k, y_k$, $1 \le k \le m$. The solutions of that system are $x_k = z_k$, $y_k = -z_k$, $1 \le k \le m$. Hence $x_k, y_k$ can be clearly define. If $\lambda = 1$, then we obtain the equality

$$\mathcal{R}(\varphi)(t_1, t_2) = \sum_{n=0}^{m} \varphi(\Lambda_{t_1 t_2}(G_n)) = \prod_{i=1}^{m} (1 + x_i t_1 + y_i t_2) = 0.$$

Hence $\varphi$ has plane zeros. □

According to the analog of Hadamard's Theorem [10] the function $\mathcal{R}(\varphi)(t_1, t_2)$ with plane zeros is of the form

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{n} \Big(1 - \big(t_1 \tfrac{\bar{a}_1^k}{|a^k|^2} + t_2 \tfrac{\bar{a}_2^k}{|a^k|^2}\big)\Big),$$

where $\{(a_1^k, a_2^k)\}$ are the zeros of $\mathcal{R}(\varphi)(t_1, t_2)$, $P(t_1, t_2)$ is analytic polynomial and we have

$$\sum_{k=1}^{n} \tfrac{1}{|a_k|} < \infty.$$

According to the Lemma 1

$$\mathcal{R}(\delta_{(x,y)})(t_1, t_2) = \prod_{i=1}^{m} (1 + x_i t_1 + y_i t_2),$$

and so the zeros of $\mathcal{R}(\delta_{(x,y)})(t_1, t_2)$ are

$$a_1^k = -\frac{\overline{x}_k}{|x_k|^2 + |y_k|^2}, \qquad a_2^k = -\frac{\overline{y}_k}{|x_k|^2 + |y_k|^2}.$$

On the other hand, if $f(t_1, t_2)$ is the function of the exponential type with plane zeros, then it can be represented as

$$\mathcal{R}(\varphi)(t_1, t_2) = \exp(P(t_1, t_2)) \prod_{i=1}^{\infty} \Big(1 - \big(t_1 \tfrac{\bar{a}_1^k}{|a^k|^2} + t_2 \tfrac{\bar{a}_2^k}{|a^k|^2}\big)\Big),$$

if

$$\sum_{k=1}^{\infty} \tfrac{1}{|a_k|} < \infty.$$

So for $\varphi \in M_{bvs}(\mathcal{X}^2)$, which we can represanted as $\varphi = \varphi_{(k,l)} \star \delta_{(x,y)}$, where $(x, y) \in \mathcal{X}^2$, $(x_k, y_k) = -\big(\frac{\bar{a}_1^k}{|a_k|^2}, \frac{\bar{a}_2^k}{|a_k|^2}\big)$ and $\varphi_{(k,l)}$ was defined in Example 1, we have that

$$\mathcal{R}(\varphi)(t_1, t_2) = f(t_1, t_2).$$

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У статті описано симетричну згортку характерів алгебри блочно-симетричних аналітичних фнкцій обмеженого типу на $\ell_1$-сумі простору $\mathbb{C}^2$. Авторами показано, що спектр такої алгебри не не збігається з множиною класів еквівалентності функціоналів значенні в точках, описано характери такої алгебри, як функції експоненціального типу з "плоскими" нулями.

*Ключові слова і фрази:* алгебраїчний базис, блочно-симетичні поліноми, блочно-симетричні аналітичні фукції, спектр, симетричний зсув, симетрична згортка.